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Conditional Association and Spin Systems

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Abstract. A 1977 theorem of T. Harris states that an attractive spin system preserves the class of associated probability measures. We study analogues of this result for measures that satisfy various conditional positive correlations properties. In particular, we show that a spin system preserves measures satisfying the FKG lattice condition (essentially) if and only if distinct spins flip independently. The downward FKG property, which has been useful recently in the study of the contact process, lies between the properties of lattice FKG and association. We prove that this property is preserved by a spin system if the death rates are constant and the birth rates are additive (e.g., the contact process), and prove a partial converse to this statement. Finally, we introduce a new property, which we call downward conditional association, which lies between the FKG lattice condition and downward FKG, and find essentially necessary and sufficient conditions for this property to be preserved by a spin system. This suggests that the latter property may be more natural than the downward FKG property.

1. Introduction.

Correlation inequalities have been used frequently in probability theory and statistical physics. In this paper, we will consider correlation inequalities for probability measures on $\{0,1\}^S$, where S is a finite set, and their connection with certain continuous time Markov chains on $\{0,1\}^S$ that are known as spin systems. The key definition is the following: the probability measure μ has positive correlations, or is associated, if

$$\int fg\,d\mu \geq \int f\,d\mu \int g\,d\mu \tag{1.1}$$

for all increasing functions f and g on $\{0,1\}^S$. Two results proved in the 1970's provide convenient ways to show that a probability measure on $\{0,1\}^S$ is associated:

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Theorem 1.1. *(FKG Theorem) Suppose μ is a probability measure on $\{0,1\}^S$ that assigns strictly positive mass to every point in $\{0,1\}^S$, and satisfies*

$$\mu(\eta \wedge \zeta)\mu(\eta \vee \zeta) \geq \mu(\eta)\mu(\zeta) \tag{1.2}$$

for all $\eta, \zeta \in \{0,1\}^S$. Then μ is associated.

Assumption (1.2) is called the FKG lattice condition, or the strong FKG condition.

For the statement of the second of these results, recall that a spin system is a continuous time Markov chain η_t on $\{0,1\}^S$ in which transitions can occur at only one site at a time. Let $\beta(x,\eta)$ and $\delta(x,\eta)$ be the rates at which the transitions $0 \to 1$ (births) and $1 \to 0$ (deaths) occur at site x if the configuration is η. (The functions $\beta(x,\eta)$ and $\delta(x,\eta)$ do not depend on $\eta(x)$.) The spin system is said to be attractive if $\beta(x,\eta)$ is an increasing function of η and $\delta(x,\eta)$ is a decreasing function of η for each x. Let $S(t)$ be the semigroup for the spin system. It acts on functions and measures in the following way:

$$S(t)f(\eta) = E^\eta f(\eta_t); \qquad \int f d[\mu S(t)] = \int S(t) f d\mu.$$

Theorem 1.2. *(Harris) If the spin system is attractive, then $\mu S(t)$ is associated whenever μ is.*

See pages 78 and 80 of Liggett (2005) for proofs of Theorems 1.1 and 1.2. The second of these implies that the stationary measure of an irreducible attractive spin system is associated. Such a measure may or may not satisfy (1.2) (It does satisfy (1.2) if the spin system is reversible and attractive, but this is a very special situation.) A major advantage of Theorem 1.2 over Theorem 1.1 is that one does not need an explicit expression for the stationary distribution in order to check that it is associated.

More recently, some conditional forms of positive correlation inequalities have been proved (Belitsky et al. (1997) and van den Berg et al. (2006a)) and applied (Liggett and Steif (2006)). We are primarily concerned here with conditional versions of Theorem 1.2. First, we state a converse to Theorem 1.2. It will be proved in Section 2.

Theorem 1.3. *Suppose a spin system has the property that $\mu S(t)$ is associated whenever μ is. Then the spin system is attractive.*

An easy consequence of Theorem 1.1 is that (1.2) is equivalent to the property that not only μ, but also all measures obtained from μ by conditioning on the values of η at any set of sites in S, are associated. (We assume here and in the sequel that all measures considered assign strictly positive probability to the events on which one is conditioning.) Thus a first natural question is, for which spin systems is it the case that $\mu S(t)$ satisfies (1.2) whenever μ does? The next result says that this occurs essentially only when the coordinate processes $\{\eta_t(x), x \in S\}$ are independent. Therefore, we see that one should not condition on too much information if one hopes to have preservation under the semigroup of some conditional positive correlations property for interesting spin systems. Say that the spin system has independent flips if for each x, $\beta(x,\eta)$ and $\delta(x,\eta)$ do not depend on η.

Theorem 1.4. *(a) Suppose that the spin system has independent flips. Then $\mu S(t)$ satisfies (1.2) whenever μ does.*

(b) Suppose that $\mu S(t)$ satisfies (1.2) *whenever μ does, and that S has at least four points. Then the spin system has independent flips.*

Theorem 1.4 will be proved in Section 3. The statement in part (b) is trivially true if S is a singleton, but is false if S has either two or three points. If S has two points, (1.2) is equivalent to association, so that by Theorems 1.2 and 1.3, property (1.2) is preserved by the semigroup if and only if the spin system is attractive – it need not have independent flips. If S has three points, the following is an example of a spin system that preserves (1.2) without having independent flips:
$\beta(x,\eta) = \delta(x,\eta) = 0$ for all η, except that

$$\beta(x,\eta) = 1 \text{ if } \eta \equiv 1 \quad \text{and} \quad \delta(x,\eta) = 1 \text{ if } \eta \equiv 0.$$

To check this, note that $\mu S(t)(\eta) = e^{-t}\mu(\eta)$, except when $\eta \equiv 0$ or $\eta \equiv 1$, and that $\mu S(t)(\eta)$ is increasing in t if $\eta \equiv 0$ or $\eta \equiv 1$. Of course, this example is reducible. To construct an irreducible example, simply add a constant to all the rates. Since the process with constant rates preserves (1.2) by Theorem 1.4(a), the modified process preserves (1.2) as well, by Proposition 1.8 below.

Following Liggett and Steif (2006), we will say that μ is downward FKG if for any $A \subset S$, the conditional measure $\mu\{\cdot \mid \eta \equiv 0 \text{ on } A\}$ is associated; i.e., conditioning is allowed only on 0's, not on 1's. This property lies between (1.2), where one is allowed to condition on any configuration on A, and association, where one is not allowed to condition at all. Van den Berg et al. (2006b) proved that the distribution of the contact process at time t is downward FKG provided that the initial distribution is deterministic. (When we refer to such properties of measures on $\{0,1\}^S$ for infinite S, we mean that all projections on finite subsets of S have the property.) This property was used by Liggett and Steif (2006) to show that the stationary distribution of the contact process on Z (and as a consequence, on many other graphs) dominates a nontrivial product measure. (The contact process stationary distribution does not satisfy the FKG lattice condition – see Liggett (1994) and van den Berg et al. (2006a), where it is shown that even conditioning on $\eta(x) = 1$ at a single site x destroys the property of association.)

Another example of a measure that is downward FKG but does not satisfy the FKG lattice condition is the following. Let π be a permutation of $\{1,...,n\}$ that is chosen uniformly at random. Define $\eta(i)$ to be the indicator of the event that $\pi(i) \neq i$, and let μ be the distribution of η. Fishburn et al. (1988) showed that μ is associated, even though it does not satisfy the FKG lattice condition. Since $\mu\{\cdot \mid \eta \equiv 0 \text{ on } A\}$ when A is of size k corresponds to the measure μ for random permutations of $n-k$ points, it follows that μ is downward FKG.

Our next result is an analogue of Theorems 1.2 and 1.3 on the one hand, and Theorem 1.4 on the other, for preservation of the downward FKG property. It will be proved in Section 4. For its statement, we need the following definition. The birth rates $\beta(x,\eta)$ are said to be additive if they can be written in the form

$$\beta(x,\eta) = \sum_{A \subset S} c(x,A) 1_{\{\eta \not\equiv 0 \text{ on } A\}} \tag{1.3}$$

with $c(x,A) \geq 0$ for all x, A. The contact process of course has additive birth rates and constant death rates.

Theorem 1.5. *(a) Suppose that the spin system satisfies*

$$\delta(x,\eta) \textit{ does not depend on } \eta \textit{ for each } x \tag{1.4}$$

and the birth rates $\beta(x,\eta)$ are additive. Then $\mu S(t)$ is downward FKG whenever μ is.

(b) Suppose that $\mu S(t)$ is downward FKG whenever μ is. Then $\delta(x,\eta)$ is constant on the set $\{\eta : \eta \not\equiv 0\}$ for each x.

(c) Suppose that all transition rates are zero, except $\beta(u,\eta)$ for a particular $u \in S$ and arbitrary η. If $\mu S(t)$ is downward FKG whenever μ is, then $\beta(u,\eta)$ is increasing in η, and satisfies

$$\beta(u,\eta\vee\zeta)+\beta(u,\eta\wedge\zeta)\le\beta(u,\eta)+\beta(u,\zeta) \tag{1.5}$$

for all η and ζ.

Note that (1.5) is an additive form of (1.2), and is satisfied whenever $\beta(u,\cdot)$ is additive. However, there is a large gap between (1.5) and additivity – we do not have a necessary and sufficient condition for preservation of the downward FKG property. Part (c) of the theorem does say that monotonicity of the birth rates alone is not sufficient.

The fact that there is significant difference between our necessary conditions and sufficient conditions for preservation of the downward FKG property suggests that this property may not be the most natural one to consider. We therefore introduce a new concept. The probability measure μ will be called downward conditionally associated (DCA) if for every strictly positive *decreasing* function h on $\{0,1\}^S$ that satisfies the lattice condition

$$h(\eta\vee\zeta)h(\eta\wedge\zeta)\ge h(\eta)h(\zeta), \tag{1.6}$$

the measure

$$\mu_h(d\eta)=h(\eta)\mu(d\eta)\Big/\int h\,d\mu$$

is associated. The implications among these properties are now:

$$\text{FKG lattice}\quad\Rightarrow\quad\text{DCA}\quad\Rightarrow\quad\text{downward FKG}\quad\Rightarrow\quad\text{association.} \tag{1.7}$$

To check the first, note that as a result of (1.6), μ_h satisfies the FKG lattice condition whenever μ does. For the second, given $A\subset S$, apply the association of μ_h for

$$h(\eta)=\prod_{x\in A}\big[1+\epsilon-\eta(x)\big],$$

and pass to the limit as $\epsilon\downarrow 0$.

One should of course ask whether the implications in (1.7) are strict. If S has two points, it is easy to check that they are all equivalences, and are equivalent to $\mu(11)\mu(00)\ge\mu(10)\mu(01)$. In Section 6, we will give necessary and sufficient conditions for each of the four properties appearing in (1.7) when S has three points. A consequence of this is that the first and third implications are strict in this case, while the middle one is an equivalence. We do not know whether the middle implication is strict for larger S. While this is an interesting question, it is in a sense not too important from the point of view of this paper. Suppose, for example, that one wanted to show that the contact process invariant measure is DCA. One approach is to use Theorem 1.7 below. Another would be to use Theorem 1.5, and then show that downward FKG implies DCA, if this is the case. Given the difficulty of showing this implication when S has three points, it seems clear that one should use Theorem 1.7 instead.

Remark 1.6. *The monotonicity assumption on h in the definition of DCA is very important. To see this, consider the fact (which was pointed out to the author by L. Chayes) that if μ_h is associated for every positive h that satisfies* (1.6)*, then μ satisfies the FKG lattice condition. To see this, suppose it does not. Then one can condition μ on the event $\{\eta : \eta \equiv \zeta \text{ on } A\}$ for some $A \subset S$ and some $\zeta \in \{0,1\}^A$ and obtain a measure that is not associated. It follows that μ_h is not associated if h is some positive perturbation of $1_{\{\eta:\eta\equiv\zeta \text{ on } A\}}$ satisfying* (1.6).

We are now in a position to state the following analogue of Theorem 1.5 for the DCA property, but with conditions that are essentially necessary and sufficient. Its proof will be given in Section 5.

Theorem 1.7. *(a) Suppose that the death rates of the spin system satisfy* (1.4) *and the birth rates are increasing and satisfy* (1.5)*. Then $\mu S(t)$ is DCA whenever μ is.*

(b) Suppose that $\mu S(t)$ is DCA whenever μ is. Then $\delta(x,\eta)$ is constant on the set $\{\eta : \eta \not\equiv 0\}$ for each x.

(c) Suppose that all transition rates are zero, except $\beta(u,\eta)$ for a particular $u \in S$. If $\mu S(t)$ is DCA whenever μ is, then $\beta(u,\eta)$ is increasing in η, and satisfies (1.5) *for all η and ζ.*

One consequence of Theorem 1.7 is that the upper invariant measure of the contact process has the presumably stronger DCA property. Another advantage over Theorem 1.5 is that it can be used to prove this property for the stationary distributions of a larger class of spin systems.

Finally. we state a result that will be used to simplify the Proofs of Theorems 1.5 and 1.7.

Proposition 1.8. *Suppose $\mathcal{L}_1, \mathcal{L}_2, S_1(t)$ and $S_2(t)$ are the generators and semigroups for two Markov chains on $\{0,1\}^S$, and that $S(t)$ is the semigroup with generator $\mathcal{L}_1 + \mathcal{L}_2$. If $\mathcal{C}$ is a closed set of functions or measures on $\{0,1\}^S$ and $S_i(t)$ maps $\mathcal{C}$ into itself for $i = 1,2$, then $S(t)$ also maps $\mathcal{C}$ into itself.*

Proof. This is an immediate consequence of the Trotter product formula (Ethier and Kurtz (1986), page 33):

$$S(t) = \lim_{n\to\infty} \left[S_1(t/n)S_2(t/n)\right]^n.$$

□

Following completion of the present paper, we learned of some closely related work. We thank Professors Chen and van den Berg for mentioning these references.

(a) Our Theorem 1.3 is Corollary 2.5 in Sun (1994). A version of this result for diffusion processes was proved by Chen and Wang (1993). Note that in Proposition 2.1 below, we prove a stronger form of Theorem 1.3.

(b) In the revision of van den Berg et al. (2006b), the authors added a result (Theorem 3.5) that is part (a) of Theorem 1.5, restricted to the contact process. Our result applies to a broader class of spin systems, and we believe that our proof is significantly simpler.

2. A converse to Harris' Theorem.

In this section we prove a stronger form of Theorem 1.3 which will also be useful in later sections. It implies Theorem 1.3 because every product measure is associated. This statement is an immediate consequence of either Theorem 1.1 or Theorem 1.2. We will use the following notation: If $\eta \in \{0,1\}^S$ and $x_1, x_2, \ldots$ are distinct elements of S, then $\eta_{x_1,x_2,\ldots}$ is the configuration with

$$\eta_{x_1,x_2,\ldots}(x) = \begin{cases} 1-\eta(x) & \text{if } x = x_i \text{ for some } i \\ \eta(x) & \text{otherwise.} \end{cases}$$

Proposition 2.1. *Suppose the semigroup for a spin system has the property that $\mu S(t)$ is associated for every product measure μ. Then the spin system is attractive.*

Proof. Fix distinct $x, y \in S$, and let γ be any configuration with $\gamma(x) = \gamma(y) = 0$. Let μ_ϵ be the product measure with marginals

$$\mu_\epsilon\{\eta : \eta(z) = 1\} = \begin{cases} \rho & \text{if } z = x, \\ \lambda & \text{if } z = y, \\ \epsilon & \text{if } z \neq x, y, \gamma(z) = 0 \\ 1-\epsilon & \text{if } z \neq x, y, \gamma(z) = 1, \end{cases}$$

where $0 < \rho, \lambda, \epsilon < 1$. By assumption, $\mu_\epsilon S(t)$ is associated for all t. Applying the definition of association to the increasing functions $f(\eta) = \eta(x)$ and $g(\eta) = \eta(y)$ gives

$$\begin{aligned} &\mu_\epsilon S(t)\{\eta(x) = 1, \eta(y) = 1\}\mu_\epsilon S(t)\{\eta(x) = 0, \eta(y) = 0\} \\ &-\mu_\epsilon S(t)\{\eta(x) = 1, \eta(y) = 0\}\mu_\epsilon S(t)\{\eta(x) = 0, \eta(y) = 1\} \geq 0\ . \end{aligned} \tag{2.1}$$

Since the left side of (2.1) is zero at $t = 0$, its derivative is nonnegative at $t = 0$. Differentiating (2.1) with respect to t, setting $t = 0$, and then letting $\epsilon \downarrow 0$ leads to the following inequality:

$$\begin{aligned} &\rho\lambda\Big[\rho(1-\lambda)\delta(x,\gamma) + (1-\rho)\lambda\delta(y,\gamma) - (1-\rho)(1-\lambda)\beta(x,\gamma) - (1-\rho)(1-\gamma)\beta(y,\gamma)\Big] \\ &\quad +(1-\rho)(1-\lambda)\Big[(1-\rho)\lambda\beta(x,\gamma_y) + \rho(1-\lambda)\beta(y,\gamma_x) - \rho\lambda\delta(x,\gamma_y) - \rho\lambda\delta(y,\gamma_x)\Big] \\ &\geq \rho(1-\lambda)\Big[(1-\rho)(1-\lambda)\beta(y,\gamma) + \rho\lambda\delta(x,\gamma_y) - (1-\rho)\lambda\beta(x,\gamma_y) - (1-\rho)\lambda\delta(y,\gamma)\Big] \\ &\quad +(1-\rho)\lambda\Big[(1-\rho)(1-\lambda)\beta(x,\gamma) + \rho\lambda\delta(y,\gamma_x) - \rho(1-\lambda)\delta(x,\gamma) - \rho(1-\lambda)\beta(y,\gamma_x)\Big]. \end{aligned}$$

Dividing by $\rho(1-\rho)\lambda(1-\lambda)$ and collecting like terms gives:

$$\begin{aligned} &\frac{\beta(x,\gamma_y) - \beta(x,\gamma)}{\rho} + \frac{\beta(y,\gamma_x) - \beta(y,\gamma)}{\lambda} \\ + \quad &\frac{\delta(x,\gamma) - \delta(x,\gamma_y)}{1-\rho} + \frac{\delta(y,\gamma) - \delta(y,\gamma_x)}{1-\lambda} \geq 0. \end{aligned}$$

Now let respectively $\rho \to 0, \lambda \to 0, \rho \to 1, \lambda \to 1$ to conclude that

$$\beta(x,\gamma) \leq \beta(x,\gamma_y), \quad \beta(y,\gamma) \leq \beta(y,\gamma_x), \quad \delta(x,\gamma_y) \leq \delta(x,\gamma), \quad \delta(y,\gamma_x) \leq \delta(y,\gamma).$$

Since this is true for all x, y, γ satisfying $\gamma(x) = \gamma(y) = 0$, it follows that the spin system is attractive. □

3. Preservation of the FKG lattice condition.

In this section, we prove Theorem 1.4. It will be convenient to recall that (1.2) holds for all η, ζ if and only if it holds whenever η and ζ differ at exactly two sites.

Proof of Theorem 1.4(a): Let $P_t(\eta,\gamma)$ be the transition probabilities for the spin system. Then

$$\mu S(t)(\gamma)=\sum_{\eta}\mu(\eta)P_t(\eta,\gamma). \tag{3.1}$$

For each $z\in S$, let $p_t(z,0,0),p_t(z,0,1),p_t(z,1,0),p_t(z,1,1)$ be the transition probabilities for the two state Markov chain that has transitions $0\to1$ and $1\to0$ at rates $\beta(z,\eta)$ and $\delta(z,\eta)$ respectively. (Recall that we are assuming that these rates do not depend on η.) Then

$$P_t(\eta,\gamma)=\prod_{z\in S}p_t(z,\eta(z),\gamma(z)). \tag{3.2}$$

Fix two distinct sites $x,y\in S$ and let γ be a configuration that satisfies $\gamma(x)=\gamma(y)=0$. We must show that if μ satisfies (1.2), then

$$\mu S(t)(\gamma_{x,y})\mu S(t)(\gamma)-\mu S(t)(\gamma_x)\mu S(t)(\gamma_y)\geq 0. \tag{3.3}$$

Using (3.1), the left side of (3.3) can be written as

$$\begin{aligned}\frac{1}{2}\sum_{\eta,\zeta}\mu(\eta)\mu(\zeta)\Big[&P_t(\eta,\gamma_{x,y})P_t(\zeta,\gamma)+P_t(\zeta,\gamma_{x,y})P_t(\eta,\gamma)\\ &-P_t(\eta,\gamma_x)P_t(\zeta,\gamma_y)-P_t(\zeta,\gamma_x)P_t(\eta,\gamma_y)\Big].\end{aligned} \tag{3.4}$$

Using (3.2), the expression in brackets in (3.4) becomes (after some cancellation)

$$\prod_{z\neq x,y}p_t(z,\eta(z),\gamma(z))p_t(z,\zeta(z),\gamma(z))\left[f(\eta)-f(\zeta)\right]\left[g(\eta)-g(\zeta)\right],$$

where

$$f(\eta)=p_t(x,\eta(x),1)\quad\text{and}\quad g(\eta)=p_t(y,\eta(y),1).$$

Note that f and g are increasing functions, so that by Theorem 1.1 applied to the measure ν, where

$$\nu(\eta)=c\mu(\eta)\prod_{z\neq x,y}p_t(z,\eta(z),\gamma(z)),$$

(3.4) is nonnegative. (Here c is a normalizing constant.) The measure ν satisfies (1.2) since μ does, and

$$\frac{\nu(\eta\wedge\zeta)\nu(\eta\vee\zeta)}{\nu(\eta)\nu(\zeta)}=\frac{\mu(\eta\wedge\zeta)\mu(\eta\vee\zeta)}{\mu(\eta)\mu(\zeta)}.$$

□

We will isolate the main part of the proof of Theorem 1.4(b) in the following proposition, since it will also be useful in Sections 4 and 5.

Proposition 3.1. *Suppose that $\mu S(t)$ is downward FKG whenever μ satisfies* (1.2). *Then $\delta(x,\eta)$ is constant on the set $\{\eta:\eta\not\equiv 0\}$ for each $x\in S$.*

Proof. Any product measure satisfies (1.2), and downward FKG implies association, so we can apply Proposition 2.1 to conclude that $\delta(x,\eta)$ is decreasing in η.

Now take three distinct sites x, y, z and let μ_ϵ be the probability measure on $\{0,1\}^S$ with respect to which $\{\eta(w), w \in S\backslash\{y,z\}\}$ are independent with $\mu_\epsilon\{\eta : \eta(w) = 1\} = \frac{1}{2}$ and independently of these, $(\eta(y), \eta(z))$ takes the following values:

$$\begin{array}{ll} (1,1) & \text{with probability } 1-3\epsilon \\ (0,1) & \text{with probability } \epsilon \\ (1,0) & \text{with probability } \epsilon \\ (0,0) & \text{with probability } \epsilon. \end{array}$$

This measure satisfies (1.2) if $4\epsilon \le 1$, which we now assume. Therefore $\mu_\epsilon S(t)$ is downward FKG for all $t \ge 0$ by assumption.

We will use the shorthand $\nu(abc)$ to mean $\nu\{\eta : \eta(x) = a, \eta(y) = b, \eta(z) = c\}$. The quantity

$$\mu_\epsilon S(t)(110)\mu_\epsilon S(t)(000) - \mu_\epsilon S(t)(100)\mu_\epsilon S(t)(010) \tag{3.5}$$

is zero at $t = 0$, and is nonnegative for $t \ge 0$ since $\mu_\epsilon S(t)$ is downward FKG. Therefore, its derivative is nonnegative at $t = 0$. To write down this derivative, let

$$\begin{aligned} \delta_{abc}(x) &= E_{\mu_\epsilon}[\delta(x,\cdot) \mid \eta(x) = a, \eta(y) = b, \eta(z) = c], \\ \delta_{abc}(y) &= E_{\mu_\epsilon}[\delta(y,\cdot) \mid \eta(x) = a, \eta(y) = b, \eta(z) = c], \\ \delta_{abc}(z) &= E_{\mu_\epsilon}[\delta(z,\cdot) \mid \eta(x) = a, \eta(y) = b, \eta(z) = c], \end{aligned}$$

with $\beta_{abc}(x), \beta_{abc}(y)$ and $\beta_{abc}(z)$ defined similarly. Then the derivative of (3.5) at $t = 0$ is

$$\begin{aligned} &\mu(110)\Big[\mu(100)\delta_{100}(x) + \mu(010)\delta_{010}(y) + \mu(001)\delta_{001}(z) - \mu(000)[\beta_{000}(x) + \beta_{000}(y) + \beta_{000}(z)]\Big] \\ &+\mu(000)\Big[\mu(010)\beta_{010}(x) + \mu(100)\beta_{100}(y) + \mu(111)\delta_{111}(z) - \mu(110)[\delta_{110}(x) + \delta_{110}(y) + \beta_{110}(z)]\Big] \\ &-\mu(100)\Big[\mu(110)\delta_{110}(x) + \mu(000)\beta_{000}(y) + \mu(011)\delta_{011}(z) - \mu(010)[\beta_{010}(x) + \delta_{010}(y) + \beta_{010}(z)]\Big] \\ &-\mu(010)\Big[\mu(000)\beta_{000}(x) + \mu(110)\delta_{110}(y) + \mu(101)\delta_{101}(z) - \mu(100)[\delta_{100}(x) + \beta_{100}(y) + \beta_{100}(z)]\Big], \end{aligned}$$

where we have omitted the subscript ϵ. Dividing this by ϵ and letting $\epsilon \downarrow 0$ yields

$$\delta_{111}(z) \ge \delta_{011}(z).$$

In other words,

$$\int_{\{\eta(x)=1\}} \delta(z,\eta)\, d\mu_0 \ge \int_{\{\eta(x)=0\}} \delta(z,\eta)\, d\mu_0.$$

Since $\delta(z,\eta)$ is decreasing in η, it follows that $\delta(z,\eta) = \delta(z,\eta_x)$ for all η such that $\eta(x) = 0, \eta(y) = 1$. Letting x and y vary, we see that $\delta(z,\eta)$ is constant on $\{\eta : \eta \not\equiv 0\}$ as required. □

Proof of Theorem 1.4(b): Since (1.2) implies downward FKG, Proposition 3.1 can be applied to conclude that $\delta(z,\eta)$ is constant on $\{\eta : \eta \not\equiv 0\}$. Interchanging the roles of 0's and 1's, we see that $\beta(z,\eta)$ is constant on $\{\eta : \eta \not\equiv 1\}$. This argument is correct, since the hypothesis of Theorem 1.4(b) is symmetric in 0's and 1's.

To complete the proof that $\delta(z,\eta)$ and $\beta(z,\eta)$ are independent of η, we need to assume that S has at least four points. Let $\delta(z)$ be the value of $\delta(z,\eta)$ on the set

$\{\eta : \eta \not\equiv 0\}$, and let $\beta(z)$ be the value of $\beta(z,\eta)$ on the set $\{\eta : \eta \not\equiv 1\}$. Take μ to be any product measure satisfying $0 < \mu\{\eta : \eta(z) = 1\} < 1$ for all $z \in S$. Fix distinct sites x, y and take nonempty sets A, B so that $A, B, \{x,y\}$ form a partition of S. Let η and ζ be the configurations

$$\eta(z) = \begin{cases} 0 & \text{on } A \cup \{x\} \\ 1 & \text{on } B \cup \{y\} \end{cases}$$

and

$$\zeta(z) = \begin{cases} 0 & \text{on } B \cup \{x\} \\ 1 & \text{on } A \cup \{y\}. \end{cases}$$

Then

$$\mu S(t)(\eta \vee \zeta)\mu S(t)(\eta \wedge \zeta) - \mu S(t)(\eta)\mu S(t)(\zeta) \tag{3.6}$$

is zero at $t = 0$, and hence by assumption, its derivative must be nonnegative at $t = 0$. Since (1.2) holds with equality for any product measure, and independent flip processes preserve the class of product measures, the derivative of (3.6) would be zero if the spin system had independent flips. This observation leads to a lot of cancellation in the derivative of (3.6) in the present case, since the process almost has independent flips. Letting 0 and 1 denote the configurations that are identically 0 and 1 respectively, we find after using this cancellation that

$$\begin{aligned} & \frac{d}{dt}\Big[\mu S(t)(\eta \vee \zeta)\mu S(t)(\eta \wedge \zeta) - \mu S(t)(\eta)\mu S(t)(\zeta)\Big]\Big|_{t=0} \\ = \;& \mu(\eta \vee \zeta)\mu(\eta \wedge \zeta)\Big[\delta(y) - \delta(y,0) + \beta(x) - \beta(x,1)\Big]. \end{aligned} \tag{3.7}$$

Since the spin system is attractive (by Proposition 2.1), $\delta(y) \le \delta(y,0)$ and $\beta(x) \le \beta(x,1)$, so it follows from the nonnegativity of (3.7) that $\delta(y) = \delta(y,0)$ and $\beta(x) = \beta(x,1)$ as required. □

4. Preservation of the downward FKG property.

This section is devoted to the proof of Theorem 1.5. We begin with a simple lemma. The inequality in (4.1) below refers to stochastic monotonicity.

Lemma 4.1. *Suppose μ is downward FKG and $A \subset B$. Then*

$$\mu\{\cdot \mid \eta \equiv 0 \ \text{on } B\} \le \mu\{\cdot \mid \eta \equiv 0 \ \text{on } A\}, \tag{4.1}$$

and any convex combination of these two conditional measures is associated.

Proof. Let f be an increasing function. By the downward FKG property, f and $1_{\{\eta\equiv 0 \text{ on } B\}}$ are negatively correlated with respect to $\mu\{\cdot \mid \eta \equiv 0 \text{ on } A\}$. This gives

$$\int f d\mu\{\cdot \mid \eta \equiv 0 \text{ on } B\} \le \int f d\mu\{\cdot \mid \eta \equiv 0 \text{ on } A\}$$

as required for (4.1). For the second statement, use Proposition 2.22 on page 83 of Liggett (2005). □

Proof of Theorem 1.5(a): By Proposition 1.8, it suffices to prove Theorem 1.5(a) for spin systems that have nonzero transition rates at only one site $u \in S$, and at that site, either $\beta(u,\eta) \equiv 0$ and $\delta(u,\eta) \equiv 1$, or $\delta(u,\eta) \equiv 0$, $\beta(x,\eta) \equiv 0$ for $x \neq u$, and $\beta(u,\eta) = 1_{\{\eta\not\equiv 0 \text{ on } A\}}$ for a fixed $A \subset S$.

We begin then by considering the spin system with $\delta(u,\eta)\equiv 1$ for a fixed u, and all other rates zero. Suppose that μ is downward FKG. If $u\notin A$, then the evolution commutes with the operation of conditioning on $\{\eta\equiv 0 \text{ on } A\}$, so that $\mu S(t)\{\cdot\mid\eta\equiv 0\text{ on } A\}$ is associated for all $t\ge 0$ by Theorem 1.2. So, we may assume that $u\in A$. Let μ_L and μ_R be defined by

$$\mu_L(\cdot)=\mu S(t)\{\cdot\mid\eta\equiv 0 \text{ on } A\}$$

and

$$\mu_R(\cdot)=\lambda\mu\{\cdot\mid\eta\equiv 0\text{ on } A\}+(1-\lambda)\mu\{\cdot\mid\eta\equiv 0\text{ on } A\backslash\{u\}\},$$

where

$$\lambda=\frac{e^{-t}\mu\{\eta\equiv 0\text{ on } A\}}{e^{-t}\mu\{\eta\equiv 0\text{ on } A\}+(1-e^{-t})\mu\{\eta\equiv 0\text{ on } A\backslash\{u\}\}}.$$

We need to show that μ_L is associated. Every function f can be written uniquely in the form

$$f(\eta)=\eta(u)f_1(\eta)+[1-\eta(u)]f_0(\eta),$$

where f_0 and f_1 do not depend on the coordinate $\eta(u)$. With this decomposition, we will write $Tf=f_0$. Note that T is linear, satisfies $T(fg)=(Tf)(Tg)$, and Tf is increasing if f is increasing. Next, we will show that

$$\int f\,d\mu_L=\int Tf\,d\mu_R.$$

To see this, write

$$\begin{aligned}\int f\,d\mu_L &= \int_{\eta\equiv 0\text{ on } A} f\,d\mu S(t)\Big/\int_{\eta\equiv 0\text{ on } A} 1\,d\mu S(t)\\ &= \int_{\eta\equiv 0\text{ on } A} Tf\,d\mu S(t)\Big/\int_{\eta\equiv 0\text{ on } A} 1\,d\mu S(t)\\ &= \int_{\eta\equiv 0\text{ on } A\backslash\{u\}} Tf[1-\eta(u)e^{-t}]\,d\mu\Big/\int_{\eta\equiv 0\text{ on } A\backslash\{u\}}[1-\eta(u)e^{-t}]\,d\mu\\ &= \int Tf\,d\mu_R.\end{aligned}$$

Since μ_R is associated by Lemma 4.1, we can now take increasing f and g and write

$$\begin{aligned}\int (fg)\,d\mu_L &= \int T(fg)\,d\mu_R=\int (Tf)(Tg)\,d\mu_R\\ &\ge \int Tf\,d\mu_R\int Tg\,d\mu_R=\int f\,d\mu_L\int g\,d\mu_L\end{aligned}$$

as required.

Turning to the second case, assume now that all flip rates are zero, except that $\beta(u,\eta)=1_{\{\eta\not\equiv 0\text{ on } A\}}$ for a particular $u\in S$ and $A\subset S\backslash\{u\}$. Let μ be downward FKG. We need to check that the measure $\mu S(t)\{\cdot\mid\eta\equiv 0\text{ on } B\}$ is associated for every $B\subset S$. If $u\notin B$, this is a consequence of Theorem 1.2, since conditioning on $\{\eta\equiv 0\text{ on } B\}$ commutes with $S(t)$. So, we may assume that $u\in B$. In this case,

$$\int_{\{\eta\equiv 0\text{ on } B\}} f\,d\mu S(t)=\int_{\{\eta\equiv 0\text{ on } B\}} f(\eta)e^{-t\beta(u,\eta)}\,d\mu$$

for every f. Writing

$$e^{-t\beta(u,\eta)}=e^{-t}+(1-e^{-t})1_{\{\eta\equiv 0\text{ on } A\}},$$

it follows that

$$\mu S(t)\{\cdot \mid \eta \equiv 0 \text{ on } B\} = \lambda\mu\{\cdot \mid \eta \equiv 0 \text{ on } B\} + (1-\lambda)\mu\{\cdot \mid \eta \equiv 0 \text{ on } A \cup B\},$$

where

$$\lambda = \frac{e^{-t}\mu\{\eta \equiv 0 \text{ on } B\}}{e^{-t}\mu\{\eta \equiv 0 \text{ on } B\} + (1-e^{-t})\mu\{\eta \equiv 0 \text{ on } A \cup B\}}.$$

Therefore $\mu S(t)\{\cdot \mid \eta \equiv 0 \text{ on } B\}$ is associated by Lemma 4.1. □

Proof of Theorem 1.5(b): Since (1.2) implies downward FKG, this follows immediately from Proposition 3.1. □

Proof Theorem 1.5(c): Since every product measure is downward FKG (by Theorem 1.1) and every measure that is downward FKG is associated, the fact that $\beta(u,\eta)$ is increasing in η is a consequence of Proposition 2.1. So, it remains to prove (1.5). It is sufficient to check it in the case that η and ζ differ at only two sites; call them x and y. Since $\beta(u,\gamma)$ does not depend on $\gamma(u)$, we may assume that x, y, u are distinct.

Take μ to be a product measure. By assumption, $\mu S(t)$ is downward FKG for all $t \geq 0$. Therefore, using the definition of downward FKG with the conditioning on $\{\eta(u) = 0\}$,

$$\begin{aligned}\mu S(t)\{\eta(x)=1,\eta(y)=1,\eta(u)=0\}\mu S(t)\{\eta(x)=0,\eta(y)=0,\eta(u)=0\} - \\ \mu S(t)\{\eta(x)=1,\eta(y)=0,\eta(u)=0\}\mu S(t)\{\eta(x)=0,\eta(y)=1,\eta(u)=0\}\end{aligned}$$

is nonnegative for all $t \geq 0$, and is zero at $t = 0$. It follows that its derivative is nonnegative at $t = 0$. Writing this out, we see that

$$\begin{aligned}E[\beta(u,\eta) \mid \eta(x)=1,\eta(y)=0] + E[\beta(u,\eta) \mid \eta(x)=0,\eta(y)=1] \geq \\ E[\beta(u,\eta) \mid \eta(x)=1,\eta(y)=1] + E[\beta(u,\eta) \mid \eta(x)=0,\eta(y)=0]\end{aligned}$$

where the conditional expectations are with respect to μ. Since μ is an arbitrary product measure, we can conclude that

$$\beta(u,\gamma_x) + \beta(u,\gamma_y) \geq \beta(u,\gamma_{x,y}) + \beta(u,\gamma)$$

for every γ such that $\gamma(x) = \gamma(y) = 0$. But this is exactly (1.5) with $\eta = \gamma_x$ and $\zeta = \gamma_y$. □

5. Preservation of the DCA property.

This section is devoted to the proof of Theorem 1.7.

Proof of Theorem 1.7(a): By Proposition 1.8, it suffices to consider a spin system with nonzero flip rates only at one site u, and at that site, only $\delta(u,\eta)$ or $\beta(u,\eta)$ is not identically zero. We assume this, and also that the nonzero rates satisfy (1.4) or (1.5) and attractiveness, in the two cases.

The proof relies on three facts:

Fact I. If f and g are increasing and h is positive, then

$$[S(t)h][S(t)(fgh)] \geq [S(t)(fh)][S(t)(gh)]. \tag{5.1}$$

Fact II. If h is positive, decreasing and satisfies (1.6), then $S(t)h$ has the same three properties.

Fact III. If f is increasing, and h is positive and satisfies (1.6), then the function f_t defined by

$$f_t(\eta) = \frac{S(t)(fh)(\eta)}{S(t)h(\eta)}$$

is increasing in η.

Remark 5.1. *To see why the proof of Theorem 1.2 is easier than the proof of Theorem 1.7(a), note that if $h \equiv 1$, then Facts II and III are immediate. (Fact I is (2.20) on page 81 of Liggett (2005).)*

First we will deduce Theorem 1.7(a) from these three statements. Suppose μ is DCA, h is positive, decreasing and satisfies (1.6), and f and g are increasing. Then $\mu_{S(t)h}$ is associated by Fact II. Applying this association to the increasing functions f_t and g_t, which are increasing by Fact III, we see that

$$\int S(t)hd\mu \int \frac{S(t)(fh)S(t)(gh)}{S(t)h} d\mu \geq \int S(t)(fh)d\mu \int S(t)(gh)d\mu.$$

Combining this with Fact I gives

$$\int S(t)hd\mu \int S(t)(fgh)d\mu \geq \int S(t)(fh)d\mu \int S(t)(gh)d\mu,$$

which can be rewritten as

$$\int hd[\mu S(t)] \int fghd[\mu S(t)] \geq \int fhd[\mu S(t)] \int ghd[\mu S(t)],$$

so that $\mu S(t)$ is DCA, as required.

We turn now to the proofs of Facts I and III. Since

(a) monotonicity of h is not assumed here,

(b) condition (1.6) is symmetric in 0's and 1's,

and

(c) the assumptions on the death rates are more stringent than those on the birth rates,

it suffices to prove this in the case that the nonzero flip rates are $\beta(u, \eta)$. For a fixed t, let $b(\eta) = e^{-t\beta(u,\eta)}$. This is the probability that $\eta_t(u) = 0$ if the initial configuration satisfies $\eta(u) = 0$. Then, for any function f,

$$S(t)f(\eta) = \begin{cases} f(\eta) & \text{if } \eta(u) = 1 \\ b(\eta)f(\eta) + (1 - b(\eta))f(\eta_u) & \text{if } \eta(u) = 0. \end{cases}$$

Therefore, the two sides of (5.1) are equal when the η at which they are evaluated satisfies $\eta(u) = 1$. If $\eta(u) = 0$, then

$$\begin{aligned} &[S(t)h](\eta)[S(t)(fgh)](\eta) - [S(t)(fh)](\eta)[S(t)(gh)](\eta) \\ &\quad = \left[b(\eta)h(\eta) + (1 - b(\eta))h(\eta_u)\right]\left[b(\eta)f(\eta)g(\eta)h(\eta) + (1 - b(\eta))f(\eta_u)g(\eta_u)h(\eta_u)\right] \\ &\qquad - \left[b(\eta)f(\eta)h(\eta) + (1 - b(\eta))f(\eta_u)h(\eta_u)\right]\left[b(\eta)g(\eta)h(\eta) + (1 - b(\eta))g(\eta_u)h(\eta_u)\right] \\ &\quad = b(\eta)(1 - b(\eta))h(\eta)h(\eta_u)[f(\eta_u) - f(\eta)][g(\eta_u) - g(\eta)] \geq 0. \end{aligned}$$

This proves Fact I.

For Fact III, we must show that for any v, if $\eta(v) = 0$, then

$$\frac{S(t)(fh)(\eta_v)}{S(t)h(\eta_v)} \geq \frac{S(t)(fh)(\eta)}{S(t)h(\eta)}. \tag{5.2}$$

If $v = u$, the difference between the left and right sides of (5.2) is

$$f(\eta_u) - \frac{b(\eta)f(\eta)h(\eta) + (1-b(\eta))f(\eta_u)h(\eta_u)}{b(\eta)h(\eta) + (1-b(\eta))h(\eta_u)} = \frac{b(\eta)h(\eta)[f(\eta_u) - f(\eta)]}{b(\eta)h(\eta) + (1-b(\eta))h(\eta_u)} \geq 0.$$

If $v \neq u$ and $\eta(u) = 1$, then the difference between the left and right sides of (5.2) is

$$f(\eta_v) - f(\eta) \geq 0.$$

If $v \neq u$ and $\eta(u) = 0$, then the difference between the left and right sides of (5.2) is

$$\frac{b(\eta_v)f(\eta_v)h(\eta_v) + (1-b(\eta_v))f(\eta_{u,v})h(\eta_{u,v})}{b(\eta_v)h(\eta_v) + (1-b(\eta_v))h(\eta_{u,v})} - \frac{b(\eta)f(\eta)h(\eta) + (1-b(\eta))f(\eta_u)h(\eta_u)}{b(\eta)h(\eta) + (1-b(\eta))h(\eta_u)}.$$

Putting this over a common denominator, the resulting numerator is

$$\begin{aligned} &b(\eta)b(\eta_v)h(\eta)h(\eta_v)[f(\eta_v) - f(\eta)] \\ &\quad +(1-b(\eta))(1-b(\eta_v))h(\eta_u)h(\eta_{u,v})[f(\eta_{u,v} - f(\eta_u)] \\ &\quad\quad +b(\eta_v)(1-b(\eta))h(\eta_u)h(\eta_v)[f(\eta_v) - f(\eta_u)] \\ &\quad\quad\quad +b(\eta)(1-b(\eta_v))h(\eta)h(\eta_{u,v})[f(\eta_{u,v} - f(\eta)]. \end{aligned} \tag{5.3}$$

Since f is increasing, all terms but the third are nonnegative. To see that the nonnegative terms compensate for the potentially negative one, we proceed as follows. The values of f at the four configurations that appear in (5.3) satisfy $f(\eta) \leq f(\eta_u), f(\eta_v) \leq f(\eta_{u,v})$. Since they appear linearly in (5.3), it is enough to check the nonnegativity of (5.3) in case $f(\eta) = 0, f(\eta_{u,v}) = 1$, and $(f(\eta_u), f(\eta_v))$ takes one of the four values $(0,0), (1,0), (0,1), (1,1)$. The summands in (5.3) are all nonnegative except in the case $f(\eta_u) = 1, f(\eta_v) = 0$. In this case, (5.3) becomes

$$b(\eta)(1-b(\eta_v))h(\eta)h(\eta_{u,v}) - b(\eta_v)(1-b(\eta))h(\eta_u)h(\eta_v).$$

But, this is nonnegative since $h(\eta)h(\eta_{u,v}) \geq h(\eta_u)h(\eta_v)$ (by (1.6)) and $b(\eta) \geq b(\eta_v)$ (since the birth rates $\beta(u, \eta)$ are increasing).

It remains to prove Fact II. We will again carry out the proof in the case that the nonzero flip rates are the birth rates at site u. To check the result for the case of nonzero death rates, simply note that the monotonicity of h is not needed in the proof that $S(t)h$ satisfies (1.6) if $\beta(u, \eta)$ is constant, so that the interchange of roles of zeros and ones can be used again. The fact that $S(t)h$ is positive is clear, and the fact that it is decreasing follows from the monotonicity of $\beta(u, \eta)$.

To verify that $S(t)h$ satisfies (1.6), it is enough to check that if η satisfies $\eta(v) = \eta(w) = 0$ for two distinct sites v and w, then

$$S(t)h(\eta_{v,w})S(t)h(\eta) \geq S(t)h(\eta_v)S(t)h(\eta_w). \tag{5.4}$$

If $w = u$, then the difference between the left and right sides of (5.4) is

$$b(\eta_v)[h(\eta)h(\eta_{u,v}) - h(\eta_u)h(\eta_v)] + [b(\eta) - b(\eta_v)]h(\eta_{u,v})[h(\eta) - h(\eta_u)],$$

which is nonnegative if b is decreasing (i.e., $\beta(u, \eta)$ is increasing), h is decreasing and satisfies (1.6). Note that the monotonicity of h is not needed if b is constant. Now suppose that u, v, w are all distinct. Again, we may assume that $\eta(u) = 0$, since otherwise (5.4) is automatic. Then the difference between the left and right sides of (5.4) is

$$\begin{aligned}&\big[b(\eta_{v,w})h(\eta_{v,w}) + (1-b(\eta_{v,w}))h(\eta_{u,v,w})\big]\big[b(\eta)h(\eta) + (1-b(\eta))h(\eta_u)\big]\\ &-\big[b(\eta_v)h(\eta_v) + (1-b(\eta_v))h(\eta_{u,v})\big]\big[b(\eta_w)h(\eta_w) + (1-b(\eta_w))h(\eta_{u,w})\big].\end{aligned}\tag{5.5}$$

We need to check that this is nonnegative whenever

$$0 < b(\eta_{v,w}) \le b(\eta_v), b(\eta_w) \le b(\eta) \le 1, \text{ and } b(\eta_v)b(\eta_w) \le b(\eta)b(\eta_{v,w}).\tag{5.6}$$

The easiest case to consider is (i), in which $\{b(\eta_v), b(\eta_w)\} = \{b(\eta), b(\eta_{v,w})\}$. In this case, we can define a probability measure σ on $\{0,1\}^3$ by

$$\begin{aligned}\sigma(111) &= (1-b(\eta_{v,w}))h(\eta_{u,v,w}), \quad &\sigma(110) &= b(\eta_{v,w})h(\eta_{v,w})\\ \sigma(101) &= (1-b(\eta_v))h(\eta_{u,v}), \quad &\sigma(100) &= b(\eta_v)h(\eta_v)\\ \sigma(011) &= (1-b(\eta_w))h(\eta_{u,w}), \quad &\sigma(010) &= b(\eta_w)h(\eta_w)\\ \sigma(001) &= (1-b(\eta))h(\eta_u), \quad &\sigma(000) &= b(\eta)h(\eta),\end{aligned}$$

and then normalizing it to sum to 1. This measure satisfies the FKG lattice condition, and hence by Theorem 1.1, is associated. But in this case, (5.5) is just a constant multiple of the covariance of the first two coordinates relative to σ, so it is nonnegative. Note that in this case, we have not used the monotonicity of h.

The next case is (ii), in which $b(\eta_v) = b(\eta_w) = b(\eta_{v,w})$ – call this common value a. Since (5.5) is linear in $b(\eta)$, it suffices to check its nonnegativity in the extreme cases $b(\eta) = 1$, and $b(\eta) = a$. The latter case is a special case of case (i). So, we may assume that $b(\eta) = 1$. In this case, (5.5) is a quadratic polynomial in a, and the coefficient of a^2 is

$$-[h(\eta_v) - h(\eta_{u,v})][h(\eta_w) - h(\eta_{u,w})].$$

This is nonpositive by the monotonicity of h. (Note that h monotone in either direction would be enough here.) So, it is enough to check the nonnegativity at the two extreme cases, $a = 0$ and $a = 1$. The case $a = 1$ is again a special case of case (i). If $a = 0$, then (5.5) becomes

$$h(\eta_{u,v,w})h(\eta) - h(\eta_{u,v})h(\eta_{u,w}),$$

which is nonnegative by (1.6) and the fact that h is *decreasing*. This completes the consideration of case (ii).

Now think of (5.5) as a function of the variables $x = b(\eta_v)$ and $y = b(\eta_w)$ for fixed values of $0 < b(\eta_{v,w}) < b(\eta) \le 1$. Since (5.5) is bilinear in these two variables, it suffices to check its nonnegativity on the boundary of the region defined by (5.6). We have already checked it at three points on the boundary, in cases (i) and (ii) above. The boundary consists of two line segments and one curve, so again by bilinearity, the nonnegativity follows on the line segments from its nonnegativity at the endpoints of those line segments. It suffices then to consider the case in which $xy = b(\eta)b(\eta_{v,w}) = A$. Replacing y by A/x in (5.5) and expanding, we see that the only dependence on x is in the terms

$$-xh(\eta_{u,w})[h(\eta_v) - h(\eta_{u,v})] - \frac{A}{x}h(\eta_{u,v})[h(\eta_w) - h(\eta_{u,w})].$$

Since h is decreasing, this is a concave function of x, so (5.5) will be proved to be nonnegative once it is nonnegative at the endpoints of the interval of x's that are relevant. But this again corresponds to case (i), so the proof is complete. □

Proof of Theorem 1.7(b): This is again a consequence of Proposition 3.5, since

$$\mu \text{ satisfies (1.2)} \Rightarrow \mu \text{ DCA } \Rightarrow \mu S(t) \text{ DCA } \Rightarrow \mu S(t) \text{ downward FKG}.$$

□

The proof of Theorem 1.7(c) is the same as that of Theorem 1.5(c).

6. The case of three sites.

In this section, we take $S = \{1, 2, 3\}$, and find necessary and sufficient conditions for μ to satisfy each of the four properties appearing in (1.7). It will follow that for three sites, the first and third implications in (1.7) are strict, while the second is an equivalence.

We will use the following notation in this section:

$$\begin{aligned} a &= \mu(111), \\ b_1 &= \mu(011), \quad b_2 = \mu(101), \quad b_3 = \mu(110), \\ c_1 &= \mu(100), \quad c_2 = \mu(010), \quad c_3 = \mu(001), \\ d &= \mu(000). \end{aligned}$$

To state the necessary and sufficient conditions for the four properties of interest, consider the following sets of inequalities:

$$\begin{aligned} a(c_2 + c_3 + d) &\geq b_1(b_2 + b_3 + c_1), \\ a(c_1 + c_3 + d) &\geq b_2(b_1 + b_3 + c_2), \\ a(c_1 + c_2 + d) &\geq b_3(b_1 + b_2 + c_3), \end{aligned} \tag{A}$$

$$\begin{aligned} d(b_2 + b_3 + a) &\geq c_1(c_2 + c_3 + b_1), \\ d(b_1 + b_3 + a) &\geq c_2(c_1 + c_3 + b_2), \\ d(b_1 + b_2 + a) &\geq c_3(c_1 + c_2 + b_3), \end{aligned} \tag{B}$$

$$\begin{aligned} (b_1 + a)(c_1 + d) &\geq (c_3 + b_2)(b_3 + c_2), \\ (b_2 + a)(c_2 + d) &\geq (c_1 + b_3)(b_1 + c_3), \\ (b_3 + a)(c_3 + d) &\geq (c_2 + b_1)(b_2 + c_1), \end{aligned} \tag{C}$$

$$b_1 d \geq c_2 c_3, \qquad b_2 d \geq c_1 c_3, \qquad b_3 d \geq c_1 c_2, \tag{D}$$

and

$$c_1 a \geq b_2 b_3, \qquad c_2 a \geq b_1 b_3, \qquad c_3 a \geq b_1 b_2. \tag{E}$$

Here are the necessary and sufficient conditions:

Proposition 6.1. *(a) μ satisfies the FKG lattice condition if and only if (D) and (E) hold.*

(b) μ satisfies the DCA property if and only if (A), (C) and (D) hold.

(c) μ satisfies the downward FKG property if and only if (A), (C) and (D) hold.

(d) μ is associated if and only if (A), (B) and (C) hold.

Remark 6.2. *For an example to show that the first implication in* (1.7) *is strict, take* ϵ *small and*

$$a = b_1 = b_2 = b_3 = \frac{1}{6(1+3\epsilon)}, \quad c_1 = c_2 = c_3 = \frac{\epsilon}{1+3\epsilon}, \quad d = \frac{1}{3(1+3\epsilon)}.$$

To show that the third implication in (1.7) *is strict, take*

$$a = \frac{1}{3(1+3\epsilon)}, \quad b_1 = b_2 = b_3 = \frac{\epsilon}{1+3\epsilon}, \quad c_1 = c_2 = c_3 = d = \frac{1}{6(1+3\epsilon)}.$$

The proof of part (a) of the proposition is immediate. We turn now to the other parts.

Proof of Proposition 6.1(d): To check that association implies (A), (B) and (C), let $f_i(\eta) = \eta(i), g_i(\eta) = \prod_{j\neq i} \eta(j)$, and $h_i(\eta)$ = the indicator of the event $\{\sum_{j\neq i} \eta(j) \geq 1\}$. Using *cov* to denote the covariance with respect to μ, we have the following:

$$\begin{aligned} cov(f_1, g_1) &= a(c_2 + c_3 + d) - b_1(b_2 + b_3 + c_1), \\ cov(f_1, h_1) &= d(b_2 + b_3 + a) - c_1(c_2 + c_3 + b_1), \\ cov(f_2, f_3) &= (b_1 + a)(c_1 + d) - (c_3 + b_2)(b_3 + c_2). \end{aligned} \tag{6.1}$$

If μ is associated, each of these covariances is nonnegative. This gives the first inequality in each of (A), (B) and (C). The others are obtained by permuting the coordinates.

The proof of the converse is longer. Assume that (A), (B) and (C) hold. By 6.1 and the corresponding inequalities obtained by permuting the coordinates, we may assume that

$$cov(f_i, g_i) \geq 0, \quad cov(f_i, h_i) \geq 0, \quad \text{and} \quad cov(f_i, f_j) \geq 0 \tag{6.2}$$

for all choices of i and j. We need to check that $cov(f, g) \geq 0$ for all increasing functions f and g. It is sufficient to check this when f and g are both increasing indicator functions, since any increasing function can be written as a positive linear combination of increasing indicator functions. Writing

$$cov(f, g) = \frac{1}{2} \sum_{\eta,\zeta} [f(\eta) - f(\zeta)][g(\eta) - g(\zeta)]\mu(\eta)\mu(\zeta),$$

we see that $cov(f, g) \geq 0$ if $[f(\eta) - f(\zeta)][g(\eta) - g(\zeta)] \geq 0$ for all η, ζ. Since we are assuming that f and g are indicators, these products can only take the values $-1, 0, +1$. Therefore, we may assume that at least one of these products is -1. The product is nonnegative whenever η and ζ are comparable. Therefore, by permuting coordinates and/or interchanging the roles of 0's and 1's, we see that there are two cases to consider:

Case 1. $[f(101) - f(011)][g(101) - g(011)] = -1$, in which case we may assume

$$f(101) = 1, \quad f(011) = 0, \quad g(101) = 0, \quad g(011) = 1.$$

The monotonicity of f and g forces f and g to also take the following values:

$$f(111) = g(111) = 1,$$

and

$$f(010) = f(001) = f(000) = g(100) = g(001) = g(000) = 0.$$

So, there are only three possibilities for f and g:

$$f = g_2, f = f_1 h_1, \text{ or } f = f_1 \text{ and } g = g_1, g = f_2 h_2, \text{ or } g = f_2.$$

The corresponding nine covariances are nonnegative by (6.3), since

$$\begin{aligned} cov(g_i, g_j) \geq cov(g_i, f_i), \quad cov(g_i, f_i h_i) \geq cov(g_i, f_i) \\ cov(f_i h_i, f_j h_j) \geq cov(f_i, f_j), \quad cov(f_i h_i, f_j) \geq cov(f_i, f_j) \end{aligned} \tag{6.3}$$

for $i \neq j$. These all follow from the easy fact that if G_i and H_i are sets satisfying $G_i \subset H_i$ for $i = 1, 2$ and $G_1 \cap G_2 = H_1 \cap H_2$, then $cov(1_{G_1}, 1_{G_2}) \geq cov(1_{H_1}, 1_{H_2})$.
Case 2. $[f(011) - f(100)][g(011) - g(100)] = -1$, in which case we may assume

$$f(011) = 1, \quad f(100) = 0, \quad g(011) = 0, \quad g(100) = 1.$$

Using the monotonicity of f and g as before, it follows that $g = f_1$ and f is one of the following:

$$g_1, \quad f_3 h_3, \quad f_3, \quad f_2 h_2, \quad h_1 h_2 h_3, \quad h_1 h_2, \quad f_2, \quad h_1 h_3, \quad h_1.$$

The nonnegativity of the covariance of each of these with f_1 follows from (6.2), (6.3),

$$cov(h_1 h_2 h_3, f_i) \geq cov(h_i, f_i) \quad \text{and} \quad cov(h_i h_j, f_i) \geq cov(h_i, f_i)$$

for $i \neq j$. □

Proof of Proposition 6.1(c): The statement that downward FKG implies (A), (C), and (D) is an immediate consequence of part (d). So, assume now that (A), (C) and (D) hold. Since (D) holds, it will be sufficient to show that μ is associated. And to do this, it suffices by part (d) to show that (B) holds. To do so, multiply the first inequality in (A) by d and then use the second and third inequalities in (D) to conclude that

$$ad(c_2 + c_3 + d) \geq db_1(b_2 + b_3 + c_1) \geq b_1 c_1 (c_2 + c_3 + d),$$

and therefore that $ad \geq b_1 c_1$. For future reference, we record the fact that

$$\text{(A) and (D) imply } ad \geq b_i c_i \text{ for } i = 1, 2, 3. \tag{6.4}$$

Now the first inequality in (B), for example, follows from (6.4) with $i = 1$, together with the last two inequalities in (D). □

Proof of Proposition 6.1(b): Now let a, b_i, c_i, d be the probabilities of the various configurations for the measure μ, and a^*, b_i^*, c_i^*, d^* be the corresponding probabilities for the measure μ_h, where h is positive, decreasing, and satisfies (1.6). We need to show that if the unstarred quantities satisfy (A), (C) and (D), then the starred quantities satisfy (A), (B) and (C). By (1.6), the starred quantities satisfy (D), since the unstarred quantities do. Therefore, once we have shown that the starred quantities satisfy (A) and (C) they will automatically satisfy (B) by parts (c) and (d) of the proposition.

We will now check that the starred quantities satisfy the first inequality in (A). To do so, define

$$\begin{aligned} x_1 = \frac{h(001)}{h(011)}, \quad x_2 = \frac{h(010)}{h(011)}, \quad x_3 = \frac{h(000)}{h(011)}, \\ y_1 = \frac{h(101)}{h(111)}, \quad y_2 = \frac{h(110)}{h(111)}, \quad y_3 = \frac{h(100)}{h(111)}. \end{aligned}$$

The starred version of the first inequality in (A) then becomes

$$a(x_2c_2 + x_1c_3 + x_3d) \geq b_1(y_1b_2 + y_2b_3 + y_3c_1). \tag{6.5}$$

Since h is decreasing, we have

$$x_1, x_2, x_3, y_1, y_2, y_3 \geq 1. \tag{6.6}$$

Since h satisfies (1.6), these quantities satisfy the following inequalities:

$$x_1 \geq y_1, x_2 \geq y_2, x_3 \geq y_3, \quad y_3 \geq y_1y_2, x_3 \geq x_1x_2, \quad y_1x_3 \geq x_1y_3, y_2x_3 \geq x_2y_3. \tag{6.7}$$

By the first three inequalities in (6.7), (6.5) will be a consequence of

$$a(y_2c_2 + y_1c_3 + y_3d) \geq b_1(y_1b_2 + y_2b_3 + y_3c_1). \tag{6.8}$$

By (6.4), $ad \geq b_1c_1$. Therefore, by the fourth inequality in (6.7), the worst case of (6.8) is that in which $y_3 = y_1y_2$. Consider this case and divide (6.8) by y_1y_2. By (6.6), we see that we need to show that

$$a(z_1c_2 + z_2c_3 + d) \geq b_1(z_2b_2 + z_1b_3 + c_1) \tag{6.9}$$

for all $0 \leq z_1, z_2 \leq 1$. Since the expressions on both sides of (6.9) are linear in z_1, z_2, it suffices to check that (6.9) holds at the four corners of this square. If $z_1 = z_2 = 0$, (6.9) is true by (6.4). If $z_1 = z_2 = 1$, (6.9) is just the first inequality in (A). The other two cases are $a(c_2 + d) \geq b_1(b_3 + c_1)$ and $a(c_3 + d) \geq b_1(b_2 + c_1)$. One is obtained from the other by permutation of coordinates, so we will prove the first of these.

To do so, note that

$$\begin{aligned}&[(b_3 + c_1)(c_2 + c_3 + d) - (b_2 + b_3 + c_1)(c_2 + d)] \\ &\qquad + [b_1(b_3 + c_1) - (c_1 + b_3)(b_1 + c_3) + b_2(c_2 + d)] = 0.\end{aligned} \tag{6.10}$$

Therefore one of the expressions in brackets is nonpositive. If it is the first one, then

$$(b_3 + c_1)(c_2 + c_3 + d) \leq (b_2 + b_3 + c_1)(c_2 + d),$$

which when combined with the first inequality in (A) implies $a(c_2+d) \geq b_1(b_3+c_1)$. If it is the second expression in brackets in (6.10) that is nonpositive, then

$$b_1(b_3 + c_1) \leq (c_1 + b_3)(b_1 + c_3) - b_2(c_2 + d),$$

which when combined with the second inequality in (C) implies $a(c_2 + d) \geq b_1(b_3 + c_1)$. Therefore, this last inequality is true in either case.

The starred version of the third inequality in (C) (we choose the third one to check instead of the first so that we can use the same x_i's and y_i's as in the previous argument) is

$$(b_3y_2 + a)(c_3x_1 + dx_3) \geq (c_2x_2 + b_1)(b_2y_1 + c_1y_3). \tag{6.11}$$

To check this inequality, start by writing

$$\begin{aligned}(b_3y_2 + a)(c_3x_1 + dx_3) \quad - &\quad (c_2x_2 + b_1)(b_2y_1 + c_1y_3) \\ = &\quad (b_3y_2 + a)(c_3y_1 + dy_3) - (c_2y_2 + b_1)(b_2y_1 + c_1y_3) \\ + &\quad c_3(b_3y_2 + a)(x_1 - y_1) + c_2(b_2y_1 + c_1y_3)\left[\frac{y_2x_3 - x_2y_3}{y_3}\right] \\ + &\quad \left[(db_3 - c_1c_2)y_2 + da - c_2b_2\frac{y_1y_2}{y_3}\right](x_3 - y_3).\end{aligned}$$

Therefore, by (D), (6.4) and (6.7), it suffices to prove

$$(b_3 y_2 + a)(c_3 y_1 + d y_3) \geq (c_2 y_2 + b_1)(b_2 y_1 + c_1 y_3). \tag{6.12}$$

By (D) and (6.4), the coefficient of y_3 on the left side of (6.12) is at least as large as the coefficient of y_3 on the right side of (6.12). Therefore, by (6.7), it suffices to consider the case $y_3 = y_1 y_2$. Canceling a common factor of y_1, we see by (6.6) that it suffices to check

$$(b_3 y_2 + a)(c_3 + d y_2) - (c_2 y_2 + b_1)(b_2 + c_1 y_2) \geq 0 \tag{6.13}$$

for $y_2 \geq 1$. Write this polynomial as $p(y_2)$. Then $p(1) \geq 0$ by (C). The coefficient of y^2 in $p(y)$ is $b_3 d - c_2 c_1$, which is nonnegative by (D). So, it suffices to check that $p'(1) \geq 0$. But this follows from (C) and (D), since

$$\begin{aligned} d(c_3 + d)p'(1) &= d^2[(b_3 + a)(c_3 + d) - (c_2 + b_1)(b_2 + c_1)] \\ &+ (c_3 + d)^2(b_3 d - c_1 c_2) + (b_1 d - c_2 c_3)(b_2 d - c_1 c_3). \end{aligned}$$

$\square$

References

V. Belitsky, P. Ferrari, N. Konno and T. M. Liggett. A strong correlation inequality for contact processes and oriented percolation. *Stoch. Proc. Appl.* **67**, 213–225 (1997).

J. van den Berg, O. Häggström and J. Kahn. Proof of a conjecture of N. Konno for the 1D contact process. In *Dynamics and Stochastics, Festschrift in Honor of Michael Keane*, MS Lecture Notes - Monograph Series. Institute of Mathematical Statistics (2006a).

J. van den Berg, O. Häggström and J. Kahn. Some conditional correlation inequalities for percolation and related processes. *Random Structures Algorithms* (2006b).

M. F. Chen and F. Y. Wang. On order-preservation and positive correlations for multidimensional diffusion processes. *Probab. Theory Related Fields* **95**, 421–428 (1993).

S. N. Ethier and T. G. Kurtz. *Markov Processes. Characterization and Convergence.* Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986).

P. C. Fishburn, P. G. Doyle and L. A. Shepp. The match set of a random permutation has the FKG property. *Ann. Probab.* **16**, 1194–1214 (1988).

T. M. Liggett. Survival and coexistence in interacting particle systems. In *Probability and Phase Transition (Cambridge, 1993)*, volume 420 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 209–226. Kluwer Acad. Publ., Dordrecht (1994).

T. M. Liggett. *Interacting Particle Systems.* Classics in Mathematics. Springer-Verlag, Berlin (2005). Reprint of the 1985 original.

T. M. Liggett and J. E. Steif. Stochastic domination: The contact process, Ising models and FKG measures. *Ann. Inst. H. Poincaré Probab. Statist.* (2006).

H. X. Sun. Stochastic monotonicity and positive correlations of spin systems (in Chinese). *Chinese J. Appl. Probab. Stat.* **10**, 225–232 (1994).