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Flows associated to Tanaka's SDE

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Abstract. Extending Tanaka's stochastic differential equation to transition kernels, we show that all the solutions can be characterized by probability measures on $[0,1]$.

1. Introduction

Tanaka's stochastic differential equation (SDE) is one the simplest example of SDE that does not have a strong solution in the usual sense. The objective of this paper is to apply to this example the theory of flows of transition kernels developed in our previous work Le Jan and Raimond (2004a), denoted hereafter by LJR, (see also Tsirelson (2004) for an improved presentation of some results). We first classify all the solutions of Tanaka's SDE, extended to transition kernels (this notion of solution of SDE was precisely defined in LJR). It is shown that they can be characterized by a probability measure on $[0,1]$. The domination and the weak domination relations (defined in LJR) between different solutions are then fully understood in terms of barycenter and balayage of the associated measures.

2. Statement of the results.

On a probability space $(\Omega,\mathcal{F},\mathsf{P})$, let $W=(W_{s,t},\ s\le t)$ be a real white noise and $K=(K_{s,t},\ s\le t)$ (resp. $\varphi=(\varphi_{s,t},\ s\le t)$) be a stochastic flow of kernels (resp. flow of measurable maps) on the real line. Recall that for all $s\le t$, $K_{s,t}:\mathbb{R}\to\mathcal{P}(\mathbb{R})$ is measurable, with $\mathcal{P}(\mathbb{R})$ denoting the set of probability measures on $\mathbb{R}$, equipped with the topology of weak convergence. We say that (see also definition 5.4 p. 1300 in LJR) (K,W) solves Tanaka's SDE if for all $s\le t$, $f\in C^2_K(\mathbb{R})$ and $x\in\mathbb{R}$,

$$K_{s,t}f(x)=f(x)+\int_s^t K_{s,u}(f'\mathsf{sign})(x)W(du)+\frac{1}{2}\int_s^t K_{s,u}(f")du, \qquad (2.1)$$

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with $\mathsf{sign}(x) = 1_{x\geq 0} - 1_{x<0}$. Note that (2.1) is a generalization of the SDE

$$dX_t = \mathsf{sign}(X_t)dW_t,$$

where $W_t = W_{0,t}1_{t\geq 0} - W_{t,0}1_{t<0}$.

In LJR (see section 6 and Remark 6.3 p. 1307), it is shown that this implies that $\sigma(W) \subset \sigma(K)$ (this is also a consequence of Lemma 3.1 below). Let N^K be the noise of K (see LJR and Tsirelson (2004) for a precise definition)[1]. The noise N^W of W is a subnoise of N^K. So we can simply say that K solves Tanaka's SDE (since W is a function of K). We say that φ solves Tanaka's SDE if δ_φ solves Tanaka's SDE. The law of a solution K is given by a Feller convolution semigroup $\nu = (\nu_t,\ t\geq 0)$, where ν_t is the law of $K_{0,t}$ (cf. LJR section 2 for a precise definition).

Two particular solutions of Tanaka's SDE are given in LJR: the coalescing solution φ^c and the Wiener solution K^W. The solution K^W is the only solution of Tanaka's SDE such that $N^K = N^W$. And φ^c is the only flow of maps solution of Tanaka's SDE. The Wiener solution can be obtained by filtering the coalescing solution: $K^W = \mathsf{E}[\delta_\varphi|W]$. An explicit expression of K^W can be given. For $x\in\mathbb{R}$, set $\tau_x = \inf\{t>0;\ W_{0,t} = -|x|\}$. Let $W^+ = (W^+_t,\ t\geq 0)$ be defined by

$$W^+_t = W_{0,t} - \inf_{s\leq t} W_{0,s}.$$

It is well known that the law of $(W^+_t)_{t\geq 0}$ and the law of $(|W_t|)_{t\geq 0}$ coincide.

Note that $W_{0,\cdot}$ can be recovered out of W^+ by Doob-Meyer decomposition. Then for $t\geq 0$,

$$K^W_{0,t}(x) = \delta_{x+\mathsf{sign}(x)W_{0,t}}1_{\{t\leq\tau_x\}} + \frac{1}{2}(\delta_{W^+_t} + \delta_{-W^+_t})1_{\{t>\tau_x\}}. \tag{2.2}$$

Let θ^W_h be the shift operator such that $W_{s,t}\circ\theta^W_h = W_{s+h,t+h}$. Then for all $s<t$, $K^W_{s,t} = K^W_{0,t-s}\circ\theta^W_s$. In LJR, φ^c was defined by the consistent families of its n-point motions, obtained by transforming the n-point motion associated with K^W into a coalescing motion. A more explicit definition can be given in this special case, as we will see in the following.

In this paper, we prove the following results.

Theorem 2.1.

a) *Each solution K of Tanaka's SDE defines a probability measure m on $[0,1]$, with mean $1/2$, which is the law of $\int_0^\infty K_{0,t}(0,dy)$ for all $t>0$.*
b) *The mapping defined in a) is a bijection between equivalence classes of solutions[2] of (2.1) and probability measures on $[0,1]$ with mean $1/2$. The Feller convolution semigroup associated with a measure m is denoted $\{\nu^m_t;\ t\geq 0\}$ or ν^m.*
c) *K^W is associated with $\delta_{1/2}$ and φ^c with $\frac{1}{2}(\delta_0+\delta_1)$.*

Definition 2.2. *Let m_1 and m_2 be probability measures on $[0,1]$.*

a) *We say m_1 is swept by m_2 if and only if for all positive convex function f, $\int f dm_2 \leq \int f dm_1$.*

[1] We recall that N^K is defined by a family of σ-fields $\mathcal{F}^K_{s,t} = \sigma(K_{u,v},\ s\leq u\leq v\leq t)$ and a shift operator θ_h such that it is P stationary and $K_{u,v}\circ\theta_h = K_{u+h,v+h}$ for all $u<v$ and $h\in\mathbb{R}$.

[2] Two solutions are equivalent when they have the same law, namely when they induce the same Feller convolution semigroup. In LJR we defined a canonical realization of a Feller convolution semigroup.

b) *We say that* m_2 *is a barycenter of* m_1 *if and only if there exists a measurable map* $\psi : [0,1] \to [0,1]$ *such that* $\psi^* m_1 = m_2$ *and* $\psi^*(I \cdot m_1) = I \cdot m_2$ *(where* I *denotes the identity function).*

It can easily be seen that a) and b) define partial order relations. The order defined by a) is the balayage order. The fact that m_2 is a barycenter of m_1 is equivalent to say that if U_1 is a random variable of law m_1, there exists a random variable U_2 of law m_2, which is $\sigma(U_1)$-measurable and satisfying $\mathsf{E}[U_1|U_2] = U_2$.

In LJR, a domination and a weak domination relation between (laws of) stochastic flow of kernels is defined: Let ν^1 and ν^2 be two Feller convolution semigroups. We recall that Definition 3.3 in LJR essentially says that ν^1 dominates ν^2 if and only if there is a joint realization (K^1, K^2) such that K^1 (resp. K^2) is a stochastic flow of kernels associated to ν^1 (resp. to ν^2), satisfying $\mathsf{E}[K^1|K^2] = K^2$ and $\sigma(K^2) \subset \sigma(K^1)$, and ν^1 weakly dominates ν^2 when only the conditional expectation assumption is verified ($\sigma(K^2)$ needs not be a sub-σ-field of $\sigma(K^1)$).

Theorem 2.3. *Let* m_1 *and* m_2 *be two probability measures on* $[0,1]$, *with mean* $1/2$.

a) ν^{m_1} *dominates* ν^{m_2} *if and only if* m_2 *is a barycenter of* m_1.
b) ν^{m_1} *weakly dominates* ν^{m_2} *if and only if* m_1 *is swept by* m_2.

Finally, note that for all x, $(\varphi^c_{0,t}(x),\ t \le \tau_x)$ is $\sigma(W)$-measurable. Indeed, for $t \le \tau_x$, we must have $\varphi^c_{0,t}(x) = x + \mathsf{sign}(x) W_{0,t}$. If θ^c is the shift operator such that $\varphi^c_{s,t} \circ \theta^c_h = \varphi^c_{s+h,t+h}$, then we also have $W_{s,t} \circ \theta^c_h = W_{s+h,t+h}$ and $(\varphi^c_{s,t}(x),\ t \le s + \tau_x \circ \theta^c_s)$ is also $\sigma(W)$-measurable. We denote by N^c the noise of φ^c.

3. The coalescing flow φ^c

Let K be a solution of Tanaka's SDE. Denote by ν its associated Feller convolution semigroup. In LJR, another Feller convolution semigroup denoted ν^c is associated to ν. A Feller convolution semigroup is determined by its family of n-point motions. The n-point motion $X^{(n),c}_t$ associated to ν^c coincides with the n-point motion of ν for $t \le T_\Delta$, with $T_\Delta = \inf\{t; X^{(n),c}_t \in \Delta\}$ and $\Delta = \{x \in \mathbb{R}^n;\ \exists i \ne j,\ x_i = x_j\}$. With the consistency of the family of n-point motions, one can see that ν^c is determined by this property. Note that the one-point motion of ν (and of ν^c) is a Brownian motion.

Lemma 3.1. *For all* $t > 0$ *and* $x \in \mathbb{R}$, $K_{0,t}(x) = \delta_{x+\mathsf{sign}(x)W_t}$ *for* $t \le \tau_x$ *where* $\tau_x = \inf\{t;\ W_{0,t} = -|x|\}$.

Proof: Take $x > 0$. Let $\tilde\tau_x = \inf\{t;\ \int_{]-\infty,0[} K_{0,t}(x, dy) > 0\}$. Then, for $t \le \tilde\tau_x$, (2.1) with $f(x) = x$ implies that $\int y K_{0,t}(x, dy) = x + W_t$ and (2.1) with $f(x) = x^2$ implies that $\int (y - x - W_t)^2 K_{0,t}(x, dy) = 0$. This proves that for $t \le \tilde\tau_x$, $K_{0,t}(x) = \delta_{x+W_t}$. The fact that $\tau_x = \tilde\tau_x$ easily follows. □

Lemma 3.2. *Let* $X^{(n)}_t$ *be the n-point motion of* ν *(or of* ν^c*) started at* $(x_1, \cdots, x_n) \notin \Delta$, *and* $x_1 = 0$. *Then for* $t \le T_\Delta$, *for* $i \ge 2$,

$$X^{(n),i}_t = x_i + \mathsf{sign}(x_i) W_t$$

where $W_t = \int_0^t \mathsf{sign}(X^{(n),1}_s) dX^{(n),1}_s$ *and* $X^{(n),1}_t$ *is a Brownian motion started at* 0.

Proof : It s an easy consequence of Lemma 3.1. □

This Lemma implies that if ν and μ are Feller convolution semigroups associated to different solutions of Tanaka's SDE, then $\nu^c = \mu^c$. This Feller convolution semigroup will be denoted ν^c in the following. Note that ν^c is associated to a coalescing flow of maps, we will denote φ^c. The law of φ^c will be denoted P^{ν^c}.

Lemma 3.3. *The flow of maps φ^c solves Tanaka's SDE. Moreover all flow of maps solution of Tanaka's SDE have the distribution P^{ν^c}.*

Proof : Let $W_{s,t} = \int_s^t \mathsf{sign}(\varphi^c_{s,u}(0))d\varphi^c_{s,u}(0)$. Then W is a real white noise. Lemma 3.2 implies that for all x,

$$\varphi^c_{s,t}(x) = (x + \mathsf{sign}(x)W_{s,t})1_{\{t\le\tau^s_x\}} + \varphi^c_{s,t}(0)1_{\{t>\tau^s_x\}},$$

with $\tau^s_x = \tau_x \circ \theta^c_s$. ¿From this it is easy to check that Tanaka's SDE

$$\varphi^c_{s,t}(x) = x + \int_s^t \mathsf{sign}(\varphi^c_{s,u}(x))W(du)$$

holds. This proves the first part of the Lemma. The second part of the Lemma is a consequence of Lemma 3.2. □

4. Proof of Theorem 2.1.

4.1. *Construction of a solution.* Let m be a probability measure on $[0,1]$ such that

$$\int_{[0,1]} xm(dx) = 1/2.$$

Let $(\epsilon_{s,t}, U_{s,t}, W_{s,t})_{s\le t}$ be a process, indexed by $\{(s,t) \in \mathbb{R}^2;\ s \le t\}$, taking its values in $\{-1,1\} \times [0,1] \times \mathbb{R}$, such that W is a real valued white noise. To describe the law of (ϵ, U) knowing W, we set $W_t = W_{0,t}1_{\{t\ge 0\}} - W_{t,0}1_{\{t<0\}}$ (so that for all $s < t$, $W_{s,t} = W_t - W_s$) and define for all $s < t$

$$\min_{s,t} = \inf\{W_u;\ u \in [s,t]\}. \tag{4.1}$$

For $s < t$ and $\{(s_i,t_i);\ 1 \le i \le n\}$ with $s_i < t_i$, the law of $(\epsilon_{s,t}, U_{s,t})$ knowing $(\epsilon_{s_i,t_i}, U_{s_i,t_i})_{1\le i\le n}$ and W is given by

$$m(du)\big(u\delta_1 + (1-u)\delta_{-1}\big) \tag{4.2}$$

when $\min_{s,t} \notin \{\min_{s_i,t_i};\ 1 \le i \le n\}$ (in particular when $s \ge \sup_{1\le i\le n} t_i$); and is given by

$$\sum_{i=1}^n \delta_{\epsilon_{s_i,t_i},U_{s_i,t_i}} \times \frac{1_{\min_{s,t}=\min_{s_i,t_i}}}{\mathrm{Card}\{i;\ \min_{s_i,t_i} = \min_{s,t}\}} \tag{4.3}$$

otherwise. Note that it defines the law of (ϵ, U, W). In particular, we have $\mathsf{P}(\epsilon_{s,t} = 1|U_{s,t}) = U_{s,t}$ and the law of $U_{s,t}$ is m. We define the filtrations $\mathcal{F}^{\epsilon,U,W}_{0,t}$ and $\mathcal{F}^{U,W}_{0,t}$ by $\mathcal{F}^{\epsilon,U,W}_{0,t} = \sigma((\epsilon_{u,v}, U_{u,v}, W_{u,v});\ 0 \le u \le v \le t)$ and $\mathcal{F}^{U,W}_{0,t} = \sigma((U_{u,v}, W_{u,v});\ 0 \le u \le v \le t)$.

For $s \in \mathbb{R}$ and $x \in \mathbb{R}$, set $\tau^x_s = \inf\{t > s;\ |x| + W_{s,t} = 0\}$. For $s < t$, set

$$W^+_{s,t} = W_t - \min_{s,t}.$$

Note that $W^+_{s,\cdot}$ is $(W_{s,\cdot})^+$. For $x \in \mathbb{R}$ and $s < t$, set

$$\begin{aligned}\varphi_{s,t}(x) &= (x+\mathsf{sign}(x)W_{s,t})1_{\{t\le\tau_s^x\}}+\epsilon_{s,t}W_{s,t}^+1_{\{t>\tau_s^x\}}; &(4.4)\\ K_{s,t}(x) &= \delta_{x+\mathsf{sign}(x)W_{s,t}}1_{\{t\le\tau_s^x\}} &(4.5)\\ &+ \left(U_{s,t}\delta_{W_{s,t}^+}+(1-U_{s,t})\delta_{-W_{s,t}^+}\right)1_{\{t>\tau_s^x\}}.\end{aligned}$$

We denote by Q^m the law of K.

Proposition 4.1.

(i) *φ is a coalescing solution of Tanaka's SDE: For all $x\in\mathbb{R}$, $s<t$,*

$$\varphi_{s,t}(x)=x+\int_s^t \mathsf{sign}(\varphi_{s,u}(x))dW_u.$$

(ii) *For all $x\in\mathbb{R}$, all $s<t$ and all bounded continuous function f,*

$$K_{s,t}f(x)=\mathsf{E}[f(\varphi_{s,t}(x))|(U,W)].$$

(iii) *K is a flow of kernels solution of Tanaka's SDE.*

Remark 4.2. Note that the one-point motion of a solution of Tanaka's SDE is a Brownian motion. If $\int xm(dx)\neq 1/2$, it cannot be the case.

Proof : (ii) is obviously checked.(i) follows from the fact that for all $s<t$, $\varphi^c_{s,t}$ and $\varphi_{s,t}$ have the same law and that φ is a flow of maps with independent increments.

Lemma 4.3. *φ is a flow of maps with independent increments.*

Proof : For $s\in\mathbb{R}$ and $x\in\mathbb{R}$, $\varphi_{s,s}(x)=x$. Let $x\in\mathbb{R}$ and $s<t<u$. We recall that $\tau_s^x=\inf\{t>s;\ W_{s,t}=-|x|\}$. In the following, it will sometimes be denoted $\tau_s(x)$. All the equalities below hold a.s.

On the event $\{u<\tau_s^x\}$, $\varphi_{s,t}(x)=x+\mathsf{sign}(x)W_{s,t}$, $\tau_t(\varphi_{s,t}(x))=\tau_s^x<u$ and

$$\begin{aligned}\varphi_{t,u}\circ\varphi_{s,t}(x) &= x+\mathsf{sign}(x)(W_{s,t}+W_{t,u})\\ &= x+\mathsf{sign}(x)W_{s,u}.\end{aligned}$$

On the event $\{\tau_s^x\in]t,u]\}$, we still have $\varphi_{s,t}(x)=x+\mathsf{sign}(x)W_{s,t}$ and $\tau_t(\varphi_{s,t}(x))=\tau_s^x\le u$, thus

$$\varphi_{t,u}\circ\varphi_{s,t}(x)=\epsilon_{t,u}W_{t,u}^+=\epsilon_{s,u}W_{s,u}^+$$

since on the event $\{\tau_s^x\in]t,u]\}$, $\min_{s,u}=\min_{t,u}$ and $W_{s,u}^+=W_u-\min_{s,u}=W_{t,u}^+$.

On the event $\{\tau_s^x\le t\}\cap\{\tau_t(W_{s,t}^+)\le u\}$, $\varphi_{s,t}(x)=\epsilon_{s,t}W_{s,t}^+$ and

$$\begin{aligned}\varphi_{t,u}\circ\varphi_{s,t}(x) &= \varphi_{t,u}(\epsilon_{s,t}W_{s,t}^+)\\ &= \epsilon_{t,u}W_{t,u}^+\\ &= \epsilon_{s,u}W_{s,u}^+\end{aligned}$$

since $W^+_{s,\tau_t(W_{s,t}^+)}=0$ and thus $\min_{s,u}=\min_{t,u}$, which implies $\epsilon_{s,u}=\epsilon_{t,u}$ and $W_{s,u}^+=W_{t,u}^+$.

On the event $\{\tau_s^x\le t\}\cap\{\tau_t(W_{s,t}^+)>u\}$, $\varphi_{s,t}(x)=\epsilon_{s,t}W_{s,t}^+$ and

$$\begin{aligned}\varphi_{t,u}\circ\varphi_{s,t}(x) &= \varphi_{t,u}(\epsilon_{s,t}W_{s,t}^+)\\ &= \epsilon_{s,t}(W_{s,t}^++W_{t,u})\\ &= \epsilon_{s,u}W_{s,u}^+\end{aligned}$$

since in this case $\min_{s,u} = \min_{s,t}$, which implies $\epsilon_{s,u} = \epsilon_{s,t}$ and

$$\begin{aligned} W^+_{s,u} &= W_u - \min_{s,u} \\ &= W_u - W_s + W_s - \min_{s,t} \\ &= W^+_{s,t} + W_{t,u}. \end{aligned}$$

Thus we have, a.s.

$$\begin{aligned} \varphi_{t,u} \circ \varphi_{s,t}(x) &= (x + \mathsf{sign}(x) W_{s,u}) \times 1_{\{u < \tau^x_s\}} \\ &\quad + \epsilon_{s,u} W^+_{s,u} \times 1_{\{\tau^x_s \le u\}} \\ &= \varphi_{s,u}(x) \end{aligned}$$

which proves that φ is a flow of maps. The fact that it has independent increments follows from the definition of φ. □

Remark 4.4. *One can also check directly that for all s and x in $\mathbb{R}$, $(\varphi_{s,s+t}(x))_{t\ge 0}$ is a Brownian motion started at x.*

Proof : Note that $\varphi_{0,t}(0) = \epsilon_{0,t} W^+_t$ and that

$$\epsilon_{0,t} W^+_t = (\epsilon_{0,s} W^+_s + W_{s,t}) 1_{\{\min_{0,s} = \min_{0,t}\}} + \epsilon_{0,t} W^+_t 1_{\{\min_{0,s} < \min_{0,t}\}}.$$

For $s > 0$, set $\mathcal{F}_s = \sigma(W_{u,v}, \epsilon_{u,v};\ 0 \le u \le v \le s)$. Then $\mathcal{F}_s = \sigma(\varphi_{u,v};\ s \le u \le v \le t\}$. The expression of the conditional distribution (4.2) implies that, on the event $\{\min_{0,s} < \min_{0,t}\}$, the law of $\epsilon_{0,t}$ conditionally to W and $\sigma(\epsilon_{u,v};\ 0 \le u \le v \le s)$ is $\frac{1}{2}(\delta_{-1} + \delta_1)$. Hence

$$\begin{aligned} &\mathsf{E}[f(\epsilon_{0,t} W^+_t) 1_{\{\min_{0,s} < \min_{0,t}\}} | \mathcal{F}_s] = \\ &\qquad \frac{1}{2} \mathsf{E}[(f(W^+_t) + f(-W^+_t)) 1_{\{\min_{0,s} < \min_{0,t}\}} | \mathcal{F}_s] \end{aligned}$$

for all bounded Borel function f.

Moreover (we denote by P_t the heat semigroup)

$$\begin{aligned} &\mathsf{E}[f(\epsilon_{0,s} W^+_s + W_{s,t}) 1_{\{\min_{0,s} = \min_{0,t}\}} | \mathcal{F}_s] \\ &= \mathsf{P}_{t-s} f(\epsilon_{0,s} W^+_s) - \mathsf{E}[f(\epsilon_{0,s} W^+_s + W_{s,t}) 1_{\{\min_{0,s} < \min_{0,t}\}} | \mathcal{F}_s] \\ &= \mathsf{P}_{t-s} f(\epsilon_{0,s} W^+_s) - \frac{1}{2} \mathsf{E}[(f(W^+_t) + f(-W^+_t)) 1_{\{\min_{0,s} < \min_{0,t}\}} | \mathcal{F}_s] \end{aligned}$$

by the reflecting principle applied to $W^+_s + W_{s,u}$ for $u \in [s,t]$. This proves that

$$\mathsf{E}[f(\epsilon_{0,t} W^+_t) | \mathcal{F}_s] = \mathsf{P}_{t-s} f(\epsilon_{0,s} W^+_s).$$

This proves the Remark. □

Remark 4.5. *One can also prove directly that φ solves Tanaka's SDE driven by W: For all $s < t$ and x,*

$$\varphi_{s,t}(x) - x = \int_s^t \mathsf{sign}(\varphi_{s,u}(x)) dW_u.$$

Proof : Set $B_t = \int_0^t \epsilon_{0,s} d\varphi_{0,s}(0)$. Then B is a Brownian motion, and $\varphi_{0,t}(0) = \int_0^t \epsilon_s dB_s$. We also have

$$|\varphi_{0,t}(0)| = B_t + L_t$$

where L is the local time at 0 of $\varphi_{0,\cdot}(0)$. But, since $|\varphi_{0,t}(0)| = W^+_{0,t}$, we have

$$|\varphi_{0,t}(0)| = W_t - m_{0,t}.$$

This implies that $B = W$ and

$$\varphi_{0,t}(0) = \int_0^t \mathsf{sign}(\varphi_{0,s}(0))dW_s.$$

For $x \in \mathbb{R}$, using the fact that for $t \geq \tau_0^x$, $\varphi_{0,t}(x) = \varphi_{0,t}(0)$,

$$\begin{aligned}\int_0^t \mathsf{sign}(\varphi_{0,s}(x))dW_s &= (\mathsf{sign}(x)W_t)1_{\{t\leq\tau_0^x\}} \\ &+ \left(\mathsf{sign}(x)W_{\tau_0^x} + \varphi_{0,t}(0) - \varphi_{0,\tau_0^x}\right)1_{\{t>\tau_0^x\}}.\end{aligned}$$

When $t \leq \tau_0^x$, $\mathsf{sign}(x)W_t = \varphi_{0,t}(x) - x$. We have $W_{\tau_0^x} = -|x|$ and $\varphi_{0,\tau_0^x}(0) = 0$, so when $t > \tau_0^x$,

$$\mathsf{sign}(x)W_{\tau_0^x} + \varphi_{0,t}(0) - \varphi_{0,\tau_0^x} = -x + \varphi_{0,t}(0).$$

This proves that

$$\int_0^t \mathsf{sign}(\varphi_{0,s}(x))dW_s = \varphi_{0,t}(x) - x.$$

This proves the Remark. □

Lemma 4.6. *K solves Tanaka's SDE.*

Proof : For all $f \in C_K^2$,

$$f(\varphi_{s,t}(x)) = f(x) + \int_s^t (\mathsf{sign} f')(\varphi_{s,u}(x))dW_u + \frac{1}{2}\int_s^t f"(\varphi_{s,u}(x))du.$$

To prove that K solves Tanaka's SDE, it remains to prove that ($\mathsf{E}^{U,W}$ denotes the conditional expectation with respect to $\sigma(U,W)$)

$$\mathsf{E}^{U,W}\left[\int_s^t (\mathsf{sign} f')(\varphi_{s,u}(x))dW_u\right] = \int_s^t \mathsf{E}^{U,W}\left[(\mathsf{sign} f')(\varphi_{s,u}(x))\right]dW_u$$

and

$$\mathsf{E}^{U,W}\left[\int_s^t f"(\varphi_{s,u}(x))du\right] = \int_s^t \mathsf{E}^{U,W}[f"(\varphi_{s,u}(x))]du.$$

The second equality is standard. For the first one we apply the following Lemma.

Lemma 4.7. *Let H_u be a bounded L^2-continuous process adapted to the filtration $\mathcal{F}_{0,t}^{\epsilon,U,W}$. Then*

$$\mathsf{E}^{U,W}\left[\int_0^t H_u dW_u\right] = \int_0^t \mathsf{E}[H_u|\mathcal{F}_u^{U,W}]dW_u.$$

Proof : We take measurable versions of H and of $K_u = \mathsf{E}[H_u|\mathcal{F}_u^{U,W}]$.

We have that as the step of the subdivision $0 = t_0 < t_1 < \cdots < t_n = t$ goes to 0 that $\sum_{i=1}^n H_{t_{i-1}}W_{t_{i-1},t_i}$ converges in L^2 towards $\int_0^t H_u dW_u$. For all i,

$$\begin{aligned}\mathsf{E}^{U,W}[H_{t_{i-1}}W_{t_{i-1},t_i}] &= \mathsf{E}[H_{t_{i-1}}|\mathcal{F}_{0,t_{i-1}}^{U,W}]W_{t_{i-1},t_i} \\ &= K_{t_{i-1}}W_{t_{i-1},t_i}.\end{aligned}$$

We also have that $\int_0^t K_u dW_u$ is the limit in L^2 of $\sum_{i=1}^n K_{t_{i-1}}W_{t_{i-1},t_i}$.

Since $\mathsf{E}^{U,W}\left[\sum_{i=1}^n H_{t_{i-1}}W_{t_{i-1},t_i}\right]$ converges in L^2 towards $\mathsf{E}[\int_0^t H_u dW_u|U,W]$, this proves the Lemma. □

Proposition 4.1 shows the existence of a stochastic flow of kernels such that for all $s < t$, the law of $K_{s,t}$ is ν_{t-s}^m, and which solves Tanaka's SDE. The weak domination of ν^m by $\nu^{1/2(\delta_{-1}+\delta_1)}$ is also proved in this Proposition, the extension of the noise of the coalescing flow being given by $(\mathcal{F}^{\epsilon,U,W}_{s,t})_{s<t}$.

Remark 4.8. The noise of K is given by $\mathcal{F}^K_{s,t} = \mathcal{F}^{U,W}_{s,t}$. To each local minimum of W is attached an independent random variable of law m. The case of φ^c is directly related to the description given in Warren (1999) for the noise of splitting. The noise of φ is given by $\mathcal{F}^{\epsilon,W}_{s,t}$. Among related works let us also mention Le Jan and Raimond (2004b) and Watanabe (2000).

4.2. *Study of a solution of Tanaka's SDE.*

4.2.1. *Properties satisfied by a solution.* In this section, K is a solution of Tanaka's SDE and θ denotes the shift operator such that $K_{s,t}\circ\theta_h = K_{s+h,t+h}$.

Lemma 4.9.

(i) $\mathsf{E}[K|W] = K^W$.

(ii) *There exists an extension* $\hat{N}$ *of* N^c *and a subnoise* N *of* $\hat{N}$ *such that* $\mathsf{E}[\delta_{\varphi^c}|N] = K$.

(iii) *For all* $s\le t$ *and* $x\in\mathbb{R}$,

$$K_{s,t}(x) = \delta_{x+\mathsf{sign}(x)W_{s,t}}1_{t\le s+\tau_x\circ\theta_s} + K_{s,t}(0)1_{t>s+\tau_x\circ\theta_s}.$$

(iv) *For all* $s<t$ *there exists a random variable* $U_{s,t}\in[0,1]$ *such that*

$$K_{s,t}(0) = U_{s,t}\delta_{W^+_{t-s}\circ\theta_s} + (1-U_{s,t})\delta_{-W^+_{t-s}\circ\theta_s}.$$

In the terminology of LJR, (i) is equivalent to say that K^W is strongly dominated by K, (ii) is equivalent to say that K is weakly dominated by φ^c (cf. LJR section 6.4 and Remark 6.3). We recall that weak and strong dominations are partial order relations and that strong domination implies weak domination (the converse is false). In (iv), the Wiener solution corresponds to the case $U_{s,t} = 1/2$ and the coalescing solution to the case $\mathsf{P}(U_{s,t}=1) = 1-\mathsf{P}(U_{s,t}=0) = 1/2$. Finally, note that (i) implies that $\mathsf{E}[U_{s,t}|W] = 1/2$.

Proof of Lemma 4.9: (i) and (ii) are proved in LJR in a more general context. We now consider the probability space associated to the noise $\hat{N}$, the extension of N^c given by (ii). On this probability space, φ^c, K and W are well defined, N^K, N^W and N^c are subnoises of $\hat{N}$. Since N^W is a subnoise of N^K and since ($\hat{\theta}$ is the shift operator of $\hat{N}$)

$$\varphi^c_{s,t}(x) = (x+\mathsf{sign}(x)W_{s,t})1_{\{t\le s+\tau_x\circ\hat{\theta}_s\}} + \varphi^c_{s,t}(0)1_{\{t>s+\tau_x\circ\hat{\theta}_s\}},$$

(ii) implies that

$$\begin{aligned} K_{s,t}(x) &= \mathsf{E}[\delta_{\varphi^c_{s,t}(x)}|N^K] \\ &= \delta_{x+\mathsf{sign}(x)W_{s,t}}1_{\{t\le s+\tau_x\circ\hat{\theta}_s\}} + \mathsf{E}[\delta_{\varphi^c_{s,t}(0)}|N^K]1_{\{t>s+\tau_x\circ\hat{\theta}_s\}} \\ &= \delta_{x+\mathsf{sign}(x)W_{s,t}}1_{\{t\le s+\tau_x\circ\hat{\theta}_s\}} + K_{s,t}(0)1_{\{t>s+\tau_x\circ\hat{\theta}_s\}}. \end{aligned}$$

This proves (iii).

To simplify the notation, take $s=0$ and denote $W_t = W_{0,t}$. Then since $\varphi^c_{0,t}(0) = \int_0^t \mathsf{sign}(\varphi^c_{0,u}(0))dW_u$,

$$W_t = \int_0^t \mathsf{sign}(\varphi^c_{0,u}(0))d\varphi^c_{0,u}(0),$$

and it is well known (see for example Revuz and Yor (1999) p. 239-240 and Ex. 1-19 p. 375) that $W^+_t = |\varphi^c_{0,t}(0)|$. Thus there exists an event A_t such that $\varphi^c_{0,t}(0) = 1_{A_t}W^+_t - 1_{A^c_t}W^+_t$. (i) and (ii) then imply that $U_{0,t} = \mathsf{P}(A_t|W)$. □

Since K solves Tanaka's SDE, K can be modified in such a way that for all $\mu \in \mathcal{P}(\mathbb{R})$ and $s \in \mathbb{R}$, $t \in]s,\infty[\mapsto \mu K_{s,t}$ is continuous in t. We will now work with this modification. If we set $U_{s,t} = \int_{]0,\infty[} K_{s,t}(0,dy)$ and $U_t = U_{0,t}$, then the process U is constant on the excursions of W^+ (since $K_{0,t}(0) = \delta_0 K_{0,t}$ charges 0 if and only if $W^+_t = 0$).

Note that for all $\mu \in \mathcal{P}(\mathbb{R})$, $(\mu K_{0,t})_{t\geq 0}$ is a Feller process taking its values in $\mathcal{P}(\mathbb{R})$. Denote by P_μ the law of this process. It satisfies the strong Markov property:

Lemma 4.10. *(strong Markov property) Let T be a finite $(\mathcal{F}_{0,t})_{t\geq 0}$ stopping time. Then the law of $(\nu K_{0,T+t},\ t \geq 0)$ knowing of $\mathcal{F}_{0,T}$ is $\mathsf{P}_{\nu K_{0,T}}$.*

If $\nu = \delta_0$ and T is such that $W^+_T = 0$ then $\mu K_{0,T} = K_{0,T}(0) = \delta_0$. Thus in this case, $(\mu K_{0,T+t},\ t \geq 0)$ is independent of $\mathcal{F}_{0,T}$ and its law is P_{δ_0}.

4.2.2. *The process $U_{0,\cdot}$ given W.* All the σ-fields considered in the following will be assumed to include all P-negligible sets. We first remark that, by Doob-Meyer decomposition, $\sigma(W^+_s;\ s \leq t) = \mathcal{F}^W_{0,t}$. Let T and L the random times defined by

$$\begin{aligned} T &= \inf\{t \geq 0;\ W^+_t = 1\} \\ L &= \sup\{t \in [0,T];\ W^+_t = 0\}. \end{aligned}$$

Then T is an $\mathcal{F}^W$ stopping time but L is not. Define $(\mathcal{H}_t)_{t\geq 0}$ to be the usual augmentation of $(\mathcal{F}^W_{0,t} \vee \sigma(L))_{t\geq 0}$. Define the following σ-fields

$$\begin{aligned} \mathcal{F}^-_L &= \sigma(X_L : X \text{ is a bounded } \mathcal{F}^W - \text{previsible process}), \\ \mathcal{F}^+_L &= \sigma(X_L : X \text{ is a bounded } \mathcal{F}^W - \text{progressive process}). \end{aligned}$$

Note that (see Jeulin (1980), p. 77) $\mathcal{F}^+_L = \mathcal{H}_L$.

Lemma 4.11. $\mathcal{F}^-_L = \mathcal{F}^+_L$.

Proof : We follow Barlow et al. (1989). Note that $(W^+_{L+t};\ t \leq T-L)$ is a BES(3) up to its first hitting time of 1, and that this process is independent of $\mathcal{F}^-_L$. For $\epsilon \in]0,1[$, let $T_\epsilon = \inf\{t;\ W^+_{L+t} = \epsilon\}$. Then $L + T_\epsilon$ is an $\mathcal{H}$-stopping time and $\mathcal{H}_{L+T_\epsilon} = \mathcal{F}^-_L \vee \sigma(W^+_{L+u};\ u \leq T_\epsilon)$. By the zero one law (for the Bessel process), $\cap_{\epsilon>0}\mathcal{H}_{L+T_\epsilon} = \mathcal{F}^-_L$, which implies that $\mathcal{F}^-_L = \mathcal{F}^+_L$. □

Lemma 4.12. *Let $f : \mathbb{R} \to \mathbb{R}$ be a bounded continuous function. Set $X_t = \mathsf{E}[f(U_t)|W]$. There exists an $\mathcal{F}^W$-progressive version of X that is constant on the excursions of W^+ out of 0.*

Proof : By induction, for all integers k and n, define the stopping times S^k_n and T^k_n by the relations $T^0_n = 0$ and for $k \geq 1$,

$$\begin{aligned} S^k_n &= \inf\{t \geq T^{k-1}_n,\ W^+_t = 2^{-n}\}; \\ T^k_n &= \inf\{t \geq S^k_n,\ W^+_t = 0\}. \end{aligned}$$

Then for $t \in [S_n^k, T_n^k[$, $U_t = U_{S_n^k}$. Set $I_n = \cup_{k\geq 1}[S_n^k, T_n^k[$. In the following U_n^k will denote $U(S_n^k)$. Note that for all $t \in [S_n^k, T_n^k]$, $\mathsf{E}[f(U_t)|W] = \mathsf{E}[f(U_n^k)|W]$ a.s., and that $\mathsf{E}[f(U_n^k)|W]$ is $\mathcal{F}^W_{S_n^k}$ measurable since $\theta_{S_n^k}(W)$ and $\mathcal{F}_{S_n^k}$ are independent.

We define a sequence X^n of $\mathcal{F}^W$-progressive processes by induction. For $t \in [S_0^k, T_0^k[$, set $X_t^0 = \mathsf{E}[f(U_0^k)|W]$ and $X_t^0 = 0$ if $t \notin I_0$. Then X^0 is $\mathcal{F}^W$-progressive. Suppose a $\mathcal{F}^W$-progressive process X^n is defined such that for $t \in [S_n^k, T_n^k[$, $X_t^n = \mathsf{E}[f(U_n^k)|W]$ and $X_t^n = 0$ if $t \notin I_n$. We now define X^{n+1}. For $t \notin I_{n+1}$, set $X_t^{n+1} = 0$. On the event $t \in [S_{n+1}^l, T_{n+1}^l[\supset [S_n^k, T_n^k[$ (note that for all k there exists l such that $]S_n^k, T_n^k] \subset]S_{n+1}^l, T_{n+1}^l]$), set $X_t^{n+1} = X^n_{S_n^k}$. This implies that $X_t^n = X_t^{n+1}$ for $t \in I_n$. On the event $t \in]S_{n+1}^l, T_{n+1}^l]$ and $I_n \cap]S_{n+1}^l, T_{n+1}^l] = \emptyset$, set $X_t^{n+1} = \mathsf{E}[f(U_{n+1}^l)|W]$. The process X^{n+1} is $\mathcal{F}^W$-progressive and for all $t \in]S_{n+1}^l, T_{n+1}^l]$, $X_t^{n+1} = \mathsf{E}[f(U_{n+1}^l)|W]$ a.s.

For all t, X_t^n is a stationary sequence. Set $\tilde{X}_t = \limsup_{n\to\infty} X_t^n$. For all $t > 0$, a.s., there exist integers k and n such that $t \in [S_n^k, T_n^k[$. Thus a.s., $\tilde{X}_t = X_t$. This proves that $\tilde{X}$ is a modification of X, and $\tilde{X}$ is $\mathcal{F}^W$-progressive. □

We now take for X this $\mathcal{F}^W$-progressive version. Then $X_T = \mathsf{E}[f(U_T)|W]$ is $\mathcal{F}_L^+$-measurable.

Lemma 4.13. $\mathsf{E}[X_T|\mathcal{F}_L^-] = \mathsf{E}[f(U_T)]$.

Proof : Let S be an $\mathcal{F}^W$-stopping time and $A \in \mathcal{F}_S^W$. Set $d_S = \inf\{t \geq S :\ W_t^+ = 0\}$. Note that a.s. on the event $\{S < L\}$, $d_S < L$. This implies that $\{S < L\} = \{d_S < T\}$ (up to some negligible set) and that $1_A 1_{\{S<L\}} = 1_A 1_{\{d_S<T\}}$, which is $\mathcal{F}^W_{d_S}$-measurable. Now

$$
\begin{aligned}
\mathsf{E}[X_T 1_A 1_{\{d_S<T\}}] &= \mathsf{E}[f(U_T) 1_A 1_{\{d_S<T\}}] \\
&= \mathsf{E}[f(\delta_0 K_{0,T}(]0,\infty[)) 1_A 1_{\{d_S<T\}}] \\
&= \mathsf{E}[f(\delta_0 K_{0,T}(]0,\infty[))] \mathsf{P}(A \cap \{d_S < T\})
\end{aligned}
$$

by the strong Markov property (Lemma 4.10) at time d_S. Since $\mathcal{F}_L^-$ is generated by the random variables $1_A 1_{\{S<L\}}$, this implies the Lemma. □

The fact that $\mathcal{F}_L^- = \mathcal{F}_L^+$ and the fact that X_T is $\mathcal{F}_L^+$ measurable imply that $X_T = \mathsf{E}[f(U_T)|W] = \mathsf{E}[f(U_T)]$ a.s. Since this holds for all bounded continuous function f, this proves

Lemma 4.14. *U_T is independent of W.*

The result of this Lemma also holds if we replace T by $\inf\{t \geq 0 :\ W_t^+ = a\}$ for all positive a. This implies that

Lemma 4.15. *For all n, $(U_n^k)_{k\geq 1}$ is a sequence of independent identically distributed random variables. Moreover, this sequence is independent of W.*

Proof : Denote by μ_n the law of $U_n^1 = U_{S_n^1}$. Lemma 4.14 shows that U_n^1 is independent of W. Using the strong Markov property at time T_n^{k-1} and Lemma 4.14, we show that the law of U_n^k knowing $\mathcal{F}_{T_n^{k-1}}$ and W is μ_n. The Lemma now easily follows. □

Lemma 4.16. *The process U is stationary. Denote by m the law of U_1, then for all positive t, on the event $\{W_t^+ \neq 0\}$, the law of U_t knowing W is m.*

Proof : We have for all positive t and all bounded continuous function $f:\mathbb{R}\to\mathbb{R}$,

$$\begin{aligned}\mathsf{E}[f(U_t)|W]1_{\{W_t^+\neq 0\}} &= \lim_{n\to\infty}\sum_k \mathsf{E}[1_{t\in[S_n^k,T_n^k[}f(U_n^k)|W] \\ &= \lim_{n\to\infty}\sum_k 1_{t\in[S_n^k,T_n^k[}\left(\int f d\mu_n\right). \\ &= \lim_{n\to\infty}\left[\left(\int f d\mu_n\right)1_{\{W_t^+\neq 0\}}+\epsilon_n(t)\right],\end{aligned}$$

where $\epsilon_n(t) = 1_{t\notin I_n}1_{\{W_t^+\neq 0\}}\left(\int f d\mu_n\right) \leq \|f\|_\infty 1_{t\notin I_n}1_{\{W_t^+\neq 0\}}$. On the event $\{W_t^+\neq 0\}$, $1_{t\notin I_n}\to 0$ as $n\to\infty$. This implies that $\epsilon_n(t)$ converges towards 0 as $n\to\infty$ and that $\left(\int f d\mu_n\right)1_{\{W_t^+\neq 0\}}$ converges towards $\mathsf{E}[f(U_t)|W]1_{\{W_t^+\neq 0\}}$. In particular this shows that $\int f d\mu_n$ converges, its limit being $\mathsf{E}[f(U_t)]$ (since $\mathsf{P}(W_t^+\neq 0)=1$). Thus U is stationary ($\lim\int f d\mu_n$ does not depend on t) and $\mathsf{E}[f(U_t)|W]1_{W_t^+\neq 0} = \left(\int f d\mu\right)1_{W_t^+\neq 0}$, where μ is the law of U_t. The Lemma is proved. □

Thus we have proved

Proposition 4.17. *Let K be a solution of Tanaka's SDE. Then there exists a probability measure m on $[0,1]$ such that $\int_{[0,1]} xm(dx) = 1/2$, a standard Brownian motion W and a random variable U_t of law m and independent of W such that*

$$K_{0,t}(x) = \delta_{x+\mathsf{sign}(x)W_t}1_{\{t\leq\tau_x\}} + (U_t\delta_{W_t^+} + (1-U_t)\delta_{-W_t^+})1_{\{t>\tau_x\}},$$

where $\tau_x = \inf\{t\geq 0 : W_t = -|x|\}$ and $W_t^+ = W_t - \inf_{s\leq t} W_s$.

This property shows that the law of $K_{0,t}$ is ν_t^m. Thus the Feller convolution semigroup associated to K is ν^m. This finishes the Proof of Theorem 2.1.

5. Proof of Theorem 2.3

5.1. *The filtration $\mathcal{F}_{0,t}^K$.* Let K be the solution of Tanaka's SDE constructed in section 4.1. Then

$$\mathcal{F}_{0,t}^K = \sigma(U_t)\vee\sigma(U_{u,v}1_{\{\min_{0,t}\neq\min_{u,v}\}};\ 0\leq u\leq v\leq t)\vee\mathcal{F}_{0,t}^W$$

and, by construction, U_t is independent of the σ-field

$$\sigma(U_{u,v}1_{\{\min_{0,t}\neq\min_{u,v}\}};\ 0\leq u\leq v\leq t)\vee\mathcal{F}_{0,t}^W.$$

5.2. *Strong domination.* We prove the first part (a) of Theorem 2.3.

Assume that ν^{m_1} dominates ν^{m_2}. Then, on N^{m_1} the noise associated to ν^{m_1}, there exists a subnoise N^2 such that $K^2 = \mathsf{E}_{\mathsf{Q}^{m_1}}[K^1|N^2]$ has law Q^{m_2} (K^1 denotes the canonical stochastic flow of kernels of law Q^{m_1}) and N^2 is the noise of K^2. Note that the noise of W is a subnoise of N^2.

Note first the following easy lemmas.

Lemma 5.1. *Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be σ-fields such that $\mathcal{A}\subset\mathcal{B}\vee\mathcal{C}$ and $\mathcal{A}\vee\mathcal{B}$ is independent of $\mathcal{C}$. Then $\mathcal{A}\subset\mathcal{B}$ a.s.*

Proof : For all A, B and C be respectively bounded $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ measurable functions,

$$\mathsf{E}[ABC] = \mathsf{E}[AB]\mathsf{E}[C] = \mathsf{E}[\mathsf{E}[A|\mathcal{B}]B]\mathsf{E}[C] = \mathsf{E}[\mathsf{E}[A|\mathcal{B}]BC]$$

which implies $A = \mathsf{E}[A|\mathcal{B}]$ a.s. □

Lemma 5.2. *Let N^1 and N^2 be two noises such that N^2 is a sub-noise of N^1. Then for all t, $\mathcal{F}^1_{-\infty,t}$ and $\mathcal{F}^1_{t,\infty}$ are independent conditionally to N^2.*

Proof : Let P^1_t (resp. F^1_t) be bounded $\mathcal{F}^1_{-\infty,t}$-measurable (resp. $\mathcal{F}^1_{t,\infty}$-measurable). Let $P^2_{-\infty,t}$ (resp. $F^2_{t,\infty}$) be bounded $\mathcal{F}^2_{-\infty,t}$-measurable (resp. $\mathcal{F}^2_{t,\infty}$-measurable). Then (using the fact that $\mathcal{F}^1_{-\infty,t}$ and $\mathcal{F}^1_{t,\infty}$ are independent, that N^2 is a subnoise of N^1 and that $\mathcal{F}^2_{-\infty,t}$ and $\mathcal{F}^2_{t,\infty}$ are independent)

$$\begin{aligned}
\mathsf{E}[P^1_t F^1_t P^2_t F^2_t] &= \mathsf{E}[P^1_t P^2_t]\mathsf{E}[F^1_t F^2_t] \\
&= \mathsf{E}[\mathsf{E}[P^1_t|N^2]P^2_t]\mathsf{E}[\mathsf{E}[F^1_t|N^2]F^2_t] \\
&= \mathsf{E}[\mathsf{E}[P^1_t|N^2]\mathsf{E}[F^1_t|N^2]P^2_t F^2_t].
\end{aligned}$$

Since $\mathcal{F}^2_{-\infty,\infty} = \mathcal{F}^2_{-\infty,t} \vee \mathcal{F}^2_{t,\infty}$, this proves that

$$\mathsf{E}[P^1_t F^1_t|N^2] = \mathsf{E}[P^1_t|N^2]\mathsf{E}[F^1_t|N^2].$$

□

Note that for all $0 < u < v < t$, on the event $\{\min_{u,v} = \min_{0,t}\}$, $U_t = U_{u,v}$ where $U = (U^1, U^2)$ and $U_t = U_{0,t}$.

Lemma 5.3. *For all positive t, $U_t = (U^1_t, U^2_t)$ is independent of*

$$\sigma((U^1_{u,v}, U^2_{u,v})1_{\{min_{0,t} \neq min_{u,v}\}} :\ 0 \le u \le v \le t) \vee \mathcal{F}^W_{0,t}.$$

Proof : Let $(u_i, v_i)_{1\le i\le n}$ be such that $0 \le u_i < v_i \le t$. Let $(f_i)_{1\le i\le N}$ be bounded measurable functions. Then the random variables of the form

$$Z = \prod_{i=1}^n \left(f_i(U_{u_i,v_i})1_{\{\min_{u_i,v_i} \neq \min_{0,t}\}}\right)$$

generate the σ-field $\sigma(U_{u,v}1_{\{\min_{u,v}\neq\min_{0,t}\}} :\ 0 \le u < v \le t)$. Let A be the complementary of $\cup_{i=1}^n [u_i, v_i]$. Then A can be written as the disjoint union of intervals: $A = \cup_{j=1}^k]s_j, t_j[$, with $k \le n+1$, $s_j \in \{0\} \cup \{v_i;\ i \le n\}$ and $t_j \in \{t\} \cup \{u_i;\ i \le n\}$. Then

$$\bigcap_{i=1}^n \{\min_{u_i,v_i} \neq \min_{0,t}\} = \bigcup_{j=1}^k \{\min_{0,t} = \min_{s_j,t_j}\}.$$

Let $F : [0,1]^2 \to \mathbb{R}$ be bounded measurable. Since, on the event $\{\min_{0,t} = \min_{s_j,t_j}\}$, $U_t = U_{s_j,t_j}$, we have

$$\mathsf{E}[F(U_t)Z|W] = \sum_{j=1}^k \mathsf{E}[Z_{s_j}F(U_{s_j,t_j})Z_{t_j,t}|W]1_{\{\min_{0,t}=\min_{s_j,t_j}\}}$$

where

$$Z_{s_j} = \prod_{\{i;\ v_i \le s_j\}} f_i(U_{u_i,v_i})$$

is $\mathcal{F}^1_{0,s_j}$-measurable and

$$Z_{t_j,t} = \prod_{\{i;\ u_i \ge t_j\}} f_i(U_{u_i,v_i})$$

is $\mathcal{F}^1_{t_j,t}$-measurable. Lemma 5.2 then implies that

$$\begin{aligned}\mathsf{E}[Z_{s_j}F(U_{s_j,t_j})Z_{t_j,t}|W] &= \mathsf{E}[Z_{s_j}|W]\mathsf{E}[F(U_{s_j,t_j})|W]\mathsf{E}[Z_{t_j,t}|W]\\ &= \mathsf{E}[F(U_{s_j,t_j})|W]\mathsf{E}[Z_{s_j}Z_{t_j,t}|W].\end{aligned}$$

Like in section 4.2.2, one can prove that for all $s<t$, $U_{s,t}$ is independent of W, its law being independent of s and t. we denote the law of $U_{s,t}$ by $m^{1,2}$. Then we have

$$\begin{aligned}\mathsf{E}[F(U_t)Z|W] &= m^{1,2}(F)\times\left(\sum_{j=1}^k \mathsf{E}[Z_{s_j}Z_{t_j,t}|W]1_{\{\min_{0,t}=\min_{s_j,t_j}\}}\right)\\ &= m^{1,2}(F)\mathsf{E}[Z|W].\end{aligned}$$

This proves the Lemma. □

We now apply Lemma 5.1 with $\mathcal{A}=\sigma(U_t^2)$, $\mathcal{B}=\sigma(U_t^1)$, $\mathcal{C}=\sigma(U_{u,v}1_{\{\min_{u,v}\neq\min_{0,t}\}}: 0\le u<v\le t)\vee\mathcal{F}^W_{0,t}$. This proves that $\sigma(U_t^2)\subset\sigma(U_t^1)$. Thus $U_t^2=\psi(U_t^1)$ for some measurable ψ. We conclude using the fact that $\mathsf{E}[U_t^1|K^2]=\mathsf{E}[U_t^1|U_t^2]=U_t^2$.

On the converse, assume that $m_2=\psi^*m_1$ and $(I\cdot m_2)=\psi^*(I\cdot m_1)$. Let W be a Brownian motion started at 0 and U an independent $[0,1]$-valued random variable of law m_1. Let $\hat{\nu}_t$ be the law of $\hat{K}_t$, where $\hat{K}_t:\mathbb{R}^2\to\mathcal{P}(\mathbb{R}^2)$ is defined by $\hat{K}_t(x,y)=K_t^1(x)\otimes K_t^2(y)$ where

$$\begin{aligned}K_t^1(x) &= \delta_{x+\mathsf{sign}(x)W_t}1_{\{t\le\tau_x\}}\\ &+ (U\delta_{W_t^+}+(1-U)\delta_{-W_t^+})1_{\{t>\tau_x\}},\\ K_t^2(y) &= \delta_{y+\mathsf{sign}(y)W_t}1_{\{t\le\tau_y\}}\\ &+ (\psi(U)\delta_{W_t^+}+(1-\psi(U))\delta_{-W_t^+})1_{\{t>\tau_x\}}.\end{aligned}$$

($W_t^+=W_t-\inf_{s\le t}W_s$ and τ_x is the first time W hits $-|x|$.) Then like in section 4.1, we prove that $\hat{\nu}$ defines a Feller convolution semigroups to which is associated a stochastic flow of kernels $\hat{K}$ on $\mathbb{R}^2$. Then $\hat{K}_{s,t}(x,y)=K^1_{s,t}(x)\otimes K^2_{s,t}(y)$, where K^1 and K^2 are respectively stochastic flows of kernels on $\mathbb{R}$ of law Q^{m_1} and Q^{m_2}. And we have

$$\begin{aligned}\mathsf{E}[K^1_{0,t}|K^2_{0,t_1},\dots,K^2_{t_n,t}] &= \mathsf{E}[K^1_{0,t_1}|K^2_{0,t_1}]\cdots\mathsf{E}[K^1_{t_n,t}|K^2_{t_n,t}]\\ &= K^2_{0,t_1}\cdots K^2_{t_n,t}\\ &= K^2_{0,t}.\end{aligned}$$

This proves part (a) of Theorem 2.3.

5.3. *Weak domination.* We prove part (b) of Theorem 2.3.

Assume ν^{m_1} weakly dominates ν^{m_2}. Then there exists an extension $\hat{N}$ of N^{m_1} the noise associated to ν^{m_1}, and a subnoise N^2 such that $K^2=\hat{\mathsf{E}}[K^1|N^2]$ has law Q^{m_2} (K^1 denotes the canonical stochastic flow of kernels of law Q^{m_1}) and N^2 is the noise of K^2. We then have, for all positive t, $\hat{\mathsf{E}}[U_t^1|K^2]=U_t^2$, and Jensen's inequality implies that m_1 is swept by m_2.

To prove the converse statement, we prove the following Lemma which is a consequence of Rost Theorem:

Lemma 5.4. *Assume m_1 is swept by m_2, then there exists a probability space $(\Omega, \mathcal{F}, \mathsf{P})$ on which exists a random variable X of law m_1 and a sub-σ-field $\mathcal{G}$ such that the law of $\mathsf{E}[X|\mathcal{G}]$ is m_2.*

Proof : (cf. Dellacherie et al. (1992), chap. XVIII p. 47): On some filtered probability space,

$$(\Omega, \mathcal{F}, (\mathcal{F}_t)_t, \mathsf{P})$$

there exists a $\mathcal{F}_t$-Brownian motion B with initial distribution m_2 and a finite stopping time T such that $B_0 = \mathsf{E}[B_T|\mathcal{F}_0]$. □

We now finish the Proof of part (b) of Theorem 2.3. Like in section 4.1 we construct (U^1, U^2, W) by replacing ϵ (in section 4.1) by U^1 and U by U^2 such that for all $s < t$, the law of $U^1_{s,t}$ is m_1 and the law of $U^2_{s,t}$ is m_2 and $\mathsf{E}[U^1_{s,t}|U^2_{s,t}] = U^2_{s,t}$. This way we construct K^1 and K^2, such that the law of K^1 is Q^{m_1}, the law of K^2 is Q^{m_2}, and $K^2 = \mathsf{E}[K^1|K^2]$. □

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