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Recurrence of Simple Random Walk on $\mathbb{Z}^2$ is Dynamically Sensitive

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Abstract. Benjamini et al. (2003) introduced the concept of a dynamical random walk. This is a continuous family of random walks, $\{S_n(t)\}_{n\in\mathbb{N},t\in\mathbb{R}}$. Benjamini et al. (2003) proved that if $d = 3$ or $d = 4$ then there is an exceptional set of t such that $\{S_n(t)\}_{n\in\mathbb{N}}$ returns to the origin infinitely often. In this paper we consider a dynamical random walk on $\mathbb{Z}^2$. We show that with probability one there exists $t \in \mathbb{R}$ such that $\{S_n(t)\}_{n\in\mathbb{N}}$ never returns to the origin. This exceptional set of times has dimension one. This proves a conjecture of Benjamini et al. (2003).

1. Introduction

We consider a dynamical simple random walk on $\mathbb{Z}^2$. Associated with each n is Poisson clock. When the clock rings the nth step of the random walk is replaced by an independent random variable with the same distribution. Thus for any fixed t the distribution of the walks at time t is that of simple random walk on $\mathbb{Z}^2$ and is almost surely recurrent.

We prove that with probability one there exists a (random) set of times t such that $S_n(t) \neq \mathbf{0}\ \forall\ n \in \mathbb{N}$. Thus we say that recurrence of simple random walk on $\mathbb{Z}^2$ is **dynamically sensitive**.

More formally let $\{Y_n^m\}_{m,n\in\mathbb{N}}$ be uniformly distributed i.i.d. random variables chosen from the set $\{(0,1),(0,-1),(1,0),(-1,0)\}$. Let $\{\tau_n^{(m)}\}_{m\geq 0,n\in\mathbb{N}}$ be an independent Poisson process of rate one and $\tau_n^{(0)} = 0$ for each n. Define

$$X_n(t) = Y_n^m$$

for all $t \in [\tau_n^{(m)}, \tau_n^{(m+1)})$. Let

$$S_n(t) = \sum_{i=1}^{n} X_i(t).$$

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Thus for each t the random variables $\{X_n(t)\}_{n\in\mathbb{N}}$ are i.i.d.

Define the exceptional set of times

$$\mathrm{Exc} = \{t:\ S_n(t) \neq \mathbf{0} \forall\ n\}.$$

Our main result is

Theorem 1.1.

$$\mathbf{P}(\mathit{Exc} \neq \emptyset) = 1.$$

Moreover, Exc has dimension 1 a.s.

Remark 1.2. *Our methods can be used to calculate a rate of escape. For any $\alpha < 1/2$ there is a set of t such that $|S_n(t)| > n^\alpha$ for all n. The limits of our method yield that with probability one there is a time t such where the rate of escape is at least*

$$|S_n(t)| > n^{.5 - 1/(\log n)^{1/4+\epsilon}}$$

for all n.

Benjamini et al. (2003) introduced the concept of dynamical random walk and showed that the strong law of large numbers and the law of iterated logarithms are satisfied for all times almost surely. Thus these properties are said to be **dynamically stable**. They also proved that in dimensions 3 and 4 that the transience of simple random walk is dynamically sensitive and in dimensions 5 and higher that transience is dynamically stable. Khoshnevisan et al. (2004), Khoshnevisan et al. (2005) have studied other properties of dynamical random walks. Häggström et al. (1997) studied similar questions of dynamic stability and sensitivity for percolation.

Dynamical random walk and the results in this paper are related to several other topics in probability. Most closely related to the work in this paper is a result of Adelman et al. (1998) about sets missed by three dimensional Brownian motion. The projection of Brownian motion on $\mathbb{R}^3$ onto a fixed plane yields Brownian motion in the plane which is neighborhood recurrent. For a fixed plane the projection of almost every Brownian path onto the plane is neighborhood recurrent. They proved that with probability one there is a (random) set of exceptional planes such that the set of times that the projected path is in any bounded set is bounded.

The questions studied about dynamical random walks and dynamical percolation have a strong resemblance to questions of quasi-everywhere properties of Brownian paths. These are properties that hold simultaneously for every cross section of a Brownian sheet with probability one. See Fukushima (1984) and Penrose (1989).

2. Outline

We start by introducing some notation. Let $s_0 = 1$ and $s_k = k^{10}2^{2k^2}$ for $k \geq 1$. This is a sequence of stopping times. Define the event $R_k(t)$ to be

$$R_k(t) = \{\exists n \in \{s_{k-1}, \dots, s_k\} \text{ such that } S_n(0) = \mathbf{0}\}.$$

For $x \in \mathbb{Z}^2$ we use the standard notation $|x| = \sqrt{(x_1)^2 + (x_2)^2}$. Define the annulus

$$A_k = \{x \in \mathbb{Z}^2:\ 2^{k^2} \leq |x| \leq k^{10}2^{k^2}\}.$$

Define the event $G_k(t)$ to be

$$G_k(t) = \{S_{s_k}(t) \in A_k\}.$$

Also define the events $G_k(0,t) = G_k(0) \cap G_k(t)$ and $R_k(0,t) = R_k(0) \cap R_k(t)$.

$$E_M(0) = \left(\cap_1^M G_k(0)\right) \setminus \left(\cup_1^M R_k(0)\right).$$
$$E_M(0,t) = E_M(0) \cap E_M(t).$$

We will show in Lemma 4.1 that there is an integrable function $f(t)$ such that for all M

$$\frac{\mathbf{P}(E_M(0,t))}{(\mathbf{P}(E_M(0)))^2} < f(t). \tag{2.1}$$

Theorem 1.1 follows from Lemma 4.1 by the second moment method.

We obtain (2.1) by multiplying together conditional probabilities. Some of our bounds will hold only when k is sufficiently large compared with $1/t$. For this reason we define $K = K(t)$ to be the unique integer such that

$$1 + |\log t| > K \geq |\log t| \tag{2.2}$$

for $t < 1$ and $K = 0$ if $t \geq 1$. Our main lemma is the following.

Lemma 2.1. *There is a positive sequence g_k such that*

(1) $\sum_1^\infty g_k < \infty$,

(2) $\Big(\mathbf{P}(E_k(0)|E_{k-1}(0))\Big)^2 > 1 - \frac{4}{k} - g_k$ *for all* $k > 1$, *and*

(3) $\mathbf{P}\Big(E_k(0,t)|E_{k-1}(0,t)\Big) < 1 - \frac{4}{k} + g_k$ *for all* $k > K$.

The next section is dedicated to proving Lemma 2.1. In the last section we show how Lemma 2.1 implies Theorem 1.1.

We end this section with a few notes about notation. We use $\log(n) = \log_2(n)$. We use C as a generic constant whose value may increase from line to line and lemma to lemma. In many of the proofs in the next section we use bounds that only hold for sufficiently large k. This causes no problem since it will be clear that we can always choose C such that the lemma is true for all k.

3. Proof of Lemma 2.1

For the rest of the paper we will use the following notation for conditional probabilities. Let

$$\mathbf{P}^{x,k-1}(*) = \mathbf{P}\left(*|\ S_{s_{k-1}}(0) = x\right)$$

and

$$\mathbf{P}^{x,y,k-1}(*) = \mathbf{P}\left(*|\ S_{s_{k-1}}(0) = x \text{ and } S_{s_{k-1}}(t) = y\right)$$

The two main parts of the Proof of Lemma 2.1 are Lemma 3.3 where we get upper and lower bounds on $\mathbf{P}^{x,k-1}(R_k(0))$ and Lemma 3.6 where we get an upper bound on $\mathbf{P}^{x,y,k-1}(R_k(0,t))$. The main tool that we use are bounds on the probability that simple random walk started at x returns to the origin before exiting the ball of radius n and center at the origin. The probability of this is calculated in Proposition 1.6.7 on page 40 of Lawler (1991). We use only a weak version of the result there.

Let η be the smallest $m > 0$ such that $S_m(0) = \mathbf{0}$ or $|S_m(0)| \geq n$.

Lemma 3.1. *There exists C such that for all x with* $0 < |x| < n$

$$\frac{\log(n) - \log|x| - C}{\log(n)} \leq \mathbf{P}(S_\eta(0) = \mathbf{0}|S_0(0) = x) \leq \frac{\log(n) - \log|x| + C}{\log(n)}.$$

We will frequently use the following standard bounds.

Lemma 3.2. *There exists C such that for all $x \in \mathbb{Z}^2$, $n \in \mathbb{N}$ and $m < \sqrt{n}$*

$$\mathbf{P}\left(\exists n' < n : S_{n'}(0) > m\sqrt{n}\right) \leq \frac{C}{m^2} \tag{3.1}$$

and

$$\mathbf{P}\left(|S_n(0) - x| < \frac{\sqrt{n}}{m}\right) \leq \frac{C}{m^2}.$$

Proof : If $|S_{n'}(0)| > m\sqrt{n}$ then one component of the random walk has absolute value bigger than $m\sqrt{n}/2$. Thus the left hand side of (3.1) is at most four times the probability that one dimensional simple random walk is ever more than $m\sqrt{n}/2$ away from the origin during the first n steps. The probability that a one dimensional simple random walk has ever been larger than $m\sqrt{n}/2$ in the first n steps is at most twice the probability that one dimensional simple random walk is greater than $m\sqrt{n}/2$ after n steps. Chebyshev's inequality then gives the first bound.

To bound the probability that $|S_n(0)-x|$ is too small we note that since $m < \sqrt{n}$ the number of $y \in \mathbb{Z}^2$ such that $|y-x| < \frac{\sqrt{n}}{m}$ is less than $\frac{10n}{m^2}$. There is C such that for any $n \in \mathbb{N}$ and $z \in \mathbb{Z}^2$ the probability that $S_n(0) = z$ is less than C/n. □

Lemma 3.3. *There exists C such that for any k and any $x \in A_{k-1}$*

$$\frac{2}{k} - \frac{C\log k}{k^2} \leq \mathbf{P}^{x,k-1}(R_k(0)) \leq \frac{2}{k} + \frac{C\log k}{k^2}.$$

Proof : If the random walk returns to $\mathbf{0}$ in less than s_k steps then either it returns to $\mathbf{0}$ before exiting the ball of radius $\sqrt{s_k}\log(s_k)$ or it exits the ball in less than s_k steps. Thus by Lemmas 3.1 and 3.2 our upper bound is

$$\begin{aligned}
&< \frac{\log(\sqrt{s_k}\log(s_k)) - \log(2^{(k-1)^2}) + C}{\log(\sqrt{s_k}\log(s_k))} + \frac{C}{(\log s_k)^2} \\
&< \frac{5\log k + k^2 + C\log k - (k-1)^2 + C}{5\log k + k^2 + \log(2k^2 + 10\log k)} + \frac{C}{k^4} \\
&< \frac{2k + C\log k}{k^2}.
\end{aligned}$$

If the random walk returns to $\mathbf{0}$ after s_{k-1} but before exiting the ball of radius $\sqrt{s_k}/\log(s_k)$ and it is outside the ball of radius $\sqrt{s_k}/\log(s_k)$ at time s_k then it has returned to $\mathbf{0}$ between times s_{k-1} and s_k. Thus by Lemmas 3.1 and 3.2 our lower bound is

$$\begin{aligned}
&> \frac{\log(\sqrt{s_k}/\log(s_k)) - \log((k-1)^{10}2^{(k-1)^2}) - C}{\log(\sqrt{s_k}/\log(s_k))} - \frac{C}{(\log(s_k))^2} \\
&> \frac{5\log k + k^2 - C\log k - 10\log(k-1) - (k-1)^2 - C}{5\log k + k^2 - \log\log s_k} - \frac{C}{k^4} \\
&> \frac{2k - C\log k}{k^2}.
\end{aligned}$$

□

Lemma 3.4. *For any k and $x \in A_{k-1}$*

$$\mathbf{P}^{x,k-1}((G_k(0))^C) \leq \frac{C}{k^{10}}.$$

Proof : This follows directly from Lemma 3.2 $\square$

Now we start to bound the probability that both walks return to the origin between times s_{k-1} and s_k. We first need the following lemma.

Lemma 3.5. *There exists C such that for all k, $n \geq s_{k-1} + s_k/2^{10k}$, for all $I \subset \{1, \ldots, n\}$ with $|I| \geq s_k/2^{10k}$ and for all $\{x_i(t)\}_{i \in \{1,\ldots,n\} \setminus I}$*

$$\mathbf{P}\left(\exists j \in \{n, \ldots, s_k\} \text{ such that } S_j(t) = \mathbf{0} \middle| \ \{x_i(t)\}_{i \in \{1,\ldots,n\} \setminus I}\right) \leq \frac{C}{k}.$$

Proof : Rearranging the first n steps of a random walk does not change the random walk after time n. The probability is largest when n and $|I|$ are as small as possible. Thus it causes no loss of generality to assume that $n = s_{k-1} + s_k/2^{10k}$ and $I = \{s_{k-1}+1, \ldots, n\}$.

If the event happens then either

(1) $|S_n(t)| \leq \frac{\sqrt{|I|}}{\log |I|}$,

(2) $|S_n(t)| > \frac{\sqrt{|I|}}{\log |I|}$ and

$$\inf\{j : j > n \text{ and } S_j(t) = \mathbf{0}\} < \inf\{j : j > n \text{ and } |S_j(t)| > \sqrt{s_k}\log(s_k)\}$$

or

(3) there exists j' such that $n < j' < s_k$ and $|S_{j'}(t) - S_n(t)| \geq .5\sqrt{s_k}\log s_k$.

The probability of the first event is bounded by the second half of Lemma 3.2 replacing n with $|I|$ and m with $\log |I|$. The probability of the second event is bounded by Lemma 3.1 replacing n by $\sqrt{s_k}\log(s_k)$ and x by $S_n(t)$. The probability of the third event is bounded by the first half of Lemma 3.2 replacing n with s_k and m with $.5\log(s_k)$. Thus our upper bound is

$$\begin{aligned}
&< \frac{C}{(\log |I|)^2} + \frac{\log(\sqrt{s_k}\log(s_k)) - \log\left(\frac{\sqrt{|I|}}{\log |I|}\right) + C}{\log(\sqrt{s_k}\log(s_k))} + \frac{C}{(\log s_k)^2} \\
&< \frac{C}{(\log |I|)^2} + \frac{.5\log(s_k) + \log(\log s_k) - \left(.5\log s_k - \log(2^{5k}) - \log(\log |I|)\right) + C}{.5\log(s_k) + \log(\log(s_k))} \\
&< \frac{C}{(2k^2 - 10k)^2} + \frac{Ck + C\log k + C}{k^2} \\
&< \frac{C}{(2k^2 - 10k)^2} + \frac{Ck}{k^2} \\
&< \frac{C}{k}.
\end{aligned}$$

$\square$

Lemma 3.6. *There exists C such that for any t, any $k > K(t)$ and any $x, y \in A_{k-1}$*

$$\mathbf{P}^{x,y,k-1}(R_k(0,t)) \leq \frac{C}{k^2}.$$

Proof : Let

$$I \subset \left\{s_{k-1}, \ldots, s_{k-1} + 2^{2(k-1)^2}/k^2\right\}$$

be the set of i such that conditioned on the Poisson process, $X_i(t)$ and $X_i(0)$ are independent. Let B be the event that there exists n such that

$$s_{k-1} \le n \le s_{k-1} + 2^{2(k-1)^2}/k^2$$

such that $S_n(0) = \mathbf{0}$. Let D be the event that there exist n and n' such that

(1) $s_{k-1} + 2^{2(k-1)^2}/k^2 < n \le n' \le s_k$
(2) $S_n(0) = S_{n'}(t) = \mathbf{0}$ and
(3) $|I| \ge s_k/2^{10k}$.

If $R_k(0,t)$ occurs then either the first return happens before step $s_{k-1}+2^{2(k-1)^2}/k^2$ or after that step. The probability that the first return is before is bounded by twice the probability of B. If the first return is after then either $|I| < s_k/2^{10k}$ or $|I| \ge s_k/2^{10k}$. The probability that the first return is after and $|I|$ is large is bounded by twice the probability of D. Thus we get that

$$\mathbf{P}^{x,y,k-1}(R_k(0,t)) \le 2\mathbf{P}^{x,y,k-1}(B) + \mathbf{P}^{x,y,k-1}(|I| < s_k/2^{10k}) + 2\mathbf{P}^{x,y,k-1}(D).$$

By (2.2) $\min(1,t) \ge 1/2^K$. As $k > K$ this implies the expected size of $|I|$ is

$$(1-e^{-t})\frac{2^{2(k-1)^2}}{k^2} > \frac{1}{2}\min(1,t)\frac{2^{2(k-1)^2}}{k^2} \ge \frac{2^{2(k-1)^2}}{2^{K+1}k^2} > \frac{2^{2k^2}}{2^{6k}} > 2\frac{s_k}{2^{10k}}. \tag{3.2}$$

Thus by Chebyshev's the probability that $|I| < s_k/2^{10k}$ is at most C/k^2.

By Lemma 3.2 the conditional probability of B is bounded by C/k^2. In order for D to happen we first need that the event $R_k(0)$ occurs. By Lemma 3.3 the probability of this is bounded by C/k. Now we condition on the following events

(1) $\{X_i(0)\}_{i\ge 0}$
(2) the Poisson process,
(3) $|I| \ge 2^{-10k}s_k$, and
(4) $\{X_i(t)\}_{i\in\{1,\dots,n\}\setminus I}$

and bound the probability that there exists $n' \in \{n,\dots,s_k\}$ with $S_{n'}(t) = \mathbf{0}$.

By the first condition in the definition of D and (3.2)

$$n > s_{k-1} + \frac{2^{(k-1)^2}}{k^2} > s_{k-1} + s_k/2^{10k}$$

and Lemma 3.5 applies. Thus the conditional probability of D given $R_k(0)$ is at most C/k.

Putting this together we get

$$\begin{aligned}
\mathbf{P}^{x,y,k-1}(R_k(0,t)) &\le 2\mathbf{P}^{x,y,k-1}(B) + \mathbf{P}^{x,y,k-1}\left(|I| < \frac{s_k}{2^{10k}}\right) + 2\mathbf{P}^{x,y,k-1}(D)\\
&\le \frac{C}{k^2} + \frac{C}{k^2} + 2\left(\frac{C}{k}\right)\left(\frac{C}{k}\right)\\
&\le \frac{C}{k^2}.
\end{aligned}$$

□

Proof of Lemma 2.1: We let

$$g_k = \frac{C\log k}{k^2}.$$

Clearly this satisfies the summability condition. Note that if $E_{k-1}(0)$ occurs then $G_{k-1}(0)$ occurs and $S_{s_{k-1}}(0) \in A_{k-1}$. Since simple random walk is Markovian, for any $x \in A_{k-1}$

$$\mathbf{P}(E_k(0)|E_{k-1}(0) \text{ and } S_{s_{k-1}}(0)=x) \quad = \quad \mathbf{P}^{x,k-1}((R_k(0))^C \cap G_k(0)).$$

Along with Lemmas 3.3 and 3.4 this tells us that

$$\begin{aligned}
\mathbf{P}(E_k(0)|E_{k-1}(0)) &\geq \min_{x\in A_{k-1}} \mathbf{P}\left(E_k(0)|E_{k-1}(0) \text{ and } S_{s_{k-1}}(0)=x\right) \\
&\geq \min_{x\in A_{k-1}} \mathbf{P}^{x,k-1}((R_k(0))^C \cap G_k(0)) \\
&\geq \min_{x\in A_{k-1}} \mathbf{P}^{x,k-1}((R_k(0))^C) - \max_{x\in A_{k-1}} \mathbf{P}^{x,k-1}\left((G_k(0))^C\right) \\
&\geq 1 - \max_{x\in A_{k-1}} \mathbf{P}^{x,k-1}(R_k(0)) - \frac{C}{k^{10}} \\
&\geq 1 - \left(\frac{2}{k} + \frac{C\log k}{k^2}\right) - \frac{C}{k^{10}} \\
&\geq 1 - \frac{2}{k} - \frac{C\log k}{k^2}.
\end{aligned}$$

Squaring both sides yields condition 2 of Lemma 2.1.

Also note that if $E_{k-1}(0,t)$ occurs then $G_{k-1}(0,t)$ occurs and

$$S_{s_{k-1}}(0), S_{s_{k-1}}(t) \in A_{k-1}.$$

Since dynamic random walk is Markovian, for any $x, y \in A_{k-1}$

$$\begin{aligned}
&\mathbf{P}\Big(E_k(0,t)|E_{k-1}(0,t) \text{ and } S_{s_{k-1}}(0)=x, S_{s_{k-1}}(t)=y\Big) \\
&\quad = \mathbf{P}^{x,y,k-1}\Big((R_k(0))^C \cap (R_k(t))^C \cap G_k(0,t)\Big).
\end{aligned}$$

Combining this with Lemmas 3.3 and 3.6 we get that

$$\begin{aligned}
&\mathbf{P}(E_k(0,t)|E_{k-1}(0,t)) \\
&\leq \max_{x,y\in A_{k-1}} \mathbf{P}^{x,y,k-1}((R_k(0))^C \cap (R_k(t))^C \cap G_k(0,t)) \\
&\leq \max_{x,y\in A_{k-1}} \mathbf{P}^{x,y,k-1}\left((R_k(0))^C \cap (R_k(t))^C\right) \\
&\leq 1 - 2\min_{x\in A_{k-1}} \mathbf{P}^{x,k-1}(R_k(0)) + \max_{x,y\in A_{k-1}} \mathbf{P}^{x,y,k-1}(R_k(0,t)) \\
&\leq 1 - 2\left(\frac{2}{k} - \frac{C\log k}{k^2}\right) + \frac{C}{k^2} \\
&\leq 1 - \frac{4}{k} + \frac{C\log k}{k^2}.
\end{aligned}$$

This proves condition 3 of Lemma 2.1. □

4. Proof of Theorem 1.1

Define

$$f(t,M) = \frac{\mathbf{P}(E_M(0,t))}{(\mathbf{P}(E_M(0)))^2}. \tag{4.1}$$

Lemma 4.1. *There exists C such that for any t and any M*

$$f(t,M) < C(1+|\log t|)^4. \tag{4.2}$$

Proof : Choose n such that

$$\frac{4}{k}+g(k)<.5$$

for all $k \geq n$. Thus we get

$$\begin{aligned}
f(t,M) &= \frac{\mathbf{P}(E_M(0,t))}{\Big(\mathbf{P}(E_M(0))\Big)^2}\\
&= \frac{\mathbf{P}(E_n(0,t))}{\Big(\mathbf{P}(E_n(0))\Big)^2}\prod_{k=n+1}^{M}\frac{\mathbf{P}\Big(E_k(0,t)\mid E_{k-1}(0,t)\Big)}{\mathbf{P}\Big(E_k(0)\mid E_{k-1}(0)\Big)^2}\\
&\leq \frac{1}{\Big(\mathbf{P}(E_n(0))\Big)^2}\prod_{k=n+1}^{K}\frac{1}{1-\frac{4}{k}-g_k}\prod_{k=K+1}^{M}\frac{1-\frac{4}{k}+g_k}{1-\frac{4}{k}-g_k}, \qquad (4.3)
\end{aligned}$$

where the first equality is repeating (4.1) and the inequality (4.3) comes from Lemma 2.1.

The inequality

$$-x^2-x<\ln(1-x)<-x$$

holds for all $x \in (0,.5)$. Thus

$$\begin{aligned}
\ln\left(\prod_{k=n+1}^{K}\frac{1}{1-\frac{4}{k}-g_k}\right) &= -\sum_{k=n+1}^{K}\ln\left(1-\frac{4}{k}-g_k\right)\\
&< \sum_{k=n+1}^{K}\frac{4}{k}+g_k+\left(\frac{4}{k}+g_k\right)^2\\
&< C+\sum_{k=n+1}^{K}\frac{4}{k}\\
&< C+4\ln(K).
\end{aligned}$$

By exponentiating both sides and (2.2) we get

$$\prod_{k=n+1}^{K}\frac{1}{1-\frac{4}{k}-g_k}\leq CK^4\leq C(1+|\log t|)^4. \qquad (4.4)$$

$$\begin{aligned}
\ln\left(\prod_{K+1}^{M}\frac{1-\frac{4}{k}+g_k}{1-\frac{4}{k}-g_k}\right) &\leq \sum_{K+1}^{\infty}\ln\left(1-\frac{4}{k}+g_k\right)-\sum_{K+1}^{\infty}\ln\left(1-\frac{4}{k}-g_k\right)\\
&\leq \sum_{K+1}^{\infty}-\frac{4}{k}+g_k-\left(-\left(\frac{4}{k}+g_k\right)-\left(\frac{4}{k}+g_k\right)^2\right)\\
&\leq \sum_{K+1}^{\infty}2g_k+\frac{16}{k^2}+\frac{8g_k}{k}+g_k^2\\
&\leq C.
\end{aligned}$$

Exponentiating both sides we get for all M

$$\prod_{K+1}^{M} \frac{1 - \frac{4}{k} + g_k}{1 - \frac{4}{k} - g_k} \leq C. \tag{4.5}$$

Putting together (4.3), (4.4) and (4.5) we get

$$f(t, M) \leq \frac{1}{(\mathbf{P}(E_n(0)))^2} C(1 + |\log t|)^4 C \leq C(1 + |\log t|)^4.$$

□

Proof of Theorem 1.1: Define

$$T_M = \{t : t \in [0, 1] \text{ and } E_M(t) \text{ occurs}\}$$

and

$$T = \cap_1^\infty \overline{T_M}.$$

Now we show that T is contained in the union of Exc and the countable set

$$\Lambda = (\cup_{n,m} \tau_n^{(m)}) \cup 1.$$

If $t \in \cap_1^\infty T_M$ then $t \in$ Exc. So if $t \in T \setminus$ Exc then t is contained in the boundary of T_M for some M. For any M the boundary of T_M is contained in Λ. Thus if $t \in T \setminus$ Exc then $t \in \Lambda$ and

$$T \subset \text{Exc} \cup \Lambda.$$

As Λ is countable, if T has dimension one with positive probability then so does Exc.

By Lemma 4.1 there exists $f(t)$ such that

$$\int_0^1 f(t)dt < \infty$$

and for all M

$$\frac{\mathbf{P}(E_M(0,t))}{(\mathbf{P}(E_M(0)))^2} < f(M, t) < f(t). \tag{4.6}$$

Let $\mathcal{L}(*)$ denote Lebesgue measure on $[0, 1]$. Then we get

$$\begin{aligned}
\mathbf{E}(\mathcal{L}(T_M)^2) &= \int_0^1 \int_0^1 \mathbf{P}(E_M(r, s))dr \times ds & (4.7)\\
&\leq \int_0^1 \int_0^1 \mathbf{P}(E_M(0, |s - r|))dr \times ds & (4.8)\\
&\leq \int_0^1 2 \int_0^1 \mathbf{P}(E_M(0, t))dr \times dt &\\
&\leq 2 \int_0^1 f(t)\mathbf{P}(E_M(0))^2 dt & (4.9)\\
&\leq 2\mathbf{P}(E_M(0))^2 \int_0^1 f(t)dt. & (4.10)
\end{aligned}$$

The equality (4.7) is true by Fubini's theorem, (4.8) is true because

$$E_M(a, b) = E_M(b, a) = E_M(0, |b - a|)$$

and (4.9) follows from (4.6).

By Jensen's inequality if $h(x) = 0$ for all $x \notin A$ then

$$\mathbf{E}(h^2) \geq \frac{\mathbf{E}(h)^2}{\mathbf{P}(A)}. \tag{4.11}$$

Then we get

$$\begin{aligned} 2\mathbf{P}(E_M(0))^2 \int_0^1 f(t)dt &\geq \mathbf{E}(\mathcal{L}(T_M)^2) && (4.12)\\ &\geq \frac{\mathbf{E}(\mathcal{L}(T_M))^2}{\mathbf{P}(T_M \neq \emptyset)} && (4.13)\\ &\geq \frac{\mathbf{P}(E_M(0))^2}{\mathbf{P}(T_M \neq \emptyset)}, \end{aligned}$$

where (4.12) is a restatement of (4.10), and (4.13) follows from (4.11) with $\mathcal{L}(T_M)$ and $T_M \neq \emptyset$ in place of h and A.

Thus for all M

$$\mathbf{P}(T_M \neq \emptyset) \geq \frac{1}{2\int_0^1 f(t)\ dt} > 0.$$

As T is the intersection of the nested sequence of compact sets T_M

$$\mathbf{P}(T \neq \emptyset) = \lim_{M\to\infty} \mathbf{P}(T_M \neq \emptyset) \geq \frac{1}{2\int_0^1 f(t)\ dt}.$$

Now we show that the dimensions of T and Exc are one. By Lemma 5.1 of Peres (1996) for any $\beta < 1$ there exists a random nested sequence of compact sets $F_k \subset [0,1]$ such that

$$\mathbf{P}(r \in F_k) \geq C(s_k)^{-\beta} \tag{4.14}$$

and

$$\mathbf{P}(r, t \in F_k) \leq C(s_k)^{-2\beta}|r-t|^{-\beta}. \tag{4.15}$$

These sets also have the property that for any set T if

$$\mathbf{P}(T \cap (\cap_1^\infty F_k) \neq \emptyset) > 0 \tag{4.16}$$

then T has dimension at least β. We construct F_k to be independent of the dynamical random walk. So by (4.1), (4.2), (4.14) and (4.15) we get

$$\begin{aligned} \frac{\mathbf{P}(r,t \in T_M \cap F_M)}{\mathbf{P}(r \in T_M \cap F_M)^2} &\leq \frac{\mathbf{P}(r,t \in T_M)\mathbf{P}(r,t \in F_M)}{\mathbf{P}(r \in T_M)^2\mathbf{P}(r \in F_M)^2}\\ &\leq \frac{\mathbf{P}(E_M(r,t))\mathbf{P}(r,t \in F_M)}{\mathbf{P}(E_M(r))^2\mathbf{P}(r \in F_M)^2}\\ &\leq C\left(1 + \Big|\log|r-t|\Big|\right)^4 |r-t|^{-\beta}. && (4.17) \end{aligned}$$

The same second moment argument as above and (4.17) implies that with positive probability T satisfies (4.16). Thus T has dimension β with positive probability. As

$$T \subset (\mathrm{Exc} \cap [0,1]) \cup \Lambda,$$

and Λ is countable, the dimension of the set of $\mathrm{Exc} \cap [0,1]$ is at least β with positive probability. By the ergodic theorem the dimension of the set of Exc is at least β with probability one. As this holds for all $\beta < 1$ the dimension of Exc is one a.s. $\square$

Finally we briefly state how to modify the proof to calculate the rate of escape mentioned in Remark 1.2. For any $\epsilon > 0$ we replace the event $R_k(t)$ with

$$R_k^\epsilon(t) = \left\{ \exists n \in \{s_{k-1}, \dots, s_k\} \text{ such that } |S_n(t)| < n^{.5 - 1/(\log(n))^{.25+\epsilon}} \right\}.$$

Instead of Lemma 3.1 we use Exercise 1.6.8 of Lawler (1991). The proof goes through with only minor modifications.

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References

O. Adelman, K. Burdzy and R. Pemantle. Sets avoided by Brownian motion. *Ann. Probab.* **26**, 429–464 (1998).

I. Benjamini, O. Häggström, Y. Peres and J. E. Steif. Which properties of a random sequence are dynamically sensitive? *Ann. Probab.* **31**, 1–34 (2003).

M. Fukushima. Basic properties of Brownian motion and a capacity on the Wiener space. *J. Math. Soc. Japan* **36**, 161–176 (1984).

O. Häggström, Y. Peres and J. E. Steif. Dynamical percolation. *Ann. Inst. H. Poincaré Probab. Statist.* **33**, 497–528 (1997).

D. Khoshnevisan, D. A. Levin and P. J. Mendez-Hernandez. Exceptional times and invariance for dynamical random walks (2004). ArXiv:math.PR/0409479. To appear in Probab. Theory Related Fields.

D. Khoshnevisan, D. A. Levin and P. J. Mendez-Hernandez. On dynamical Gaussian random walks. *Ann. Probab.* **33**, 1452–1478 (2005).

G. F. Lawler. *Intersections of Random Walks.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA (1991).

M. D. Penrose. On the existence of self-intersections for quasi-every Brownian path in space. *Ann. Probab.* **17**, 482–502 (1989).

Y. Peres. Intersection-equivalence of Brownian paths and certain branching processes. *Comm. Math. Phys.* **177**, 417–434 (1996).