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The localized phase of disordered copolymers with adsorption

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Abstract. We analyze the localized phase of a general model of a directed polymer in the proximity of an interface that separates two solvents. Each monomer unit carries a charge, ω_n, that determines the type (attractive or repulsive) and the strength of its interaction with the solvents. In addition, there is a polymer–interface interaction and we want to model the case in which there are impurities $\widetilde{\omega}_n$, called again charges, at the interface. The charges are distributed in an inhomogeneous fashion along the chain and at the interface: more precisely the model we consider is of quenched disorder type.
It is well known that such a model undergoes a localization/delocalization transition. We focus on the localized phase, where the polymer sticks to the interface. Our new results include estimates on the exponential decay of correlations, the proof that the free energy is infinitely differentiable away from the transition and estimates on finite–size corrections to the thermodynamic limit of the free energy per unit site. Other results we prove, instead, generalize earlier works that typically deal either with the case of copolymers near an homogeneous interface ($\widetilde{\omega} \equiv 0$) or with the case of disordered pinning, where the only polymer–environment interaction is at the interface ($\omega \equiv 0$). Moreover, with respect to most of the previous literature, we work with rather general distributions of charges (we will assume only a suitable concentration inequality) and we allow more freedom on the law of the underlying random walk.

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1. Introduction

1.1. *Copolymers, selective solvents and adsorption.* Polymers are repetitive chains of elementary blocks called monomers (or monomer units). Copolymers are inhomogeneous polymers, in the sense that each monomer unit carries a charge, and the charge is distributed along the chain in a *disordered* way. It is well known that when the medium surrounding the copolymer is made of two solvents, separated for example by an interface, and the solvents interact with the monomers according to the value of the charge, the typical behavior of the copolymer may differ substantially from the case in which the medium is homogeneous. On copolymers there is an extremely extended literature, given above all their practical relevance, see for example Garel et al. (1989) and Giacomin (2004) and references therein. Moreover, for a realistic model of the interface, one should consider the possibility of the presence of impurities or fluctuations in the interface layer, as in Soteros and Whittington (2004). As an extreme, but very important example, one could consider also the case in which the interactions at the interface are essentially the only relevant ones, see Derrida et al. (1992); Forgacs et al. (1986).

In order to be more concrete, let us introduce a specific model, which is just a particular example of the general class we consider. It is based on the process $S^{\mathsf{RW}} = \{S_n^{\mathsf{RW}}\}_{n=0,1,\ldots}$, a simple random walk on $\mathbb{Z}$, with $S_0^{\mathsf{RW}} = 0$ and $\{S_j^{\mathsf{RW}} - S_{j-1}^{\mathsf{RW}}\}_{j\in\mathbb{N}}$, $\mathbb{N} := \{1, 2, \ldots\}$, a sequence of IID random variables with $\mathbf{P}\left(S_1^{\mathsf{RW}} = 1\right) = \mathbf{P}\left(S_1^{\mathsf{RW}} = -1\right) = 1/2$.

The process S^{RW} has to be interpreted as a directed polymer in (1+1)–dimension, and $\mathbf{P}$ as its law in absence of any interaction with the environment (*free polymer*). The polymer–environment interaction depends on the *charges* $\underline{\omega} = (\omega, \widetilde{\omega}) \in \mathbb{R}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ and on four real parameters $\underline{v} := (\lambda, h, \widetilde{\lambda}, \widetilde{h})$: without loss of generality we will assume λ, h and $\widetilde{\lambda}$ to be non–negative. Let us set $S_n = S_{2n}^{\mathsf{RW}}/2$ and let us introduce the family of Boltzmann measures indexed by $N \in \mathbb{N}$

$$\frac{\mathrm{d}\mathbf{P}_{N,\underline{\omega}}^{\underline{v}}}{\mathrm{d}\mathbf{P}}(S^{\mathsf{RW}}) = \frac{1}{\widetilde{Z}_{N,\underline{\omega}}^{\underline{v}}} \exp\left(\lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}\,(S_n) + \widetilde{\lambda} \sum_{n=1}^{N} \left(\widetilde{\omega}_n + \widetilde{h}\right) \mathbf{1}_{\{S_n=0\}}\right) \mathbf{1}_{\{S_N=0\}}, \quad (1.1)$$

with the convention that $\mathrm{sign}\,(S_n) = \mathrm{sign}\left(S_{2n-1}^{\mathsf{RW}}\right)$ for any n such that $S_n = 0$. The superscript $\underline{v}$ will be often omitted.

The model is completely defined once $\underline{\omega}$ is given: being interested in the disordered case, we choose for instance $\underline{\omega}$ an realization of an IID family, of law $\mathbb{P}$, of symmetric variables taking value ± 1.

We invite the reader to look at Figure 1 in order to get an intuitive idea on this model.

1.2. *The model.* An important observation on the model we have introduced is that its Hamiltonian may be easily rewritten in terms of the sequence $\tau := \{\tau_i\}_i$, defined by setting $\tau_0 := 0$ and, for $i \in \mathbb{N}$, $\tau_i := \inf\{n > \tau_{i-1} : S_n = 0\}$ ($\tau_i < +\infty$ with probability one for every i since S^{RW} is recurrent) and in terms of the sign of the excursions, that, conditionally on τ, are just an independent sequence of IID

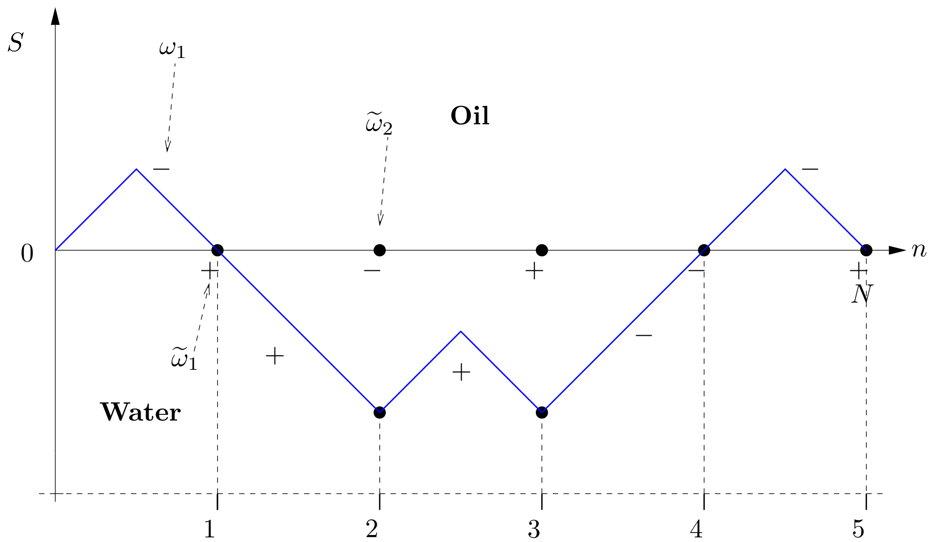

FIGURE 1. The polymer is at the interface between two solvents, say oil and water, situated in the positive and negative half-planes, respectively. Along the (one–dimensional) interface $S = 0$ are placed random charges $\widetilde{\omega}_n$, positive or negative. The polymer prefers to touch the interface whenever $\widetilde{\omega}_n + \widetilde{h} > 0$. On the other hand, a random charge ω_n is associated to the n^{th} monomer of the chain, i.e., to the portion of the polymer contained between $n-1$ and n. This is responsible for the polymer-solvent interaction: if $\omega_n + h > 0$ the n^{th} monomer prefers to be in oil, otherwise it prefers to be in water. The non–negative parameters $\lambda, \widetilde{\lambda}$ may be thought of as two effective inverse temperatures, and h as a measure of the asymmetry between the two solvents. Note that by convention, if $S_n = 0$ then $\text{sign}(S_n)$ coincides with the sign of $S_{n-(1/2)}$, which is unambiguously defined. For instance, $\text{sign}(S_1) = +1$ and $\text{sign}(S_4) = -1$. In this picture, $N = 5$ and in fact the endpoint S_N is pinned at zero.

symmetric variables taking values ± 1. Of course the (strong) Markov property of S^{RW} immediately yields also that $\{\tau_j - \tau_{j-1}\}_{j \in \mathbb{N}}$ is an IID sequence.

We are now going to introduce the general class of models that we consider. These models are based on a real–valued free process $S := \{S_n\}_{n=0,1,\ldots}$, with law $\mathbf{P}$ that satisfies the following properties:

(1) The sequence $\tau = \{\tau_j\}_{j=0,1,\ldots}$ of successive returns to 0 is an infinite sequence with $\tau_0 = 0$, so S is a process starting from 0 and for which 0 is a recurrent state. Moreover τ is a renewal sequence, that is $\{\tau_j - \tau_{j-1}\}_{j=1,2,\ldots}$ is a sequence of IID random variables, and we set $K(n) := \mathbf{P}(\tau_1 = n)$.
(2) For some $\alpha \geq 1$ and for some function $L(\cdot)$ which is slowly varying at infinity (see below for the definition and properties of slowly varying functions)
$$K(n) \;=\; \frac{L(n)}{n^{\alpha}}. \tag{1.2}$$
In particular $K(n) > 0$ for every $n \in \mathbb{N}$.
(3) For $i \in \mathbb{N}$ such that $\tau_i - \tau_{i-1} > 1$ we set $s_i := \text{sign}(S_{\tau_i - 1})$. Then conditionally on τ, $\{s_i\}_{i:\, \tau_i - \tau_{i-1} > 1}$ is an IID sequence of symmetric random variables

taking values ± 1. Conventionally we complete the sequence $\{s_i\}_{i\in\mathbb{N}}$ (i.e., we choose the s_i for i such that $\tau_{i+1}-\tau_i=1$) by tossing independent fair coins and, always by convention, we stipulate that $\mathrm{sign}(S_{\tau_i})=s_i$. Of course, conditionally on τ, $\{s_i\}_{i\in\mathbb{N}}$ is just independent fair coin tossing.

(4) Conditionally on τ, $\{(S_{\tau_i+1},\ldots,S_{\tau_{i+1}})\}_{i=0,1,\ldots}$ is an independent sequence of random vectors. Moreover the law of $(S_{\tau_i+1},\ldots,S_{\tau_{i+1}})$, conditionally on τ, depends on τ only via the value of $\tau_{i+1}-\tau_i$. Note that this property implies that, conditionally on $S_m=0$, the process $\{S_n\}_{n\geq m}$ has the same distribution as the original process $\{S_n\}_{n\geq 0}$. With some abuse of language, we will call this property the *renewal property of* S.

The free process S is put in interaction with an environment via the *charges* $\underline{\omega}=(\omega,\widetilde{\omega})\in\mathbb{R}^{\mathbb{N}}\times\mathbb{R}^{\mathbb{N}}$. The definition of $\mathbf{P}^{\underline{v}}_{N,\underline{\omega}}$ is still as in (1.1), provided one suppresses the superscript RW in the left–hand side. The process S constructed in Section 1.1 starting from S^{RW} corresponds to the case of $\alpha=3/2$ and $\lim_{n\to\infty}L(n)=1/(2\sqrt{\pi})$.

We recall that a function $L(\cdot)$ is slowly varying (at infinity) if $L(\cdot)$ is a measurable function from $(0,\infty)$ to $(0,\infty)$ such that $\lim_{r\to\infty}L(xr)/L(r)=1$ for every $x>0$. One of the properties of slowly varying functions is that both $L(r)$ and $1/L(r)$ are $o(r^{\epsilon})$ for every $\epsilon>0$. An example of slowly varying function is $r\mapsto(\log(r+1))^b$, for $b\in\mathbb{R}$, but also $r\mapsto\exp((\log(r+1))^b)$, for $b<1$, as well as any positive function for which $\lim_{r\to\infty}L(r)>0$. A complete treatment of slowly varying functions is found in Feller (1971), but these functions are needed for us to in order to work in a reasonably general and well defined set–up: we will use no fine property of slowly varying function since in most of the cases rough bounds on $K(\cdot)$ will suffice.

Remark 1.1. In short, one may think of building S by first assigning the return times to zero according to a renewal process. The excursions are then *glued* to the renewal points, but essentially the only relevant aspect of the excursions for us is the sign, that is chosen by repeated tossing of a fair coin. Note again that the energy of the model depends only on τ and on $\{s_i\}_i$, and not on the details of the excursions in the upper or lower half plane. The last property in the list above, the renewal property of S, makes a bit more precise what the excursions of the process really are: this property is very useful to have a nice pictorial vision of the process, but it is rather inessential for us. We will use it only in stating Theorem 2.2 since this way it turns out to be somewhat nicer and more intuitive, but the essence of our analysis lies in τ. Note also that if $\lambda=0$ the energy does not depend on $\{s_i\}_i$. In this case we are dealing with a pure pinning model and insisting on S taking values in $\mathbb{R}$ is a useless limitation.

Remark 1.2. We have chosen 0 to be a recurrent state of S because it simplifies a little the notations, but everything carries over to the case in which 0 is transient.

The sequence ω is chosen as a typical realization of an IID sequence of random variables, still denoted by $\omega=\{\omega_n\}_n$. We make the very same assumptions on $\widetilde{\omega}$ and, in addition, ω and $\widetilde{\omega}$ are independent. The law of $\underline{\omega}$ will be denoted by $\mathbb{P}$, and the corresponding expectation by $\mathbb{E}$. A further assumption on $\underline{\omega}$ is the following *concentration inequality*: there exists a positive constant C such that for every N, for every Lipschitz and convex function $g:\mathbb{R}^{2N}\to\mathbb{R}$ with $g(\underline{\omega}):=$

$g(\omega_1,\ldots,\omega_N,\widetilde{\omega}_1,\ldots,\widetilde{\omega}_N) \in L^1(\mathbb{P})$ and for $t \geq 0$

$$\mathbb{P}\left(\left|g(\underline{\omega}) - \mathbb{E}\left[g(\underline{\omega})\right]\right| \geq t\right) \leq C\exp\left(-\frac{t^2}{C\|g\|_{\mathrm{Lip}}^2}\right), \tag{1.3}$$

where $\|g\|_{\mathrm{Lip}}$ is the Lipschitz constant of g with respect to the Euclidean distance on $\mathbb{R}^{2N}$.

Of course such an inequality implies that ω_1 and $\widetilde{\omega}_1$ are exponentially integrable: without loss of generality we assume ω_1 and $\widetilde{\omega}_1$ to be centered and of unit variance.

Remark 1.3. In practice, it is sufficient to check that the concentration inequality holds for ω and for $\widetilde{\omega}$ separately. In fact, suppose that

$$\mathbb{P}\left(\left|G(\omega) - \mathbb{E}\left[G(\omega)\right]\right| \geq t\right) \leq C\exp\left(-\frac{t^2}{C\|G\|_{\mathrm{Lip}}^2}\right), \tag{1.4}$$

for every $G : \mathbb{R}^N \to \mathbb{R}$ and $G(\omega) := G(\omega_1,\ldots,\omega_N)$, and similarly for $\widetilde{\omega}$. Then,

$$\mathbb{P}\left(\left|g(\underline{\omega}) - \mathbb{E}[g(\underline{\omega})]\right| \geq u\right) \leq \\ \mathbb{P}\left(\left|\mathbb{E}[g(\underline{\omega})|\omega] - \mathbb{E}[g(\underline{\omega})]\right| \geq \frac{u}{2}\right) + \mathbb{P}\left(\left|g(\underline{\omega}) - \mathbb{E}[g(\underline{\omega})|\omega]\right| \geq \frac{u}{2}\right). \tag{1.5}$$

Using twice (1.4), once for $G_1(\omega) := \mathbb{E}[g(\underline{\omega})|\omega]$ and once for $G_2^\omega(\widetilde{\omega}) := g(\omega,\widetilde{\omega})$, noting that $\max(||G_1||_{\mathrm{Lip}}, ||G_2^\omega||_{\mathrm{Lip}}) \leq ||g||_{\mathrm{Lip}}$ uniformly in ω, one obtains immediately (1.3).

The concentration inequality (1.3) is known to hold with a certain generality: its validity for the Gaussian case and for the case of bounded random variables is by now a classical result (Ledoux (2001)). While of course such an inequality requires $\mathbb{E}\left[\exp\left(\varepsilon\left(\omega_1^2+\widetilde{\omega}_1^2\right)\right)\right] < \infty$ for some $\varepsilon > 0$, a complete characterization of the distributions for which (1.3) holds is, to our knowledge, still lacking, but among these distributions there are, for example, the cases in which the laws of ω_1 and $\widetilde{\omega}_1$ satisfy the log–Sobolev inequality, see Talagrand (1996), Ledoux (2003) and Villani (2003), therefore, in particular, whenever ω_1 and $\widetilde{\omega}_1$ are continuous variables with positive densities of the form $\exp(-V(\cdot))$, with $V \in C^2$ and $V''(\cdot) \geq c > 0$ when restricted to $(-\infty,-K) \cup (K,\infty)$, for some $K > 0$.

Remark 1.4. We have chosen to work assuming concentration because it provides a unified rather general framework in which proofs are at several instances much shorter (and, possibly, more transparent). However, as it will be clear, several results hold under weaker assumptions. On the other hand we deal only with the polymer pinned at the endpoint, that is, constrained to $S_N = 0$: we have chosen this case for the sake of conciseness, but we could have decided for example to leave the endpoint free.

1.3. *Free energy, localization and delocalization.* Under the above assumptions on the disorder the *quenched free energy* of the system exists, namely the limit

$$f(\underline{v}) := \lim_{N\to\infty} \frac{1}{N} \log \widetilde{Z}_{N,\underline{\omega}}^{v}, \tag{1.6}$$

exists $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–almost surely and in the $\mathbb{L}^1(\mathbb{P})$ sense. The existence of this limit can be proven via standard super–additivity arguments (we refer for example to

Giacomin (2004) for details). We stress that the concentration inequality implies immediately that $f(\underline{v})$ is not random, but such a result may be proven under much weaker assumptions, see e.g. Giacomin (2004).

A simple but fundamental observation is that

$$f(\underline{v}) \, \geq \, \lambda h. \tag{1.7}$$

The proof of this is elementary: if we set $\Omega_N^+ = \{S \in \Omega : S_n > 0$ for $n = 1, 2, \ldots, N-1$ and $S_N = 0\}$ we have

$$\begin{aligned} &\frac{1}{N}\mathbb{E}\log \widetilde{Z}^{\underline{v}}_{N,\underline{\omega}} \geq \\ &\quad \frac{1}{N}\mathbb{E}\log \mathbf{E}\left[\exp\left(\lambda\sum_{n=1}^{N}(\omega_n + h)\,\mathrm{sign}\,(S_n) + \widetilde{\lambda}\sum_{n=1}^{N}\left(\widetilde{\omega}_n + \widetilde{h}\right)\mathbf{1}_{\{S_n=0\}}\right); \Omega_N^+\right] \\ &\qquad = \frac{\lambda}{N}\sum_{n=1}^{N}\left(\mathbb{E}\left[\omega_1\right] + h\right) + \frac{\widetilde{\lambda}}{N}\left(\mathbb{E}\left[\widetilde{\omega}_N\right] + \widetilde{h}\right) + \frac{1}{N}\log \mathbf{P}\left(\Omega_N^+\right) \stackrel{N\to\infty}{\longrightarrow} \lambda h, \end{aligned} \tag{1.8}$$

where we have applied the fact that, by (1.2), $\log \mathbf{P}\left(\Omega_N^+\right) = \log((1/2)K(N)) = o(N)$, for $N \to \infty$.

The observation (1.7), above all if viewed in the light of its proof, suggests the definition $\textsc{f}(\underline{v}) := f(\underline{v}) - \lambda h$ and the following partition of the parameter space (or *phase diagram*):

- The localized region: $\mathcal{L} = \{\underline{v} : \textsc{f}(\underline{v}) > 0\}$;
- The delocalized region: $\mathcal{D} = \{\underline{v} : \textsc{f}(\underline{v}) = 0\}$.

Along with this definition we observe that

$$\frac{\mathrm{d}\mathbf{P}^{\underline{v}}_{N,\underline{\omega}}}{\mathrm{d}\mathbf{P}}(S) \, \propto \, \exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n + h)\,\Delta_n + \widetilde{\lambda}\sum_{n=1}^{N}\left(\widetilde{\omega}_n + \widetilde{h}\right)\delta_n\right)\delta_N, \tag{1.9}$$

where

$$\Delta_n := \mathbf{1}_{\{\mathrm{sign}(S_n)=-1\}} = \left(1 - \mathrm{sign}(S_n)\right)/2 \quad \text{ and } \quad \delta_n := \mathbf{1}_{\{S_n=0\}}. \tag{1.10}$$

Of course the normalization constant

$$Z^{\underline{v}}_{N,\underline{\omega}} \, := \, \mathbf{E}\left[\exp\left(-2\lambda\sum_{n=1}^{N}(\omega_n + h)\,\Delta_n + \widetilde{\lambda}\sum_{n=1}^{N}\left(\widetilde{\omega}_n + \widetilde{h}\right)\delta_n\right)\delta_N\right], \tag{1.11}$$

changes, but $Z^{\underline{v}}_{N,\underline{\omega}} = \exp\left(-\lambda\sum_{n=1}^{N}(\omega_n + h)\right)\widetilde{Z}^{\underline{v}}_{N,\underline{\omega}}$ so that the $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. and $\mathbb{L}^1(\mathbb{P})$ asymptotic behavior of $(1/N)\log Z^{\underline{v}}_{N,\underline{\omega}}$ are given by $\textsc{f}(\underline{v})$. We will always work with $Z^{\underline{v}}_{N,\underline{\omega}}$, in order to conform with most of the previous mathematical literature.

1.4. *Discussion of the model.* For the model we introduced there is a very vast literature, mostly in chemistry, physics and bio–physics. It often goes under the name of *copolymer with adsorption*, see e.g. Soteros and Whittington (2004) and references therein, and such a name clearly reflects the superposition of two distinct polymer–environment interactions:

- The monomer–solvent interaction, associated to the charges ω. Some monomers prefer one solvent and some prefer the other one. Since the charges are placed in an inhomogeneous way along the chain, energetically favored trajectories need to stick close to the interface. Whether pinning actually takes place or not depends on the interplay between energetic gain and entropic loss associated to localization (trajectories that stay close to the interface have a much smaller entropy than those which wander away).

  If $\lambda > 0$ and $\widetilde{\lambda} = 0$ only this interaction is present and we will call the model simply *copolymer*.
- The monomer–interface interaction, associated to $\widetilde{\omega}$. This interaction leads to a pinning (or depinning) phenomenon with a more direct mechanism: trajectories are energetically favored if they touch the interface *as often as possible* at points where $\widetilde{\omega}_n + \widetilde{h} > 0$, avoiding at the same time the points in which $\widetilde{\omega}_n + \widetilde{h} < 0$. Also in this case, a non–trivial energy/entropy competition is responsible for the localization/delocalization transition.

  When $\lambda = 0$ and $\widetilde{\lambda} > 0$ we will refer to the model as *pinning* model.

The copolymer model has received a lot of attention: we mention in particular Garel et al. (1989), in which it was first introduced and the replica method was applied in order to investigate the transition. Rigorous work started with Sinai (1993), followed by Albeverio and Zhou (1996): these works deal with the case $h = 0$ and ω_1 symmetric and taking only the values ± 1 (*binary charges*). It turns out that in such a case there is no transition and the model is localized for every $\lambda > 0$.

In general, one may distinguish between results concerning the free energy and results on pathwise behavior of the polymer. About the first point, we mention that in Bolthausen and den Hollander (1997) the model with $h \geq 0$ has been considered, still with the choice of binary charges, and the existence of a transition has been established, along with estimates on the critical curve and remarkable limiting properties of the free energy of the model in the limit of weak coupling (λ small). Improved estimates on free energy and critical curve may be found in Bodineau and Giacomin (2004). In the physical literature one can find a number of conjectures, mostly on the free energy behavior, that are far from clarifying the phase diagram. In this respect, it is interesting to mention that recent numerical simulations (Caravenna et al. (2005)) show that the critical line is different from that predicted in the theoretical physics literature, which means that the localization mechanism is still poorly understood.

Disordered pinning has been extensively studied in the physical literature, see e.g. Derrida et al. (1992) and Forgacs et al. (1986) (see also Giacomin and Toninelli (2005b) for more recent references), but much less in the mathematical one. However the model has started attracting attention lately, see Alexander and Sidoravicius (2005), Petrelis (2005) and Giacomin and Toninelli (2005b). We should stress that there is no agreement in the physical literature on several important issues for disordered pinning. For instance, it is still unclear whether the critical curve coincides with the so–called annealed curve.

About the study of path behavior, there is a basic difference between the localized and the delocalized phase: in the first case, since $\mathrm{F}(\underline{v}) > 0$, the interaction produces an exponential modification of the free polymer measure, and therefore

Large Deviation techniques apply very naturally. The path behavior of the copolymer model in the localized phase has been considered in Sinai (1993); Albeverio and Zhou (1996), for $h = 0$ and binary charges, while in Biskup and den Hollander (1999) also the case $h > 0$ is taken into account. In Biskup and den Hollander (1999) the focus is on the Gibbsian characterization of the infinite volume polymer measure (in the localized phase). The delocalized phase is more subtle, due to the fact that $\textsc{f}(\underline{v}) = 0$, and path delocalization estimates involve estimates on *Moderate Deviations* of the free energy. Results on the path behavior in the delocalized phase have been obtained only recently in Giacomin and Toninelli (2005a), both for the copolymer and for the disordered pinning model.

2. Main results

In the present work, we consider the localized phase of the general model defined in Section 1.2 and formula (1.1). In addition to giving new results, our approach provides also a setting to reinterpret in a simpler way known results for copolymer and pinning models.

2.1. *Smoothness of the free energy and decay of correlations.* The free energy is everywhere continuous and almost everywhere differentiable, by convexity, so in particular $\mathcal{L}$ is an open set. However, one can go much beyond that, as our first theorem shows:

Theorem 2.1. $\textsc{f}$ *is infinitely differentiable in* $\mathcal{L}$*.*

An interesting problem is to study the regularity properties of $\textsc{f}(\cdot)$ at the boundary between $\mathcal{L}$ and $\mathcal{D}$, where it is non–analytic. This corresponds to investigating the order of the localization/delocalization transition. Recently, an important step in this direction was performed in Giacomin and Toninelli (2005b) and Giacomin and Toninelli (2006) where it was proved, in particular, that the first derivatives of $\textsc{f}(\cdot)$ are continuous on the boundary. In other words, the (de)localization transition is at least of second order.

As it will become clear in Section 4, the smoothness of the free energy in $\mathcal{L}$ boils down to a property of exponential decay of (average) correlations. For this implication we essentially rely on von Dreifus et al. (1995), where a similar result has been proven in the context of disordered Ising models.

Let us therefore state the decay of correlation property. We say that A is a bounded local observable if A is a real bounded measurable local function of the path configurations. In the sequel $\mathcal{S}(A)$ will denote the *support of* A, that is the intersection of all the subsets $I \subset \mathbb{N}$ of the form $I = \{\ell, \dots, k\}$, with $k, \ell \in \mathbb{N}$, such that A is measurable with respect to the σ–algebra $\sigma(S_n : n \in I)$.

We have

Theorem 2.2. *For every* $\underline{v} \in \mathcal{L}$ *there exist finite constants* $c_1(\underline{v}), c_2(\underline{v}) > 0$ *such that the following holds for every* $N \in \mathbb{N}$*:*

- *(Exponential decay of correlations.) For every couple of bounded local observables A and B we have*

$$\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(AB)-\mathbf{E}_{N,\underline{\omega}}(A)\mathbf{E}_{N,\underline{\omega}}(B)\right|\right] \le c_1\|A\|_\infty\|B\|_\infty \exp(-c_2 d(\mathcal{S}(A),\mathcal{S}(B))) \quad (2.1)$$

  where $d(I,J)=\min\{|i-j|, i\in I, j\in J\}$ if $I,J\subset\mathbb{N}$.
- *(Influence of the boundary.) For every bounded local observable A and $k\in\mathbb{N}$, such that $\mathcal{S}(A)\subset\{1,\ldots,k\}$, we have*

$$\sup_{N>k}\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(A)-\mathbf{E}_{k,\underline{\omega}}(A)\right|\right] \le c_1\|A\|_\infty \exp(-c_2 d(\mathcal{S}(A),\{k\})). \quad (2.2)$$

- *For every bounded local observable A the following limit exists $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–almost surely:*

$$\lim_{N\to\infty}\mathbf{E}_{N,\underline{\omega}}(A) =: \mathbf{E}_{\infty,\underline{\omega}}(A). \quad (2.3)$$

Remark 2.3. From (2.1) one may easily extract an almost sure statement. Choose two bounded local observables A and B and set $B_k(S)=B(\theta^k S)$, i.e. $(\theta S)_n = S_{n+1}$. Take the limit $N\to\infty$ in (2.1) to obtain

$$\mathbb{E}\left[\exp(c_2k/2)\left|\mathbf{E}_{\infty,\underline{\omega}}(AB_k)-\mathbf{E}_{\infty,\underline{\omega}}(A)\mathbf{E}_{\infty,\underline{\omega}}(B_k)\right|\right] \le \exp(-c_2k/4), \quad (2.4)$$

for k sufficiently large. Therefore the Fubini–Tonelli theorem and (2.4) yield

$$\mathbb{E}\left[\sum_k \exp(c_2k/2)\left|\mathbf{E}_{\infty,\underline{\omega}}(AB_k)-\mathbf{E}_{\infty,\underline{\omega}}(A)\mathbf{E}_{\infty,\underline{\omega}}(B_k)\right|\right] < \infty. \quad (2.5)$$

The series appearing in the left–hand side is therefore $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. convergent. This implies that there exists a random variable $C_{A,B}(\underline{\omega})$, $C_{A,B}(\underline{\omega})<\infty$ $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s., such that

$$\left|\mathbf{E}_{\infty,\underline{\omega}}(AB_k)-\mathbf{E}_{\infty,\underline{\omega}}(A)\mathbf{E}_{\infty,\underline{\omega}}(B_k)\right| \le C_{A,B}(\underline{\omega})\exp(-c_2k/2), \quad (2.6)$$

for every k.

2.2. *Path localization and maximal excursions.* We consider now the question of whether or not knowing that $\underline{v}\in\mathcal{L}$ does mean that the path of the polymer is really tight to the interface. Even if this question has not been treated for the general model we are considering here, the techniques used in Sinai (1993), Albeverio and Zhou (1996) and Biskup and den Hollander (1999), see also Giacomin (2004), may be applied directly and one obtains for example that, in the case $S_n = S^{\mathsf{RW}}_{2n}/2$, for every $\underline{v}\in\mathcal{L}$ there exist finite constants $c_1, c_2>0$ such that for every N, $s\in\mathbb{N}$, and $0\le k\le N$

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(|S_k|\ge s\right)\le c_1 e^{-c_2 s}, \quad (2.7)$$

or one can obtain an analogous $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. result, which is a bit more involved to state Sinai (1993).

The reason for revisiting this type of results, besides generalizing them to our case, is that they are only bounds and we would like to find estimates that are sharp to leading order. A notable exception is the case of some of the results in Albeverio and Zhou (1996, Th. 5.3 and Th. 6.1) where the precise asymptotic size of the largest excursion (and of the maximum displacement of the chain from the interface) is obtained. This result is a bit surprising since it depends on a certain annealed decay exponent. This exponent turns out to be different from the decay exponent one finds for $\mathbb{P}(\mathrm{d}\underline{\omega})$–a.s. estimates, see discussion after our Theorem 2.5. The argument of the crucial point of the proof of Albeverio and Zhou (1996, Th. 5.3) looks obscure to us and we propose here a different one, based on decay of correlations.

Of course we present our results in terms of *excursion lengths*. Recall the definition of the return times τ in Section 1.2. For every $k \in \{1, \ldots, N-1\}$ let us set $\mathbf{\Delta}_N(k) = \tau_{\iota(k)+1} - \tau_{\iota(k)}$, with $\iota(k)$ equal to the value i such that $k \in \{\tau_i, \ldots, \tau_{i+1}-1\}$. So $\iota(k)$ is the left margin of the excursion to which k belongs and $\mathbf{\Delta}_N(k)$ is the length of such an excursion.

Two distinct questions can be posed concerning polymer excursions: one may be interested about the typical length of a given excursion or, more globally, about the typical length of the *longest* excursion, $\mathbf{\Delta}_N := \max_k \mathbf{\Delta}_N(k)$. Denote by θ the left shift on ω, like for S: $\{\theta\omega\}_k = \omega_{k+1}$. About the first problem, we can prove:

Proposition 2.4. *Take $\underline{v} \in \mathcal{L}$ and let $\underline{\omega}$ be the two–sided sequence of IID random variables, $\underline{\omega} := (\omega, \widetilde{\omega}) \in \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$, with law $\mathbb{P}$. For every $\varepsilon > 0$ there exist random variables $C_\varepsilon^{(j)}(\underline{\omega})$, $j = 1, 2$, such that $\mathbb{P}(0 < C_\varepsilon^{(j)}(\underline{\omega}) < \infty) = 1$ and*

$$\begin{aligned} \mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N(k) = s\right) &\geq C_\varepsilon^{(1)}(\theta^k \underline{\omega}) \exp\left(-\left(\mathrm{F}(\underline{v}) + \varepsilon\right) s\right), \\ \mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N(k) = s\right) &\leq C_\varepsilon^{(2)}(\theta^k \underline{\omega}) \exp\left(-\left(\mathrm{F}(\underline{v}) - \varepsilon\right) s\right), \end{aligned} \tag{2.8}$$

for every N, every $k \in \{1, \ldots N-1\}$ and every $s \in \mathbb{N}$: for the first inequality, the lower bound, we require also $s < N/2$.

Note that, in the definition of $\mathbf{P}_{N,\underline{\omega}}$, the fact that $\underline{\omega}$ is a doubly infinite sequence is completely irrelevant, since the polymer measure depends only on $\underline{\omega}_1, \ldots, \underline{\omega}_N$. The introduction of two–sided disorder sequences, which might seem a bit unnatural, is needed here to have the almost–sure result uniformly in N, k and s.

About the maximal excursion, we have:

Theorem 2.5. *The following holds:*

(1) *the limit*

$$-\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[\frac{1 + e^{-2\lambda \sum_{n=1}^N (\omega_n + h)}}{Z_{N,\underline{\omega}}^{\underline{v}}}\right] := \mu(\underline{v}), \tag{2.9}$$

exists and satisfies the bounds $0 \leq \mu(\underline{v}) \leq \mathrm{F}(\underline{v})$. Moreover, for every $c_1 > 0$ there exists $c_2 > 0$ such that $c_2 \left(\mathrm{F}(\underline{v})\right)^2 \leq \mu(\underline{v})$, for $\mathrm{F}(\underline{v}) \leq 1$ and $\max(\lambda, \widetilde{\lambda}) \leq c_1$.

(2) *Fix $\underline{v} \in \mathcal{L}$. For every $\varepsilon \in (0,1)$ we have that*

$$\lim_{N\to\infty} \mathbf{P}_{N,\underline{\omega}}^{\underline{v}}\left(\frac{\mathbf{\Delta}_N}{\log N} < \frac{1-\varepsilon}{\mu(\underline{v})}\right) = 0, \quad \mathbb{P}(\mathrm{d}\underline{\omega})\text{–a.s.}, \tag{2.10}$$

and

$$\lim_{N\to\infty} \mathbf{P}_{N,\underline{\omega}}^{\underline{v}}\left(\frac{\mathbf{\Delta}_N}{\log N} > \frac{1+\varepsilon}{\mu(\underline{v})}\right) = 0, \quad \text{in probability.} \tag{2.11}$$

In the localized region, under rather general conditions on the law $\mathbb{P}$, we can prove that $\mu(\underline{v}) < \mathrm{F}(\underline{v})$, see Appendix B.1. Notice therefore the gap between (2.10) and the result in Proposition 2.4, so that the largest excursion appears to be achieved in atypical regions.

Remark 2.6. When the law of ω_1 is symmetric, one can also prove (see Appendix B.2) that $\mu(\underline{v})$ is equivalently given by

$$\mu(\underline{v}) = -\lim_{N\to\infty} \frac{1}{N} \log \mathbb{E}\left[\frac{1}{Z^{\underline{v}}_{N,\underline{\omega}}}\right]. \tag{2.12}$$

This coincides with the expression given in Albeverio and Zhou (1996), where only the case $\widetilde{\lambda} = 0$, $h = 0$ and ω_1 taking values in $\{-1, 1\}$ was considered.

2.3. *Finite–size corrections and central limit theorem for the free energy.* In this section, we investigate finite–size corrections to the infinite volume limit of the quenched average of the free energy, and the behavior of its disorder fluctuations. About the first point, it is quite easy to prove (see also Albeverio and Zhou (1996, Proposition 2.6))

Proposition 2.7. *There exists $c_1 < \infty$ such that, for every $\underline{v}$ and $N \in \mathbb{N}$, one has*

$$\left|\mathrm{F}(\underline{v}) - \frac{1}{N}\mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}}\right| \le c_1 \frac{\log N}{N} + \frac{\widetilde{\lambda}}{N}(1 + |\widetilde{h}|). \tag{2.13}$$

This bound is somehow optimal in general, in the sense that it is possible to prove a lower bound of order $\log N/N$ if $\underline{v}$ is in the *annealed region*, i.e., the sub–region of $\mathcal{D}$ where $(1/N)\log \mathbb{E} Z^{\underline{v}}_{N,\underline{\omega}} \to 0$. However, in the localized region we can go much farther:

Theorem 2.8. *Assume that $\underline{v} \in \mathcal{L}$. Then, there exists $c(\underline{v}) < \infty$ such that, for every $N \in \mathbb{N}$, one has*

$$\left|\mathrm{F}(\underline{v}) - \frac{1}{N}\mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}}\right| \le \frac{c(\underline{v})}{N}. \tag{2.14}$$

About fluctuations, in Albeverio and Zhou (1996) it was proven that, for $\widetilde{\lambda} = 0$, $h = 0$ and ω_1 taking values in $\{-1, 1\}$, the free energy satisfies in the large volume limit a central limit theorem on the scale $1/\sqrt{N}$. Here, we generalize this result to the entire localized region. Our proof employs basically the same idea as in Albeverio and Zhou (1996); however, the use of concentration of measure ideas, plus a more direct way to show that the limit variance is not degenerate, allow for remarkable simplifications. The precise result is the following:

Theorem 2.9. *If $\underline{v} \in \mathcal{L}$ the following limit in law holds:*

$$\frac{1}{\sqrt{N}}\left(\log Z^{\underline{v}}_{N,\underline{\omega}} - \mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}}\right) \stackrel{N\to\infty}{\longrightarrow} \mathcal{N}(0, \sigma^2), \tag{2.15}$$

with $\sigma^2 = \sigma^2(\underline{v}) > 0$.

2.4. *Some notational conventions.* For compactness we introduce a notation for the Hamiltonian

$$\mathcal{H}_{N,\underline{\omega}}(S) \;=\; -2\lambda \sum_{n=1}^{N} (\omega_n + h)\,\Delta_n + \widetilde{\lambda} \sum_{n=1}^{N} \Big(\widetilde{\omega}_n + \widetilde{h}\Big)\,\delta_n, \tag{2.16}$$

and for $\overline{\Omega} \in \sigma\,(S_n : n \in \mathbb{N})$ we set

$$Z_{N,\underline{\omega}}\left(\overline{\Omega}\right) \;=\; \mathbf{E}\Big[\exp\left(\mathcal{H}_{N,\underline{\omega}}(S)\right);\,\{S_N = 0\} \cap \overline{\Omega}\Big]. \tag{2.17}$$

Moreover, we set

$$\zeta(x) = e^{\widetilde{\lambda}(x+\widetilde{h})}. \tag{2.18}$$

3. Decay of correlations

The proof of Theorem 2.2 is based on the following lemma, which is somehow similar in spirit to Lemmas 4 and 5 in Biskup and den Hollander (1999).

Lemma 3.1. *For every $\underline{v} \in \mathcal{L}$, there exist constants $0 < c_1(\underline{v}), c_2(\underline{v}) < \infty$ such that, for every $N \in \mathbb{N}$, $1 \le a < b \le N$, $k \le (b-a-1)$ and $a < i_1 < i_2 \ldots < i_k < b$,*

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(\{a \le j \le b : S_j = 0\} = \{a, i_1, i_2, \ldots, i_k, b\}) \le c_1^{k+1} e^{-c_2(b-a)}. \tag{3.1}$$

Moreover, let S^1, S^2 be two independent copies of the copolymer, distributed according to the product measure $\mathbf{P}^{\otimes 2}_{N,\underline{\omega}}$. Then,

$$\mathbb{E}\,\mathbf{P}^{\otimes 2}_{N,\underline{\omega}}\left(\nexists\, j :\, a < j < b, S^1_j = S^2_j = 0\right) \le c_1 e^{-c_2(b-a)}. \tag{3.2}$$

Remark 3.2. It will be clear from the proof that the constant c_2 in (3.1) may be chosen smaller, but arbitrarily close to $\mu(\underline{v})$. The quantitative estimate on the constant c_2 in (3.2) that one can extract from the proof given below is instead substantially worse and certainly not optimal.

Proof of Theorem 2.2. Let $\mathcal{S}(A) = \{a_1, \ldots, a_2\}$ and $\mathcal{S}(B) = \{b_1, \ldots, b_2\}$. We assume that $d(\mathcal{S}(A), \mathcal{S}(B)) > 0$, otherwise (2.1) holds trivially with $c_1 = 2$. Without loss of generality, we take $b_1 > a_2$. Then, letting E be the event

$$\mathrm{E} = \{\nexists\, j :\, a_2 < j < b_1, S^1_j = S^2_j = 0\},$$

one can write

$$\mathbf{E}_{N,\underline{\omega}}(AB) - \mathbf{E}_{N,\underline{\omega}}(A)\mathbf{E}_{N,\underline{\omega}}(B) = \mathbf{E}^{\otimes 2}_{N,\underline{\omega}}\left\{(A(S_1)B(S_1) - A(S_1)B(S_2))\,\mathbf{1}_{\mathrm{E}}\right\} \tag{3.3}$$

since, by a simple symmetry argument based on the renewal property of S, one can show that the above average vanishes, if conditioned to the complementary of the event E. At this point, using (3.2) one obtains

$$\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(AB) - \mathbf{E}_{N,\underline{\omega}}(A)\mathbf{E}_{N,\underline{\omega}}(B)\right|\right] \le 2c_1 \|A\|_\infty \|B\|_\infty e^{-c_2(b_1-a_2)}, \tag{3.4}$$

which is statement (2.1) of the theorem.

As for (2.2) we observe that, since we are assuming that $\mathcal{S}(A) \subset \{1, \ldots, k\}$, one has the identity

$$\mathbf{E}_{k,\underline{\omega}}\,(A) \;=\; \frac{\mathbf{E}_{N,\underline{\omega}}\,(A\,\delta_k)}{\mathbf{E}_{N,\underline{\omega}}\,(\delta_k)}, \tag{3.5}$$

where we recall that $\delta_k = \mathbf{1}_{\{S_k=0\}}$. Therefore, there exist positive constants c and c' such that

$$\begin{aligned}\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(A) - \mathbf{E}_{k,\underline{\omega}}(A)\right|\right] &= \mathbb{E}\left[\left|\frac{\mathbf{E}_{N,\underline{\omega}}(A)\,\mathbf{E}_{N,\underline{\omega}}(\delta_k) - \mathbf{E}_{N,\underline{\omega}}(A\,\delta_k)}{\mathbf{E}_{N,\underline{\omega}}(\delta_k)}\right|\right] \\ &\le ck^c\mathbb{E}\left[\zeta(\widetilde{\omega}_k)\left|\mathbf{E}_{N,\underline{\omega}}(A)\,\mathbf{E}_{N,\underline{\omega}}(\delta_k) - \mathbf{E}_{N,\underline{\omega}}(A\,\delta_k)\right|\right] \\ &\le ck^c\left(\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(A)\,\mathbf{E}_{N,\underline{\omega}}(\delta_k) - \mathbf{E}_{N,\underline{\omega}}(A\,\delta_k)\right|^2\right]\mathbb{E}\left[\zeta(\widetilde{\omega}_k)^2\right]\right)^{1/2} \\ &\le c'k^c\|A\|_\infty e^{-c_2 d(\mathcal{S}(A),\{k\})}, \end{aligned} \tag{3.6}$$

where in the first inequality we have applied Lemma A.1, in the second the Cauchy–Schwarz inequality and in the third Theorem 2.2, formula (2.1). Since A is a local observable, $d(\mathcal{S}(A),\{k\}) = k(1+o(1))$ as $k\to\infty$ and therefore the proof of (2.2) is complete.

Finally, (2.3) is a consequence of the decay of the influence of boundary conditions expressed by (2.2). Note in fact that (2.2) states that $\left\{\mathbf{E}_{n,\underline{\omega}}(A)\right\}_n$ is a Cauchy sequence in $L^1(\mathbb{P})$. Therefore it convergence in L^1 toward a limit random variable that we denote $\mathbf{E}_{\infty,\underline{\omega}}(A)$. Therefore (2.2) holds if we set $N=\infty$ and, by the Fubini–Tonelli Theorem, this clearly implies that

$$\mathbb{E}\left[\sum_k \left|\mathbf{E}_{\infty,\underline{\omega}}(A) - \mathbf{E}_{k,\underline{\omega}}(A)\right|\right] < \infty, \tag{3.7}$$

so the series in the expectation is $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. convergent, which implies the almost sure convergence of $\left\{\mathbf{E}_{n,\underline{\omega}}(A)\right\}_n$, that is (2.3).

Theorem 2.2
□

Proof of Lemma 3.1, Equation (3.1). It is immediate to realize that, letting $i_0 = a$ and $i_{k+1} = b$,

$$\begin{aligned}&\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(\{a\le j\le b : S_j = 0\} = \{i_0, i_1, \ldots, i_{k+1}\}) \\ &\qquad\le \prod_{\ell=0}^{k}\mathbb{E}\left[\frac{K(i_{\ell+1}-i_\ell)\zeta\left(\widetilde{\omega}_{i_{\ell+1}}\right)}{2Z^{\underline{v}}_{i_{\ell+1}-i_\ell,\theta^{i_\ell}\underline{\omega}}}\left(1+e^{-2\lambda\sum_{j=i_\ell+1}^{i_{\ell+1}}(\omega_j+h)}\right)\right],\end{aligned} \tag{3.8}$$

where $K(\cdot)$ was defined in Section 1.3 and θ is the left shift. Indeed, it suffices to apply Lemma A.2 with $m=k+1$ and $A_j = \{S_i \neq 0, i_j < i < i_{j+1}\}$. Thanks to Part 1 of Theorem 2.5 and to the exponential integrability of $\widetilde{\omega}_1$, one has

$$-\lim_{N\to\infty}\frac{1}{N}\log\mathbb{E}\left[\frac{\zeta(\widetilde{\omega}_N)}{Z^{\underline{v}}_{N,\underline{\omega}}}\left(1+e^{-2\lambda\sum_{j=1}^{N}(\omega_j+h)}\right)\right] = \mu(\underline{v}) =: \mu > 0,$$

so that one obtains from (3.8)

$$\begin{aligned}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(\{a\le j\le b : S_j = 0\} = \{i_0, i_1, \ldots, i_{k+1}\}) &\le \prod_{\ell=0}^{k} c_3 e^{-\mu(i_{\ell+1}-i_\ell)/2} \\ &= c_3^{k+1}e^{-\mu(b-a)/2}.\end{aligned} \tag{3.9}$$

Lemma 3.1, Eq. (3.1)
□

Proof of Lemma 3.1, Equation (3.2). In this proof the positive constants, typically dependent on $\underline{v}$, will be denoted by $d_0, d_1, \ldots$. The constants c_1 and c_2 are taken from (3.1), but since c_2 is repeated several times we set $\mathrm{C} := c_2$. Let us first define, for $s = 1, 2$, $\eta_0^{(s)} = \sup\{0 \le j \le a : S_j^s = 0\}$, $\eta_i^{(s)} = \inf\{j > \eta_{i-1}^{(s)} : S_j^s = 0\}$ for $i \ge 1$ and $r^{(s)} = \sup\{j : \eta_j^{(s)} < b\}$. Then, for $j = 0, \ldots, r^{(s)}$ we let $\chi_j^{(s)} = \eta_{j+1}^{(s)} - \eta_j^{(s)} \ge 1$. We will refer to the interval $\{\eta_j^{(s)}, \cdots, \eta_{j+1}^{(s)}\}$ as to the j^{th} excursion from zero of the walk S^s, and to $\chi_j^{(s)}$ as to its length, see Figure 2. (Note that the 0^{th} and the $(r^{(s)})^{th}$ excursions may have an endpoint outside $\{a, \ldots, b\}$.)

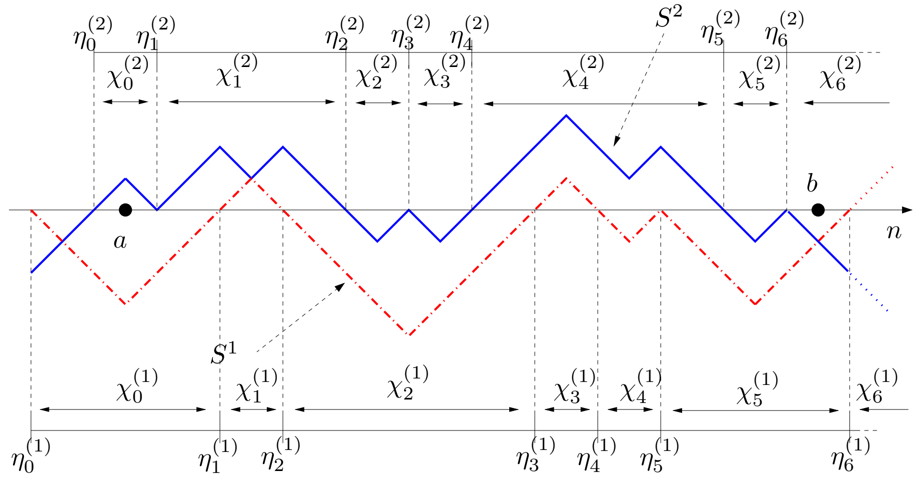

FIGURE 2. In the interval $\{a+1, \ldots, b-1\}$ the walk S^1 (dashed line) and the walk S^2 (full line) never touch zero at the same time. We have marked the returns to zero $\eta_j^{(s)}$ and the lengths of the excursions $\chi_j^{(s)}$. In this example, $r^{(1)} = 5$ and $r^{(2)} = 6$. If for instance we choose $\kappa_2/\mathrm{C} = 4$, then in this example S^1 makes three *short excursions*, and S^2 four. Note that the two walks can meet and cross away from the line $S = 0$.

The basic observation is that, as we will prove in a moment, there exists $\kappa_1, \kappa_2 > 0$ independent of $\underline{v}$ such that

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\sum_{0 \le j \le r^{(1)} : \chi_j^{(1)} \ge \kappa_2 \mathrm{C}^{-1}} \chi_j^{(1)} > \frac{b-a}{4} \right) \le d_1 e^{-\kappa_1 \mathrm{C}(b-a)}, \tag{3.10}$$

uniformly in $N \in \mathbb{N}$, $1 \le a < b \le N$, for some finite constant $d_1 := d_1(\underline{v}) > 0$. In words, this means that with high probability at least $3/4$ of the interval $\{a, \ldots, b\}$ is covered by excursions of S whose length is smaller than $\kappa_2 \mathrm{C}^{-1}$ (we will call them *short excursions*). To prove (3.10), set $x = \chi_0^{(1)}$, $y = \chi_{r^{(1)}}^{(1)}$ and

$$u := \sum_{0 < j < r^{(1)} : \chi_j^{(1)} \ge \kappa_2 \mathrm{C}^{-1}} \chi_j^{(1)}, \tag{3.11}$$

so that the condition in the probability in the left–hand side of (3.10) reads $u + x\mathbf{1}_{x \ge \kappa_2 \mathrm{C}^{-1}} + y\mathbf{1}_{y \ge \kappa_2 \mathrm{C}^{-1}} \ge (b-a)/4$. Obviously, the number ℓ of excursions entirely contained in $\{a, \ldots, b\}$ and of length at least $\kappa_2 \mathrm{C}^{-1}$ (*long excursions*) is at most $\lfloor u\,\mathrm{C}/\kappa_2 \rfloor$, and one can (very roughly) bound above the number of possible ways one can place them in the stretch $\{a, \dots, b\}$ by

$$\left(\begin{array}{c} (b-a) \\ \ell \end{array} \right)^2 . \tag{3.12}$$

These facts, together with a simple application of Lemma A.2 and Eq. (3.1), allow to bound above the left–hand side of (3.10) by

$$\begin{aligned} &\sum_{\substack{x,y\geq 0, u\leq (b-a): \\ \left(x\mathbf{1}_{x\geq \frac{\kappa_2}{\mathrm{C}}}+y\mathbf{1}_{y\geq \frac{\kappa_2}{\mathrm{C}}}+u\right)\geq \frac{b-a}{4}}} (b-a)^2 e^{-\mathrm{C}(u+x+y)} \sum_{\ell=0}^{\lfloor \frac{u\mathrm{C}}{\kappa_2}\rfloor} \left(\begin{array}{c} (b-a) \\ \ell \end{array} \right)^2 c_1^{\ell+1} \\ &\leq \; d_0 (b-a)^3 e^{-\mathrm{C}\frac{b-a}{4}} \max(c_1,1)^{\frac{\mathrm{C}(b-a)}{\kappa_2}} \left(\begin{array}{c} (b-a) \\ \lfloor \frac{\mathrm{C}(b-a)}{\kappa_2} \rfloor \end{array} \right)^2 \leq d_1 e^{-\kappa_1 \mathrm{C}(b-a)}, \end{aligned} \tag{3.13}$$

where the last inequality follows, if κ_2 is sufficiently large, from the Stirling formula. We stress that the constant c_1 is the one appearing in (3.1). The factor $(b-a)^2$ in (3.13) just takes care of the possible location of $\eta_1^{(1)}$ and $\eta_{r^{(1)}}^{(1)}$ in $\{a+1, \dots, b-1\}$.

For ease of notation, let $\mathrm{B}_{a,b}$ be the event $\mathrm{B}_{a,b} = \{\nexists j : a < j < b, S_j^1 = S_j^2 = 0\}$ that the two walks do not touch zero at the same time between a and b, and for $s = 1, 2$ let $\mathrm{C}^{(s)}$ be the event

$$\mathrm{C}^{(s)} = \left\{ \sum_{0\leq j\leq r^{(s)}: \chi_j^{(s)} < \kappa_2 \mathrm{C}^{-1}} \chi_j^{(s)} > \frac{3}{4}(b-a) \right\}. \tag{3.14}$$

Then, from Eq. (3.10), if $\bar{C} = \mathrm{C}^{(1)} \cap \mathrm{C}^{(2)} \cap \mathrm{B}_{a,b}$, one has

$$\mathbb{E}\, \mathbf{P}_{N,\underline{\omega}}^{\otimes 2} (\mathrm{B}_{a,b}) \leq 2 d_1 e^{-\kappa_1 \mathrm{C}(b-a)} + \mathbb{E}\, \mathbf{P}_{N,\underline{\omega}}^{\otimes 2} \left(\bar{C}\right). \tag{3.15}$$

To estimate the last term in (3.15) let us notice that if the event $\bar{C}$ occurs then, denoting by V^s the union of the short excursions of S^s, the set $V^1 \cap V^2 \cap \{a, \dots, b\}$ contains at least $(b-a)/4$ sites. As a consequence, recalling that short excursions do not exceed $\kappa_2 \mathrm{C}^{-1}$ in length, $V := \{j \in V^1 \cap V^2 \cap \{a, \dots, b\} : S_j^1 S_j^2 = 0\}$ contains at least $\lfloor \mathrm{C}(b-a)/(8\kappa_2) \rfloor$ sites. In words, if $j \in V$ then either $S_j^1 = 0$ and j belongs to a short excursion of S^2, or the same holds interchanging the roles of S^1 and S^2. One can rewrite V as the disjoint union $V = W^1 \cup W^2$, where $W^s = \{j \in V : S_j^s = 0\}$ and, of course, at least one among W^1 and W^2 contains $\lfloor \mathrm{C}(b-a)/(16\kappa_2) \rfloor$ points. Therefore, using also the symmetry between S^1 and S^2, one has

$$\begin{aligned} \mathbf{P}_{N,\underline{\omega}}^{\otimes 2} \left(\bar{C}\right) \;&\leq\; 2\, \mathbf{E}_{N,\underline{\omega}}^{\otimes 2} \left(\mathbf{1}_{\bar{C}\cap \{|W^1| \geq \lfloor \mathrm{C}(b-a)/(16\kappa_2)\rfloor\}}\right) \\ &\leq\; 2\, \mathbf{E}_{N,\underline{\omega}}^{\otimes 2} \left(\mathbf{1}_{\bar{C}\cap \{|\hat{W}^1| \geq \lfloor \mathrm{C}(b-a)/(16\kappa_2)\rfloor\}}\right), \end{aligned} \tag{3.16}$$

where $\hat{W}^1$ is the subset of $\{a, \dots, b\}$ which satisfies the following properties:

(1) $S_j^1 = 0$ for every $j \in \hat{W}^1$;
(2) for every $j \in \hat{W}^1$ there exist $a \leq x_j < j < y_j \leq b$ such that
- $S_{x_j}^2 = S_{y_j}^2 = 0$
- $0 < y_j - x_j < \kappa_2 \mathrm{C}^{-1}$
- $S_i^2 \neq 0$ if $\{x_j < i < y_j$ and $S_i^1 \neq 0\}$.

Note that, since we are working on $\bar{C}$, $S_i^2 \neq 0$ when $S_i^1 = 0$: this prescription is not contained in the definition of $\hat{W}^1$. One can now write (see also Fig. 3)

$$\begin{aligned} \mathbf{P}_{N,\underline{\omega}}^{\otimes 2}\left(\bar{C}\right) &\le 2\,\mathbf{E}_{N,\underline{\omega}}\left(\mathbf{E}_{N,\underline{\omega}}^{\otimes 2}\left(\mathbf{1}_{\{\bar{C}\}}\mathbf{1}_{\{|\hat{W}^1|\ge\lfloor \mathrm{C}(b-a)/(16\kappa_2)\rfloor\}}\Big|\, S^1\right)\right) \\ &= 2\,\mathbf{E}_{N,\underline{\omega}}\left(\sum_{\hat{W}}\mathbf{E}_{N,\underline{\omega}}^{\otimes 2}\left(\mathbf{1}_{\{\bar{C}\}}\mathbf{1}_{\{\hat{W}^1=\hat{W}\}}\Big|\, S^1\right)\right) \\ &= 2\,\mathbf{E}_{N,\underline{\omega}}\left(\mathbf{1}_{\{\mathrm{C}^{(1)}\}}\sum_{\hat{W}}\mathbf{E}_{N,\underline{\omega}}^{\otimes 2}\left(\mathbf{1}_{\{\mathrm{C}^{(2)}\}}\mathbf{1}_{\{\mathrm{B}_{a,b}\}}\mathbf{1}_{\{\hat{W}^1=\hat{W}\}}\Big|\, S^1\right)\right). \end{aligned} \tag{3.17}$$

In the second step we have decomposed the probability by summing over all *a priori* admissible configurations $\hat{W}$ of the set $\hat{W}^1$, i.e., all possible subsets of $\{a, \ldots, b\}$ containing at least $\lfloor \mathrm{C}(b-a)/(16\kappa_2)\rfloor$ sites. We are now going to relax the constraint given by $\mathbf{1}_{\{\mathrm{B}_{a,b}\}}$, but estimating the corresponding ratio of probabilities. We claim in fact that, given $k, \ell \le \kappa_2\mathrm{C}^{-1}$ and $0 < i_1 < \ldots < i_k < \ell$, we have

$$\frac{\mathbf{P}_{\ell,\underline{\omega}}(S_j \neq 0, 0<j<\ell)}{\mathbf{P}_{\ell,\underline{\omega}}(S_j \neq 0, j\notin\{0,i_1,\ldots,i_k,\ell\})} \le \prod_{r=1}^{k}\eta\left(\widetilde{\omega}_{i_r}\right), \tag{3.18}$$

where

$$\eta(\widetilde{\omega}) := \left(1+d_2\zeta\left(\widetilde{\omega}\right)\right)^{-1}, \tag{3.19}$$

where d_2 is a positive constant that depends on $K(\cdot)$ and on the value of $\kappa_2\mathrm{C}^{-1}$. The bound (3.18), which will be applied with $S = S^2$, is proven as follows. First observe that

$$\begin{aligned} &\frac{\mathbf{P}_{\ell,\underline{\omega}}(S_j \neq 0, 0<j<\ell)}{\mathbf{P}_{\ell,\underline{\omega}}(S_j \neq 0, j\notin\{0,i_1,\ldots,i_k,\ell\})} \\ &\le \frac{\mathbf{P}_{\ell,\underline{\omega}}(S_j \neq 0, 0<j<\ell)}{\mathbf{P}_{\ell,\underline{\omega}}(S_j > 0, j\notin\{0,i_1,\ldots,i_k,\ell\})+\mathbf{P}_{\ell,\underline{\omega}}(S_j < 0, j\notin\{0,i_1,\ldots,i_k,\ell\})}, \end{aligned} \tag{3.20}$$

and in the ratio in the right–hand side we may factor the expression containing the copolymer energy term, that is we can set $\lambda = 0$ and we can restrict ourselves to considering $S \ge 0$. We write:

$$\begin{aligned} &\frac{\mathbf{P}_{\ell,\underline{\omega}}\left(S_j > 0, j\notin\{0,i_1,\ldots,i_k,\ell\}\right)}{\mathbf{P}_{\ell,\underline{\omega}}\left(S_j > 0, 0<j<\ell\right)} \\ &\quad = \sum_{A\subset\{i_1,\ldots,i_k\}}\frac{\mathbf{P}_{\ell,\underline{\omega}}\left(S_j = 0, j\in A,\, S_j>0, j\in\{1,\ldots,\ell-1\}\setminus A\right)}{K(\ell)/2} \\ &\quad \ge \sum_{A\subset\{i_1,\ldots,i_k\}}\prod_{j\in A}\epsilon_\ell\zeta\left(\widetilde{\omega}_j\right) = \prod_{j\in\{i_1,\ldots,i_k\}}\left(1+\epsilon_\ell\zeta\left(\widetilde{\omega}_j\right)\right), \end{aligned} \tag{3.21}$$

where ϵ_ℓ may be chosen, with a very rough estimate, equal to $\min_{j=1,\ldots,\ell} K(j)^2/4$. Since $\ell \le \kappa_2\mathrm{C}^{-1}$, we obtain (3.18).

Therefore from (3.17), using (3.18), we can extract

$$\mathbf{P}_{N,\underline{\omega}}^{\otimes 2}\left(\bar{C}\right) \le 2\,\mathbf{E}_{N,\underline{\omega}}\left(\mathbf{1}_{\{\mathrm{C}^{(1)}\}}\sum_{\hat{W}}\mathbf{E}_{N,\underline{\omega}}^{\otimes 2}\left(\mathbf{1}_{\{\mathrm{C}^{(2)}\}}\mathbf{1}_{\{\hat{W}^1=\hat{W}\}}\Big|\, S^1\right)\prod_{j\in\hat{W}}\eta(\widetilde{\omega}_j)\right). \tag{3.22}$$

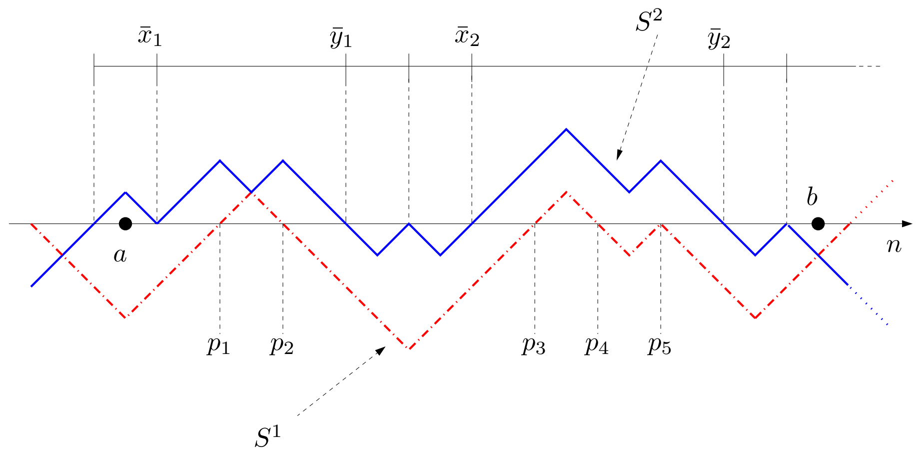

FIGURE 3. Consider again the example of Fig. 2. If, say, $\kappa_2 \text{C}^{-1} = 9$, then $\hat{W} = \{p_1, \ldots, p_5\}$. For $j = 1, 2$ one has $(x_j, y_j) = (\bar{x}_1, \bar{y}_1)$, while $(x_j, y_j) = (\bar{x}_2, \bar{y}_2)$ if $j = 3, 4, 5$.

The remaining problem now is that $\eta(\widetilde{\omega}_n) < 1$ is random and can get arbitrarily close to 1. This however does not happen too often: in fact there exists κ_3 such that, for every $\underline{v} \in \mathcal{L}$,

$$\mathbb{P}(U) := \mathbb{P}\left(|\{a < i < b : \omega_i < -\kappa_3\}| \geq \left\lfloor \frac{\text{C}(b-a)}{32\kappa_2} \right\rfloor \right) \leq d_3 e^{-d_4(b-a)}, \qquad (3.23)$$

for every value of $(b-a)$. The bound (3.23) follows from a direct (large deviation) estimate on the binomial random variable with parameters $p := \mathbb{P}(\widetilde{\omega}_i < -\kappa_3)$, which can be made arbitrarily small, and $b-a$. It suffices for example that $2p \leq \text{C}/32\kappa_2$.

Thanks to $|W| \geq \lfloor \text{C}(b-a)/(16\kappa_2) \rfloor$, (3.22) implies that, on the complementary of U,

$$\mathbf{P}^{\otimes 2}_{N,\underline{\omega}}\left(\bar{C}\right) \leq (1 + d_2 \zeta(-\kappa_3))^{-\frac{\text{C}(b-a)}{32\kappa_2}} \leq d_5 e^{-d_6(b-a)}. \qquad (3.24)$$

Together with (3.15) and (3.23), this completes the proof of (3.2). Lemma 3.1, (3.2) □

4. Regularity of the free energy

Proof of Theorem 2.1. Thanks to the Ascoli-Arzelà Theorem, it is sufficient to show that, for every integer k, the k^{th} derivative of $(1/N)\mathbb{E} \log Z^{\underline{v}}_{N,\underline{\omega}}$ with respect to any of the parameters $\lambda, h, \widetilde{\lambda}, \widetilde{h}$ is bounded above uniformly in N. For definiteness, let us show this property for

$$\frac{\partial^k}{\partial \widetilde{\lambda}^k} \frac{1}{N} \mathbb{E} \log Z^{\underline{v}}_{N,\underline{\omega}}. \qquad (4.1)$$

The above derivative is given by

$$\frac{\partial^k}{\partial \widetilde{\lambda}^k} \frac{1}{N} \mathbb{E} \log Z^{\underline{v}}_{N,\underline{\omega}} = \frac{1}{N} \sum_{1 \leq n_1, n_2, \ldots, n_k \leq N} \mathbb{E}\left\{ \widetilde{\omega}_{n_1} \ldots \widetilde{\omega}_{n_k} \mathbf{E}_{N,\underline{\omega}}(\delta_{n_1}; \delta_{n_2}; \ldots; \delta_{n_k}) \right\}, \qquad (4.2)$$

which is expressed through truncated correlation functions (Ursell functions) defined as

$$\mathbf{E}_{N,\underline{\omega}}\left(A_1;\ldots;A_k\right)=\sum_{\mathcal{P}}(-1)^{|\mathcal{P}|-1}(|\mathcal{P}|-1)!\prod_{P\in\mathcal{P}}\mathbf{E}_{N,\underline{\omega}}\left(\prod_{p\in P}A_p\right), \tag{4.3}$$

where the sum runs over all partitions $\mathcal{P}$ of $\{1,\ldots,k\}$ into subsets P. Starting from the property (2.1) of decay of correlations for every pair of bounded local observables, one can prove by induction over $k\geq 2$ that

$$\begin{aligned}&\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(A_1;\ldots;A_k)\right|\right]\\&\quad\leq c_1^{(k)}\|A_1\|_\infty\ldots\|A_k\|_\infty\exp(-c_2^{(k)}d(\mathcal{S}(A_1),\ldots,\mathcal{S}(A_k))),\end{aligned} \tag{4.4}$$

for some finite positive constants $c_1^{(k)},c_2^{(k)}$ where, if I is the smallest interval including the supports of $A_1,\ldots,A_k$, then $d(\mathcal{S}(A_1),\ldots,\mathcal{S}(A_k))\equiv|I|-\sum_{\ell=1}^k|\mathcal{S}(A_\ell)|$. A proof that the exponential decay of average two–points correlations implies exponential decay of the average n–point truncated correlations can be found for instance in von Dreifus et al. (1995), in the context of disordered d–dimensional spin systems in the high temperature or large magnetic field regime (see Remark 4.1 below for a sketch of the proof). In von Dreifus et al. (1995), explicit bounds for the constants $c_1^{(k)},c_2^{(k)}$ are also given, which is not needed in our case. The proof in von Dreifus et al. (1995), which is basically an application of Hölder's inequality, can be transposed to the present context almost without changes, to yield (4.4). Then, after an application of the Cauchy-Schwarz inequality, it is immediate to realize that the sum in (4.2) converges uniformly in N, for every k.

Theorem 2.1
□

Remark 4.1. A sketch of how (4.4) is deduced from (2.1) goes as follows. Consider the case $k=3$ and assume that $A_i=\delta_{n_i}$ (which is just the case we need in view of (4.2)) and, without loss of generality, let $n_1\leq n_2\leq n_3$. Then consider the simple identities

$$\begin{aligned}&\mathbf{E}_{N,\underline{\omega}}\big(\delta_{n_1};\delta_{n_2};\delta_{n_3}\big)\\&\quad=\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1}\delta_{n_2};\delta_{n_3})-\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1})\mathbf{E}_{N,\underline{\omega}}(\delta_{n_2};\delta_{n_3})-\mathbf{E}_{N,\underline{\omega}}(\delta_{n_2})\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1};\delta_{n_3})\\&\quad=\mathbf{E}_{N,\underline{\omega}}(\delta_{n_2}\delta_{n_3};\delta_{n_1})-\mathbf{E}_{N,\underline{\omega}}(\delta_{n_2})\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1};\delta_{n_3})-\mathbf{E}_{N,\underline{\omega}}(\delta_{n_3})\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1};\delta_{n_2}).\end{aligned} \tag{4.5}$$

From the first identity and (2.1) we obtain

$$\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1};\delta_{n_2};\delta_{n_3})\right|\right]\leq 3c_1e^{-c_2(n_3-n_2)}, \tag{4.6}$$

while from the second identity we obtain the bound $3c_1\exp(-c_2(n_2-n_1))$ on the same quantity. Therefore

$$\mathbb{E}\left[\left|\mathbf{E}_{N,\underline{\omega}}(\delta_{n_1};\delta_{n_2};\delta_{n_3})\right|\right]\leq 3c_1e^{-c_2((n_3-n_2)+(n_2-n_1))/2}, \tag{4.7}$$

which is just the statement of (4.4) in this specific case.

5. On the maximal excursion

5.1. *Proof of Theorem 2.5. Part 1.* The existence of the limit follows from the subadditivity of $\left\{\log \mathbb{E}\left[(1+\exp(-2\lambda\sum_{n=1}^N(\omega_n+h))/Z^{\underline{v}}_{N,\underline{\omega}}\right]\right\}_N$, which is an immediate consequence of the renewal property (of $\mathbf{P}$) and of the IID property (of $\underline{\omega}$): for every $M\in\mathbb{N}$, $M<N$, we have in fact

$$\begin{aligned}\mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}}{Z^{\underline{v}}_{N,\underline{\omega}}}\right] &\le \mathbb{E}\left[\frac{(1+e^{-2\lambda\sum_{n=1}^M(\omega_n+h)})(1+e^{-2\lambda\sum_{n=M+1}^N(\omega_n+h)})}{Z^{\underline{v}}_{M,\underline{\omega}}Z^{\underline{v}}_{N-M,\theta^M\underline{\omega}}}\right]\\ &= \mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^M(\omega_n+h)}}{Z^{\underline{v}}_{M,\underline{\omega}}}\right]\mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^{N-M}(\omega_n+h)}}{Z^{\underline{v}}_{N-M,\underline{\omega}}}\right].\end{aligned} \tag{5.1}$$

The inequality $\mu(\underline{v})\ge 0$ is an immediate consequence of the elementary lower bound

$$Z_{N,\underline{\omega}} \ge \zeta(\widetilde{\omega}_N)\left(1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}\right)\mathbf{P}\left(\Omega_N^+\right). \tag{5.2}$$

A more refined lower bound on $\mu(\underline{v})$, valid in the localized region, follows from the concentration inequality: call E_N the event

$$E_N=\left\{\underline{\omega}:\ -(4\lambda/N)\sum_{n=1}^N(\omega_n+h)<\textsc{f}(\underline{v})/2<(1/N)\log Z^{\underline{v}}_{N,\underline{\omega}}\right\}.$$

Since for N sufficiently large $\mathbb{P}\left(E_N^\complement\right)\le\kappa_1\exp(-\kappa_2 N\textsc{f}(\underline{v})^2/\max(\lambda,\widetilde{\lambda})^2)$ with κ_1 and κ_2 suitable positive constants, one has

$$\begin{aligned}\mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}}{Z^{\underline{v}}_{N,\underline{\omega}}}\right] &\le 2\exp\left(-N\textsc{f}(\underline{v})/4\right)+\frac{\mathbb{E}\left[\zeta(\widetilde{\omega}_N)^{-1}\mathbf{1}_{\{E_N^\complement\}}\right]}{\mathbf{P}\left(\Omega_N^+\right)}\\ &\le 2\exp\left(-N\textsc{f}(\underline{v})/4\right)+\kappa_1'\exp\left(-\kappa_2'N\textsc{f}(\underline{v})^2/\max(\lambda,\widetilde{\lambda})^2\right),\end{aligned}$$

which immediately implies $\mu(\underline{v})>0$ and, for $\textsc{f}(\underline{v})$ sufficiently small, $2\mu(\underline{v})>\kappa_2'\textsc{f}(\underline{v})^2/\max(\lambda,\widetilde{\lambda})^2$.

Finally, $\mu(\underline{v})\le\textsc{f}(\underline{v})$ is an immediate consequence of Jensen's inequality.

Theorem 2.5 part 1 □

Part 2. Throughout this proof $\underline{v}\in\mathcal{L}$, so that $\mu(\underline{v})>0$, and we set $a_N^\pm := \lfloor(1\pm\varepsilon)\log N/\mu(\underline{v})\rfloor$. We start with the proof of

$$\lim_{N\to\infty}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N>a_N^+\right)\;=\;0, \tag{5.3}$$

which clearly implies (2.11). For $n=0,1,2,\dots$ we set

$$E_n\;:=\;\left\{S:\ S_n=0,\ S_{n+j}\neq 0 \text{ for } j=1,2,\dots,a_N^+\right\}, \tag{5.4}$$

so that $\{\mathbf{\Delta}_N>a_N^+\}=\cup_n E_n$, n ranging up to $N-a_N^+-1$, and we have

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N>a_N^+\right)\;\le\;\sum_{n:\,n\le N-a_N^+-1}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(E_n\right). \tag{5.5}$$

Let us estimate the terms in the sum by writing first E_n as the disjoint union of the sets

$$E_{n,\ell} := \{S : S_n = 0,\, S_{n+j} \neq 0 \text{ for } j = 1, 2, \ldots, \ell - 1,\, S_{n+\ell} = 0\}, \tag{5.6}$$

with $\ell \in \mathbb{N}$ and $a_N^+ < \ell \leq N - n$. Recalling the notations of Section 2.4, we have the bound

$$\begin{aligned} \mathbf{P}_{N,\underline{\omega}}(E_{n,\ell}) &\leq \frac{Z^{\underline{v}}_{N,\underline{\omega}}(E_{n,\ell})}{Z^{\underline{v}}_{N,\underline{\omega}}(S_n = S_{n+\ell} = 0)} \\ &\leq \frac{1 + \exp\left(-2\lambda \sum_{j=1}^{\ell} ((\theta^n\omega)_j + h)\right)}{2Z^{\underline{v}}_{\ell,\theta^n\underline{\omega}}}\zeta(\widetilde{\omega}_{n+\ell}). \end{aligned} \tag{5.7}$$

This is an immediate consequence of the renewal property of S. Notice that once we take the expectation of both sides of (5.7) we may set $n = 0$ in the right–hand side and therefore, by (2.9), for every n and every $\ell \geq \ell_0$, ℓ_0 sufficiently large, we have that

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(E_{n,\ell}) \leq \exp\left(-\ell\mu(\underline{v})(1 - (\varepsilon/2))\right). \tag{5.8}$$

Indeed, the factor $\zeta(\widetilde{\omega}_{n+\ell}) = \exp(\widetilde{\lambda}(\widetilde{\omega}_{n+\ell} + \widetilde{h}))$ is negligible for ℓ large since, with probability at least of order $1 - O(\exp(-c\ell^{3/2})$, $|\widetilde{\omega}_{n+\ell}|$ does not exceed $\ell^{3/4}$, see (1.3), so that it does not modify the exponential behavior (5.8).

Therefore, for N sufficiently large and ε sufficiently small

$$\begin{aligned} \mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(E_n) &= \sum_{\substack{\ell\in\mathbb{N}: \\ a_N^+ < \ell \leq N-n}} \mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(E_{n,\ell}) \\ &\leq \sum_{\ell > a_N^+} \exp\left(-\ell\mu(\underline{v})(1 - (\varepsilon/2))\right) \leq N^{-1-(\varepsilon/4)}. \end{aligned} \tag{5.9}$$

Going back to (5.5) we see that $\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N > a_N^+\right) \leq N^{-\varepsilon/4}$ and (5.3) is proven.

Remark 5.1. It is immediate to realize by looking at the proof that

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}(\boldsymbol{\Delta}_N > C\log N) \leq N^{-L(C)}, \tag{5.10}$$

with $L(C) \nearrow \infty$ as $C \nearrow \infty$.

Let us then turn to proving

$$\lim_{N\to\infty} \mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N \geq a_N^-\right) = 1, \quad \mathbb{P}(\mathrm{d}\underline{\omega}) - \text{a.s.}. \tag{5.11}$$

Let us set $j_N := \lfloor N/(\log N)^2 \rfloor - 1$ and $n_j = \lfloor Nj/(j_N + 1) \rfloor$ for $j = 1, 2, \ldots, j_N$.

We now consider the family of events $\left\{E_{n_j,a_N^-}\right\}_{j=1,2,\ldots,j_N}$, recall the definition of $E_{n,\ell}$ in (5.6), and we observe that $\cup_j E_{n_j,a_N^-} \subset \{\boldsymbol{\Delta}_N \geq a_N^-\}$. In words, we are simply going to compute the probability that the walk makes at least one excursion of length exactly equal to a_N^- at j_N prescribed locations: the excursions have to start at n_j for some j, see Figure 4. Therefore

$$\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N < a_N^-\right) \leq \mathbf{P}_{N,\underline{\omega}}\left(\cap_{j=1}^{j_N} E^{\complement}_{n_j,a_N^-}\right) = \prod_{j=1}^{j_N}\left(1 - \mathbf{P}_{N,\underline{\omega}}\left(E_{n_j,a_N^-}\right)\right) + R_N(\underline{\omega}), \tag{5.12}$$

where the last step defines $R_N(\underline{\omega})$ (we anticipate that, by Theorem 2.2, formula (2.1), $R_N(\underline{\omega})$ is negligible, details are postponed to Lemma 5.2 below). We need therefore a lower bound on $\mathbf{P}_{N,\omega}\left(E_{n_j,a_N^-}\right)$: we will find a lower bound on this quantity that depends on $\underline{\omega}$ only via ω_n and $\widetilde{\omega}_n$ with $n = n_j, n_j+1, \ldots, n_j + a_N^-$ and this will allow the direct use of independence when taking the expectation with respect to $\mathbb{P}$. We use the explicit formula

$$\mathbf{P}_{N,\underline{\omega}}\left(E_{n_j,a_N^-}\right) = \frac{Z^{\underline{v}}_{n_j,\underline{\omega}} K\left(a_N^-\right)\zeta(\widetilde{\omega}_{n_j+a_N^-})\left(1+\exp\left(-2\lambda\sum_{n=n_j+1}^{n_j+a_N^-}(\omega_n+h)\right)\right) Z^{\underline{v}}_{N-n_j-a_N^-,\theta^{n_j+a_N^-}\underline{\omega}}}{2Z^{\underline{v}}_{N,\underline{\omega}}}. \tag{5.13}$$

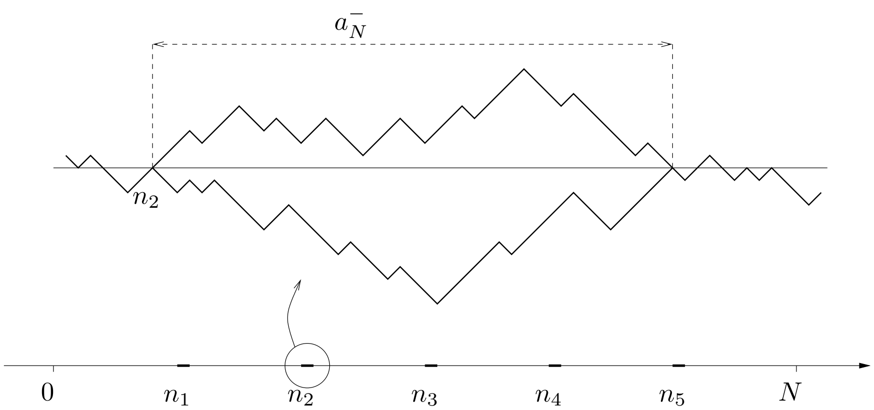

FIGURE 4. The lower bound on the length of the maximal excursion is achieved by focusing on what happens right after $j_N = N(1+o(1))/(\log N)^2$ sites n_j, equal to $n_j = j(\log N)^2$ (modulo lattice corrections). In the Figure, $j_N = 5$. One looks at the probability of finding the (positive or negative) excursion that starts exactly at n_j and comes back to zero at $n_j + a_N^-$, at least for one j. Note that $a_N^- = (1-\varepsilon)(\mu(\underline{v}))^{-1}(1+o(1))\log N$ is much smaller than $n_j - n_{j-1}$ and this allows decoupling of these occurrences.

We proceed by finding an upper bound on the denominator and we first observe that, by (5.3), the event $A_N := \left\{\underline{\omega} :\, Z^{\underline{v}}_{N,\underline{\omega}}(\mathbf{\Delta}_N \le C\log N)/Z^{\underline{v}}_{N,\underline{\omega}} \ge 1-\delta\right\}$ has $\mathbb{P}(\mathrm{d}\underline{\omega})$–probability tending to 1 for $C > 1/\mu(\underline{v})$ and $\delta > 0$: this is simply due to the fact that $Z^{\underline{v}}_{N,\underline{\omega}}(\mathbf{\Delta}_N \le C\log N)/Z^{\underline{v}}_{N,\underline{\omega}} = \mathbf{P}_{N,\underline{\omega}}(\mathbf{\Delta}_N \le C\log N)$. We will actually choose C larger, so that to guarantee that $\mathbb{P}[A_N] \ge 1 - N^{-2}$ for N large, cf. Remark 5.1. Of the requirements defining the event $\mathbf{\Delta}_N \le C\log N$ we now take advantage only of the fact that there exists a return to zero at distance at most $C\log N$ from both sites n_j and $n_j + a_N^-$, for every j. By Lemma A.1 we have that there exists $c > 0$ such that for $\underline{\omega} \in A_N$

$$\begin{aligned} Z^{\underline{v}}_{N,\underline{\omega}} &\le \frac{1}{1-\delta} Z^{\underline{v}}_{N,\underline{\omega}}\left(\min_i |\tau_i - n_j|, \min_i |\tau_i - (n_j + a_N^-)| \le C\log N\right) \\ &\le c(\log N)^c Z^{\underline{v}}_{n_j,\underline{\omega}} Z^{\underline{v}}_{a_N^-,\theta^{n_j}\underline{\omega}} Z^{\underline{v}}_{N-n_j-a_N^-,\theta^{n_j+a_N^-},\underline{\omega}} \zeta(|\widetilde{\omega}_{n_j}|)\zeta(|\widetilde{\omega}_{n_j+a_N^-}|), \end{aligned} \tag{5.14}$$

so that from (5.13) there exist $c, c' > 0$ such that

$$\mathbf{P}_{N,\underline{\omega}}\left(E_{n_j,a_N^-}\right) \ge c(\log N)^{-c'} \frac{1+\exp\left(-2\lambda\sum_{n=n_j+1}^{n_j+a_N^-}(\omega_n+h)\right)}{\zeta(|\widetilde{\omega}_{n_j}|)\zeta(|\widetilde{\omega}_{n_j+a_N^-}|)\, Z^{\underline{v}}_{a_N^-,\theta^{n_j}\underline{\omega}}} =: Q_{N,j}(\underline{\omega}), \tag{5.15}$$

where the inequality holds for N sufficiently large. Notice that the random variables $\{Q_{N,j}(\underline{\omega})\}_j$ are independent, since $Q_{N,j}(\underline{\omega})$ depends on $(\omega_n, \widetilde{\omega}_n)$ with n only in $\{n_j, n_j+1, \dots, n_j+a_N^-\}$. Going back to (5.12), we have

$$\begin{aligned}\mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N < a_N^-\right) &= \mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N < a_N^-\right)\left(\mathbf{1}_{A_N}(\underline{\omega}) + \mathbf{1}_{A_N^\complement}(\underline{\omega})\right) \\ &\le \prod_{j=1}^{j_N}\left(1-Q_{N,j}(\underline{\omega})\right) + \mathbf{1}_{A_N^\complement}(\underline{\omega}) + R_N(\underline{\omega}).\end{aligned} \tag{5.16}$$

We take now the $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–expectation of both sides of (5.16): by independence, by the choice of C in the definition of A_N and by Lemma 5.2

$$\begin{aligned}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N < a_N^-\right) &\le \prod_{j=1}^{j_N}\left(1-\mathbb{E}\left[Q_{N,j}(\underline{\omega})\right]\right) + \mathbb{P}\left[A_N^\complement\right] + \mathbb{E}\left[|R_N(\underline{\omega})|\right] \\ &= \prod_{j=1}^{j_N}\left(1-\mathbb{E}\left[Q_{N,j}(\underline{\omega})\right]\right) + O(1/N^2).\end{aligned} \tag{5.17}$$

We are reduced to estimating $\mathbb{E}[Q_{N,j}(\underline{\omega})]$. By (2.9), for ε sufficiently small and N sufficiently large, we have

$$\mathbb{E}\left[Q_{N,j}(\underline{\omega})\right] \ge \exp\left(-\mu(\underline{v})(1+\varepsilon/2)a_N^-\right) \ge N^{-1+\varepsilon/4}. \tag{5.18}$$

From this one gets

$$\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\mathbf{\Delta}_N < a_N^-\right) \le \left(1-N^{-1+\varepsilon/4}\right)^{j_N} + O(1/N^2) = O(1/N^2). \tag{5.19}$$

By applying the Markov inequality and Borel–Cantelli lemma we conclude the proof of (5.11).

Theorem 2.5 part 2
□

Lemma 5.2. *For every $m > 0$ we have*

$$\lim_{N\to\infty} N^m\,\mathbb{E}\left[|R_N(\underline{\omega})|\right] = 0, \tag{5.20}$$

with $R_N(\underline{\omega})$ defined in (5.12).

Proof. Observe that a direct application of Theorem (2.2), formula (2.1), yields that for $k = 2, 3, \dots, j_N$ we have

$$\begin{aligned}\mathbb{E}\left[\left|\mathbf{P}_{N,\underline{\omega}}\left(\cap_{j=1}^{k} E^\complement_{n_j,a_N^-}\right) - \mathbf{P}_{N,\underline{\omega}}\left(\cap_{j=1}^{k-1} E^\complement_{n_j,a_N^-}\right)\mathbf{P}_{N,\underline{\omega}}\left(E^\complement_{n_k,a_N^-}\right)\right|\right] \\ \le c_1\exp\left(-c_2\frac{(\log N)^2}{2}\right).\end{aligned} \tag{5.21}$$

One now applies iteratively this inequality starting from $k = j_N$, down to $k = 2$, obtaining that $\mathbb{E}\left[|R_N(\underline{\omega})|\right]$ is bounded above by $c_1 N\exp\left(-c_2(\log N)^2/2\right)$.

Lemma 5.2
□

5.2. *Proof of Proposition 2.4.* It makes use of the identity

$$\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N(k)=s\right)=$$
$$K(s)\sum_{\substack{l\in\mathbb{N}\cup\{0\},r\in\mathbb{N}:\\ l+r=s,\,k-l\geq 0,\,k+r\leq N}}\frac{Z^{\underline{v}}_{k-l,\underline{\omega}}\left(1+e^{-2\lambda\sum_{n=k-l+1}^{k+r}(\omega_n+h)}\right)\zeta(\widetilde{\omega}_{k+r})Z^{\underline{v}}_{N-k-r,\theta^{k+r}\underline{\omega}}}{2Z^{\underline{v}}_{N,\underline{\omega}}}. \tag{5.22}$$

Upper bound. We first observe that, reasoning as in Section 5.1, we have

$$\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N(k)=s\right)\leq\sum_{\substack{l\in\mathbb{N}\cup\{0\},r\in\mathbb{N}:\\ l+r=s}}G_{l,r}\left(\theta^k\underline{\omega}\right), \tag{5.23}$$

where

$$G_{l,r}\left(\underline{\omega}\right)\,=\,K(l+r)\frac{1+\exp\left(-2\lambda\sum_{n=-l+1}^{r}(\omega_n+h)\right)}{2Z_{l+r,\theta^{-l}\underline{\omega}}}\zeta(\widetilde{\omega}_r), \tag{5.24}$$

(recall that we are working here with two–sided disorder sequences, so that the $\underline{\omega}$ variables may have negative indices). Now we claim that for every $\varepsilon>0$ there exists $s_0:=s_0\left(\underline{\omega}\right)$, $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. finite, such that

$$G_{l,r}\left(\underline{\omega}\right)\,\leq\,\exp\left(-s(\mathrm{F}(\underline{v})-\varepsilon/2)\right), \tag{5.25}$$

for every l and r such that $l+r=s$ and every $s\geq s_0$. Of course (5.25) implies that for every $\varepsilon>0$

$$C^{(2)}_\varepsilon(\underline{\omega})\,:=\sum_{s\in\mathbb{N}}\sum_{\substack{l\in\mathbb{N}\cup\{0\},r\in\mathbb{N}:\\ l+r=s}}G_{l,r}\left(\underline{\omega}\right)\exp\left(s(\mathrm{F}(\underline{v})-\varepsilon)\right)\,<\infty,\quad\mathbb{P}(\,\mathrm{d}\underline{\omega})-\text{a.s.}. \tag{5.26}$$

By combining (5.23) and (5.26) one directly obtains the upper bound in (2.8).

We are therefore left with the proof of (5.25). This follows by observing first that

$$Z^{\underline{v}}_{l+r,\theta^{-l}\underline{\omega}}\geq Z^{\underline{v}}_{l,R\underline{\omega}}Z^{\underline{v}}_{r,\underline{\omega}}e^{\widetilde{\lambda}(\widetilde{\omega}_0-\widetilde{\omega}_{-l})},$$

where $R\underline{\omega}$ is the disorder sequence reflected around the origin: $(R\underline{\omega})_n=\underline{\omega}_{-n}$. Therefore, with $s=l+r$, we have

$$\frac{1}{s}\log G_{l,r}\left(\underline{\omega}\right)\,\leq\,\frac{1}{s}\log K(s)+\frac{\widetilde{\lambda}\left(\widetilde{\omega}_r-\widetilde{\omega}_0+\widetilde{\omega}_{-l}+\widetilde{h}\right)}{s}+$$
$$\frac{1}{s}\log\left(\frac{1+\exp\left(-2\lambda\sum_{n=-l+1}^{r}(\omega_n+h)\right)}{2}\right)-\frac{1}{s}\log Z^{\underline{v}}_{l,R\underline{\omega}}-\frac{1}{s}\log Z^{\underline{v}}_{r,\underline{\omega}}. \tag{5.27}$$

The leading terms are the last two: by definition of $\mathrm{F}(\underline{v})$, for every $\varepsilon>0$ there exists $n_0(\underline{\omega})$, $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–almost surely finite, such that for l and r larger than $n_0(\underline{\omega})$ both $(1/l)\log Z^{\underline{v}}_{l,R\underline{\omega}}$ and $(1/r)\log Z^{\underline{v}}_{r,\underline{\omega}}$ are larger than $\mathrm{F}(\underline{v})-\varepsilon/2$ and this easily yields the existence of $s_1(\underline{\omega})$ such that

$$\frac{1}{s}\log Z^{\underline{v}}_{l,R\underline{\omega}}+\frac{1}{s}\log Z^{\underline{v}}_{r,\underline{\omega}}\,\geq\mathrm{F}(\underline{v})-\varepsilon/2, \tag{5.28}$$

for every $s\geq s_1(\underline{\omega})$. The remaining term in the last line of (5.27) is treated by an analogous splitting of the sum and by applying the law of large numbers one sees that it gives a negligible contribution for s sufficiently large. Finally the first two terms in the right–hand side of (5.27) are both vanishing as s diverges by the properties of $K(\cdot)$, cf. (1.2), and by the fact that $\widetilde{\omega}_1$ is integrable, so that $\widetilde{\omega}_n/n \stackrel{n\to\infty}{\longrightarrow} 0$, $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s..

Proposition 2.4, upper bound
□

Lower bound. By selecting in the sum in (5.22) only the *excursion* from k to $k+s$ we have that we can write for $k+s \le N$

$$\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N(k)=s\right) \,\ge\, K(s)\frac{Z^{\underline{v}}_{k,\underline{\omega}}\left(1+e^{-2\lambda\sum_{n=k+1}^{k+s}(\omega_n+h)}\right)\zeta(\widetilde{\omega}_{k+s})Z^{\underline{v}}_{N-k-s,\theta^{k+s}\underline{\omega}}}{2Z^{\underline{v}}_{N,\underline{\omega}}}, \tag{5.29}$$

and we will use also

$$\frac{1}{Z^{\underline{v}}_{N,\underline{\omega}}}=\frac{1}{Z_{N,\underline{\omega}}(O_{k,s})}(1-\mathbf{P}_{N,\underline{\omega}}(O^{\complement}_{k,s})), \tag{5.30}$$

where $O_{k,s}$ is the event

$$O_{k,s}=\left\{\exists\, n\in\{k-s^2,\ldots,k+s^2\}:\ S_n=0\right\} \tag{5.31}$$

(with the conventions of Section 2.4 for $Z_{N,\underline{\omega}}(\bar{\Omega})$). Thanks to Lemma A.1, one can write for some $c>0$

$$Z_{N,\underline{\omega}}(O_{k,s}) \le c\, s^c Z^{\underline{v}}_{k,\underline{\omega}} Z^{\underline{v}}_{s,\theta^k\underline{\omega}} Z^{\underline{v}}_{N-k-s,\theta^{k+s}\underline{\omega}}\zeta(|\widetilde{\omega}_k|)\zeta(|\widetilde{\omega}_{k+s}|) \tag{5.32}$$

which, together with (5.29) and (5.30), implies

$$\mathbf{P}_{N,\underline{\omega}}\left(\boldsymbol{\Delta}_N(k)\,=\,s\right) \ge c\, s^{-c} G_{0,s}(\theta^k\underline{\omega})\left[\zeta(|\widetilde{\omega}_k|)\zeta(|\widetilde{\omega}_{k+s}|)\right]^{-1}-\mathbf{P}_{N,\underline{\omega}}(O^{\complement}_{k,s}). \tag{5.33}$$

Proceeding in analogy with the proof of the upper bound, and using in addition Lemma A.1 to bound $Z^{\underline{v}}_{s,\underline{\omega}}$ *above*, one can show that $G_{0,s}(\underline{\omega})\exp((\mathrm{F}(\underline{v})+\varepsilon)s)$ diverges $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. as $s\to\infty$, for every $\varepsilon>0$. Therefore, keeping in mind that $k+s\le N$, the first term on the r.h.s. of (5.33) is bounded *below* by $\exp(-(\mathrm{F}(\underline{v})+\varepsilon)s)$ if s is larger than $s_0(\theta^k\underline{\omega})$, where $s_0(\underline{\omega})$ is a suitable $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. finite number, and the same quantity may then be bounded below by

$$C(\theta^k\underline{\omega})\exp(-(\mathrm{F}(\underline{v})+\varepsilon)s), \tag{5.34}$$

where $C(\underline{\omega})$ is a $\mathbb{P}(\,\mathrm{d}\underline{\omega})$–a.s. positive random variable. On the other hand, from the upper bound in Proposition 2.4 one easily obtains

$$\mathbf{P}_{N,\underline{\omega}}(O^{\complement}_{k,s}) \le\; cC^{(2)}_{\varepsilon}(\theta^k\underline{\omega})\exp(-(\mathrm{F}(\underline{v})-\varepsilon)s^2). \tag{5.35}$$

Taking ε small enough, this immediately implies the lower bound in Proposition 2.4 for $0\le k\le N-s$.

The restrictions on k is of course due to having chosen in the first step of the proof (5.29) the S–*excursion* from k to $k+s$. But we may as well choose the S–*excursion* from $k-s+1$ to $k+1$, and the argument may be repeated yielding the same bound, except for different $\underline{\omega}$ dependent constants, for every k ranging from $s-1$ to $N-1$. Therefore the proof is complete for k in $\{0,1,2,\ldots,N-s\}\cup\{s-1,s,\ldots,N-1\}$ and this is the whole set of sites smaller than N if $s<N/2$.

Proposition 2.4, lower bound
□

6. Finite–size corrections to the infinite–volume free energy

Proof of Proposition 2.7. Just note that

$$0 \le \frac{1}{2N}\mathbb{E}\log Z^{\underline{v}}_{2N,\underline{\omega}} - \frac{1}{N}\mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}} = -\frac{1}{2N}\mathbb{E}\log \mathbf{P}^{\underline{v}}_{2N,\underline{\omega}}(S_N=0) \\ \le c\frac{\log N}{N} + \frac{\widetilde{\lambda}}{N}\mathbb{E}|\widetilde{\omega}_N + \widetilde{h}|, \tag{6.1}$$

where the first inequality is a consequence of the renewal property of S and the last one of Lemma A.1. Inequality (2.13) then immediately follows.

Proposition 2.7
□

Proof of Theorem 2.8. It is sufficient to prove that there exists $c(\underline{v})$ such that, for every $N \in \mathbb{N}$,

$$\frac{1}{2N}\mathbb{E}\log Z^{\underline{v}}_{2N,\underline{\omega}} - \frac{1}{N}\mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}} \le \frac{c(\underline{v})}{N}. \tag{6.2}$$

It is convenient to define, for $x \ge 0$,

$$\hat{F}(\underline{v},x) = \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log \hat{Z}^{\underline{v},x}_{N,\underline{\omega}} := \lim_{N\to\infty}\frac{1}{N}\mathbb{E}\log \mathbf{E}\left(e^{\mathcal{H}_{N,\underline{\omega}}(S)+x\sum_{n=1}^N \delta_n}\delta_N\right) \tag{6.3}$$

which, in view of (1.11), just corresponds to $\hat{\mathrm{F}}(\underline{v},x) := \mathrm{F}(\lambda,h,\widetilde{\lambda},\widetilde{h}+x/\widetilde{\lambda})$ if $\widetilde{\lambda}\neq 0$. Note that, since $\underline{v}\in\mathcal{L}$, one has $\partial_x\, \hat{\mathrm{F}}(\underline{v},x)|_{x=0} > 0$ (cf. Lemma 7.3 below) and that

$$\hat{\mathrm{F}}(\underline{0},x) \ge \mathrm{F}(\underline{v}) > 0, \tag{6.4}$$

provided that $x \ge (2\lambda + \widetilde{\lambda}(1+\widetilde{h}))/(\partial_x\, \hat{\mathrm{F}}(\underline{v},x)|_{x=0})$. Indeed, since $\hat{\mathrm{F}}$ is convex in x one has

$$\hat{\mathrm{F}}(\underline{v},x) \ge \hat{\mathrm{F}}(\underline{v},0) + x\partial_x\, \hat{\mathrm{F}}(\underline{v},x)|_{x=0} \ge \mathrm{F}(\underline{v}) + 2\lambda + \widetilde{\lambda}(1+\widetilde{h})$$

from which (6.4) immediately follows, since $\mathbb{E}\,\omega_1^2 = \mathbb{E}\,\widetilde{\omega}_1^2 = 1$. Essentially the same argument shows that there esists a smooth (e.g. differentiable) path $(\underline{v}_t, x_t)$, with $0\le t\le 1$, such that $(\underline{v}_0,x_0) = (\underline{0},x)$, $(\underline{v}_1,x_1) = (\underline{v},0)$, and such that $\hat{\mathrm{F}}(\underline{v}_t,x_t) \ge \mathrm{F}(\underline{v})$ for every t. Note that at $(\underline{0},x)$ the desorder dependence disappears and we have simply a homogenous pinning model which, thanks to (6.4), is in the localized phase. For this model it is easy to prove (this can be extracted, for instance, from Appendix A of Giacomin and Toninelli (2005b)) that

$$\frac{1}{2N}\log \hat{Z}^{\underline{0},x}_{2N,\underline{\omega}} - \frac{1}{N}\log \hat{Z}^{\underline{0},x}_{N,\underline{\omega}} \le \frac{c_0(\underline{v})}{N}. \tag{6.5}$$

On the other hand, we will prove in a moment that

$$\left|\frac{d}{dt}\left(\frac{1}{2N}\mathbb{E}\log \hat{Z}^{\underline{v}_t,x_t}_{2N,\underline{\omega}} - \frac{1}{N}\mathbb{E}\log \hat{Z}^{\underline{v}_t,x_t}_{N,\underline{\omega}}\right)\right| \le \frac{c_1(\underline{v})}{N} \tag{6.6}$$

uniformly for $0\le t\le 1$. Of course, Eqs. (6.5)-(6.6) immediately give (6.2). To prove (6.6), let us compute for instance the derivative with respect to x: with obvious notations,

$$\left|\frac{d}{dx}\left(\frac{1}{2N}\mathbb{E}\log \hat{Z}^{\underline{w},x}_{2N,\underline{\omega}} - \frac{1}{N}\mathbb{E}\log \hat{Z}^{\underline{w},x}_{N,\underline{\omega}}\right)\right| \\ = \left|\frac{1}{2N}\sum_{n=1}^{2N}\mathbb{E}\,\hat{\mathbf{E}}^{\underline{w},x}_{2N,\underline{\omega}}(\delta_n) - \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}\,\hat{\mathbf{E}}^{\underline{w},x}_{N,\underline{\omega}}(\delta_n)\right| \tag{6.7}$$

which, thanks to Eq. (2.2), is bounded above by

$$\frac{c_3(\underline{w},x)}{N}\sum_{n=1}^{N} e^{-c_4(\underline{w},x)(N-n)} \leq \frac{c_5(\underline{w},x)}{N}, \tag{6.8}$$

c_5 being bounded above uniformly for $(\underline{w},x)$ belonging to the path. Similar estimates hold for the derivatives with respect to $\underline{w}$ and therefore (6.6) follows.

Theorem 2.8 □

7. A Central Limit Theorem

The first step in the proof of Theorem 2.9 is to show that the variance of $N^{-1/2}\log Z^{\underline{v}}_{N,\underline{\omega}}$ is not trivial in the infinite volume limit:

Lemma 7.1. *If $\underline{v} \in \mathcal{L}$, then*

$$0 < \liminf_{N\to\infty} \frac{1}{N}\mathrm{Var}(\log Z^{\underline{v}}_{N,\underline{\omega}}) \leq \limsup_{N\to\infty} \frac{1}{N}\mathrm{Var}(\log Z^{\underline{v}}_{N,\underline{\omega}}) < \infty. \tag{7.1}$$

Proof of Lemma 7.1. The upper bound is an immediate consequence of the deviation inequality (1.3) applied to $g(\underline{\omega}) = N^{-1/2}\log Z^{\underline{v}}_{N,\underline{\omega}}$. Indeed, it is immediate to verify that in this case $\|g\|_{Lip} \leq c$, for some constant c.

To obtain the lower bound, we employ a martingale method analogous to that developed in Aizenman and Wehr (1990). Suppose first that $\widetilde{\lambda} \neq 0$. Observe that, if $X_{N,\widetilde{\omega}} = \mathbb{E}[\log Z^{\underline{v}}_{N,\underline{\omega}}|\widetilde{\omega}]$ and

$$Y_k^{(N)} = \mathbb{E}\left[X_{N,\widetilde{\omega}}|\widetilde{\omega}_1,\ldots,\widetilde{\omega}_k\right] - \mathbb{E}\left[X_{N,\widetilde{\omega}}|\widetilde{\omega}_1,\ldots,\widetilde{\omega}_{k-1}\right], \tag{7.2}$$

then

$$\mathrm{Var}(\log Z^{\underline{v}}_{N,\underline{\omega}}) \geq \mathrm{Var}(X_{N,\widetilde{\omega}}) = \sum_{k=1}^{N}\mathbb{E}\left[\left(Y_k^{(N)}\right)^2\right] \geq \sum_{k=1}^{N}\mathbb{E}\left[\left(\mathbb{E}\left(Y_k^{(N)}|\widetilde{\omega}_k\right)\right)^2\right], \tag{7.3}$$

with the convention that $\mathbb{E}\left[X_{N,\widetilde{\omega}}|\widetilde{\omega}_1,\ldots,\widetilde{\omega}_{k-1}\right] = \mathbb{E}\left[X_{N,\widetilde{\omega}}\right]$ if $k=1$. Next, observe that

$$\partial_{\widetilde{\omega}_k}\mathbb{E}\left[Y_k^{(N)}|\widetilde{\omega}_k\right] = \widetilde{\lambda}\,\mathbb{E}\left[\mathbf{E}_{N,\underline{\omega}}(\delta_k)|\widetilde{\omega}_k\right] \geq 0, \tag{7.4}$$

and that

$$\left|\partial^2_{\widetilde{\omega}_k}\mathbb{E}\left[Y_k^{(N)}|\widetilde{\omega}_k\right]\right| \leq \widetilde{\lambda}^2. \tag{7.5}$$

At this point, one can employ the following

Lemma 7.2. *(Aizenman and Wehr (1990)) Let $0 \leq M \leq 1$, $\beta > 0$ and let V_β be the set of functions*

$$V_\beta = \{g:\mathbb{R}\to\mathbb{R} : g\in C^2, 0\leq g'(\cdot)\leq 1, |g''(\cdot)|\leq\beta\}, \tag{7.6}$$

and, for every probability law ν on $\mathbb{R}$,

$$\gamma_\nu(M,\beta) = \inf\left\{\sqrt{\int g^2(\eta)\nu(d\eta)}\,\middle|\, g\in V_\beta, \int g'(\eta)\nu(d\eta) = M\right\}. \tag{7.7}$$

Then, $\gamma_\nu(M,\beta)$ is convex in M and $\gamma_\nu(M,\beta) > 0$ for $M > 0$, provided that ν is not concentrated on a single point.

Identifying $\beta = \widetilde{\lambda}$, $g_k(\widetilde{\omega}_k) = \widetilde{\lambda}^{-1}\mathbb{E}\left[Y_k^{(N)}|\widetilde{\omega}_k\right]$ and ν with the law of $\widetilde{\omega}_1$, using the convexity of γ_ν and recalling (7.3), one obtains immediately

$$\mathrm{Var}(\log Z_{N,\underline{\omega}}^{\underline{v}}) \geq N\widetilde{\lambda}^2 \left(\gamma_\nu\left(\frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\,\mathbf{E}_{N,\underline{\omega}}(\delta_k), \widetilde{\lambda}\right)\right)^2 . \tag{7.8}$$

To conclude the proof of Lemma 7.1, it suffices therefore to show that

Lemma 7.3. *If* $\underline{v} \in \mathcal{L}$,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\,\mathbf{E}_{N,\underline{\omega}}(\delta_k) > 0. \tag{7.9}$$

Proof of Lemma 7.3. It is enough to prove that there exists ε sufficiently small such that

$$\lim_{N\to\infty}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(|\{j : S_j = 0\}| \leq \varepsilon N\right) = 0. \tag{7.10}$$

Indeed, we have

$$\begin{aligned}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}&\left(|\{j : S_j = 0\}| \leq \varepsilon N\right) = \\ &\sum_{\ell=0}^{\lfloor \varepsilon N\rfloor}\sum_{1\leq i_1<\ldots<i_\ell\leq N}\mathbb{E}\,\mathbf{P}_{N,\underline{\omega}}\left(\{j : S_j = 0\} = \{i_1, i_2, \ldots, i_\ell\}\right) \\ &\leq \sum_{\ell=0}^{\lfloor \varepsilon N\rfloor}\begin{pmatrix} N \\ \ell\end{pmatrix} c_1^{\varepsilon N}e^{-c_2 N} \leq c_3(\varepsilon N+1)\begin{pmatrix} N \\ \lfloor \varepsilon N\rfloor\end{pmatrix} e^{-c_4 N},\end{aligned} \tag{7.11}$$

where in the first inequality we employed (3.1). Using Stirling's approximation, it is clear that, if ε is small enough, the right–hand side of (7.11) decays exponentially in N.

Lemma 7.3 □

In the case $\widetilde{\lambda} = 0$, the condition $\underline{v} \in \mathcal{L}$ implies $\lambda \neq 0$ and the proof of Lemma 7.1 proceeds analogously, with the role of $\widetilde{\omega}$ being played by ω and (7.9) replaced by

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N}\mathbb{E}\,\mathbf{E}_{N,\underline{\omega}}(\Delta_k) > 0. \tag{7.12}$$

Lemma 7.1 □

The second step in the proof of Theorem 2.9 is the observation that (this follows for instance from Lemma A.1)

$$Z_{N_1,\underline{\omega}}^{\underline{v}} Z_{N_2,\theta^{N_1}\underline{\omega}}^{\underline{v}} \leq Z_{N,\underline{\omega}}^{\underline{v}} \leq cN^c e^{\widetilde{\lambda}|\widetilde{\omega}_{N_1}+\widetilde{h}|} Z_{N_1,\underline{\omega}}^{\underline{v}} Z_{N_2,\theta^{N_1}\underline{\omega}}^{\underline{v}}, \tag{7.13}$$

for some $c < \infty$ independent of $\underline{\omega}$. Therefore, keeping also into account the upper bound in (7.1), one obtains the following approximate subadditivity property:

$$\mathrm{Var}(\log Z_{N,\underline{\omega}}^{\underline{v}}) \leq \mathrm{Var}(\log Z_{N_1,\underline{\omega}}^{\underline{v}}) + \mathrm{Var}(\log Z_{N_2,\underline{\omega}}^{\underline{v}}) + c'\sqrt{N}\log N, \tag{7.14}$$

for some constant c' and N large enough. This, together with (7.1), implies that the following limit exists:

$$\lim_{N\to\infty}\frac{1}{N}\mathrm{Var}(\log Z_{N,\underline{\omega}}^{\underline{v}}) = \sigma^2(\underline{v}) > 0. \tag{7.15}$$

Next, employing repeatedly (7.13), one obtains the decomposition

$$\left|\log Z^{\underline{v}}_{N,\underline{\omega}} - \sum_{\ell=0}^{\lfloor N^{1/4}\rfloor -1} \log Z^{\underline{v}}_{\lfloor N^{3/4}\rfloor, \theta^{\ell \lfloor N^{3/4}\rfloor}\underline{\omega}}\right| \le N^{1/4}\log(cN^c) + \widetilde{\lambda} \sum_{\ell=1}^{\lfloor N^{1/4}\rfloor -1} |\widetilde{\omega}_{\ell \lfloor N^{3/4}\rfloor}| \tag{7.16}$$

for every $\underline{\omega}$ (we assumed for simplicity that N is multiple of $\lfloor N^{1/4}\rfloor$). Therefore, if $U_{N,\underline{\omega}} = N^{-1/2}(\log Z^{\underline{v}}_{N,\underline{\omega}} - \mathbb{E}\log Z^{\underline{v}}_{N,\underline{\omega}})$,

$$U_{N,\underline{\omega}} = \frac{1}{N^{1/8}} \sum_{\ell=0}^{\lfloor N^{1/4}\rfloor -1} U_{\lfloor N^{3/4}\rfloor, \theta^{\ell \lfloor N^{3/4}\rfloor}\underline{\omega}} + Q_{N,\underline{\omega}}, \tag{7.17}$$

where $Q_{N,\underline{\omega}}$ tends to zero in law and $\mathbb{P}(\mathrm{d}\underline{\omega})$–a.s.. The summands in (7.17) are IID random variables and Lyapunov's condition, which in this case reads simply

$$\lim_{N\to\infty} \frac{1}{N^{\delta/8}} \mathbb{E}\left[(U_{\lfloor N^{3/4}\rfloor,\underline{\omega}})^{2+\delta}\right] = 0$$

for some $\delta > 0$, follows immediately since the deviation inequality (1.3) implies that $\mathbb{E}[(U_{N,\underline{\omega}})^{2+\delta}]$ is bounded uniformly in N. The central limit theorem is proven.

Theorem 2.9

□

Appendix A. Some technical estimates

Recall the notations in Section 2.4.

Lemma A.1. *There exists $c > 0$ such that for every $k, N \in \mathbb{N}$, $k \le N$ and every $\underline{\omega}$ we have that*

$$\mathbf{P}_{N,\underline{\omega}}(S_k = 0) \ \ge\ \frac{c}{(k \wedge (N-k))^c} \zeta(|\widetilde{\omega}_k|)^{-1}. \tag{A.1}$$

Proof. We write

$$\mathbf{P}_{N,\underline{\omega}}(S_k = 0) \ =\ \frac{Z^{\underline{v}}_{k,\underline{\omega}} Z^{\underline{v}}_{N-k,\theta^k\underline{\omega}}}{Z^{\underline{v}}_{N,\underline{\omega}}} \tag{A.2}$$

and

$$Z^{\underline{v}}_{N,\underline{\omega}} = Z^{\underline{v}}_{k,\underline{\omega}} Z^{\underline{v}}_{N-k,\theta^k\underline{\omega}} + \sum_{\substack{0\le j<k \\ k<\ell\le N}} Z^{\underline{v}}_{j,\underline{\omega}} Z^{\underline{v}}_{N-\ell,\theta^\ell\underline{\omega}} \zeta(\widetilde{\omega}_\ell) K(\ell - j) \frac{1+e^{-2\lambda\sum_{n=j+1}^{\ell}(\omega_n+h)}}{2}. \tag{A.3}$$

Using (1.2) and the definition of slow variation for $L(\cdot)$, one finds that

$$\frac{K(\ell-j)}{K(k-j)K(\ell-k)} \le c((\ell-k)\wedge(k-j))^{2\alpha} \le c(k\wedge(N-k)))^{2\alpha}, \tag{A.4}$$

uniformly in j, ℓ, so that the expression (A.3) can be bounded above by

$$c' Z^{\underline{v}}_{k,\underline{\omega}} Z^{\underline{v}}_{N-k,\theta^k\underline{\omega}} \zeta(|\widetilde{\omega}_k|)(k\wedge(N-k)))^{2\alpha}, \tag{A.5}$$

for some constant c' independent of $\underline{\omega}$, which directly yields (A.1).

Lemma A.1

□

Lemma A.2. *Let* $m \in \mathbb{N}$, $0 \le i_0 < i_1 \ldots < i_m \le N$ *with* $i_\ell \in \mathbb{N}$, *and let* A_j, $j = 0, \ldots, m-1$ *be events depending only on* $S_{i_j+1}, \ldots, S_{i_{j+1}-1}$, *i.e.,* $A_j \in \sigma(S_n : i_j < n < i_{j+1})$. *Then,*

$$\mathbf{P}_{N,\underline{\omega}}\left(S_{i_j} = 0 \text{ for } 0 \le j \le m; \cap_{j=0}^{m-1} A_j\right) \le \prod_{j=0}^{m-1} \mathbf{P}_{i_{j+1}-i_j, \theta^{i_j}\underline{\omega}}(A_j). \tag{A.6}$$

Proof. This is elementary: just rewrite the probability in (A.6) as

$$\frac{\mathbf{E}\left(e^{\mathcal{H}_{N,\underline{\omega}}(S)} \mathbf{1}_{\{S_{i_j}=0 \text{ for } 0 \le j \le m\}} \mathbf{1}_{\{\cap_{j=0}^{m-1} A_j\}} \mathbf{1}_{\{S_N=0\}}\right)}{Z^{\underline{v}}_{N,\underline{\omega}}}, \tag{A.7}$$

and notice that one obtains an upper bound for it if one constrains the walk to touch zero at $i_0, \ldots, i_m$ in the denominator $Z^{\underline{v}}_{N,\underline{\omega}}$. At that point, the probability factorizes thanks to the renewal property of the random walk and one obtains just the right–hand side of (A.6).

Lemma A.2 □

Appendix B. Some remarks on $\mu(\underline{v})$

B.1. In this section we sketch a proof of the fact that, under some reasonable conditions on the law $\mathbb{P}$, the strict inequality $\mu(\underline{v}) < \textsc{f}(\underline{v})$ holds in the localized region.

Since $\lambda^2 + \widetilde{\lambda}^2 > 0$ in $\mathcal{L}$, let us assume for definiteness that $\widetilde{\lambda} > 0$ (in the alternative case, the role of $\widetilde{\omega}$ in the following is played by ω). In analogy with Giacomin and Toninelli (2005b), we assume that one of the following two conditions holds for the IID sequence $\widetilde{\omega}$:

- **C1**: *Continuous random variables.* The law of $\widetilde{\omega}_1$ has a density $P(\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}$, and the function

$$I : \mathbb{R} \ni x \longrightarrow I(x) = \int_{\mathbb{R}} P(y) \log P(y+x)\, \mathrm{d}y, \tag{B.1}$$

  exists and is at least twice differentiable in a neighborhood of $x = 0$. This is true in great generality whenever $P(\cdot)$ is positive, for instance in the case of $P(\cdot) = \exp(-V(\cdot))$, with $V(\cdot)$ a polynomial bounded below.
- **C2**: *Bounded random variables.* The random variable $\widetilde{\omega}_1$ is bounded,

$$|\widetilde{\omega}_1| \le M < \infty. \tag{B.2}$$

Assume first that condition **C1** holds. Given $\varepsilon > 0$, let $\widetilde{\mathbb{P}}_N$ be the law obtained from $\mathbb{P}$ shifting the law of $\widetilde{\omega}_1, \ldots, \widetilde{\omega}_N$ so that $\widetilde{\mathbb{E}}_N[\widetilde{\omega}_\ell] = -\varepsilon$. If ε is small enough, thanks to the assumed regularity of the function $I(\cdot)$ in (B.1) one has

$$\mathrm{H}(\widetilde{\mathbb{P}}_N | \mathbb{P}) := \mathbb{E}\left(\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}} \log \frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}}\right) \le K N \varepsilon^2, \tag{B.3}$$

for some finite constant K. Then, applying the Jensen inequality,

$$\frac{1}{N}\log\mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}}{Z^{\underline{v}}_{N,\underline{\omega}}}\right] = \\ \frac{1}{N}\log\widetilde{\mathbb{E}}_N\left[\frac{\left(1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}\right)}{Z^{\underline{v}}_{N,\underline{\omega}}}e^{\log\left(\mathrm{d}\mathbb{P}/\mathrm{d}\widetilde{\mathbb{P}}_N\right)}\right] \\ \geq \frac{1}{N}\widetilde{\mathbb{E}}_N\left(\log\frac{\mathrm{d}\mathbb{P}}{\mathrm{d}\widetilde{\mathbb{P}}_N}\right) - \frac{1}{N}\mathbb{E}\log Z^{\underline{v}'}_{N,\underline{\omega}}, \quad \text{(B.4)}$$

so that

$$\mu(\underline{v}) \leq K\varepsilon^2 + \textsc{f}(\underline{v}'), \tag{B.5}$$

where $\underline{v}'$ is obtained from $\underline{v}$ replacing $\widetilde{h}$ with $\widetilde{h}-\varepsilon$. Since $\textsc{f}(\cdot)$ is smooth in the localized region and the derivative $\partial_{\widetilde{h}}\textsc{f}(\underline{v}) \geq 0$ is not zero (cf. (7.9)), for ε sufficiently small one has

$$\mu(\underline{v}) \leq K\varepsilon^2 + \textsc{f}(\underline{v}) - \frac{1}{2}\partial_{\widetilde{h}}\textsc{f}(\underline{v})\varepsilon \leq \textsc{f}(\underline{v}) - \frac{1}{4}\partial_{\widetilde{h}}\textsc{f}(\underline{v})\varepsilon < \textsc{f}(\underline{v}). \tag{B.6}$$

In the case of condition **C2**, the proof of (B.6) is slightly more complicated. Instead of shifting the law of $\widetilde{\omega}_1,\ldots,\widetilde{\omega}_N$, $\widetilde{\mathbb{P}}_N$ is obtained by tilting it:

$$\frac{\mathrm{d}\widetilde{\mathbb{P}}_N}{\mathrm{d}\mathbb{P}}(\underline{\omega}) \propto \exp\left(-\varepsilon\sum_{n=1}^N\widetilde{\omega}_n\right). \tag{B.7}$$

The estimate (B.5) on the relative entropy is still valid, with a different constant K, while the proof that

$$\lim_{N\to\infty}\frac{1}{N}\widetilde{\mathbb{E}}_N\log Z^{\underline{v}}_{N,\underline{\omega}} \leq \textsc{f}(\underline{v}) - c\varepsilon, \tag{B.8}$$

for some positive c, although rather intuitive in view of the previous case (note in fact that, for ε small, the tilting (B.7) produces a shift of $-\varepsilon+O(\varepsilon^2)$ of the average of $\widetilde{\omega}_1,\ldots,\widetilde{\omega}_N$), still requires some care. We do not report here other details, which the interested reader can easily reconstruct from the proof of Lemma 3.3 in Giacomin and Toninelli (2005b).

□

B.2. *The case* $\omega_1 \sim -\omega_1$. Here we prove that, if the law of ω_1 is symmetric, then the definition (2.9) of $\mu(\underline{v})$ is equivalent to (2.12). In fact,

$$\begin{aligned}
&\mathbb{E}\frac{e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}}{Z^{\underline{v}}_{N,\underline{\omega}}}\\
&\quad= \mathbb{E}\frac{1}{\mathbf{E}\left[\exp\left(2\lambda\sum_{n=1}^N(\omega_n+h)(1-\Delta_n)+\widetilde{\lambda}\sum_{n=1}^N(\widetilde{\omega}_n+\widetilde{h})\delta_n\right)\delta_N\right]}\\
&\quad\overset{S\overset{\mathbf{P}}{\sim}-S}{=} \mathbb{E}\frac{1}{\mathbf{E}\left[\exp\left(2\lambda\sum_{n=1}^N(\omega_n+h)\Delta_n+\widetilde{\lambda}\sum_{n=1}^N(\widetilde{\omega}_n+\widetilde{h})\delta_n\right)\delta_N\right]}\\
&\quad\le \mathbb{E}\frac{1}{\mathbf{E}\left[\exp\left(2\lambda\sum_{n=1}^N(\omega_n-h)\Delta_n+\widetilde{\lambda}\sum_{n=1}^N(\widetilde{\omega}_n+\widetilde{h})\delta_n\right)\delta_N\right]} \overset{\omega\overset{\mathbb{P}}{\sim}-\omega}{=} \mathbb{E}\frac{1}{Z^{\underline{v}}_{N,\underline{\omega}}},
\end{aligned} \tag{B.9}$$

where in the inequality we simply used the fact that $\lambda, h \ge 0$, see Section 1.2. Therefore,

$$\begin{aligned}
\frac{1}{N}\log\mathbb{E}\left[\frac{1}{Z^{\underline{v}}_{N,\underline{\omega}}}\right] &\le \frac{1}{N}\log\mathbb{E}\left[\frac{1+e^{-2\lambda\sum_{n=1}^N(\omega_n+h)}}{Z^{\underline{v}}_{N,\underline{\omega}}}\right]\\
&\le \frac{1}{N}\log\mathbb{E}\left[\frac{1}{Z^{\underline{v}}_{N,\underline{\omega}}}\right]+\frac{\log 2}{N},
\end{aligned} \tag{B.10}$$

and the claim follows in the limit $N \to \infty$.

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