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Uniqueness of percolation on products with $\mathbb{Z}$

Itai Benjamini and Gady Kozma

Department of mathematics, Weizmann Institute of Science,
Herzl 234, Rehovot 76100, Israel
E-mail address: gady.kozma@weizmann.ac.il
E-mail address: itai.benjamini@weizmann.ac.il
URL: http://www.wisdom.weizmann.ac.il/~itai/

Abstract. We show that there exists a connected graph G with subexponential volume growth such that critical percolation on $G \times \mathbb{Z}$ has infinitely many infinite clusters. We also give some conditions under which this cannot occur.

This paper begins with the observation that if G is any connected graph and p is any number in $[0, 1]$, then the number of infinite clusters in p-percolation[1] on $G \times \mathbb{Z}$ is deterministic, and is either 0, 1 or ∞. The proof is an easy consequence of the fact that one can take any finite set of vertices and translate it along the $\mathbb{Z}$ axis and get a set of variables disjoint from the one you started with.

In view of this, Sznitman asked[2] whether the argument of Burton and Keane (1989) applies. Namely, assume G is amenable, does it follow that $G \times \mathbb{Z}$ has only finitely many infinite clusters? The definition of amenability used here is that the Cheeger constant is 0, namely, for every $\epsilon > 0$ there is some finite set of vertices A such that $|\partial A| \le \epsilon |A|$ where ∂A is the edge boundary of A.

As stated the answer is no. A binary tree with an infinite path added at the root serves as a counterexample. We suggest a slight modification of this question.

Say that G is *strongly amenable* if G contains no nonamenable subgraph.

Assume G is strongly amenable, can one find an interval $[p_1, p_2]$ such that percolation on $G \times \mathbb{Z}$ has infinitely many infinite clusters for every p in this interval? What if we further assume that G has polynomial volume growth?

Our main result is to construct an example of a strongly amenable graph of the form $G \times \mathbb{Z}$ with non uniqueness at $p_c(G \times \mathbb{Z})$. We do not see yet any example of such a graph in which no percolation occurs at $p_c(G \times \mathbb{Z})$, but non uniqueness occurs for some $p > p_c(G \times \mathbb{Z})$.

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[1]For background on percolation see Grimmett (1999) or Lyons with Peres (2012).
[2]Personal communication.

It is tempting to reformulate this question as $p_c = p_u$. Recall that for a *transitive* graph there is a unique transition between the regime of infinitely many infinite clusters and the regime of a unique infinite cluster, i.e. a value p_u such that for all $p > p_u$ there is a unique infinite cluster, while for $p < p_u$ the number of infinite clusters is either 0 or ∞. See Lyons with Peres (2012). However, such a reformulation will be misleading as graphs of the type $G \times \mathbb{Z}$ do not necessary enjoy monotonicity of uniqueness. To see an example, connect the root of a $\mathbb{Z}^{99}$ lattice to the root of a 10 regular tree T, denote this graph by G. The parameters were chosen so as to satisfy

$$p_c(\mathbb{Z}^{100}) < p_c(T \times \mathbb{Z}) < p_u(T \times \mathbb{Z}),$$

(the first inequality follows from the fact that $p_c(\mathbb{Z}^d) \le C/d$, see Kesten, 1990 or Alon et al., 2004, §4; and from the bound $p_c(T \times \mathbb{Z}) \ge \frac{1}{11}$ which holds for any graph with degree 12. The second inequality follows from Schonmann, 2001). It is not hard to see that for small p no percolation occurs on $G \times \mathbb{Z}$. Then between $p_c(\mathbb{Z}^{100})$ and $p_c(T \times \mathbb{Z})$ there is a unique infinite cluster. Between $p_c(T \times \mathbb{Z})$ and $p_u(T \times \mathbb{Z})$ there are infinitely many infinite clusters. Finally, above $p_u(T \times \mathbb{Z})$ again one has a unique infinite cluster. This example can be generalized to an arbitrary (even infinite) number of transitions.

This note has two results on this problem. The first is a counterexample:

Theorem 1. *There exists a connected graph G with subexponential volume growth such that critical percolation on $G \times \mathbb{Z}$ has infinitely many infinite clusters.*

(note that a graph with subexponential volume growth is strongly amenable)

The second is a positive result, a family of graphs G for which we can prove that $G \times \mathbb{Z}$ does not have infinitely many infinite clusters at any p. The result is not very satisfying, and calls for strengthening.

Theorem 2. *Let G be a connected graph such that each finite set can be disconnected from infinity by removing a bounded number of edges. Then $G \times \mathbb{Z}$ does not have infinitely many infinite components.*

In other words, we require from G that there exists some constant K such that for every finite set of vertices A one can find K edges $e_1(A), \dots, e_K(A)$ such that removing these edges will make all the components of all $v \in A$ finite.

Let us finish this introduction with a question unrelated to percolation. There is no known example of an exponentially growing Cayley graph which is strongly amenable and the existence of such is still open, (see de Cornulier and Tessera, 2008 for recent related work and a review of what is known for groups). Recall that a graph G has uniform growth if all balls with the same radius have the same size up to a fixed multiplicative constant. Is there a graph with a uniform exponential growth which is strongly amenable?

Proof of theorem 1

Let d be some sufficiently large number to be fixed later. The graph is constructed as follows. Take a tree of degree $4d$. Let $l_1 = 1$ and $l_{n+1} = l_n + \lceil d^2 \log(n+1) \rceil$. Now, for each $n \ge n_0$ (n_0 to be fixed later too, depending on d) and for each edge (x, y) where x is in level $l_n - 1$ and y is in level l_n, disconnect (x, y) and instead take a copy of $\mathbb{Z}^d$ (considered as a graph with the usual structure) and connect x

with the vertex $(0,\dots,0)$ and y with the vertex $(n,\dots,n)$. All copies of $\mathbb{Z}^d$ (for all such (x,y)) are disjoint. This terminates the definition of the graph G.

We will show that at $p=p_c(\mathbb{Z}^{d+1})$ the graph $G\times\mathbb{Z}$ has infinitely many infinite clusters. One can rather easily convince oneself that in fact below $p_c(\mathbb{Z}^{d+1})$ our graph $G\times\mathbb{Z}$ has no infinite clusters, so $p=p_c(G\times\mathbb{Z})$, but we will not do it here. Note that $p=(1+o(1))/2d$ where $o(1)$ is as $d\to\infty$, see Kesten (1990).

Subexponential growth. Examine the ball B of radius r around the root of the tree we started with. Now, for any x in level l_n of the tree, $d(x,0)\approx n^2$ because the shortest path wastes k steps between levels l_k-1 and l_k for each $k<n$. Therefore B contains tree elements up to level l_h for $h\approx\sqrt{r}$. Since $l_h\approx\sqrt{r}\log r$ we get that B contains $\le\exp(C\sqrt{r}\log r)$ tree vertices. The non-tree vertices of B are contained in $\le\exp(C\sqrt{r}\log r)$ copies of a d-dimensional ball of radius r, so all in all we get

$$|B|\le Cr^de^{C\sqrt{r}\log r}\le Ce^{C\sqrt{r}\log r}$$

which is subexponential, as needed.

Existence. We now turn to show that there are infinite clusters. Let γ be some path in $G\times\mathbb{Z}$. We say that γ is "between levels l_{n-1} and l_n" if for each vertex (v,n) of γ, either v is in the tree, and its level is between l_{n-1} and l_n, or v belongs to one of the copies of $\mathbb{Z}^d$ that were connected between levels l_n-1 and l_n. Further we require that only the first and last vertices of γ may have their v in levels l_{n-1} and l_n. The interior vertices need to be in levels strictly between, or in the copies of $\mathbb{Z}^d$. With this definition we have

Lemma 3. *Let d be sufficiently large, $n>n_0$ and let x be a tree element in level l_{n-1}. Let Z be the set of vertices z in level l_n such that $(z,0)$ is connected to $(x,0)$ by an open path between l_{n-1} and l_n. Then $|Z|$ stochastically dominates a variable U, independent of n, with $\mathbb{E}U>1$.*

Proof: Examine the set Y of vertices y of G in level l_n-1 such that $(x,0)\in G\times\mathbb{Z}$ is connected to $(y,0)$ inside the "slice" $G\times\{0\}$. This is just a problem on supercritical branching processes (for d sufficiently large p_c of the tree, which is $1/(4d-1)$, is smaller than $p_c(\mathbb{Z}^{d+1})=(1+o(1))/2d$) and a standard second moment argument gives that

$$\mathbb{P}\left(|Y|>\tfrac{1}{2}((4d-1)p)^{d^2\log n}\right)>c,$$

where the term $d^2\log n$ is simply l_n-1-l_{n-1}, the height of the tree we are examining. Here and below c denotes positive constants which are allowed to depend only on d. For d sufficiently large we may replace the term $(4d-1)p$ with $3/2$ and drop the the $\frac{1}{2}$ before it. We get

$$\mathbb{P}(|Y|>n^{cd^2})>c.$$

Examine next the set $Z(y)$ of vertices z in level l_n such that $(y,0)$ is connected to $(z,0)$ by an open path that starts by moving from $(y,0)$ into an element $(0,\dots,0)$ in one of the copies of $\mathbb{Z}^d\times\mathbb{Z}$ "below" it, then winds around in that copy and finally takes the last step from $(n,\dots,n,0)$ to $(z,0)$ (this time we allow the path to use the extra dimension, i.e. it is not restricted to the slice $G\times\{0\}$). By Hara (2008), the probability that $(0,\dots,0)\leftrightarrow(n,\dots,n,0)$ in $\mathbb{Z}^d\times\mathbb{Z}=\mathbb{Z}^{d+1}$ is $\ge cn^{2-d}$, recall that we are examining $p_c(\mathbb{Z}^{d+1})$. Further, all these events (for different y)

are independent, because they examine disjoint copies of $\mathbb{Z}^{d+1}$. With the argument of the previous paragraph we see that $(x,0)$ has probability $> c$ to have n^{cd^2} "children" in level $l_n - 1$ and each one has probability $> cn^{2-d}$ to have a child in level l_n, independently. In other words, $Z = \cup_{y\in Y} Z(y)$ dominates a random variable which is with probability $1-c$ empty, and with probability c a sum of n^{cd^2} independent Bernoulli variables with probability cn^{2-d}. Hence, for n sufficiently large, Z dominates a variable U which is empty with probability $1 - c/2$ and with probability $c/2$ is $4/c$. The lemma is thus proved. □

The existence of an infinite cluster at p now follows. Examine a vertex $(x,0)$ with x in level n for $n > n_0$. Define inductively sets of vertices X_i with $X_0 = \{(x,0)\}$ and X_i being all vertices $(y,0)$ with y in level l_{n+i} which are connected to some vertex in X_{i-1} by an open path between levels l_{n+i-1} and l_{n+i}. By lemma 3, the number of elements in X_i which are connected to a given element in X_{i-1}, stochastically dominates the variable U. Further, all these connection events are independent i.e. if x and x' are different elements in $\bigcup X_i$ then the set of their descendants are independent events, because the connections use different edges. Hence the process X_i dominates an independent branching process with offspring distribution U. By lemma 3, $\mathbb{E}U > 1$ so by standard results, a branching process with distribution U survives with positive probability. Hence the process X_i also survives with positive probability. But if X_i survives to infinity then the cluster of $(x,0)$ is infinite. So the probability that an infinite cluster exists is positive. As remarked above, this is a 0-1 event, so in fact the probability is 1.

Below we will also need that the probability has a uniform lower bound, so let us note it now: there exists some constant $c > 0$ such that

$$\mathbb{P}((x,k) \text{ is in an infinite cluster}) > c \qquad \forall x \in l_n\ \forall n > n_0\ \forall k. \tag{1}$$

Non-uniqueness. To see that there are infinitely many clusters we apply the approach of Benjamini and Schramm (1996, Theorem 4) of comparing to a branching random walk. We will not use usual branching random walk but a slightly different process. Let us describe it.

1. If we have a particle in some vertex (x,k) for x in the tree, it sends one particle to each neighbour of (x,k) in $G \times \mathbb{Z}$ with probability p. In particular, if x is in level $l_n - 1$ then a particle is sent to each copy of $\mathbb{Z}^{d+1}$ "below" it, and if it is in level l_n then one particle is sent to the copy of $\mathbb{Z}^{d+1}$ "above" it.

2. Now assume we have a particle in $(0,\dots,0,k)$ in some copy of $\mathbb{Z}^{d+1}$ in the n^{th} level. It sends two kinds of particles. First, one particle with probability p to its tree neighbour (x,k) (which is "above" it in level $l_n - 1$). Second, it sends particles to all vertices (y,l) with y being in the same copy of $\mathbb{Z}^d$ and equal to either $(0,\dots,0)$ or $(n,\dots,n)$, with the distribution of descendants identical to that of vertices connected to $(0,\dots,0,k)$ by independent percolation at p in $\mathbb{Z}^{d+1}$.

3. A particle in $(n,\dots,n,k)$ does the same, sending one particle with probability p to its tree neighbour (x,k) "below" it, and extra particles to $(0,\dots,0,l)$ and $(n,\dots,n,l)$ with the percolation distribution.

This ends the description of the process. Denote the set of particles at time t by X_t. The proof of non-uniqueness now follows from the following two claims

Lemma 4. *For any $x \in G \times \mathbb{Z}$, the set of vertices visited by X_t at some t stochastically dominates $\mathscr{C}(x) \setminus Z$ where Z is the union of all copies of $\mathbb{Z}^{d+1}$.*

(here and below $\mathscr{C}(x)$ denotes the cluster of x).

Lemma 5. *For d and n_0 sufficiently large (n_0 depending on d), X_t is transient i.e. the expected number of returns to the starting point is finite.*

Proof of nonuniqueness given lemmas 4 and 5: Assume by contradiction that there is only one infinite cluster. That would imply, for any x and y in G.

$$\begin{aligned}\mathbb{P}((x,0) \leftrightarrow (y,0)) \geq \mathbb{P}(\{|\mathscr{C}((x,0))| = \infty\} \cap \{|\mathscr{C}((y,0))| = \infty\}) \geq \\ \geq \mathbb{P}(|\mathscr{C}((x,0))| = \infty)\mathbb{P}(|\mathscr{C}((y,0))| = \infty),\end{aligned}$$

where the second inequality follows from FKG (see Grimmett, 1999, §2.2). Assuming x and y belong to level l_n (not necessarily the same n for x and y), (1) would give

$$\mathbb{P}((x,0) \leftrightarrow (y,0)) \geq c.$$

On the other hand, our process X is transient (lemma 5) and symmetric, i.e. the probability to reach x from y is the same as reaching y from x. Any such process must satisfy that, when we fix the starting point of X,

$$\lim_{y\to\infty} \mathbb{P}\left(y \in \bigcup X_t\right) = 0 \tag{2}$$

since otherwise you will have a sequence $y_n \to \infty$ such that you can return to your the starting point with probability $> c$ after visiting y_n. This clearly contradicts transience.

Now apply the domination result. If we also assume that y is not in the copies of $\mathbb{Z}^{d+1}$ then we get

$$\lim_{y\to\infty} \mathbb{P}((x,0) \leftrightarrow (y,0)) = 0. \tag{3}$$

We have reached a contradiction, demonstrating that one cannot have a unique infinite cluster, and thus proving the theorem. □

Proof of lemma 4: This is completely standard: one simply explores the cluster using breadth-first search and note that the "past" of the algorithm only blocks you from exploring some vertices, while the branching process has no such restriction. One has to adapt the breadth-first search to our branching process i.e. when it enters a copy of $\mathbb{Z}^{d+1}$, search all neighbours in the two lines which connect outside in one step, but other than that there is no change necessary in the standard proof (see e.g. Benjamini and Schramm, 1996). □

Proof of lemma 5: We will show that even the projection of X_t on G is transient. Since X_t avoids the copies of $\mathbb{Z}^d$ (except for the points directly connected to the tree), let us consider the graph H which is the tree of degree $4d$, with every edge between level $l_n - 1$ and level l_n "stretched" i.e. replaced by a line with three edges and two vertices. The projection of X_t to G is equivalent to a process on H that, from every particle, sends particles to all neighbours with probability p, sends an average of $2p$ particles to itself; and sometimes sends additional particles to itself and to one of its neighbours (above or below, depending on whether you are in one of the stretched levels, and where exactly you are in them). By lemma 6 below, the expected number of particles that remain in place (whether additional or not) is $< \frac{1}{2}$, if only d is chosen sufficiently large. As for the expected number of the additional particles sent to the neighbouring vertices, it can be bounded directly from the two-point function i.e. from $\mathbb{P}(0 \leftrightarrow x) \le C(d)|x|^{2-d}$, see Hara (2008), and if only n_0 is sufficiently large (as a function of d), the expectation of these can be bounded by $1/4d$.

As a final simplification, embed H in a $4d$-regular tree by "filling" the two sparse rows between $l_n - 1$ and l_n. Namely, level $l_n - 1$ of H goes to a subset of level $l_n - 1 + 2(n - n_0)$ of the tree, the vertices on the stretched edges go to levels $l_n + 2(n - n_0)$ and $l_n + 1 + 2(n - n_0)$ of the tree, level l_n of H goes to level $l_n + 2 + 2(n - n_0)$ of the tree etc. The process on H is now stochastically dominated by a process on the tree, which sends from *each* vertex of the tree to each of its neighbours a particle with probability p (like the "usual" vertices of H), and also sends to itself and to *all* of its neighbors additional particles (like the vertices of H on the stretched edges). In short, each vertex sends both the particles it would have sent if it were a vertex of the tree and the particles it would have sent if it were a vertex of the stretched edges.

We have now reached a very well-understood process: a branching random walk on a $4d$-regular tree, where each particle sends to each of its neighbours an expected number $p + 1/4d < 1/d$ of offspring, and an expected number $< \frac{1}{2}$ of offspring remain in place. Showing that this process is transient can be done with a straightforward calculation. Fix some t and examine the number of particles still at the origin at time t. Any such particles must have done s steps on the tree (for some $s \le t$) and stayed in place $t - s$ steps. For a fixed t and s the expected number of offspring is less than or equal to

$$\#\{\text{paths in the tree of length } s \text{ returning to } x\} \cdot \frac{1}{d^s} \cdot \frac{1}{2^{t-s}} = (4d)^{s/2} \cdot \frac{1}{d^s} \cdot \frac{1}{2^{t-s}}$$

and summing over all t and s shows that the process is transient, if only d is sufficiently large (this last step requires $d > 4$, but we also rely on $p_c(\mathbb{Z}^d)$ being sufficiently close to $1/2d$ and on lemma 6, both which require larger d). □

Lemma 6. *Let d be sufficiently large. Then critical percolation on $\mathbb{Z}^d$ satisfies*

$$\sum_{n \neq 0} \mathbb{P}(\vec{0} \leftrightarrow (0, \ldots, 0, n)) < \frac{C}{\sqrt{d}}$$

(here $\vec{0} = (0, \ldots, 0) \in \mathbb{Z}^d$ and C is a constant independent of the dimension).

We assume that the correct asymptotic behaviour is C/d, but we do not need it in this paper. It is well known that in $d = 2$ this sum is ∞, for example, it follows from the estimate $\mathbb{P}(0 \leftrightarrow \partial B(n)) \ge cn^{-1/3}$, see Kesten (1987, equation (5.1)). We will not give more details on this fact, as it will take us too far off course.

Proof: We follow Heydenreich et al. (2008). Let us recall formula (1.41) ibid., in their notation:

$$\widehat{G_{z_c}}(k) = \frac{1 + O(\beta)}{1 - \widehat{D}(k)}. \tag{4}$$

Let us explain the notation. The z_c is $2d \cdot p_c$ (ibid., §1.2.3) and G is the connection probability

$$G_z(x) = \mathbb{P}(0 \leftrightarrow x)$$

where the probability is with respect to percolation at $z/2d$ (ibid., §1.2.3 and equation 1.19) so G_{z_c} is the critical connection probability. $\beta = K/d$ where K is some absolute constant (ibid., first line of §1.3) and the constant implicit in the $O(\cdot)$ is also dimension independent. $D(x) = \frac{1}{2d}\mathbf{1}_{\{|x|=1\}}$ (ibid., equation 1.1). Finally $\widehat{\cdot}$ is the usual Fourier transform and $k \in [-\pi, \pi)^d$. In particular

$$\widehat{D}(k) = \frac{1}{d}\sum_{i=1}^{d} \cos(k_i).$$

With (4) explained, let us calculate first the sum including the term $\mathbb{P}(\vec{0} \leftrightarrow \vec{0})$,

$$\begin{aligned}\sum_n \mathbb{P}(\vec{0} \leftrightarrow (0,\dots,0,n)) &= \frac{1}{(2\pi)^{d-1}}\int \widehat{G_{z_c}}(k_1,\dots,k_{d-1},0)\,dk_1\cdots dk_{d-1}\\ &\le \frac{1}{(2\pi)^{d-1}}\int \frac{1+C/d}{\frac{d-1}{d} - \frac{1}{d}\sum_{i=1}^{d-1}\cos(k_i)}\,dk_1\cdots dk_{d-1}\\ &= \frac{d+C}{d-1}\cdot\frac{1}{(2\pi)^{d-1}}\int \frac{dk}{1-\widehat{D_{d-1}}(k)}.\end{aligned}$$

Removing the term $\mathbb{P}(\vec{0} \leftrightarrow \vec{0}) = 1 = \frac{1}{(2\pi)^{d-1}}\int 1$ gives

$$\begin{aligned}\sum_{n\neq 0} &\le \frac{d+C}{d-1}\cdot\frac{1}{(2\pi)^{d-1}}\int \frac{1}{1-\widehat{D_{d-1}}(k)} - \frac{d-1}{d+C}\,dk\\ &= \frac{d+C}{d-1}\cdot\frac{1}{(2\pi)^{d-1}}\int \frac{\widehat{D_{d-1}}(k)}{1-\widehat{D_{d-1}}(k)}\,dk + \frac{C+1}{d-1}.\end{aligned} \tag{5}$$

To estimate the integral, apply Cauchy-Schwarz and get

$$\int \frac{\widehat{D}}{1-\widehat{D}} \le \left(\int \widehat{D}^2\right)^{1/2}\left(\int \frac{1}{(1-\widehat{D})^2}\right)^{1/2}.$$

The first integral (with the $(2\pi)^{1-d}$ which we have omitted from the formula above) is simply the probability that simple random walk returns to zero after two steps, and hence it is simply $1/2(d-1)$. The second integral is shown in Heydenreich et al. (2008, (3.4)-(3.6)) to be bounded independently of the dimension (in that paper there is an additional parameter in the calculation, s, which in our case is 1). All in all we get

$$\frac{1}{(2\pi)^{d-1}}\int \frac{\widehat{D_{d-1}}(k)}{1-\widehat{D_{d-1}}(k)}\,dk \le \left(\frac{1}{2(d-1)}\right)^{1/2} C^{1/2}$$

which we plug into (5) and get

$$\sum_{n\neq 0}\mathbb{P}(\vec{0} \leftrightarrow (0\dots,0,n)) \le \frac{d+C}{d-1}\cdot\frac{C}{\sqrt{2(d-1)}} + \frac{C+1}{d-1} \le \frac{C}{\sqrt{d}}$$

as required. □

Proof of theorem 2

Fix some vertex v and examine the number $\mathscr{N}$ of infinite components that intersect $\{v\}\times\mathbb{Z}$. $\mathscr{N}$ is invariant to the translations of $\mathbb{Z}$ so it is constant almost surely. Further, the standard modification argument, Burton and Keane (1989), shows that $\mathscr{N}$ cannot take any finite value > 1. The main step is to preclude the possibility that $\mathscr{N}=\infty$. An infinite cluster which intersects $\{v\}\times\mathbb{Z}$ could intersect it either at finitely many vertices or at infinitely many vertices. We start with the first case.

Lemma 7. *For every vertex v of G, the probability that there exists an infinite cluster intersecting $\{v\}\times\mathbb{Z}$ only at finitely many vertices is 0.*

Proof: The idea is simple: if such clusters exist then they have some positive density. However, as you trace the cluster from v further and further in G, its boundary must increase, eventually increasing beyond $K/$(the density), leading to a contradiction to the disconnection property of G.

Let us make this more formal. Using the disconnection property of G repeatedly, one may find a sequence of $v\in Q_1\subset Q_2\subset\cdots$ with $|\partial Q_i|\le K$ and such that every vertex w which belong to some edge in ∂Q_i belongs to Q_{i+1}. Denote by ∂_{v} the internal vertex boundary i.e. $\partial_{\mathrm{v}}X$ is the set of all vertices in X with neighbours outside X. Then the events

$$E_i := \{(v,0)\leftrightarrow\partial_{\mathrm{v}}Q_i\times\mathbb{Z}\}\setminus\{(v,0)\leftrightarrow\partial_{\mathrm{v}}Q_{i+1}\times\mathbb{Z}\}$$

are disjoint. Hence $\sum\mathbb{P}(E_i)\le 1$, in particular $\mathbb{P}(E_i)\to 0$ as $i\to\infty$. Fix now some L and let F_i be the event that

$$0<|\{n\in\mathbb{Z}:\exists x\in\partial_{\mathrm{v}}Q_i\text{ s.t. }(v,0)\leftrightarrow(x,n)\text{ in }Q_i\times\mathbb{Z}\}|\le L$$

(as usual, "$a\leftrightarrow b$ in X" means that there exists an open path from a to b using only vertices in X). Then if F_i happened then there are at most KL edges through which the cluster may continue, and if they are all closed this would imply E_i. Hence $\mathbb{P}(E_i)\ge(1-p)^{LK}\mathbb{P}(F_i)$. Hence $\mathbb{P}(F_i)\to 0$ as $i\to\infty$.

Assume now that the probability that the cluster of $(v,0)$ is infinite, but its intersection with $\{v\}\times\mathbb{Z}$ is finite, is positive. Let r be some number such that

$$q:=\mathbb{P}\Big(|\mathscr{C}(v,0)|=\infty,(\mathscr{C}(v,0)\cap\{v\}\times\mathbb{Z})\subset\{v\}\times[-r,r)\Big)>0.$$

Fix $L=8Kr/q$ (recall that K is the constant in the disconnection property of G) and with this L define the event F_i above. Since $\mathbb{P}(F_i)\to 0$, let i be sufficiently large such that $\mathbb{P}(F_i)<\frac14 q$. Subtracting we get

$$\mathbb{P}\begin{pmatrix}|\mathscr{C}(v,0)|=\infty,\\(\mathscr{C}(v,0)\cap\{v\}\times\mathbb{Z})\subset\{v\}\times[-r,r),\\|\mathscr{C}(v,0)\cap\partial_{\mathrm{v}}Q_i\times\mathbb{Z}|>L\end{pmatrix}>\tfrac34 q$$

(we used here that if $\mathscr{C}(0)$ is infinite then it cannot be contained in $Q_i\times\mathbb{Z}$, except with probability 0, because $p_c(\text{finite graph}\times\mathbb{Z})=1$). Finally, strengthen the last requirement to $|\mathscr{C}(0)\cap\partial_{\mathrm{v}}Q_i\times[-N,N]|>L$ for some N so large so that it only decreases the probability by $\frac14 q$. We get

$$\mathbb{P}\begin{pmatrix}|\mathscr{C}(v,0)|=\infty,\\(\mathscr{C}(v,0)\cap\{v\}\times\mathbb{Z})\subset\{v\}\times[-r,r),\\|\mathscr{C}(v,0)\cap\partial_{\mathrm{v}}Q_i\times[-N,N]|>L\end{pmatrix}>\tfrac12 q.$$

Denote this event by B and by B_n its translation by n. We finish by ergodicity of the translations by $2r\mathbb{Z}$. Indeed, we know that

$$|\{n\in 1,\dots,a:B_{2rn}\text{ occurred}\}|>\tfrac12 aq$$

for a sufficiently large (random). But this means that in $\partial_{\mathrm{v}}Q_i\times[2r-N,2ar+N]$ there are $>\frac12 aq\cdot L=4Kar$ distinct points, since each cluster has $>L$ points, and the clusters are disjoint since their intersections with $\{v\}\times\mathbb{Z}$ belong to disjoint intervals. But there is no room for $4Kar$ points, only to $K(2ar-2r+1+2N)=2Kar+o(a)$, for a sufficiently large, leading to a contradiction and establishing the lemma. □

Lemma 8. *It is not possible for infinitely many clusters to intersect $\{v\}\times\mathbb{Z}$ at infinitely many vertices each.*

Proof: The idea is as follows: we construct a forest with minimal degree 3 of trifurcation points in $\{v\}\times\mathbb{Z}$ (as was done in Benjamini et al., 1999), and show that this contradicts the amenability of $\mathbb{Z}$. Let us give the details.

We first show that there are trifurcation points. Following Burton and Keane (1989) we note that if there are infinitely many clusters that intersect $\{v\}\times\mathbb{Z}$, then there are trifurcation points. Indeed, for some $r\geq 1$ the probability that three different infinite components intersect $\{v\}\times[-r,r]$ is positive. Let y_1,y_2,y_3 be three different points in $\{v\}\times[-r,r]$ which are connected to infinity by three simple open paths $\gamma_1,\gamma_2,\gamma_3$, respectively, which do not intersect $\{v\}\times[-r,r]$ again and are contained in three different components. Now modify the environment as follows: Open all edges of $\{v\}\times[-r,r]$ and close all edges of $\{e\}\times[-r,r]$ for all edges $e\ni v$ except the three edges which connect each y_i to the next vertex in γ_i. It is clear that after this modification one of the y_i (the middle one) is a trifurcation point. We have just showed that the probability that there is a trifurcation point in $\{v\}\times[-r,r]$ is positive and hence by translation invariance the probability that (v,n) is a trifurcation point is positive for any n.

We remark that, from lemma 7 we can deduce that for any trifurcation point, each of the clusters one would get by removing the trifurcation point must intersect $\{v\}\times\mathbb{Z}$ at infinitely many vertices.

We now follow Benjamini et al. (1999, §4), which shows that under these conditions one may find a forest with minimal degree 3 of trifurcation points. Since the argument is clearly explained there, here we be will brief. The first step is

Claim 9. *If (v,n) is a trifurcation point then each infinite cluster left after removal of (v,n) has at least one other trifurcation point (v,m).*

Proof: Define a mass transport function $M(n,m)$ for any m, $n\in\mathbb{Z}$ as follows: $M(m,n)=1$ if $(v,n)\leftrightarrow(v,m)$ and if (v,n) is the unique closest trifurcation point to (v,m) (the distance is the graph distance on the cluster). Let $M=0$ otherwise. Then each m sends at most 1 unit of mass, and by the mass transport principle the expected amount of mass received by n should also be no more than 1 (we are using the mass transport principle on $\mathbb{Z}$ here). But the negation of the statement of the claim means that n receives an infinite amount of mass (here is where we use the previous remark, that each of the three remaining clusters after the removal of (v,n) is not just infinite, but intersects $\mathbb{Z}$ at infinitely many vertices), so this must happen with probability 0. □

As promised, we now construct an auxiliary graph T over the trifurcation points as follows: m will be connected to n (denoted by $m\sim n$) if both are trifurcation points and if (v,m) is the closest trifurcation point to (v,n) in one of the remaining clusters after the removal of (v,n), or vice versa. Again "closest" means in the graph distance, and we break ties by adding i.i.d. variables X_n uniform in $[0,1]$ and choosing the one with the larger value X. As in Benjamini et al. (1999), T is a forest, and by claim 9 each vertex has degree ≥ 3.

Finally we derive a contradiction to the amenability of $\mathbb{Z}$. Denote $q = \mathbb{P}(0$ is a trifurcation). Let N be some parameter and define the event $E(N)$ that 0 is a trifurcation point and in additional the three $v \sim 0$ satisfy $v \in [-N, N]$. Taking N to be sufficiently large one may assume that $\mathbb{P}(E(N)) > \frac{3}{4}q$. We fix N at this value (and remove it from the notation $E(N)$, so that we just denote it by E from now). Denote by E_n the translation of E by n i.e. the event that n is a trifurcation point with the condition on the neighbours as above.

Using ergodicity we know that as $r \to \infty$, the number of trifurcation points in $[-r, r)$ is $2r(q + o(1))$, while the number of E_n for $n \in [-r, r)$ is $> 2r \cdot \frac{3}{4}q$. Denote by A the set of these trifurcation points. Since T is a forest with minimal degree ≥ 3 we know that

$$|\{v \in T \setminus A \text{ such that } \exists w \in A, w \sim v\}| \geq |A| + 2.$$

But this is clearly impossible, since any such v must be in $[-r - N, r + N]$, and there are $< 2N + 2rq(\frac{1}{4} + o(1))$ trifurcation points in this interval which are not in A. This is a contradiction and the lemma is proved. With lemma 7, and since v was arbitrary, the theorem has been demonstrated. □

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