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Stochastic Schrödinger equations and applications to Ehrenfest-type theorems

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Abstract. We study stochastic evolution equations describing the dynamics of open quantum systems. First, using resolvent approximations, we obtain a sufficient condition for regularity of solutions to linear stochastic Schrödinger equations driven by cylindrical Brownian motions applying to many physical systems. Then, we establish well-posedness and norm conservation property of a wide class of open quantum systems described in position representation. Moreover, we prove Ehrenfest-type theorems that describe the evolution of the mean value of quantum observables in open systems. Finally, we give a new criterion for the existence and uniqueness of weak solutions to non-linear stochastic Schrödinger equations. We apply our results to physical systems such as fluctuating ion traps and quantum measurement processes of position.

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1. Introduction

Stochastic Schrödinger equations are frequently used to describe quantum measurement processes (see, e.g., Barchielli and Gregoratti (2009); Wiseman and Milburn (2010)) and, in general, quantum systems that are sensitive to the environment influence (see, e.g., Gardiner and Zoller (2004); Carmichael (2008)). Moreover, non-linear stochastic Schrödinger equations are becoming an established tool for numerical simulation of the evolution of open quantum systems (see, e.g., Breuer and Petruccione (2002); Percival (1998)). This motivates the study of mathematical properties of stochastic Schrödinger equations allowing us to obtain information on physical phenomena. In this research direction, we first investigate regularity of solutions to linear and non-linear stochastic Schrödinger equations arising in the study of quantum systems with continuous variables, namely having $L^2\left(\mathbb{R}^d, \mathbb{C}\right)$ as state space. Then, we prove a version of Ehrenfest's theorem for open quantum systems. As a concrete physical application, we deduce rigorously the linear heating in a Paul trap.

In Section 2, we first focus on open quantum systems described by the linear stochastic evolution equation in a complex separable Hilbert space $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$:

$$X_t\left(\xi\right) = \xi + \int_0^t G\left(s\right) X_s\left(\xi\right) ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell\left(s\right) X_s\left(\xi\right) dW_s^\ell, \tag{1.1}$$

see, e.g., Barchielli and Gregoratti (2009); Barchielli and Holevo (1995); Bassi et al. (2010); Belavkin (1989); Breuer and Petruccione (2002); Gehm et al. (1998); Gough and Sobolev (2004); Grotz et al. (2006); Halliwell and Zoupas (1995); Schneider and Milburn (1999); Singh and Rost (2007) and the references therein. The driving noise $\left(W^\ell\right)_{\ell\geq 1}$ is a sequence of real valued independent Wiener processes on a filtered complete probability space $\left(\Omega, \mathfrak{F}, \left(\mathfrak{F}_t\right)_{t\geq 0}, \mathbb{P}\right)$, the solution X is a pathwise continuous adapted stochastic processes taking values in $\mathfrak{h}$, $\xi \in L^2(\Omega,\mathbb{P})$, and $\left(G\left(t\right)\right)_{t\geq 0}, \left(L_\ell\left(t\right)\right)_{t\geq 0}$ are given families of linear operators on $\mathfrak{h}$ satisfying

$$G\left(t\right) = -iH\left(t\right) - \frac{1}{2}\sum_{\ell=1}^{\infty} L_\ell\left(t\right)^* L_\ell\left(t\right) \tag{1.2}$$

on suitable common domain with $H\left(t\right)$ symmetric operator. The relation (1.2) is a necessary condition for mean norm square conservation of $X_t\left(\xi\right)$, an important physical property that must hold in the application to open quantum systems.

In Subsection 2.1 we establish a sufficient condition for regularity of solutions to (1.1), closely adapted to its special structure. Regular solutions are essentially solutions with finite energy, indeed, regularity of $X_t\left(\xi\right)$ is characterized through $\mathbb{E}\left\|CX_t(\xi)\right\|^2 < \infty$ for suitable non-negative operators C on $\mathfrak{h}$, with a domain contained in the domains of $G(t)$ and $L_\ell(t)$, allowing us to control unboundedness of these operators. Taking inspiration from resolvent approximation methods developed in Fagnola and Wills (2003), we strengthen results of Mora (2004) and Mora and Rebolledo (2007, 2008) and improve their applicability to open quantum systems with infinite dimensional state space in coordinate representation (see Section 2.1.1 for a review of previous works). Moreover, we prove that regularity of X implies the mean norm square conservation property, namely $\mathbb{E}\left\|X_t(\xi)\right\|^2 = \mathbb{E}\left\|\xi\right\|^2$ for all $t \geq 0$.

In Subsection 2.2, we report our careful verification that existence and uniqueness of the regular solution to (1.1) yields existence and uniqueness of the regular solution to

$$Y_t = Y_0 + \int_0^t G(s, Y_s)\, ds + \sum_{\ell=1}^{\infty} \int_0^t \left(L_\ell(s)\, Y_s - \Re \langle Y_s, L_\ell(s)\, Y_s \rangle\, Y_s \right) dW_s^\ell \tag{1.3}$$

with

$$G(s, y) = G(s)\, y + \sum_{\ell=1}^{\infty} \left(\Re \langle y, L_\ell(s)\, y \rangle\, L_\ell(s)\, y - \frac{1}{2} \Re^2 \langle y, L_\ell(s)\, y \rangle\, y \right).$$

We thus get from Subsection 2.1 a sufficient condition for well-posedness of (1.3). This non-linear stochastic Schrödinger equation is a fundamental tool for modeling the dynamics of states in quantum measurement processes (see, e.g., Barchielli and Holevo (1995); Barchielli and Gregoratti (2009); Bassi et al. (2010); Belavkin (1989); Breuer and Petruccione (2002); Gough and Sobolev (2004)), as well as numerical simulation of the evolution of mean values of quantum observables (see, e.g., Breuer and Petruccione (2002); Mora (2005); Percival (1998)), which are represented by $\mathbb{E}\langle Y_t, AY_t \rangle$.

Mathematics of closed quantum systems is well established, on the contrary, only a few papers deal with open quantum systems whose state space $\mathfrak{h}$ contains, among its components, $L^2(\mathbb{R}^d, \mathbb{C})$ (see, e.g., Bassi et al. (2010); Chebotarev and Fagnola (1998); Kolokol′tsov (1998); Gough and Sobolev (2004); Mora and Rebolledo (2008); Mora (2013) and references therein). However, important physical phenomena are realistically described by open quantum systems involving continuous variables such as position (see, e.g., D'Agosta and Di Ventra (2008); Gough and Sobolev (2004); Halliwell and Zoupas (1995); Haroche and Raimond (2006); Wiseman and Milburn (2010)). This motivates Section 3 where we use our general results as the starting point for investigating well-posedness and norm conservation property of physical systems described in position representation with Hilbert space $\mathfrak{h} = L^2(\mathbb{R}^d, \mathbb{C})$, Hamiltonian

$$H(t) = -\alpha\Delta + i \sum_{j=1}^{d} \left(A^j(t,\cdot)\partial_j + \partial_j A^j(t,\cdot) \right) + V(t,\cdot) \tag{1.4}$$

and noise coefficients

$$L_\ell(t) = \begin{cases} \sum_{j=1}^{d} \sigma_{\ell j}(t,\cdot)\, \partial_j + \eta_\ell(t,\cdot), & \text{if } 1 \leq \ell \leq m \\ 0, & \text{if } \ell > m \end{cases}, \tag{1.5}$$

where $t \geq 0$, $m \in \mathbb{N}$, α is a non-negative real constant, ∂_j denotes the partial derivative with respect to the j^{th}-coordinate, $V, A^j : [0,+\infty[\times\mathbb{R}^d \to \mathbb{R}$ and $\sigma_{\ell j}$, $\eta_\ell : [0,+\infty[\times\mathbb{R}^d \to \mathbb{C}$ are measurable smooth functions. We thus include in our study concrete physical situations like: continuous measurements of position Bassi and Dürr (2008); Dürr et al. (2011); Gough and Sobolev (2004); Kolokol′tsov (1998), atoms in interaction with polarized lasers Singh and Rost (2007), quantum systems in fluctuating traps Grotz et al. (2006); Schneider and Milburn (1999) and collisions of heavy-ions Alicki (1982); Chebotarev and Fagnola (1998). The main difficulties in the study of stochastic partial differential equations (1.1) and (1.3) with Hamiltonian (1.4) and noise operators (1.5) lies in the unboundedness of partial derivatives ∂_j in the noise coefficients as well as in the magnetic fields terms $A^j(t,\cdot)\partial_j + \partial_j A^j(t,\cdot)$, a possible linear growth of functions η_ℓ and the possible quadratic behavior of the potential V; solving (1.1) and (1.3) we must cope with all of them at the same time. We overcome these difficulties by using the reference operator $C = -\Delta + |x|^2$, together with non-trivial algebraic and analytic manipulations.

In Section 4, we derive rigorously Ehrenfest-type theorems for open quantum systems. Indeed, assuming that (1.1) has a unique C-regular solution, we prove, roughly speaking, that the mean value of a C-bounded observable A satisfies:

$$\begin{aligned}\mathbb{E}\left\langle X_t(\xi), AX_t(\xi)\right\rangle &= \mathbb{E}\left\langle \xi, A\xi\right\rangle + \int_0^t \mathbb{E}\left\langle A^* X_s(\xi), G(s) X_s(\xi)\right\rangle ds \\ &\quad + \int_0^t \mathbb{E}\left\langle G(s) X_s(\xi), AX_s(\xi)\right\rangle ds \\ &\quad + \int_0^t \left(\sum_{\ell=1}^{\infty} \mathbb{E}\left\langle L_\ell(s) X_s(\xi), AL_\ell(s) X_s(\xi)\right\rangle\right) ds.\end{aligned} \tag{1.6}$$

States of quantum systems are described by density operators, i.e., positive operators on $\mathfrak{h}$ with unit trace. Under, for instance, the Born-Markov approximation, the density operator at time t is given (in Dirac notation) by

$$\rho_t = \mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$$

whenever the initial density operator is $\mathbb{E}\left|\xi\right\rangle\left\langle\xi\right|$ (see, e.g., Barchielli and Gregoratti (2009); Breuer and Petruccione (2002); Mora (2013); Percival (1998)). Hence the mean value of a C-bounded observable A is well-defined by $tr(\rho_t A)$, which is equal to $\mathbb{E}\langle X_t(\xi), AX_t(\xi)\rangle$ (see, e.g., Mora (2013)), and (1.6) becomes

$$\frac{d}{dt} tr(\rho_t A) = tr\left(\rho_t\left(-i\left[A, H(t)\right] + \frac{1}{2} L_\ell(t)^*\left[A, L_\ell(t)\right] + \frac{1}{2}\left[L_\ell(t)^*, A\right] L_\ell(t)\right)\right), \tag{1.7}$$

where $[\cdot,\cdot]$ stands for the commutator between two operators and $tr(\cdot)$ denotes the trace operation.

Ehrenfest-type theorems describe the rate of change of mean values of quantum observables. In the physical literature on open quantum systems, the generalized Ehrenfest equations (1.6) and (1.7) have been used, for example, to demonstrate connections between quantum and classical mechanics (see, e.g., Percival (1998)), and to estimate the behavior of the expected value of important quantum observables Breuer and Petruccione (2002); Englert and Morigi (2002); Halliwell and Zoupas (1995); Hupin and Lacroix (2010); Salmilehto et al. (2012). Nevertheless, (1.6) and (1.7) have not been rigorously examined from the mathematical viewpoint. This motivates Section 4 where we present the first, to the best of our knowledge, rigorous proof of the Ehrenfest equations (1.6) and (1.7) for open quantum systems with infinite-dimensional state space $\mathfrak{h}$. We would like to point out here that Ehrenfest-type theorems for closed quantum systems have been recently proved by Friesecke and Koppen (2009); Friesecke and Schmidt (2010); our results also generalize this work.

In Section 4, we also introduce sufficient conditions for validity of (1.6) and (1.7) applied to the system with Hamiltonian (1.4) and noise operators (1.5). This, together with Section 3, provides a sound framework for studying open quantum systems in coordinate representation with smooth potentials.

As a concrete physical application we consider ions traps (see, e.g., Wineland et al. (1998) for a description). Quadrupole ion traps were initially developed by Hans Georg Dehmelt and Wolfgang Paul who were awarded the Nobel Prize in Physics for this work having a great impact in quantum information. Experiments show that these traps lose coherence, because the coupling with the environment is relatively strong (see, e.g., Grotz et al. (2006); Leibfried et al. (2003); Wineland et al. (1998) and references therein). This drastically reduces life times of trapped atoms. Here, we prove rigorously the linear heating in a model of a Paul trap whenever the initial density operator is regular enough, providing a mathematically rigorous presentation of the arguments given by Schneider and Milburn (1999).

1.1. *Notation.* In this article, $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$ is a separable complex Hilbert space whose scalar product $\langle\cdot,\cdot\rangle$ is linear in the second variable and anti-linear in the first one. We write $\mathcal{D}(A)$ for the domain of A, whenever A is a linear operator in $\mathfrak{h}$. If $\mathfrak{X}$, $\mathfrak{Z}$ are normed spaces, then we denote by $\mathfrak{L}(\mathfrak{X},\mathfrak{Z})$ the set of all bounded operators from $\mathfrak{X}$ to $\mathfrak{Z}$ and we define $\mathfrak{L}(\mathfrak{X}) = \mathfrak{L}(\mathfrak{X},\mathfrak{X})$. We set $[A,B] = AB - BA$ when A, B are operators in $\mathfrak{h}$. By $\mathcal{B}(\mathfrak{Y})$ we mean the set of all Borel set of the topological space $\mathfrak{Y}$.

Suppose that C is a self-adjoint positive operator in $\mathfrak{h}$. For any $x, y \in \mathcal{D}(C)$ we define the graph scalar product $\langle x,y\rangle_C = \langle x,y\rangle + \langle Cx, Cy\rangle$ and the graph norm $\|x\|_C = \sqrt{\langle x,x\rangle_C}$. We use the symbol $L^2(\mathbb{P},\mathfrak{h})$ to denote the set of all square integrable random variables from $(\Omega,\mathfrak{F},\mathbb{P})$ to $(\mathfrak{h},\mathfrak{B}(\mathfrak{h}))$. Moreover, $L^2_C(\mathbb{P},\mathfrak{h})$ stands for the set of all $\xi \in L^2(\mathbb{P},\mathfrak{h})$ such that $\xi \in \mathcal{D}(C)$ a.s. and $\mathbb{E}\|\xi\|_C^2 < \infty$. We define $\pi_C : \mathfrak{h} \to \mathfrak{h}$ by $\pi_C(x) = x$ if $x \in \mathcal{D}(C)$ and $\pi_C(x) = 0$ if $x \notin \mathcal{D}(C)$.

In case $g : \mathbb{R}^n \mapsto \mathbb{C}$ is Borel measurable, $\lceil g \rceil$ stands for the multiplication operator in $L^2(\mathbb{R}^n,\mathbb{C})$ given by $f \mapsto gf$. We abbreviate $\lceil g \rceil$ to g when no confusion can arise. We denote by $C^k(\mathbb{R}^d,\mathbb{K})$ with $\mathbb{K} = \mathbb{R},\mathbb{C}$, the set of all functions from $\mathbb{R}^d$ to $\mathbb{K}$ whose partial derivatives up to order k are with continuous. Moreover, $C_c^\infty(\mathbb{R}^d,\mathbb{C})$ is the set of all functions of $C^\infty(\mathbb{R}^d,\mathbb{C})$ having compact support. If $f : \mathbb{R}^d \mapsto \mathbb{C}$, then $\partial_k f$ denotes the partial derivative of f with respect to its k-th argument, ∇f stands for the gradient of f and Δf is the Laplacian of f.

In what follows, the letter K denotes generic constants. We will write $K(\cdot)$ for different non-decreasing non-negative functions on the interval $[0,\infty[$ when no confusion is possible.

2. Stochastic Schrödinger equations

2.1. *Linear stochastic Schrödinger equation.*

2.1.1. *Previous works.* In the autonomous case, Holevo (1996) obtained the existence and uniqueness of the weak (topological) solution to (1.1) whenever G is the infinitesimal generator of a contraction semigroup. A drawback of such weak solutions is that they may not preserve the mean value of $\|X_t(\xi)\|^2$ (see, e.g., Holevo (1996)). Rozovskiĭ (1990) proved the existence and uniqueness of variational solutions for a class dissipative linear stochastic evolution equations on real Hilbert spaces, where the regularity of $X_t(\xi)$ is essentially characterized through a strictly positive operator C. In particular, approximating $G(s)$ by $G(s) - \epsilon C^2$ in (1.1), Rozovskiĭ (1990) obtained a solution of (1.1) as a limit of solutions to coercive stochastic evolution equations that are treated using the Galerkin method. This indirect proof makes it difficult to address some properties of the SSEs as time-global estimates needed for establishing the existence of regular invariant measures for (1.3), and time-local estimates appearing in the numerical solution of (1.1) and (1.3) (see, e.g., Mora (2004)). Using Galerkin approximations, Grecksch and Lisei (2011) proved the existence and uniqueness of variational solutions to

$$dX_t = \left(i\left(-H_0 X_t + f\left(t, X_t\right)\right)\right) dt + ig\left(t, X_t\right) dW_t, \tag{2.1}$$

where W is a cylindrical Brownian motion with values in a separable real Hilbert space, f, g are locally Lipschitz functions and $-H_0$ is a coercive operator with discrete spectrum. These conditions are strong in case (2.1) becomes linear.

Applying directly the Galerkin method, together with a priori estimates of the graph norm of the approximating solutions with respect to the reference positive operator C, Mora (2004) and Mora and Rebolledo (2007) proved that (1.1) has a unique strong regular solution, in the autonomous case. The assumptions of Mora (2004) and Mora and Rebolledo (2007) include the existence of an orthonormal basis $\left(e_n\right)_n$ of $(\mathfrak{h}, \langle\cdot,\cdot\rangle)$ that satisfies, for instance, $\sup_{n\in\mathbb{Z}_+}\|CP_n x\| \le \|Cx\|$ for all x belonging to the domain of C, where P_n is the orthogonal projection of $\mathfrak{h}$ over the linear manifold spanned by $e_0, \dots e_n$ and summability of the series $\sum_\ell \left\|L_\ell^* e_n\right\|^2$ together with some domain hypotheses on the adjoint G^* of G. Summability of the series $\sum_\ell \left\|L_\ell^* e_n\right\|^2$, in particular, is a strong mathematical requirement that may not hold even when the operators G and L_ℓ are bounded. In Section 2.1.2, we prove the well-posedness of (1.1), as well as the regularity of its solution, under hypotheses that do not involve the orthogonal basis $\left(e_n\right)_n$, the summability condition and technical hypotheses on adjoints of G and L_ℓ (that now are also time-dependent). Then, we obtain stronger results with simplified proofs and wider range of applications.

Finally, the non-commutative version of (1.1) has been treated using resolvent approximations and a priori estimates by Fagnola and Wills (2003).

2.1.2. *Main results.* We start by making precise the notion of strong regular solution to (1.1).

Hypothesis 1. *Let C be a self-adjoint positive operator in $\mathfrak{h}$ such that:*

(H1.1) *For any $\ell \ge 1$ and $t \ge 0$, $\mathcal{D}(C) \subset \mathcal{D}\left(L_\ell(t)\right)$ and $L_\ell(\cdot)\circ\pi_C$ is measurable as a function from $\left([0,\infty[\times\mathfrak{h}, \mathcal{B}\left([0,\infty[\times\mathfrak{h}\right)\right)$ to $\left(\mathfrak{h}, \mathcal{B}(\mathfrak{h})\right)$.*

(H1.2) *For all $t \ge 0$, $\mathcal{D}(C) \subset \mathcal{D}(G(t))$. Moreover,*

$$G(\cdot)\circ\pi_C : \left([0,\infty[\times\mathfrak{h}, \mathcal{B}\left([0,\infty[\times\mathfrak{h}\right)\right) \to \left(\mathfrak{h}, \mathcal{B}(\mathfrak{h})\right)$$

is measurable.

Definition 2.1. Let Hypothesis 1 hold. Assume that $\mathbb{I}$ is either $[0,\infty[$ or the interval $[0,T]$, with $T \in \mathbb{R}_+$. An $\mathfrak{h}$-valued adapted process $\left(X_t(\xi)\right)_{t\in\mathbb{I}}$ with continuous sample paths is called strong C-solution of (1.1) on $\mathbb{I}$ with initial datum ξ if and only if, for all $t \in \mathbb{I}$:

- $\mathbb{E}\left\|X_t(\xi)\right\|^2 \le \mathbb{E}\|\xi\|^2$, $X_t(\xi) \in \mathcal{D}(C)$ a.s. and $\sup_{s\in[0,t]}\mathbb{E}\left\|CX_s(\xi)\right\|^2 < \infty$.
- $X_t(\xi) = \xi + \int_0^t G(s)\,\pi_C\left(X_s(\xi)\right) ds + \sum_{\ell=1}^\infty \int_0^t L_\ell(s)\,\pi_C\left(X_s(\xi)\right) dW_s^\ell$ $\mathbb{P}$-a.s.

The lemma below guarantees that Hypothesis 1 is valid in many physical models.

Lemma 2.2. *Consider the self-adjoint positive operator* $C : \mathcal{D}(C) \subset \mathfrak{h} \to \mathfrak{h}$. *Suppose that each family of linear operators* $(G(t))_{t\geq 0}$ *and* $(L_\ell(t))_{t\geq 0}$, *with* $\ell \in \mathbb{N}$, *can be written as*

$$\left(\sum_{k=1}^{n} f_k(t)\,\Phi_k\right)_{t\geq 0},$$

where $f_1, \ldots, f_n : ([0,\infty[\,, \mathcal{B}([0,\infty[)) \to (\mathbb{C}, \mathcal{B}(\mathbb{C}))$ *are measurable and* $\Phi_1, \ldots, \Phi_n$ *belong to* $\mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. *Then Hypothesis 1 is fulfilled.*

Proof: Deferred to Subsection 5.1. □

Remark 2.3. Assume that (1.1) is autonomous, i.e., $G(t)$ and $L_\ell(t)$ do not depend on t. From Lemma 2.2 we have that Hypothesis 1 holds in case $G, L_\ell \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, where C is a self-adjoint positive operator on $\mathfrak{h}$.

The following theorem provides a new general sufficient condition for the existence and uniqueness of strong C-solutions to (1.1).

Hypothesis 2. *Let* C *satisfy Hypothesis 1. In addition assume that:*

(H2.1) *For all* $t \geq 0$ *and* $x \in \mathcal{D}(C)$, $\|G(t)x\|^2 \leq K(t)\|x\|_C^2$.

(H2.2) *For every natural number* ℓ *there exists a non-decreasing function* K_ℓ *on* $[0,\infty[$ *satisfying* $\|L_\ell(t)x\|^2 \leq K_\ell(t)\|x\|_C^2$ *for all* $x \in \mathcal{D}(C)$ *and* $t \geq 0$.

(H2.3) *There exists a non-decreasing non-negative function* α *such that*

$$2\Re\left\langle C^2 x, G(t)x\right\rangle + \sum_{\ell=1}^{\infty}\|CL_\ell(t)x\|^2 \leq \alpha(t)\|x\|_C^2$$

for all $t \geq 0$ *and any* x *belonging to a core* $\mathfrak{D}_1$ *of* C^2.

(H2.4) *There exists a core* $\mathfrak{D}_2$ *of* C *such that* $2\Re\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty}\|L_\ell(t)x\|^2 \leq 0$ *for all* $x \in \mathfrak{D}_2$ *and* $t \geq 0$.

Theorem 2.4. *Let Hypothesis 2 hold and assume that* $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$ *is* $\mathfrak{F}_0$*-measurable. Then (1.1) has a unique strong* C*-solution* $(X_t(\xi))_{t\geq 0}$ *with initial datum* ξ. *Moreover,*

$$\mathbb{E}\|CX_t(\xi)\|^2 \leq \exp(t\alpha(t))\left(\mathbb{E}\|C\xi\|^2 + t\alpha(t)\,\mathbb{E}\|\xi\|^2\right).$$

Proof: Deferred to Subsection 5.2. □

Remark 2.5. Under the assumptions and notation of Theorem 2.4, we can prove the Markov property of $X_t(\xi)$ by techniques of well-posed martingale problems (see, e.g., Mora and Rebolledo (2008)).

The next lemma provides an equivalent formulation of Condition H2.3, stated in terms of random variables.

Lemma 2.6. *Suppose that* C *is a self-adjoint positive operator in* $\mathfrak{h}$ *such that* $G(t)$ *and* $CL_\ell(t)$ *belong to* $\mathfrak{L}\left(\left(\mathcal{D}\left(C^2\right), \|\cdot\|_{C^2}\right), \mathfrak{h}\right)$ *for all* $t \geq 0$ *and* $\ell \in \mathbb{N}$. *We define* $\mathcal{L}^+(t,x)$ *to be the positive part of* $2\Re\left\langle C^2 x, G(t)x\right\rangle + \sum_{\ell=1}^{\infty}\|CL_\ell(t)x\|^2$ *whenever* $t \geq 0$ *and* $x \in \mathcal{D}\left(C^2\right)$. *Assume that* $\mathfrak{D}_1$ *is a a core of* C^2. *Then, Condition H2.3 holds if and only if:*

(H2.3') *For all* $\zeta \in L_C^2(\mathbb{P}, \mathfrak{h})$ *satisfying* $\zeta \in \mathfrak{D}_1$ *and* $\|\zeta\| = 1$, *the function*

$$t \mapsto \mathbb{E}\left(\mathcal{L}^+(t,\zeta)\right)$$

is bounded on any interval $[0,T]$, *with* $T > 0$.

Proof: Deferred to Subsection 5.3. □

Under Hypothesis 2 and Condition H3.1 below, we can obtain the mean norm square conservation of $X_t(\xi)$, a crucial physical property of the quantum systems.

Hypothesis 3. *Let Hypothesis 1 hold together with Condition H2.1. Suppose that:*

(H3.1) *For all* $t \geq 0$ *and* $x \in \mathcal{D}(C)$, $2\Re\langle x, G(t)x\rangle + \sum_{\ell=1}^{\infty} \|L_\ell(t)x\|^2 = 0$.

(H3.2) *For any initial datum* ξ *belonging to* $L^2_C(\mathbb{P}, \mathfrak{h})$, *(1.1) has a unique strong* C*-solution on any bounded interval.*

Theorem 2.7. *Assume that Hypothesis 3 holds, together with* $\xi \in L^2_C(\mathbb{P}; \mathfrak{h})$. *Then* $\left(\|X_t(\xi)\|^2\right)_t$ *is a martingale. In particular* $\mathbb{E}\|X_t(\xi)\|^2 = \|\xi\|^2$ *for all* $t \geq 0$.

Proof: Deferred to Subsection A.1. □

Remark 2.8. Condition H3.1 is a quadratic form version of (1.2). It arises from physical situations where we can expect that the solutions of the quantum master equations have trace 1 at any time. Nevertheless, (1.2) is not a sufficient condition for a minimal quantum dynamical semigroup to be identity preserving (see, e.g., Fagnola (1999)).

Remark 2.9. Hypothesis 2, together with Condition H3.1, constitutes a generalized version of non-explosion criteria used to guarantee the conservation of the probability mass of minimal quantum dynamical semigroups (see, e.g., Chebotarev and Fagnola (1998); Chebotarev et al. (1998); Fagnola (1999)). This can be verified in a wide range of applications.

Remark 2.10. The operator C in Theorem 2.4 plays the role of superharmonic (or excessive) functions in the Lyapunov condition for non-explosion of classical minimal Markov processes. For simplicity, suppose that $G(t)$ and $L_\ell(t)$ are time-independent. In this case Condition H2.3 of Hypothesis 2 formally reads as

$$\mathcal{L}\left(C^2\right) \leq \alpha\left(C^2 + I\right),$$

where $\alpha > 0$ and $\mathcal{L}(X) := G^*X + XG + \sum_{\ell=1}^{\infty} L_\ell^* X L_\ell$. Here $\mathcal{L}(X)$ represents the infinitesimal generator of the Markov process X_t applied to the function $x \mapsto \langle x, Xx\rangle$. Actually, we can choose C satisfying $\mathcal{L}\left(C^2\right) \leq \alpha C^2$, hence

$$\frac{d}{dt}\exp(-\alpha t)\, C^2 + \mathcal{L}\left(\exp(-\alpha t)\, C^2\right) \leq 0.$$

Thus, $\phi(t,x) := \exp(-\alpha t)\|Cx\|^2$ is, roughly speaking, an α-excessive function. Therefore, applying formally Itô's formula we obtain that $\exp(-\alpha t)\|CX_t\|^2$ is a supermartingale. Heuristically, ϕ helps us to prove that X_t does not escape from the domain of C, like the existence of superharmonic functions prevents finite explosion times in classical Markov processes.

2.2. *Non-linear stochastic Schrödinger equations.* Using the linear stochastic Schrödinger equation (1.1), Barchielli and Holevo (1995) construct a weak probabilistic solution of (1.3) provided that G and $L_1, L_2, \ldots$ are bounded operators; they actually considered driven noises with jumps in place of some W^ℓ. In the case where $\mathfrak{h}$ is finite-dimensional and at most a finite number of L_k are different from 0, the existence and uniqueness of the strong solution of (1.3) was obtained in Lemma 5 of Mora (2005) by classical methods for stochastic differential equations with locally

Lipschitz coefficients, see also Barchielli and Gregoratti (2009); Pellegrini (2008, 2010).

Gatarek and Gisin (1991) established the existence and pathwise uniqueness of solutions of (1.3) in the following two examples:

- $H = 0$, L_1 self-adjoint and $L_\ell = 0$ for all $\ell \geq 2$.
- Let $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$. Choose $H = -\Delta$, $L_1 f(x) = x f(x)$, and $L_2 = L_3 = \cdots = 0$.

To handle the uniqueness property, Gatarek and Gisin (1991) used strongly that L_1 is a self-adjoint operator. Mora and Rebolledo (2008) obtained the existence and weak uniqueness of regular solutions to (1.3) under the assumptions of Mora and Rebolledo (2007), which were discussed in Section 2.1.1. In the preparation of this paper, we verified that applying the same arguments of the proof of Theorem 1 of Mora and Rebolledo (2008) we can prove Theorem 2.12, asserting the existence and uniqueness of solutions to the non-linear stochastic Schrödinger equation (1.3) under Hypothesis 3. We thus get that Theorem 2.4 provides a sufficient condition for the existence and uniqueness of weak (in the probabilistic sense) regular solution to (1.3).

Definition 2.11. Let C satisfy Hypothesis 1. Suppose that $\mathbb{I}$ is either $[0, +\infty[$ or $[0, r]$ with $r \in \mathbb{R}_+$. We say that $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{I}}, \mathbb{Q}, (Y_t)_{t\in\mathbb{I}}, (W_t^\ell)_{t\in\mathbb{I}}^{\ell\in\mathbb{N}}\right)$ is a solution of class C of (1.3) with initial distribution θ on the interval $\mathbb{I}$ if and only if:

- $(W^\ell)_{\ell\geq 1}$ is a sequence of real valued independent Brownian motions on the filtered complete probability space $\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in\mathbb{I}}, \mathbb{Q}\right)$.
- $(Y_t)_{t\in\mathbb{I}}$ is an $\mathfrak{h}$-valued process with continuous sample paths such that the law of Y_0 coincides with θ and $\mathbb{Q}(\|Y_t\| = 1 \text{ for all } t \in \mathbb{I}) = 1$. Moreover, for every $t \in \mathbb{I}$, $Y_t \in \mathcal{D}(C)$ $\mathbb{Q}$-$a.s.$ and $\sup_{s\in[0,t]} \mathbb{E}_{\mathbb{Q}} \|CY_s\|^2 < \infty$.
- $\mathbb{Q}$-$a.s.$, for all $t \in \mathbb{I}$,

$$\begin{aligned} Y_t &= Y_0 + \int_0^t G(s, \pi_C(Y_s))\, ds \\ &\quad + \sum_{\ell=1}^{\infty} \int_0^t (L_\ell(s)\, \pi_C(Y_s) - \Re\langle Y_s, L_\ell(s)\, \pi_C(Y_s)\rangle\, Y_s)\, dW_s^\ell. \end{aligned}$$

We shall say, for short, that $\left(\mathbb{Q}, (Y_t)_{t\in\mathbb{I}}, (W_t)_{t\in\mathbb{I}}\right)$ is a C-solution of (1.3).

Theorem 2.12. *Let C satisfy Hypothesis 3. Assume that θ is a probability measure on $\mathfrak{h}$ concentrated on $\mathcal{D}(C) \cap \{y \in \mathfrak{h} : \|y\| = 1\}$ such that $\int_{\mathfrak{h}} \|Cx\|^2\, \theta(dx) < \infty$. Then (1.3) has a unique C-solution $\left(\mathbb{Q}, (Y_t)_{t\geq 0}, (W_t)_{t\geq 0}\right)$ with initial law θ.*

Proof: Theorem 2.7 allows us to use arguments of Theorem 1 in Mora and Rebolledo (2008) to show our statement. □

Remark 2.13. Let the assumptions of Theorem 2.12 hold, and let $(X_t(\xi))_{t\geq 0}$ be the strong C-solution of (1.1), where ξ is distributed according to θ. For a given $T \in]0, +\infty[$, we define $\mathbb{Q} = \|X_T(\xi)\|^2 \cdot \mathbb{P}$,

$$B_t^\ell = W_t^\ell - \int_0^t \frac{1}{\|X_s(\xi)\|^2} d\left[W^\ell, \|X(\xi)\|^2\right]_s,$$

and

$$Y_t = \begin{cases} X_t(\xi) / \|X_t(\xi)\|, & \text{if } X_t(\xi) \neq 0 \\ 0, & \text{if } X_t(\xi) = 0 \end{cases},$$

where $t \in [0,T]$ and $\ell \in \mathbb{N}$. By Theorem 2.7, proceeding along the same lines as in the proof of Proposition 1 of Mora and Rebolledo (2008) we can obtain that

$$\left(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t\in[0,T]}, \mathbb{Q}, (Y_t)_{t\in[0,T]}, (B_t^\ell)_{t\in[0,T]}^{\ell\in\mathbb{N}}\right)$$

is a C-solution of (1.3) with initial distribution θ.

3. Open quantum systems in coordinate representation

We now focus on the model given by (1.4) and (1.5), with the functions $\sigma_{\ell h}$ satisfying

$$\sum_{\ell\geq 1} \sigma_{\ell k}(t,x)\,(\partial_j \overline{\sigma}_{\ell h})(t,x) = \sum_{\ell \geq 1} \overline{\sigma}_{\ell k}(t,x)\,(\partial_j \sigma_{\ell h})(t,x) \tag{3.1}$$

for all j,h,k. It is worth noticing that (3.1) obviously holds when functions $\sigma_{\ell k}$ do not depend on x and also when they are real valued or can be transformed into real valued functions by a suitable change of phase. A counterexample due to Fagnola and Pantaleo Martnez (2012) shows that mean norm square conservation may fail when (3.1) does not hold and phases of $\sigma_{\ell k}$ depend on the space variable x. We next collect our smoothness assumptions on the functions involved in (1.4) and (1.5).

Hypothesis 4. *Let $L_\ell(t)$ be the operator (1.5) and assume that (3.1) holds. For all $t \geq 0$, define $G(t) = -iH(t) - \frac{1}{2}\sum_{\ell=1}^m L_\ell^*(t) L_\ell(t)$, where $H(t)$ is as in (1.4). Suppose that there exists a continuous increasing function $K : [0,+\infty[\to]0,+\infty[$ such that:*

(H4.1) *For all $t \geq 0$ and $1 \leq j \leq d$, $V(t,\cdot) \in C^2(\mathbb{R}^d, \mathbb{R})$, $A^j(t,\cdot) \in C^3(\mathbb{R}^d,\mathbb{R})$. Moreover,* $\max\left\{|V(t,x)|, |\Delta V(t,x)|, \left|\partial_j(\Delta A^j)\right|\right\} \leq K(t)\left(1+|x|^2\right)$,

$$\max\left\{|\partial_j V(t,x)|, \left|A^j(t,x)\right|, \left|(\partial_{j'}\partial_j A^j)(t,x)\right|\right\} \leq K(t)(1+|x|)$$

and $\left|\partial_{j'} A^j(t,x)\right| \leq K(t)$, where $t \geq 0$, $x \in \mathbb{R}^d$ and $1 \leq j,j' \leq d$.

(H4.2) *For all $1 \leq \ell \leq m$ and $t \geq 0$ we have $|\sigma_{\ell k}(t,\cdot)| \leq K(t)$, with $1 \leq k \leq d$, $\eta_\ell(t,\cdot) \in C^3(\mathbb{R}^d,\mathbb{C})$ and the absolute values of all the partial derivatives of $\eta_\ell(t,\cdot)$ from the first up to the third order are bounded by $K(t)$. Moreover, at least one of the following conditions holds:*

(H4.2.a) *For all $1 \leq \ell \leq m$, $1 \leq k \leq d$ and $t \geq 0$ we have $|\eta_\ell(t,\cdot)| \leq K(t)$, $\sigma_{\ell k}(t,\cdot) \in C^3(\mathbb{R}^d,\mathbb{C})$, and the absolute values of all partial derivatives of $\sigma_{\ell k}(t,\cdot)$ up to the third order are dominated by $K(t)$.*

(H4.2.b) *For any $1 \leq \ell \leq m$ and $1 \leq k \leq d$, the function $(t,x) \mapsto \sigma_{\ell k}(t,x)$ does not depend on x and $|\eta_\ell(t,0)| \leq K(t)$.*

Note that condition (H4.2.b) allows linear growth in x of $\eta(t,x)$ while (H4.2.a) does not. Theorems 2.4 and 2.7 help us to establish the following result.

Theorem 3.1. *Suppose that Hypothesis 4 holds and set $C = -\Delta + |x|^2$. Let ξ be a $\mathfrak{F}_0$ - measurable random variable taking values in $L^2(\mathbb{R}^d,\mathbb{C})$ such that $\mathbb{E}\|\xi\|^2 = 1$ and $\mathbb{E}\|C\xi\|^2 < \infty$. Then (1.1) has a unique strong C-solution with initial datum ξ. Moreover, $\mathbb{E}\|X_t(\xi)\|^2 = \|\xi\|^2$ for all $t > 0$. If in addition $\|\xi| = 1$ a.s., then (1.3) has a unique C-solution whose initial distribution coincides with that of ξ.*

Proof: Deferred to Subsection 5.4. □

Theorem 3.1 applies in a number of physical models like those listed below, which, for simplicity, are restricted to $\mathfrak{h} = L^2(\mathbb{R}, \mathbb{C})$ and $m = 1$.

(E.1) Choose $\alpha = 1/(2M)$, $A^1(t,x) = cx$, $\sigma_{11}(t,x) = b$, and $\eta_1(t,x) = ax$, where $a, b, c \in \mathbb{R}$ and $M > 0$. Moreover, the potential V is a smooth function. This describes a large particle coupled to a bath of harmonic oscillators in thermal equilibrium (see, e.g., Halliwell and Zoupas (1995)).

(E.2) Let $\alpha = 1/(2M)$, with $M > 0$. Moreover, we take $A^1(t,x) = \sigma_{11}(t,x) = 0$ and $\eta_1(t,x) = \eta x$, where η is a real number. This model describes the dynamics of the continuous measurement of position of a free quantum particle subject to a time-dependent potential $V(t,\cdot)$ (see, e.g., Bassi and Dürr (2008); Gough and Sobolev (2004)), a process that can be observed with detectors.

(E.3) Singh and Rost (2007) modeled the application of intense linearly polarized laser to the hydrogen atom by means of: $\alpha = 1/2$, $A^1(t,\cdot) = \sigma_{11}(t,\cdot) = 0$, $\eta_1(t,x) = -i\eta x$, and

$$V(t,x) = V_0(x) + xF(t),$$

where $V_0(x) = -1/\left(x^2 + \epsilon^2\right)^{1/2}$ and

$$F(t) = F_0 \sin(\beta t + \delta) \cdot \begin{cases} \sin(\pi t/(2\tau)), & \text{if } t < \tau \\ 1, & \text{if } \tau \leq t \leq T - \tau \\ \cos^2(\pi(t + \tau - T)/(2\tau)), & \text{if } T - \tau \leq t \leq T \end{cases}.$$

Here $\beta, \eta, \delta \in \mathbb{R}$ and ϵ, F_0, τ, T are positive constants. This simulates the evolution of the electron of the hydrogen atom under the influence of a laser field $F(t)$. The soft core potential V approximates the Coulomb potential of the atom.

(E.4) To describe the evolution of a quantum system in a parabolic fluctuating trap, we follow Grotz et al. (2006) and Schneider and Milburn (1999) in assuming $\alpha = 1/(2M)$, $A^1(t,x) = \sigma_{11}(t,x) = 0$, $V(t,x) = \frac{1}{2}M\omega^2 x^2$ and $\eta_1(t,x) = -i\eta x$, where $M, \eta > 0$ and $\omega \in \mathbb{R}$.

(E.5) A free particle confined by a moving Gaussian well, in interaction with a heat bath, is simulated by $\alpha = 1/(2M)$, $A^1(t,x) = 0$,

$$V(t,x) = -V_0 \exp\left(-\alpha\,(x - r(t))^2\right),$$

$\sigma_{11}(t,x) = b$ and $\eta_1(t,x) = ax$, where $a, b \in \mathbb{R}$ and $M, V_0, \alpha > 0$. The measurable bounded function $r : [0,\infty[\to \mathbb{R}$ represents the displacement of the trap's center.

4. Ehrenfest's theorem

4.1. *Markovian open quantum systems.* The next theorem provides a rigorous derivation of a version of Ehrenfest's equations for open quantum systems in Lindblad form.

Hypothesis 5. *Let C satisfy Hypothesis 3. Suppose that:*

(H5.1) *For all* $t \geq 0$ *and any* x *belonging to a core of* C,

$$\sum_{\ell=1}^{\infty} \left\| C^{1/2} L_{\ell}(t) x \right\|^2 \leq K(t) \|x\|_C^2 .$$

Let $A = B_1^* B_2$, *where* B_1, B_2 *are operators in* $\mathfrak{h}$ *such that:*

(H5.2) *For all* $x \in \mathcal{D}\left(C^{1/2}\right)$, $\max\{\|B_1 x\|^2, \|B_2 x\|^2\} \leq K \|x\|_{C^{1/2}}^2$.

(H5.3) $\max\left\{\|Ax\|^2, \|A^* x\|^2\right\} \leq K \|x\|_C^2$ *whenever* $x \in \mathcal{D}(C)$.

Theorem 4.1. *Let Hypothesis 5 hold, together with* $\xi \in L_C^2(\mathbb{P}; \mathfrak{h})$. *Then, for all* $t \geq 0$ *we have*

$$\begin{aligned} \mathbb{E}\left\langle X_t(\xi), A X_t(\xi)\right\rangle = {} & \mathbb{E}\langle \xi, A\xi\rangle + \int_0^t \mathbb{E}\left\langle A^* X_s(\xi), G(s) X_s(\xi)\right\rangle ds \\ & + \int_0^t \mathbb{E}\left\langle G(s) X_s(\xi), A X_s(\xi)\right\rangle ds \\ & + \int_0^t \left(\sum_{\ell=1}^{\infty} \mathbb{E}\left\langle B_1 L_{\ell}(s) X_s(\xi), B_2 L_{\ell}(s) X_s(\xi)\right\rangle \right) ds. \end{aligned} \tag{4.1}$$

Proof: Deferred to Subsection 5.5. □

Suppose that $X_t(\xi)$ is the unique strong C-solution of (1.1). Set

$$\rho_t := \mathbb{E}\left|X_t(\xi)\right\rangle \left\langle X_t(\xi)\right|,$$

where we use Dirac notation. Then ρ_t is a C-regular density operator and

$$tr\left(\rho_t A\right) = \mathbb{E}\left\langle X_t(\xi), A X_t(\xi)\right\rangle,$$

provided that A is C-bounded (see Mora (2013) for details). In the homogeneous case, from Mora (2013) we have that ρ_t is the unique solution of the quantum master equation

$$\frac{d}{dt}\rho_t = G\rho_t + \rho_t G^* + \sum_{\ell=1}^{\infty} L_{\ell}\rho_t L_{\ell}^*, \qquad \rho_0 = \mathbb{E}\left|\xi\right\rangle\left\langle\xi\right| .$$

We now combine Theorem 4.1 with Theorem 3.2 of Mora (2013) to deduce the following corollary, which asserts which asserts the validity (1.7) whenever essentially AL_{ℓ} is C-bounded. To this end, we use basic properties of the adjoints of unbounded operators (see, e.g., Kato (1976)).

Corollary 4.2. *In addition to Hypothesis 5 and* $\xi \in L_C^2(\mathbb{P}; \mathfrak{h})$, *suppose that the operators* $G(t), B_1 L_1(t), B_2 L_2(t), \ldots$ *are cerrable for all* $t \geq 0$. *Then*

$$\begin{aligned} tr\left(A\rho_t\right) = {} & tr\left(A\rho_0\right) + \int_0^t \left(tr\left(G(s)\rho_s A\right) + tr\left(A\rho_s G(s)^*\right)\right) ds \\ & + \int_0^t \left(\sum_{\ell=1}^{\infty} tr\left(B_2 L_{\ell}(s)\rho_s L_{\ell}(s)^* B_1^*\right)\right) ds, \end{aligned} \tag{4.2}$$

where $t \geq 0$ *and* $\rho_t := \mathbb{E}\left|X_t(\xi)\right\rangle\left\langle X_t(\xi)\right|$.

4.2. *Applications.* We begin by applying Theorem 4.1 to the model given by (1.4) and (1.5).

Theorem 4.3. *Assume the context of (1.4) and (1.5), together with Hypothesis 4. Let* $A = B_1^* B_2$*, where* B_1 *and* B_2 *satisfy one of the following conditions:*

- $B_1 = \lceil c_1 \rceil$ *and* $B_2 = \lceil c_2 \rceil$ *provided that* $c_1, c_2 : \mathbb{R}^d \to \mathbb{R}$ *are Borel measurable functions such that* $|c_j(x)| \leq K(1+|x|)$ *for all* $x \in \mathbb{R}^d$ *and* $j = 1, 2$.
- *For any* $j = 1, 2$, B_j *is either* $\partial_k \lceil a_j \rceil$, $\lceil b_j \rceil \partial_k$ *or* $\lceil c_j \rceil$, *where* $k = 1, \ldots, d$, $a_j \in C^2(\mathbb{R}^d, \mathbb{R})$ *and* $b_j, c_j \in C^1(\mathbb{R}^d, \mathbb{R})$. *Moreover, for all* $x \in \mathbb{R}^d$ *and* $l, k = 1, \ldots, d$ *we have:* $\max\{|a_j(x)|, |b_j(x)|\} \leq K$,
$$\max\{|c_j(x)|, |\partial_l a_j(x)|, |\partial_l b_j(x)|\} \leq K(1+|x|),$$
and $\max\{|\partial_l c_j(x)|, |\partial_k \partial_l a_j(x)|\} \leq K\left(1+|x|^2\right)$.

Then (4.1) and (4.2) hold in case $\xi \in L^2_{-\Delta+|x|^2}(\mathbb{P}; \mathfrak{h})$.

Proof: Deferred to Subsection 5.6. □

Using Theorem 4.3 we can obtain expressions describing the evolution of some important observables, which sometimes are closed systems of ordinary differential equations. For instance, the following theorem makes mathematically rigorous computations given in Schneider and Milburn (1999), which establish the linear heating of a Paul trap due to fluctuating electrical fields that change the center of this ion trap (see also Gehm et al. (1998); Grotz et al. (2006)).

Corollary 4.4. *Consider (1.4) and (1.5) with* $d = 1$, $\alpha = 1/(2M)$, $A^j(t,x) = 0$, $V(t,x) = V(x)$, $\sigma_{\ell k}(t,x) = 0$ *and* $\eta_1(t,x) = -i\eta x$, *where* $M, \eta > 0$ *and* $V \in C^2(\mathbb{R}, \mathbb{R})$. *Suppose that for any* $x \in \mathbb{R}$, $|V(x)| \leq K\left(1+|x|^2\right)$, $|V'(x)| \leq K(1+|x|)$ *and* $|V''(x)| \leq K\left(1+|x|^2\right)$. *Then for all* $t \geq 0$,
$$\mathbb{E}\langle X_t(\xi), HX_t(\xi)\rangle = \mathbb{E}\langle \xi, H\xi\rangle + \frac{1}{2M}\eta^2 t. \tag{4.3}$$

Proof: Deferred to Subsection 5.7. □

Remark 4.5. Schneider and Milburn (1999) restricted their attention to
$$V(x) = M\omega^2 x^2/2.$$

5. Proofs

5.1. *Proof of Lemma 2.2.* We first characterize the domain of C by means of Yosida approximations of $-C$.

Lemma 5.1. *Let* C *be a self-adjoint positive operator in* $\mathfrak{h}$. *Then*
$$\mathcal{D}(C) = \{x \in \mathfrak{h} : (CR_n x)_n \text{ converges}\} = \left\{x \in \mathfrak{h} : \sup_{n\in\mathbb{N}} \|CR_n x\| < \infty\right\},$$
where $R_n = n(n+C)^{-1}$.

Proof: Since $-C$ is dissipative and self-adjoint, for all $x \in \mathcal{D}(C)$ we have
$$CR_n x \longrightarrow_{n\to\infty} Cx$$
(see, e.g., Pazy (1983)). Thus $\mathcal{D}(C) \subset \{x \in \mathfrak{h} : (CR_n x)_n \text{ converges}\}$.

Now, assume that $(\|CR_n x\|)_{n\in\mathbb{N}}$ is bounded. Using the Banach-Alaoglu theorem we deduce that there exists a subsequence $(CR_{n_k}x)_{k\in\mathbb{N}}$ which converges weakly to a vector $z \in \mathfrak{h}$. Since $R_n x \longrightarrow_{n\to\infty} x$, for any $y \in \mathcal{D}(C)$ we have

$$\langle x, Cy\rangle = \lim_{k\to\infty} \langle R_{n_k}x, Cy\rangle = \lim_{k\to\infty} \langle CR_{n_k}x, y\rangle = \langle z, y\rangle .$$

Hence $x \in \mathcal{D}(C^*)$ $(= \mathcal{D}(C))$, and so $\{x \in \mathfrak{h} : \sup_{n\in\mathbb{N}} \|CR_n x\| < \infty\} \subset \mathcal{D}(C)$. □

The assertion of Lemma 2.2 follows straightforward from the next lemma.

Lemma 5.2. *Let C be a self-adjoint positive operator on $\mathfrak{h}$. Suppose that $L \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$. Then $L \circ \pi_C : (\mathfrak{h}, \mathcal{B}(\mathfrak{h})) \to (\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable.*

Proof: Let R_n be as in Lemma 5.1. Using Lemma 5.1 we obtain that $\mathcal{D}(C)$ is a Borel set of $\mathfrak{h}$ since $CR_n \in \mathfrak{L}(\mathfrak{h})$, and so $\pi_C : (\mathfrak{h}, \mathcal{B}(\mathfrak{h})) \to (\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ is measurable. Since the range of R_n is a subset of $\mathcal{D}(C)$ and $L \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$, $LR_n \in \mathfrak{L}(\mathfrak{h})$. Hence $LR_n \circ \pi_C$ is measurable. It follows from $R_n \longrightarrow_{n\to\infty} I$ and

$$CR_n x \longrightarrow_{n\to\infty} Cx$$

that $LR_n \circ \pi_C \longrightarrow_{n\to\infty} L \circ \pi_C$, which implies the measurability of $L \circ \pi_C$. □

5.2. *Proof of Theorem 2.4*. First, we extend the inequality given in Condition H2.3 to $\mathcal{D}(C^2)$.

Remark 5.3. Let L be a closable operator in $\mathfrak{h}$ such that $\mathcal{D}(C) \subset \mathcal{D}(L)$, with C self-adjoint positive operator in $\mathfrak{h}$. Applying the closed graph theorem gives $L \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$.

Lemma 5.4. *Suppose that C satisfies Conditions H2.1 - H2.3 of Hypothesis 2. If x belongs to $\mathcal{D}(C^2)$ and $t \geq 0$, then $L_\ell(t)x \in \mathcal{D}(C)$ for any $\ell \in \mathbb{N}$, and*

$$2\Re\langle C^2 x, G(t)x\rangle + \sum_{\ell=1}^{\infty} \|CL_\ell(t)x\|^2 \leq \alpha(t)\|x\|_C^2 . \tag{5.1}$$

Proof: Since $\mathfrak{D}_1$ is a core of C^2, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $\mathfrak{D}_1$ converging to x such that $C^2 x_n \longrightarrow_{n\to\infty} C^2 x$. Using Remark 5.3 and Condition H2.1 we deduce that $G(t) \in \mathfrak{L}((\mathcal{D}(C^2), \|\cdot\|_{C^2}), \mathfrak{h})$, and so Condition H2.3 leads to

$$\lim_{n,n'\to\infty} \sum_{\ell=1}^{\infty} \|CL_\ell(t)(x_n - x_{n'})\|^2 = 0. \tag{5.2}$$

By C is closed, from (5.2) we have $L_\ell(t)x \in \mathcal{D}(C)$ and $CL_\ell(t)x_n \to CL_\ell(t)x$ as $n \to \infty$. Then (5.1) follows immediately, because (5.1) is true for x_n for all n.

□

The inequality of Condition H2.4 can be immediately extended to $\mathcal{D}(C)$, by the definition of core and Fatou's lemma, following the lines of the proof of Lemma 5.4.

Lemma 5.5. *Under Conditions H2.1, H2.2 and H2.4, for all x in $\mathcal{D}(C)$ we have*

$$2\Re\langle x, G(t)x\rangle + \sum_{k=1}^{\infty} \|L_\ell(t)x\|^2 \leq 0.$$

In contrast to Mora and Rebolledo (2007), where we used the Galerkin method, in the proof of Theorem 2.4 we obtain $X_t(\xi)$ as the $L^2(\mathbb{P}, \mathfrak{h})$-weak limit of the solutions to the sequence of stochastic evolution equations (5.3) given below.

Definition 5.6. Let Hypothesis 1 hold, together with Conditions H2.1 and H2.2. Suppose that ξ is a $\mathfrak{F}_0$-measurable random variable belonging to $L^2(\mathbb{P}, \mathfrak{h})$. For each natural number n, we define X^n to be the unique continuous solution of

$$X_t^n = \xi + \int_0^t G^n(s) X_s^n ds + \sum_{\ell=1}^n \int_0^t L_\ell^n(s) X_s^n dW_s^\ell, \tag{5.3}$$

where $G^n(s) = \widetilde{R}_n G(s) \widetilde{R}_n$ and $L_\ell^n(s) = L_\ell(s) \widetilde{R}_n$ with $\widetilde{R}_n = n\left(n + C^2\right)^{-1}$.

Remark 5.7. Recall that $C^2\widetilde{R}_n \in \mathfrak{L}(\mathfrak{h})$ and $\left\|\widetilde{R}_n\right\| \leq 1$. As a consequence, X^n is well-defined because H2.1 and H2.2 imply that $G^n(t)$ and $L_\ell^n(t)$ are bounded operators in $\mathfrak{h}$ whose norms are uniformly bounded on compact time intervals.

Though the next three estimates for X^n essentially coincide with those given in Lemma 2.3 of Mora and Rebolledo (2007), the infinite-dimensional nature of (5.3) forces us to use a more refined analysis.

Lemma 5.8. *Adopt Hypothesis 1, together with Conditions H2.1, H2.2 and H2.4. Then for any $t \geq 0$, $\mathbb{E}\|X_t^n\|^2 \leq \mathbb{E}\|\xi\|^2$. Moreover, for all $x \in \mathfrak{h}$ and $t \geq 0$ we have*

$$2\Re\langle x, G^n(t) x\rangle + \sum_{\ell=1}^\infty \|L_\ell^n(t) x\|^2 \leq 0. \tag{5.4}$$

Proof: Since the range of $\widetilde{R}_n$ is a subset of $\mathcal{D}\left(C^2\right)$, Lemma 5.5 leads to (5.4). Using complex Itô's formula we obtain

$$\|X_t^n\|^2 \leq \|\xi\|^2 + \sum_{\ell=1}^n \int_0^t 2\Re\langle X_s^n, L_\ell^n(s) X_s^n\rangle dW_s^\ell. \tag{5.5}$$

Set $\tau_j = \inf\{t \geq 0 : \|X_t^n\| > j\}$. Then $\tau_j \nearrow \infty$ as $j \to \infty$, because X^n is pathwise continuous. By (5.5), Fatou's lemma yields $\mathbb{E}\|X_t^n\|^2 \leq \liminf_{j\to\infty} \mathbb{E}\left\|X_{t\wedge\tau_j}^n\right\|^2 \leq \mathbb{E}\|\xi\|^2$. □

Lemma 5.9. *Let Hypothesis 2 hold. If $\xi \in L_C^2(\mathbb{P}, \mathfrak{h})$, then*

$$\mathbb{E}\|CX_t^n\|^2 \leq \exp(t\alpha(t))\left(\mathbb{E}\|C\xi\|^2 + t\alpha(t)\,\mathbb{E}\|\xi\|^2\right). \tag{5.6}$$

Proof: Combining Condition H2.1 with Lemma 5.4 we obtain that $CG^n(t)$ and $CL_\ell^n(t)$ are bounded operators on $\mathfrak{h}$ whose norms are uniformly bounded on compact intervals. Lemma 5.8 gives $\mathbb{E}\|CG^n(t) X_t^n\|^2 \leq \|CG^n(t)\|^2 \mathbb{E}\|\xi\|^2$ and

$$\mathbb{E}\|CL_\ell^n(t) X_t^n\|^2 \leq \|CL_\ell^n(t)\|^2 \mathbb{E}\|\xi\|^2.$$

Therefore $CX_t^n = Y_t^n$ *a.s.* for any $t \geq 0$, where

$$Y^n = C\xi + \int_0^\cdot CG^n(s) X_s^n ds + \sum_{\ell=1}^n \int_0^\cdot CL_\ell^n(s) X_s^n dW_s^\ell.$$

This follows from, for instance, Propositions 1.6 and 4.15 of Da Prato and Zabczyk (1992).

Since $\widetilde{R}_n$ commutes with both C and C^2, using Lemma 5.4 and $\left\|\widetilde{R}_n\right\| \leq 1$ we deduce that for any $x \in \mathcal{D}\left(C^2\right)$ and $t \geq 0$,

$$\begin{aligned} & 2\Re\left\langle Cx, CG^n\left(t\right)x\right\rangle + \sum_{\ell=1}^{n}\left\|CL_\ell^n\left(t\right)x\right\|^2 \\ \leq \quad & 2\Re\left\langle C^2\widetilde{R}_n x, G\left(t\right)\widetilde{R}_n x\right\rangle + \sum_{\ell=1}^{\infty}\left\|CL_\ell\left(t\right)\widetilde{R}_n x\right\|^2 \leq \alpha\left(t\right)\left\|\widetilde{R}_n x\right\|_C^2 \leq \alpha\left(t\right)\left\|x\right\|_C^2 . \end{aligned}$$

As $\mathcal{D}\left(C^2\right)$ is a core of C, by a passage to the limit we get that for all $y \in \mathcal{D}\left(C\right)$ and $t \geq 0$,

$$2\Re\left\langle Cy, CG^n\left(t\right)y\right\rangle + \sum_{\ell=1}^{n}\left\|CL_\ell^n\left(t\right)y\right\|^2 \leq \alpha\left(t\right)\left\|y\right\|_C^2 . \tag{5.7}$$

Finally, choose $\tau_j = \inf\left\{t \geq 0 : \left\|Y_t^n\right\| > j\right\}$. Applying Itô's formula yields

$$\mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 = \mathbb{E}\left\|C\xi\right\|^2 + \mathbb{E}\int_0^{t\wedge\tau_j}\left(2\Re\left\langle Y_s^n, CG^n\left(s\right)X_s^n\right\rangle + \sum_{\ell=1}^{n}\left\|CL_\ell^n\left(s\right)X_s^n\right\|^2\right)ds,$$

because $\mathbb{E}\left|\Re\left\langle Y_{s\wedge\tau_j}^n, CL_\ell^n\left(s\right)X_s^n\right\rangle\right|^2 \leq j^2\left\|CL_\ell^n\left(s\right)\right\|^2\mathbb{E}\left\|\xi\right\|^2$ by Lemma 5.8. Since $Y_s^n = CX_s^n$ a.s., combining (5.7) with Lemma 5.8 we have

$$\mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 \leq \mathbb{E}\left\|C\xi\right\|^2 + \alpha\left(t\right)\int_0^t\mathbb{E}\left\|CX_s^n\right\|^2 ds + t\alpha\left(t\right)\mathbb{E}\left\|\xi\right\|^2,$$

and so

$$\mathbb{E}\left\|Y_t^n\right\|^2 \leq \liminf_{j\to\infty}\mathbb{E}\left\|Y_{t\wedge\tau_j}^n\right\|^2 \leq \mathbb{E}\left\|C\xi\right\|^2 + t\alpha\left(t\right)\mathbb{E}\left\|\xi\right\|^2 + \alpha\left(t\right)\int_0^t\mathbb{E}\left\|Y_s^n\right\|^2 ds.$$

The Gronwall-Bellman lemma now leads to (5.6). □

Lemma 5.10. *Fix $T > 0$. Under the assumptions of Theorem 2.4,*

$$\mathbb{E}\left\|X_t^n - X_s^n\right\|^2 \leq K_{T,\xi}\left(t-s\right), \tag{5.8}$$

where $0 \leq s \leq t < T$ and $K_{T,\xi}$ is a constant depending of T and ξ.

Proof: Consider $\tau_j = \inf\left\{t \geq 0 : \left\|X_t^n\right\| > j\right\}$. According to Itô's formula we have

$$\begin{aligned} & \mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \\ & = \mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j}\left(2\Re\left\langle X_r^n - X_{s\wedge\tau_j}^n, G^n\left(r\right)X_r^n\right\rangle + \sum_{\ell=1}^{n}\left\|L_\ell^n\left(r\right)X_r^n\right\|^2\right)ds, \end{aligned}$$

and hence (5.4) leads to

$$\mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \leq -\mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j} 2\Re\left\langle X_s^n, G^n\left(r\right)X_r^n\right\rangle dr.$$

From Condition H2.1, $\left\|\widetilde{R}_n\right\| \leq 1$ and $\widetilde{R}_n C \subset C\widetilde{R}_n$ we deduce that $\left\|G^n\left(t\right)x\right\|^2 \leq K\left(t\right)\left\|x\right\|_C^2$ for all $x \in \mathcal{D}\left(C\right)$. Therefore

$$\mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \leq K\left(t\right)\mathbb{E}\int_{s\wedge\tau_j}^{t\wedge\tau_j}\left\|X_s^n\right\|\left\|X_r^n\right\|_C dr$$

by $X_s^n \in \mathcal{D}(C)$ a.s., and so Fatou's lemma implies

$$\mathbb{E}\left\|X_t^n - X_s^n\right\|^2 \leq \liminf_{j\to\infty} \mathbb{E}\left\|X_{t\wedge\tau_j}^n - X_{s\wedge\tau_j}^n\right\|^2 \leq K(t)\int_s^t \sqrt{\mathbb{E}\left\|X_r^n\right\|_C^2}\sqrt{\mathbb{E}\left\|X_s^n\right\|^2}dr.$$

Applying Lemmata 5.8 and 5.9 we obtain (5.8). □

We next obtain a strong C-solution of (1.1) by means of a limit procedure.

Definition 5.11. For any natural number n, we define $\left(\mathfrak{G}_s^{\xi,n}\right)_{s\geq 0}$ to be the filtration that satisfies the usual hypotheses generated by ξ and $W^1,\dots,W^n$. Let t be a non-negative real number. By $\mathfrak{G}_t^{\xi,W}$ we mean the σ-algebra generated by $\cup_{n\in\mathbb{N}}\mathfrak{G}_t^{\xi,n}$. As usual, $\mathfrak{G}_{t+}^{\xi,W} = \cap_{\epsilon>0}\mathfrak{G}_{t+\epsilon}^{\xi,W}$.

Lemma 5.12. *Let the assumptions of Theorem 2.4 hold. Fix $T>0$. Then, we can extract from any subsequence of $(X^n)_{n\in\mathbb{N}}$ a subsequence $(X^{n_k})_{k\in\mathbb{N}}$ for which there exists a $\left(\mathfrak{G}_{t+}^{\xi,W}\right)_{t\in[0,T]}$-predictable process $(Z_t)_{t\in[0,T]}$ such that for any $t\in[0,T]$,*

$$X_t^{n_k} \longrightarrow_{k\to\infty} Z_t \qquad \text{weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right). \tag{5.9}$$

Proof: By Lemmata 5.8 and 5.10, using a compactness method in the same way as in the proof of Lemma 2.4 of Mora and Rebolledo (2007) we obtain our assertion (see Subsection A.2 for details). □

In contrast with Mora and Rebolledo (2007), in the following steps we do not make assumptions about the adjoints of $G(t)$ and $L_\ell(t)$, which even may not exist.

Lemma 5.13. *Adopt the assumptions of Theorem 2.4, together with the notation of Lemma 5.12. Let $t\in[0,T]$. Then $\mathbb{E}\|Z_t\|^2 \leq \mathbb{E}\|\xi\|^2$, $Z_t\in Dom(C)$ a.s., and*

$$\mathbb{E}\|CZ_t\|^2 \leq \exp(\alpha(t)t)\left(\mathbb{E}\|C\xi\|^2 + \alpha(t)t\mathbb{E}\|\xi\|^2\right). \tag{5.10}$$

Moreover, $G^{n_k}(t)X_t^{n_k} \longrightarrow_{k\to\infty} G(t)Z_t$ weakly in $L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$, and for all $\ell\in\mathbb{N}$,

$$L_\ell^{n_k}(t)X_t^{n_k} \longrightarrow_{k\to\infty} L_\ell(t)Z_t \quad \text{weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right). \tag{5.11}$$

Proof: By (5.9), Lemma 5.8 leads to $\mathbb{E}\|Z_t\|^2 \leq \mathbb{E}\|\xi\|^2$.

For a given $U\in L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$, the dominated convergence theorem yields $\widetilde{R}_nU \longrightarrow_{n\to\infty} U$ in $L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$, and so using (5.9) we get

$$x\mathbb{E}\left\langle U,\widetilde{R}_{n_k}X_t^{n_k}\right\rangle = \mathbb{E}\left\langle \widetilde{R}_{n_k}U, X_t^{n_k}\right\rangle \longrightarrow_{k\to\infty} \mathbb{E}\langle U,Z_t\rangle. \tag{5.12}$$

Suppose that $L\in\mathfrak{L}((\mathcal{D}(C),\|\cdot\|_C),\mathfrak{h})$, and define $L^n = L\widetilde{R}_n$. Since $\widetilde{R}_nC\subset C\widetilde{R}_n$ and $\left\|\widetilde{R}_n\right\|\leq 1$, applying Lemma 5.9 and the Banach-Alaoglu theorem we deduce that any subsequence of $(n_k)_{k\in\mathbb{N}}$ contains a subsequence denoted (to shorten notation) by $(l)_{l\in\mathbb{N}}$ such that $(L^lX_t^l)_{l\in\mathbb{N}}$ and $\left(C\widetilde{R}_lX_t^l\right)_{l\in\mathbb{N}}$ are weakly convergent in $L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}\right)$. By (5.12),

$$\left(\widetilde{R}_lX_t^l, L^lX_t^l, C\widetilde{R}_lX_t^l\right) \text{ converges weakly in } L^2\left(\left(\Omega,\mathfrak{G}_t^{\xi,W},\mathbb{P}\right),\mathfrak{h}^3\right).$$

The set $\mathcal{D}(C) \times L(\mathcal{D}(C)) \times C(\mathcal{D}(C))$ is closed in $\mathfrak{h}^3$, because L is relatively bounded with respect to C. Then the set of all triple $(\eta, A\eta, L\eta)$, with $\eta \in L^2_C\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}\right)$, is a closed linear linear manifold of $L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}^3\right)$, and hence closed with respect to the weak topology of $L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}^3\right)$. Using (5.12) we now get $\left(\widetilde{R}_l X^l_t, L^l X^l_t, C\widetilde{R}_l X^l_t\right)$ converges weakly in $L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}^3\right)$ to (Z_t, LZ_t, CZ_t) as $l \to \infty$, which implies

$$L^{n_k} X^{n_k}_t \longrightarrow_{k\to\infty} LZ_t \quad \textit{weakly in } L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}\right), \tag{5.13}$$

and so (5.11) holds by Condition H2.2. Taking $L = C$ in (5.13) and using Lemma 5.9 we get (5.10).

Condition H2.1, together with (5.13), shows that $G(t)\widetilde{R}_{n_k} X^{n_k}_t$ converges to $G(t) Z_t$ weakly in $L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}\right)$ as $n \to \infty$. It follows that for any $U \in L^2\left(\left(\Omega, \mathfrak{G}^{\xi,W}_t, \mathbb{P}\right), \mathfrak{h}\right)$,

$$\mathbb{E}\left\langle U, \widetilde{R}_{n_k} G(t)\widetilde{R}_{n_k} X^{n_k}_t\right\rangle = \mathbb{E}\left\langle \widetilde{R}_{n_k} U, G(t)\widetilde{R}_{n_k} X^{n_k}_t\right\rangle \longrightarrow_{k\to\infty} \mathbb{E}\left\langle U, G(t) Z_t\right\rangle. \quad \square$$

By Lemma 5.13, as in Lemma 2.5 of Mora and Rebolledo (2007) we establish that $Z_t(\xi)$ satisfies (1.1) a.s. using the following predictable representation.

Remark 5.14. Let $\chi \in L^2\left(\left(\Omega, \mathfrak{G}^{\xi,m}_t, \mathbb{P}\right), \mathbb{C}\right)$, with $t \in [0,T]$. Then, there exist $\left(\mathfrak{G}^{\xi,m}_s\right)_s$ - predictable processes $H^1, \cdots, H^m$ such that: (i) $H^1, \cdots, H^m \in L^2\left(\left([0,T]\times\Omega, dt\otimes\mathbb{P}\right), \mathbb{C}\right)$; and (ii) $\chi = \mathbb{E}\left(\chi | \mathfrak{G}^{\xi,m}_0\right) + \sum_{j=1}^m \int_0^t H^j_s dW^j_s$.

Lemma 5.15. *Assume the setting of Theorem 2.4. Suppose that* $\left(X^{n_k}\right)_{k\in\mathbb{N}}$ *and* χ *are as in Lemma 5.12 and Remark 5.14 respectively. If* $x \in \mathfrak{h}$*, then*

$$\lim_{k\to\infty} \mathbb{E}\left\langle \chi x, \sum_{\ell=1}^{n_k} \int_0^t L^{n_k}_\ell(s) X^{n_k}_s dW^j_s\right\rangle = \mathbb{E}\left\langle \chi x, \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\pi(Z_s) dW^\ell_s\right\rangle.$$

Proof: Throughout this proof, $H^1, \cdots, H^m$ are as in Remark 5.14. First, using Lemma 5.8, basic properties of stochastic integrals and Fubini's theorem we deduce that for all $n \geq m$,

$$\mathbb{E}\chi\left\langle x, \sum_{\ell=1}^{n} \int_0^t L^n_\ell(s) X^n_s dW^\ell_s\right\rangle = \sum_{\ell=1}^{m} \int_0^t \mathbb{E}\, H^\ell_s \left\langle x, L^n_\ell(s) X^n_s\right\rangle ds.$$

By $\left\|\widetilde{R}_n\right\| \leq 1$ and $\widetilde{R}_n C \subset C\widetilde{R}_n$, combining (5.11), Lemmata 5.8 and 5.9, and the dominated convergence theorem we obtain that for any $\ell = 1, \ldots, m$,

$$\int_0^t \mathbb{E}\, H^\ell_s \left\langle x, L^{n_k}_\ell(s) X^{n_k}_s\right\rangle ds \longrightarrow_{k\to\infty} \int_0^t \mathbb{E}\, H^\ell_s \left\langle x, L_\ell(s)\pi(Z_s)\right\rangle ds,$$

and so

$$\lim_{k\to\infty} \mathbb{E}\left\langle \chi x, \sum_{\ell=1}^{n_k} \int_0^t L^{n_k}_\ell(s) X^{n_k}_s dW^\ell_s\right\rangle = \sum_{\ell=1}^{m} \int_0^t \mathbb{E}\, H^\ell_s \left\langle x, L_\ell(s)\pi(Z_s)\right\rangle ds.$$

Second, Lemmata 5.8 and 5.9, together with Condition H2.2, yield

$$\sum_{\ell=1}^{m} \int_0^t \mathbb{E}\, H_s^\ell \left\langle x, L_\ell(s)\, \pi(Z_s) \right\rangle ds = \sum_{\ell=1}^{n} \mathbb{E}\, \chi \int_0^t \left\langle x, L_\ell(s)\, \pi(Z_s) \right\rangle dW_s^\ell$$

whenever $n \geq m$. Condition H2.1 and Lemma 5.5 show that $\sum_{k=1}^{\infty} \left\| L_\ell(t)\, y \right\|^2 \leq K(t) \left\| y \right\|_C^2$ for all y in $\mathcal{D}(C)$ and $t \geq 0$. Therefore $\sum_{\ell=1}^{n} \int_0^t L_\ell(s)\, \pi(Z_s)\, dW_s^\ell$ converges in $L^2(\mathbb{P}, \mathfrak{h})$ to $\sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\, \pi(Z_s)\, dW_s^\ell$, which implies that

$$\sum_{\ell=1}^{m} \int_0^t \mathbb{E}\, H_s^\ell \left\langle x, L_\ell(s)\, \pi(Z_s) \right\rangle ds = \sum_{\ell=1}^{\infty} \mathbb{E}\, \chi \int_0^t \left\langle x, L_\ell(s)\, \pi(Z_s) \right\rangle dW_s^\ell.$$

□

Lemma 5.16. *Adopt the assumptions of Theorem 2.4. Let T and Z be defined as in Lemma 5.12. Then for all $t \in [0, T]$ we have*

$$Z_t = \xi + \int_0^t G(s)\, \pi_C(Z_s)\, ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\, \pi_C(Z_s)\, dW_s^\ell \quad a.s.$$

Proof: Consider $x \in \mathfrak{h}$ and let $\left(X^{n_k}\right)_{k \in \mathbb{N}}$ be as in Lemma 5.12. According to Lemma 5.15 we have

$$\lim_{k \to \infty} \mathbb{E} \left\langle \chi x, \sum_{\ell=1}^{n_k} \int_0^t L_\ell^{n_k}(s)\, X_s^{n_k} dW_s^\ell \right\rangle = \mathbb{E} \left\langle \chi x, \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\, \pi(Z_s)\, dW_s^\ell \right\rangle.$$

Using Lemmata 5.8, 5.9 and 5.13 and the dominated convergence theorem we obtain

$$\int_0^t \mathbb{E}\left[\left\langle x, G^{n_k}(s)\, X_s^{n_k} \right\rangle \mathbb{E}\left[\chi | \mathfrak{G}_s^{\xi, W} \right] \right] ds \to_{k \to \infty} \int_0^t \mathbb{E}\left[\left\langle x, G(s)\, \pi(Z_s) \right\rangle \mathbb{E}\left[\chi | \mathfrak{G}_s^{\xi, W} \right] \right] ds,$$

since $\mathbb{E}\left[\chi \,|\, \mathfrak{G}_s^{\xi, W} \right] \in L^2(\mathbb{P}, \mathbb{C})$. Thus, combining (5.9) with the definition of X^n yields

$$\mathbb{E}\chi \left\langle x, Z_t \right\rangle = \mathbb{E}\chi \left\langle x, \xi + \int_0^t G(s)\, \pi(Z_s)\, ds + \sum_{\ell=1}^{\infty} \int_0^t \left\langle x, L_\ell(s)\, \pi(Z_s) \right\rangle dW_s^\ell \right\rangle. \tag{5.14}$$

As in Mora and Rebolledo (2007), using a monotone class theorem (e.g., Th. I.21 of Dellacherie and Meyer (1978)) we extend the range of validity of (5.14) from $\chi \in L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi, m}, \mathbb{P} \right), \mathbb{C} \right)$ to any bounded $\chi \in L^2\left(\left(\Omega, \mathfrak{G}_t^{\xi, W}, \mathbb{P} \right), \mathbb{C} \right)$, which completes the proof. □

We are now in a position to finish the proof of Theorem 2.4 by classical arguments.

Proof of Theorem 2.4: Consider $T > 0$. First, we combine Lemma 5.5 with Itô's formula to deduce that there exists at most one strong C-solution of (1.1) on $[0, T]$ (see proof of Lemma 2.2 of Mora and Rebolledo (2007) for details). Second, for all $t \in [0, T]$, we set

$$Z_t^T = \xi + \int_0^t G(s)\, \pi_C(Z_s)\, ds + \sum_{\ell=1}^{\infty} \int_0^t L_\ell(s)\, \pi_C(Z_s)\, dW_s^\ell,$$

where Z is as in Lemma 5.12. Using Lemma 5.16 we see that Z^T is a continuous version of Z. Hence Z^T is a strong C-solution of (1.1) on $[0,T]$, and so Z^T is the unique one.

Define $\widetilde{\Omega}$ to be the set of all ω satisfying $Z_t^n(\omega) = Z_t^{n+1}(\omega)$ for all $n \in \mathbb{N}$ and any $t \in [0,n]$. For any $t \in [0,n]$ with $n \in \mathbb{N}$, we choose $X_t(\xi)(\omega) = Z_t^n(\omega)$ whenever $\omega \in \widetilde{\Omega}$. Set $X(\xi) \equiv 0$ in the complement of $\widetilde{\Omega}$. Thus $X(\xi)$ is the unique strong C-solution of (1.1) on $[0,\infty[$. □

5.3. *Proof of Lemma 2.6.*

Proof: According to Lemma 5.2 we have that $G(t) \circ \pi_{C^2}$ and $CL_\ell(t) \circ \pi_{C^2}$ are measurable functions from $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$ to $(\mathfrak{h}, \mathcal{B}(\mathfrak{h}))$, and so $\mathcal{L}^+(t,\zeta)$ is a positive random variable for every $\zeta \in L_C^2(\mathbb{P},\mathfrak{h}) \bigcap \mathfrak{D}_1$. Hence $\mathbb{E}(\mathcal{L}^+(t,\zeta))$ is well-defined.

Condition H2.3 leads straightforward to Condition H2.3'. In the other direction, we assume from now on that H2.3' holds. Fix $t \geq 0$. To obtain a contradiction, suppose that for any $n \in \mathbb{N}$ there exists $x_n \in \mathfrak{D}_1$ such that

$$2\Re\left\langle C^2 x_n, G(t)\, x_n\right\rangle + \sum_{\ell=1}^{\infty} \left\|CL_\ell(t)\, x_n\right\|^2 > n \left\|x_n\right\|_C^2 . \tag{5.15}$$

We can consider a random variable ζ defined by

$$\mathbb{P}(\zeta = y_n) = \frac{p}{n^2\left(1 + \left\|Cy_n\right\|^2\right)},$$

where $y_n = x_n / \left\|x_n\right\|$ and

$$1/p = \sum_{n=1}^{\infty} \frac{1}{n^2\left(1+\left\|Cy_n\right\|^2\right)} < \infty.$$

Then $\|\zeta\| = 1$, $\zeta \in \mathfrak{D}_1$, and $\mathbb{E}\left\|\zeta\right\|_C^2 = p\sum_{n=1}^{\infty} 1/n^2 < \infty$. Using (5.15) yields

$$\mathbb{E}\mathcal{L}^+(t,\zeta) \geq p \sum_{n=1}^{\infty} 1/n = \infty,$$

which contradicts Condition H2.3'. Therefore, there exists a constant $\beta(t) \geq 0$ such that for all $x \in \mathfrak{D}_1$,

$$2\Re\left\langle C^2 x, G(t)\, x\right\rangle + \sum_{\ell=1}^{\infty} \left\|CL_\ell(t)\, x\right\|^2 \leq \beta(t) \left\|x\right\|_C^2 . \tag{5.16}$$

By abuse of notation, we denote by $\beta(t)$ the smallest $\beta(t)$ satisfying (5.16).

Suppose, contrary to H2.3, that $\sup_{t\in[0,T]} \beta(t) = \infty$ for some $T > 0$. Then, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ of different elements of $[0,T]$ such that

$$2\Re\left\langle C^2 z_n, G(s_n)\, z_n\right\rangle + \sum_{\ell=1}^{\infty} \left\|CL_\ell(s_n)\, z_n\right\|^2 > n^3 \left\|z_n\right\|_C^2 \tag{5.17}$$

for some $z_n \in \mathfrak{D}_1$ satisfying $\|z_n\| = 1$. Similarly to the paragraph above, we can choose a random variable ζ defined by

$$\mathbb{P}(\zeta = z_n) = \frac{p}{n^2\left(1+\left\|Cz_n\right\|^2\right)},$$

with $1/p = \sum_{n=1}^{\infty} 1/\left(n^2\left(1+\|Cz_n\|^2\right)\right) < \infty$. Hence $\mathbb{E}\,\|\zeta\|_C^2 < \infty$. From (5.17) we deduce that

$$\mathbb{E}\mathcal{L}^+\left(s_n,\zeta\right) \geq \mathcal{L}^+\left(s_n,z_n\right) \geq np,$$

which contradicts Condition H2.3'. Taking

$$\alpha\left(t\right) = \sup_{t\in[0,t]} \beta\left(s\right) < \infty.$$

we can assert that Condition H2.3 holds. □

5.4. *Proof of Theorem 3.1.* Throughout this subsection, C denotes the operator in $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$ given by $C = -\Delta + |x|^2$. Moreover, $\|\cdot\|$ stands for the norm in $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$, and we shall often use Einstein summation convention (each index can appear at most twice in any term, repeated indexes are implicitly summed over).

Since $|x|^2$ is locally in $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$, the operator $-\Delta+|x|^2$ is essentially self-adjoint on $C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ (see, e.g., Th. X.29 of Reed and Simon (1975)). The Hermite functions (i.e., Hermite polynomials multiplied by $\mathrm{e}^{-x^2/2}$) are the eigenfunctions of the operator in $L^2\left(\mathbb{R},\mathbb{C}\right)$ given by $-d^2/dx^2+x^2$, and hence the Schwarz space of rapidly decreasing functions is an essential domain for $\left(-d^2/dx^2+x^2\right)^2$. We can now use standard approximation arguments to show that $C_c^\infty\left(\mathbb{R},\mathbb{C}\right)$ is a core for $\left(-d^2/dx^2+x^2\right)^2$, which implies that $C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ is a core for C^2 (see, e.g., Th. VIII.33 of Reed and Simon (1980)), and so $C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ is an essential domain for C. As $-\Delta+|x|^2 = \sum_{j=1}^d(\partial_j+x_j)^*(\partial_j+x_j)+dI$, the operator C is bounded from below by d times the identity operator I.

We next provide some relative bounds on C, Δ and the multiplication operator by $|x|^2$.

Lemma 5.17. *Let $f\in C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$. Then*

$$\|Cf\|^2 = \|\Delta f\|^2 + \left\|\,|x|^2 f\right\|^2 + 2\sum_{j=1}^d \left\langle \partial_j f, |x|^2\partial_j f\right\rangle - 2d\,\|f\|^2\,, \tag{5.18}$$

$$\sum_{j=1}^d \|(1+|x|)\partial_j f\|^2 \leq 4\,\|Cf\|^2\,, \tag{5.19}$$

$$\left\|\,(1+|x|^2)f\right\|^2 \leq 8\,\|Cf\|^2\,. \tag{5.20}$$

Proof: Using integration by parts yields

$$\|Cf\|^2 = \sum_{j,k=1}^d \left(\left\langle -\partial_j^2 f, -\partial_k^2 f\right\rangle + \left\langle x_j^2 f, x_k^2 f\right\rangle\right) - \sum_{j,k=1}^d \left\langle f, (\partial_j^2 x_k^2 + x_j^2\partial_k^2) f\right\rangle. \tag{5.21}$$

A short computation based on the commutation relation $[\partial_j, x_j] = I$ gives

$$(\partial_j^2 x_j^2 + x_j^2\partial_j^2)f = 2\partial_j x_j^2\partial_j f + 2(\partial_j x_j - x_j\partial_j)f = 2\partial_j x_j^2 \partial_j f + 2f. \tag{5.22}$$

For any $j\neq k$ we have $\partial_j^2 x_k^2 + x_j^2\partial_k^2 = \partial_j x_k^2 \partial_j + \partial_k x_j^2\partial_k$ on $C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$, which together with (5.21) and (5.22) implies (5.18).

We now check inequality (5.19). Combining (5.18) with the inequality

$$\sum_{j=1}^d \|(1+|x|)\partial_j f\|^2 \leq 2\sum_{j=1}^d \left\langle \partial_j f, (1+|x|^2)\partial_j f\right\rangle = 2\left\langle f, -\Delta f\right\rangle + 2\sum_{j=1}^d \left\langle \partial_j f, |x|^2\partial_j f\right\rangle,$$

we obtain $\sum_{j=1}^{d} \|(1+|x|)\partial_j f\|^2 \leq 2\langle f, -\Delta f\rangle - \|\Delta f\|^2 - \left\| |x|^2 f\right\|^2 + \|Cf\|^2 + 2d\|f\|^2$, and hence

$$\sum_{j=1}^{d} \|(1+|x|)\partial_j f\|^2 \leq -\|(\Delta+1)f\|^2 + \|Cf\|^2 + (2d+1)\|f\|^2. \tag{5.23}$$

By $C^2 \geq d^2 1$, $(2d+1)\|f\|^2 \leq (2d^{-1}+d^{-2})\|Cf\|^2 \leq 3\|Cf\|^2$ since $d \geq 1$. Then, (5.19) follows from (5.23).

According to (5.18), we have

$$\left\|(1+|x|^2)f\right\|^2 \leq 2\|f\|^2 + 2\left\||x|^2 f\right\|^2 \leq 2(2d+1)\|f\|^2 + 2\|Cf\|^2.$$

Then $\left\|(1+|x|^2)f\right\|^2 \leq 2(1+2d^{-1}+d^{-2})\|Cf\|^2$, which leads to (5.20). □

Lemma 5.18. *Under Hypothesis 4, the operators $G(t)$ and $L_\ell(t)$ satisfy Hypothesis 1, as well as Conditions H2.1 and H2.2 of Hypothesis 2.*

Proof: For all $f \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ we have (Einstein summation convention on j)

$$\|H(t)f\| \leq \alpha\|\Delta f\| + 2\left\|A^j(t,\cdot)\partial_j f\right\| + \left\|f\partial_j A^j(t,\cdot)\right\| + \|V(t,\cdot)f\|.$$

Combining the Schwarz inequality with (5.19) gives

$$\left\|A^j(t,\cdot)\partial_j f\right\| \leq K(t)\left(d\sum_{j=1}^{d}\|(1+|x|)\,\partial_j f\|^2\right)^{1/2} \leq 2d^{1/2}K(t)\|Cf\|.$$

By (5.20), $\|V(t,\cdot)f\| \leq 8^{1/2}K(t)\|Cf\|$. Moreover, Condition H4.1 implies

$$\left\|f\partial_j A^j(t,\cdot)f\right\| \leq d\,K(t)\|f\|,$$

and the identity (5.18) yields $\|\Delta f\| \leq \|Cf\| + (2d)^{1/2}\|f\|$. Therefore

$$\|H(t)f\| \leq \left(\alpha + \left(2d^{1/2}+8^{1/2}\right)K(t)\right)\|Cf\| + \left(\alpha(2d)^{1/2} + dK(t)\right)\|f\|. \tag{5.24}$$

A straightforward computation yields

$$L_\ell(t)^* L_\ell(t) = -\overline{\sigma}_{\ell j}\sigma_{\ell k}\partial_j\partial_k - \left(\overline{\sigma}_{\ell k}\eta_\ell - \overline{\eta}_\ell\sigma_{\ell k} + \overline{\sigma}_{\ell j}\left\lceil\partial_j\sigma_{\ell k}\right\rceil\right)\partial_k + \left(\overline{\eta}_\ell\eta_\ell - \overline{\sigma}_{\ell j}\left\lceil\partial_j\eta_\ell\right\rceil\right)$$

Since $1 \leq \ell \leq m$, using Condition H4.2 and the Schwarz inequality we deduce that

$$\|\overline{\sigma}_{\ell j}\sigma_{\ell k}\partial_j\partial_k f\| \leq md\,K(t)^2\left(\sum_{j,k=1}^{d}\|\partial_j\partial_k f\|^2\right)^{1/2} = md\,K(t)^2\|\Delta f\|.$$

From $|\eta_\ell| \leq K(t)(1+|x|)$, $|\sigma_{\ell j}| \leq K(t)$ and $|\partial_j\sigma_{\ell j}| \leq K(t)$ we have

$$\begin{aligned}\left\|\left(\overline{\eta}_\ell\eta_\ell - \overline{\sigma}_{\ell j}\left\lceil\partial_j\eta_\ell\right\rceil\right)f\right\| &\leq \|\overline{\eta}_\ell\eta_\ell f\| + \left\|\overline{\sigma}_{\ell j}\left\lceil\partial_j\eta_\ell\right\rceil f\right\| \\ &\leq 2mK(t)^2\left\|(1+|x|^2)f\right\| + 2mdK(t)^2\|(1+|x|)f\| \\ &\leq 2m(2d+1)K(t)^2\left\|(1+|x|^2)f\right\|.\end{aligned}$$

Similarly, combining the Schwarz inequality with (5.19) yields

$$\begin{aligned}\left\|\left(\overline{\sigma}_{\ell j}\ \left\lceil\partial_j\sigma_{\ell k}\right\rceil - \overline{\eta}_\ell\sigma_{\ell k} + \eta_\ell\overline{\sigma}_{\ell k}\right)\partial_k f\right\| &\leq 4mK(t)^2\sum_{k=1}^{d}\|(1+|x|)\partial_k f\| \\ &\leq 8md^{1/2}\,K(t)^2\|Cf\|.\end{aligned}$$

Summing up, $\sum_{\ell=1}^{m} \|L_\ell(t)^* L_\ell(t) f\|$ is less than or equal to $mK(t)^2$ times

$$\begin{aligned} & 8d^{1/2}\|Cf\| + d\|\Delta f\| + 2(2d+1)\left\||x|^2 f\right\| + 2(2d+1)\|f\| \\ \leq\ & 8d^{1/2}\|Cf\| + 2(2d+1)\left(\|\Delta f\| + \left\||x|^2 f\right\|\right) + 2(2d+1)\|f\| \\ \leq\ & 8d^{1/2}\|Cf\| + 2(2d+1)\left(\|Cf\| + (2d)^{1/2}\|f\|\right) + 2(2d+1)\|f\|. \end{aligned}$$

This, together with (5.24), shows that $G(t)$ satisfies Condition H2.1 since $C_c^\infty(\mathbb{R}^d;\mathbb{C})$ is a core for C. In a similar, but simpler, way (we deal now with first order differential operators), we can prove that the operators $L_\ell(t)$ satisfy Condition H2.2.

Let $g \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$. Since $\sigma_{\ell k}(t,\cdot)$ is continuous, using Fubini's theorem we deduce the measurability of $t \mapsto \left\langle \phi, \overline{\sigma_{\ell k}(t,\cdot)} g \right\rangle$ for all $\phi \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$, and so $t \mapsto \overline{\sigma_{\ell k}(t,\cdot)} g$ is measurable. Combining Lemma 5.2 with (5.19) yields the measurability of $f \mapsto \partial_k \pi_C(f)$ as a map from $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$ to $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$. Therefore $(t,f) \mapsto \left\langle \overline{\sigma_{\ell k}(t,\cdot)} g, \partial_k \pi_C(f) \right\rangle$ is measurable, which implies the measurability of $(t,f) \mapsto \sigma_{\ell k}(t,\cdot)\,\partial_k \pi_C(f)$ as a function from $[0,\infty[\, \times L^2\left(\mathbb{R}^d,\mathbb{C}\right)$ to $L^2\left(\mathbb{R}^d,\mathbb{C}\right)$. In the same manner we can see that $(t,f) \mapsto \eta_{\ell k}(t,\cdot)\,\pi_C(f)$ is measurable, hence Condition H1.1 holds. Similarly, we can obtain that $G(t)$ satisfies Condition H1.2. □

We now verify Condition H2.3 with $\mathfrak{D}_1 = C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$, which is the most complicated step of our proof. The key inequality in Condition H2.3 at a formal purely algebraic level reads as $\mathcal{L}(C^2) \leq \alpha(t)\left(C^2 + I\right)$, where $\mathcal{L}$ is the formal time-dependent Lindbladian associated with the operators $G(t)$ and $L_\ell(t)$, namely $\mathcal{L}(X) = G(t)^* X + \sum_\ell L_\ell(t)^* X L_\ell(t) + X G(t)$. Decomposing $\mathcal{L}$ as the sum of a Hamiltonian part $i[H(t),\cdot]$ and a dissipative part $\mathcal{L}_0(X) = \mathcal{L}(X) - i[H(t),X]$ we check separately that $i[H(t),C^2] \leq K(t)\left(C^2+I\right)$ (Lemma 5.19) and $\mathcal{L}_0(C^2) \leq K(t)\left(C^2+I\right)$ (Lemmata 5.20, 5.21 and 5.22) in the quadratic form sense.

Lemma 5.19. *Suppose that Hypothesis 4 holds. Then, for all $f \in C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ we have*

$$2\Re\left\langle C^2 f, iH(t) f\right\rangle \leq K(t)\left(\|Cf\|^2 + \|f\|^2\right). \tag{5.25}$$

Proof: Since $f \in C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$,

$$\begin{aligned} & \left\langle C^2 f, iH(t) f\right\rangle + \left\langle iH(t) f, C^2 f\right\rangle \\ =\ & \langle Cf, i(H(t)C + [C,H(t)])f\rangle + \langle i(H(t)C + [C,H(t)])f, Cf\rangle \\ =\ & -i\langle Cf, [H(t),C]f\rangle + i\langle [H(t),C]f, Cf\rangle. \end{aligned}$$

Then $\left|2\Re\left\langle C^2 f, iH(t) f\right\rangle\right| = |2\Im\langle Cf,[H(t),C]f\rangle| \leq 2\|Cf\|\cdot\|[H(t),C]f\|$. A computation allows us to write the commutator $[H(t),C]$ as

$$\begin{aligned} & 4i\left\lceil \partial_k A^j\right\rceil \partial_k\partial_j + \left(2\left\lceil\partial_j V\right\rceil - 4\alpha x_j + 2i\left\lceil \Delta A^j\right\rceil + i\left\lceil \partial_j\partial_k A^k\right\rceil\right)\partial_j \\ & + \left(\lceil \Delta V\rceil - 2\alpha d + i\left\lceil \partial_j \Delta A^j\right\rceil + 4ix_j A^j\right). \end{aligned}$$

Using Condition H4.1 and the Schwarz inequality we obtain that the norm of the first term, acting on a function f, is less than or equal to

$$4K(t)\sum_{j,k=1}^{d}\|\partial_k\partial_j f\| \leq 4K(t)d\left(\sum_{j,k=1}^{d}\|\partial_k\partial_j f\|^2\right)^{1/2} = 2K(t)d\,\|\Delta f\|.$$

The norm of the second term, with first order partial derivatives of f, is upper bounded by $K(t)\sum_{k=1}^{d}\|(1+|x|)\partial_k f\|$, which is less than or equal to

$$d^{1/2}K(t)\left(\sum_{k=1}^{d}\|(1+|x|)\partial_k f\|^2\right)^{1/2}.$$

The third term is not bigger than $K(t)\left\|(1+|x|^2)f\right\|$. We now use Lemma 5.17 to get (5.25). □

Starting from the formal algebraic equality

$$\mathcal{L}_0(C^2)=C\mathcal{L}_0(C)+\mathcal{L}_0(C)C+\sum_{\ell=1}^{d}[C,L_\ell(t)]^*[C,L_\ell(t)]$$

written in the quadratic form sense, we now establish an estimate of $\left|\langle f,\mathcal{L}_0(C^2)f\rangle\right|$ (formally the left-hand side of the inequality (5.26) given below). Note that $\mathcal{L}_0(C^2)$ does not make sense as a sixth (or fourth, after simplifications) order differential operator acting on $f\in C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ because $\sigma_{\ell k},\eta_\ell$ are only three-times differentiable, but $\mathcal{L}_0(C)$ does. The right-hand side of (5.26), however, can be written rigorously as $\sum_{\ell=1}^{m}\|[C,L_\ell]f\|^2+2\|Cf\|\cdot\|\mathcal{L}_0(C)f\|$.

Lemma 5.20. *For all $f\in C_c^\infty\left(\mathbb{R}^d,\mathbb{C}\right)$ we have*

$$\sum_{\ell=1}^{m}\left(\|CL_\ell f\|^2-\Re\left\langle C^2f,L_\ell^*L_\ell f\right\rangle\right)\le\sum_{\ell=1}^{m}\|[C,L_\ell]f\|^2 \qquad (5.26)$$
$$+\|Cf\|\left\|\sum_{\ell=1}^{m}\left(L_\ell^*[C,L_\ell]+[L_\ell^*,C]L_\ell\right)f\right\|.$$

Proof: Rearranging terms we have that for all $\ell=1,\dots,m$,

$$\begin{aligned}&\langle CL_\ell f,CL_\ell f\rangle-\left\langle C^2f,L_\ell^*L_\ell f\right\rangle\\ =\;&\langle(L_\ell C+[C,L_\ell])f,(L_\ell C+[C,L_\ell])f\rangle-\langle Cf,(L_\ell^*C+[C,L_\ell^*])L_\ell f\rangle\\ =\;&\langle L_\ell Cf,L_\ell Cf\rangle+\langle L_\ell Cf,[C,L_\ell]f\rangle+\langle[C,L_\ell]f,L_\ell Cf\rangle+\langle[C,L_\ell]f,[C,L_\ell]f\rangle\\ &-\langle L_\ell Cf,CL_\ell f\rangle-\langle Cf,[C,L_\ell^*]L_\ell f\rangle\end{aligned}$$

Note that the sum of the first, second and fifth term vanishes and the third is equal to $\langle L_\ell^*[C,L_\ell]f,Cf\rangle$. We find then

$$\langle CL_\ell f,CL_\ell f\rangle-\left\langle C^2f,L_\ell^*L_\ell f\right\rangle=\|[C,L_\ell]f\|^2+\langle L_\ell^*[C,L_\ell]f,Cf\rangle+\langle Cf,[L_\ell^*,C]L_\ell f\rangle,$$

and so taking the real part we can write

$$\begin{aligned}\langle CL_\ell f,CL_\ell f\rangle-\Re\left\langle C^2f,L_\ell^*L_\ell f\right\rangle&=\|[C,L_\ell]f\|^2+\frac{1}{2}\langle L_\ell^*[C,L_\ell]f,Cf\rangle\\ &+\frac{1}{2}\langle Cf,L_\ell^*[C,L_\ell]f\rangle+\frac{1}{2}\left(\langle Cf,[L_\ell^*,C]L_\ell f\rangle+\langle[L_\ell^*,C]L_\ell f,Cf\rangle\right),\end{aligned}$$

which implies

$$\begin{aligned}\langle CL_\ell f,CL_\ell f\rangle-\Re\left\langle C^2f,L_\ell^*L_\ell f\right\rangle&=\|[C,L_\ell]f\|^2+\frac{1}{2}\langle L_\ell^*[C,L_\ell]f,Cf\rangle\\ &+\frac{1}{2}\langle[L_\ell^*,C]L_\ell f,Cf\rangle+\frac{1}{2}\left(\langle Cf,[L_\ell^*,C]L_\ell f\rangle+\langle Cf,L_\ell^*[C,L_\ell]f\rangle\right).\end{aligned}$$

The conclusion follows summing up over ℓ and applying the Schwarz inequality. □

We now show that $\mathcal{L}_0(C)$ is a second order differential operator with well-behaved coefficients allowing us to prove that $\mathcal{L}_0(C)$ is relatively bounded with respect to C.

Lemma 5.21. *Under the Hypothesis 4, for all $f \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$ we have*

$$\|\mathcal{L}_0(C)f\| \le K(t)\,\|Cf\|. \tag{5.27}$$

Proof: Simple algebraic computations yield

$$\begin{aligned} 2\mathcal{L}_0(x_j) = (L_\ell^*[x_j, L_\ell] + [L_\ell^*, x_j]L_\ell) &= (\partial_k \overline{\sigma}_{\ell k} - \overline{\eta}_\ell)\,\sigma_{\ell j} - \overline{\sigma}_{\ell j}\,(\sigma_{\ell k}\partial_k + \eta_\ell) \\ &= \partial_k(\sigma^*\sigma)_{kj} - (\sigma^*\sigma)_{jk}\partial_k - \overline{\eta}_\ell\sigma_{\ell j} - \overline{\sigma}_{\ell j}\eta_\ell, \end{aligned}$$

which implies $2\mathcal{L}_0(x_j) = ((\sigma^*\sigma)_{kj} - (\sigma^*\sigma)_{jk})\,\partial_k + \lceil \partial_k(\sigma^*\sigma)_{kj}\rceil - \overline{\eta}_\ell\sigma_{\ell j} - \eta_\ell\overline{\sigma}_{\ell j}$. From $\mathcal{L}_0(|x|^2) = x_j\mathcal{L}_0(x_j) + \mathcal{L}_0(x_j)x_j + [x_j, L_\ell^*][L_\ell, x_j]$ it follows that

$$\mathcal{L}_0(|x|^2) = x_j\,((\sigma^*\sigma)_{kj} - (\sigma^*\sigma)_{kj})\,\partial_k + x_j\,\lceil \partial_k(\sigma^*\sigma)_{kj}\rceil - 2\Re(\overline{\eta}_\ell\sigma_{\ell j})x_j + (\sigma^*\sigma)_{jj}. \tag{5.28}$$

In a similar way $-\Delta = \sum_j -\partial_j\partial_j$, and

$$\begin{aligned} 2\mathcal{L}_0(\partial_j) &= L_\ell^*[\partial_j, L_\ell] + [L_\ell^*, \partial_j]L_\ell \\ &= (-\partial_h\overline{\sigma}_{\ell h} + \overline{\eta}_\ell)\,(\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_k + \lceil \partial_j\eta_\ell\rceil) \\ &\quad + (\partial_h\,\lceil \partial_j\overline{\sigma}_{\ell h}\rceil - \lceil \partial_j\overline{\eta}_\ell\rceil)\,(\sigma_{\ell k}\partial_k + \eta_\ell)\,. \end{aligned}$$

In the above differential operator, second order terms cancel. In fact, we can write the expression $-\partial_h\overline{\sigma}_{\ell h}\,\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_k + \partial_h\,\lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\sigma_{\ell k}\partial_k$, by an exchange of summation indexes h, k in the second term, in the form

$$\begin{aligned} &\partial_k\sigma_{\ell h}\,\lceil \partial_j\overline{\sigma}_{\ell k}\rceil\,\partial_h - \partial_h\,\overline{\sigma}_{\ell h}\,\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_k = (\sigma_{\ell h}\,\lceil \partial_j\overline{\sigma}_{\ell k}\rceil - \overline{\sigma}_{\ell h}\,\lceil \partial_j\sigma_{\ell k}\rceil)\,\partial_h\partial_k \\ &+ (\lceil \partial_k\sigma_{\ell h}\rceil\,\lceil \partial_j\overline{\sigma}_{\ell k}\rceil + \sigma_{\ell h}\,\lceil \partial_j\partial_k\overline{\sigma}_{\ell k}\rceil)\,\partial_h - (\lceil \partial_h\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\sigma_{\ell k}\rceil + \overline{\sigma}_{\ell h}\,\lceil \partial_h\partial_j\sigma_{\ell k}\rceil)\,\partial_k. \end{aligned}$$

This is a first order differential operator because both the second order coefficient vanishes by (3.1) and $2\mathcal{L}(\partial_j)$ is equal to

$$\begin{aligned} &(\lceil \partial_k\sigma_{\ell h}\rceil\,\lceil \partial_j\overline{\sigma}_{\ell k}\rceil + \sigma_{\ell h}\,\lceil \partial_j\partial_k\overline{\sigma}_{\ell k}\rceil + \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\eta_\ell - \overline{\sigma}_{\ell h}\,\lceil \partial_j\eta_\ell\rceil)\,\partial_h \\ &- (\lceil \partial_h\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\sigma_{\ell k}\rceil + \overline{\sigma}_{\ell h}\,\lceil \partial_h\partial_j\sigma_{\ell k}\rceil + \lceil \partial_j\overline{\eta}_\ell\rceil\,\sigma_{\ell k} - \overline{\eta}_\ell\,\lceil \partial_j\sigma_{\ell k}\rceil)\,\partial_k \\ &+ \lceil \partial_j\partial_h\overline{\sigma}_{\ell h}\rceil\,\eta_\ell + \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_h\eta_\ell\rceil - \lceil \partial_h\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\eta_\ell\rceil - \overline{\sigma}_{\ell h}\,\lceil \partial_j\partial_h\eta_\ell\rceil + 2i\Im\,(\overline{\eta}_\ell\,\lceil \partial_j\eta_\ell\rceil)\,. \end{aligned}$$

Therefore $\mathcal{L}_0(\partial_j) = \nu_{jk}\partial_k + \xi_j$, where $\nu_{jk} := \Re\,(\lceil \partial_j\sigma_{\ell k}\rceil\,\overline{\eta}_\ell - \sigma_{\ell k}\,\lceil \partial_j\overline{\eta}_\ell\rceil)$ and

$$\begin{aligned} 2\xi_j \quad := \quad &\lceil \partial_j\partial_h\overline{\sigma}_{\ell h}\rceil\,\eta_\ell + \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_h\eta_\ell\rceil - \lceil \partial_h\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\eta_\ell\rceil - \overline{\sigma}_{\ell h}\,\lceil \partial_j\partial_h\eta_\ell\rceil \\ &+ 2i\Im\,(\overline{\eta}_\ell\,\lceil \partial_j\eta_\ell\rceil)\,. \end{aligned}$$

Since $\mathcal{L}_0(\Delta) = \partial_j\mathcal{L}_0(\partial_j) + \mathcal{L}_0(\partial_j)\partial_j + [\partial_j, L_\ell^*][\partial_j, L_\ell]$,

$$\begin{aligned} \mathcal{L}_0(\Delta) \quad = \quad &\partial_j\,(\nu_{jh}\partial_h + \xi_j) + (\nu_{jk}\partial_k + \xi_j)\,\partial_j \\ &+ (\lceil \partial_j\overline{\eta}_\ell\rceil - \partial_h\,\lceil \partial_j\overline{\sigma}_{\ell h}\rceil)\,(\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_k + \lceil \partial_j\eta_\ell\rceil)\,. \end{aligned}$$

This gives

$$\begin{aligned} \mathcal{L}_0(\Delta) = \; &2\nu_{jk}\partial_j\partial_k + \lceil \partial_j\nu_{jk}\rceil\,\partial_k + 2\xi_j\partial_j + \lceil \partial_j\xi_j\rceil - \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_h\partial_k \\ &- (\lceil \partial_h\partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\sigma_{\ell k}\rceil + \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_h\partial_j\sigma_{\ell k}\rceil)\,\partial_k + \lceil \partial_j\overline{\eta}_\ell\rceil\,\lceil \partial_j\sigma_{\ell k}\rceil\,\partial_k \\ &- \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\eta_\ell\rceil\,\partial_h - \lceil \partial_h\partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_j\eta_\ell\rceil - \lceil \partial_j\overline{\sigma}_{\ell h}\rceil\,\lceil \partial_h\partial_j\eta_\ell\rceil \\ &+ \lceil \partial_j\overline{\eta}_\ell\rceil\,\lceil \partial_j\eta_\ell\rceil\,. \end{aligned} \tag{5.29}$$

By Condition H4.2, combining $\mathcal{L}_0(C) = \mathcal{L}_0(-\Delta) + \mathcal{L}_0(|x|^2)$ with (5.28) and (5.29) we deduce that $\mathcal{L}_0(C)$ is a second order differential operator of the form $\sum_{|\mu|\le 2} a_\mu\partial_\mu$

(in multiindex notation $\mu = (\mu_1, \dots, \mu_d)$, $|\mu| = \mu_1 + \cdots + \mu_d$, $\partial_\mu = \partial_{\mu_1} \cdots \partial_{\mu_d}$) with: a_μ bounded for $|\mu| = 2$, $|a_\mu| \le K(t)(1+|x|)$ for $|\mu| = 1$ and $|a_\mu| \le K(t)(1+|x|^2)$ for $|\mu| = 0$. The conclusion follows them from applications of Lemma 5.17 with some long but straightforward computations. □

Lemma 5.22. *Under the Hypothesis 4, Condition H2.3 holds.*

Proof: Since $[C, L_\ell] = ((-\partial_j + x_j)[\partial_j + x_j, L_\ell] + [-\partial_j + x_j, L_\ell](-\partial_j + x_j))$,

$$\begin{aligned}[C, L_\ell] = {} & (-\partial_j + x_j)\left(\lceil \partial_j \sigma_{\ell k} \rceil \partial_k + \lceil \partial_j \eta_\ell \rceil - \delta_{jk}\sigma_{\ell k}\right) \\ & + \left(- \lceil \partial_j \sigma_{\ell k} \rceil \partial_k - \lceil \partial_j \eta_\ell \rceil - \delta_{jk}\sigma_{\ell k}\right)(-\partial_j + x_j),\end{aligned}$$

where δ_{jk} is the Kronecker delta. The term with two partial derivatives writes as

$$-\partial_j \lceil \partial_j \sigma_{\ell k} \rceil \partial_k + \lceil \partial_j \sigma_{\ell k} \rceil \partial_k \partial_j = 2 \lceil \partial_j \sigma_{\ell k} \rceil \partial_k \partial_j - \left\lceil \partial_j^2 \sigma_{\ell k} \right\rceil \partial_k,$$

terms with a single partial derivative are

$$\begin{aligned} & -\partial_j \lceil \partial_j \eta_\ell \rceil + \delta_{jk}\partial_j \sigma_{\ell k} + x_j \lceil \partial_j \sigma_{\ell k} \rceil \partial_k + \lceil \partial_j \eta_\ell \rceil \partial_j + \delta_{jk}\sigma_{\ell k}\partial_j - \lceil \partial_j \sigma_{\ell k} \rceil \partial_k x_j \\ = {} & - \lceil \Delta \eta_\ell \rceil + 2\delta_{jk}\sigma_{\ell k}\partial_j,\end{aligned}$$

and terms with no partial derivatives sum up $-2\delta_{jk} x_j \sigma_{\ell k}$. Therefore

$$[C, L_\ell] = 2 \lceil \partial_j \sigma_{\ell k} \rceil \partial_j \partial_k - \left\lceil \partial_j^2 \sigma_{\ell k} \right\rceil \partial_k + 2\sigma_{\ell j}\partial_j - 2x_j \sigma_{\ell j} - \lceil \Delta \eta_\ell \rceil.$$

We now use Condition H4.2 and Lemma 5.17, together with straightforward inequalities and estimates, to obtain $\|[C, L_\ell(t)]f\|^2 \le K(t)\left(\|Cf\|^2 + \|f\|^2\right)$. Thus, the claimed inequality in Condition H2.3 follows from Lemmata 5.19, 5.20 and 5.21. □

Proof of Theorem 3.1: Hypothesis 1 and Conditions H2.1, H2.2 of Hypothesis 2 hold by Lemma 5.18. In Lemma 5.22 we verify Condition H2.3. According the definition of $G(t)$ we have $2\Re\langle f, G(t) f\rangle + \sum_{\ell=1}^{\infty} \|L_\ell(t) f\|^2 = 0$ for all $f \in C_c^\infty(\mathbb{R}^d;\mathbb{C})$. Therefore, Condition H3.1 (stronger form of H2.4) holds, because $C_c^\infty(\mathbb{R}^d;\mathbb{C})$ is a core for C and the operators $G(t), L_\ell(t)$ are relatively bounded with respect to C with bound uniform for t in bounded intervals $[0,T]$. Hence, applying Theorems 2.4, 2.7 and 2.12 we get the assertions of the theorem. □

5.5. *Proof of Theorem 4.1.*

Proof of Theorem 4.1: Consider the stopping time

$$\tau_m = \inf\{t \ge 0 : \|X_t(\xi)\| > m\} \wedge T,$$

where $T > 0$ and $m \in \mathbb{N}$. For any $n \in \mathbb{N}$, we set $A_n = R_n A R_n$, with $R_n = n(n+C)^{-1}$. From $A \in \mathfrak{L}((\mathcal{D}(C), \|\cdot\|_C), \mathfrak{h})$ it follows that $A_n \in \mathfrak{L}(\mathfrak{h})$, and so using the complex Itô formula we obtain

$$\langle X_{t\wedge\tau_m}(\xi), A_n X_{t\wedge\tau_m}(\xi)\rangle = \langle \xi, A_n \xi\rangle + \int_0^{t\wedge\tau_m} \mathcal{L}(s, A_n, X_s(\xi))\, ds + M_{t\wedge\tau_m}, \tag{5.30}$$

where $t \in [0,T]$,

$$M_t = \sum_{\ell=1}^{\infty} \int_0^t \left(\langle X_s(\xi), A_n L_\ell(s) X_s(\xi)\rangle + \langle L_\ell(s) X_s(\xi), A_n X_s(\xi)\rangle\right) dW_s^\ell,$$

and for all $x \in \mathcal{D}(C)$,

$$\mathcal{L}(s, A_n, x) = \langle x, A_n G(s) x \rangle + \langle G(s) x, A_n x \rangle + \sum_{\ell=1}^{\infty} \langle L_\ell(s) x, A_n L_\ell(s) x \rangle .$$

Combining Lemma 5.5 with the Cauchy-Schwarz inequality we get

$$\begin{aligned} &\mathbb{E} \sum_{\ell=1}^{\infty} \int_0^{t \wedge \tau_m} \left| \langle X_s(\xi), A_n L_\ell(s) X_s(\xi) \rangle + \langle L_\ell(s) X_s(\xi), A_n X_s(\xi) \rangle \right|^2 ds \\ &\leq 8m^3 \|A_n\|^2 \int_0^T \mathbb{E} \|G(s) X_s(\xi)\| \, ds, \end{aligned}$$

which together with Condition H2.1, yields $\mathbb{E} M_{t \wedge \tau_m} = 0$. Then, (5.30) leads to

$$\mathbb{E} \langle X_{t \wedge \tau_m}(\xi), A_n X_{t \wedge \tau_m}(\xi) \rangle = \mathbb{E} \langle \xi, A_n \xi \rangle + \mathbb{E} \int_0^{t \wedge \tau_m} \mathcal{L}(s, A_n, X_s(\xi)) \, ds. \quad (5.31)$$

Since $\mathbb{E} \sup_{s \in [0,T]} \|X_s(\xi)\|^2 < +\infty$, applying the dominated convergence theorem gives

$$\lim_{m \to \infty} \mathbb{E} \langle X_{t \wedge \tau_m}(\xi), A_n X_{t \wedge \tau_m}(\xi) \rangle = \mathbb{E} \langle X_t(\xi), A_n X_t(\xi) \rangle .$$

Letting $m \to \infty$ in (5.31) we deduce, using the dominated convergence theorem, that

$$\mathbb{E} \langle X_t(\xi), A_n X_t(\xi) \rangle = \mathbb{E} \langle \xi, A_n \xi \rangle + \mathbb{E} \int_0^t \mathcal{L}(s, A_n, X_s(\xi)) \, ds,$$

and so from Fubini's theorem we obtain

$$\mathbb{E} \langle X_t(\xi), A_n X_t(\xi) \rangle = \mathbb{E} \langle \xi, A_n \xi \rangle + \int_0^t \mathbb{E} \mathcal{L}(s, A_n, X_s(\xi)) \, ds. \quad (5.32)$$

Let $x \in \mathcal{D}(C)$. By Conditions H2.2 and H5.1, analysis similar to that in the proof of Lemma 5.4 shows that $L_\ell(s) x \in \mathcal{D}(C^{1/2})$ and

$$\sum_{\ell=1}^{\infty} \left\| C^{1/2} L_\ell(s) x \right\|^2 \leq K(s) \|x\|_C^2 . \quad (5.33)$$

Since $R_n C \subset C R_n$, $C^{1/2}$ commutes with R_n, and so Condition H5.2 leads to

$$\begin{aligned} &\|B_j R_n L_\ell(s) x - B_j L_\ell(s) x\|^2 \\ &\leq K \left(\left\| R_n C^{1/2} L_\ell(s) x - C^{1/2} L_\ell(s) x \right\|^2 + \|R_n L_\ell(s) x - L_\ell(s) x\|^2 \right), \end{aligned}$$

with $j = 1, 2$. This implies

$$B_j R_n L_\ell(s) x \longrightarrow_{n \to \infty} B_j L_\ell(s) x. \quad (5.34)$$

Moreover, using $R_n C^{1/2} \subset C^{1/2} R_n$ we deduce that

$$\begin{aligned} \|B_j R_n L_\ell(s) x\|^2 &\leq K \left(\left\| R_n C^{1/2} L_\ell(s) x \right\|^2 + \|R_n L_\ell(s) x\|^2 \right) \\ &\leq K \left(\left\| C^{1/2} L_\ell(s) x \right\|^2 + \|L_\ell(s) x\|^2 \right). \end{aligned}$$

Lemma 5.5 and Condition H2.1 lead to $\sum_{\ell=1}^{\infty} \|L_\ell(s)x\|^2 \leq K(s)\|x\|_C^2$. Therefore, applying the dominated convergence theorem, together with (5.33) and (5.34), yields

$$\int_0^t \mathbb{E} \sum_{\ell=1}^{\infty} \langle B_1 R_n L_\ell(s) X_s(\xi), B_2 R_n L_\ell(s) X_s(\xi)\rangle ds$$
$$\longrightarrow_{n\to\infty} \int_0^t \mathbb{E} \sum_{\ell=1}^{\infty} \langle B_1 L_\ell(s) X_s(\xi), B_2 L_\ell(s) X_s(\xi)\rangle ds.$$

Hence

$$\int_0^t \mathbb{E} \sum_{\ell=1}^{\infty} \langle B_1 R_n L_\ell(s) X_s(\xi), B_2 R_n L_\ell(s) X_s(\xi)\rangle ds \tag{5.35}$$
$$\longrightarrow_{n\to\infty} \int_0^t \sum_{\ell=1}^{\infty} \mathbb{E} \langle B_1 L_\ell(s) X_s(\xi), B_2 L_\ell(s) X_s(\xi)\rangle ds.$$

According to $R_n^* = R_n$, for any $x \in \mathcal{D}(C)$ we have

$$\begin{aligned} \mathcal{L}(s, A_n, x) &= \langle R_n A^* R_n x, G(s)x\rangle + \langle G(s)x, R_n A R_n x\rangle \\ &\quad + \sum_{\ell=1}^{\infty} \langle B_1 R_n L_\ell(s)x, B_2 R_n L_\ell(s)x\rangle. \end{aligned}$$

By (5.35) and Condition H5.3, letting $n \to \infty$ in (5.32) we get (4.1). □

5.6. *Proof of Theorem 4.3.*

Proof of Theorem 4.3: Let $C = -\Delta + |x|^2$. According to Theorem 3.1, (1.1) has a unique strong C-solution with initial datum in $L_C^2(\mathbb{P}; \mathfrak{h})$. Moreover, in the proof of Theorem 3.1 we verify that C satisfies Hypothesis 2.

Suppose that f belongs to $C_c^\infty(\mathbb{R}^d, \mathbb{C})$, which is a core for C. Then, for any $\ell = 1, \ldots, m$ and $t \geq 0$ we have

$$\left\| C^{1/2} L_\ell(t) f \right\|^2 = \sum_{j=1}^{d} \|i\partial_j(L_\ell(t)f)\|^2 + \||x| L_\ell(t)f\|^2.$$

Since

$$\sum_{j,k=1}^{d} \|\partial_j \partial_k f\|^2 = \|-\Delta f\|^2, \tag{5.36}$$

combining Hypothesis 4 with Lemma 5.17 yields Condition H5.1.

Consider $f \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$. Then $\|\lceil c_j \rceil f\|^2 \leq K\left(\|f\|^2 + \left\langle f, |x|^2 f\right\rangle\right)$ and

$$\|\lceil b_j \rceil \partial_k f\|^2 \leq K\left\langle f, -\partial_k^2 f\right\rangle.$$

In addition, $\|\partial_k \lceil a_j \rceil f\|^2 \leq 2\|\lceil \partial_k a_j \rceil f\|^2 + 2\|\lceil a_j \rceil \partial_k f\|^2$. Therefore

$$\|B_j f\|^2 \leq K\left(\|f\|^2 + \langle f, Cf\rangle\right) = K\|f\|_{C^{1/2}}^2,$$

and so B_j satisfies Condition H5.2, because $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ is a core for $C^{1/2}$.

We now take $B_1 = \lceil b_1 \rceil \partial_\ell$ and $B_2 = \partial_k \lceil a_2 \rceil$, and so $A = -\partial_\ell \lceil \bar{b_1} \rceil \partial_k \lceil a_2 \rceil$. For any $f \in C_c^\infty(\mathbb{R}^d, \mathbb{C})$,

$$\begin{aligned} Af &= -\left(\partial_\ell \bar{b_1}\right)(\partial_k a_2) f - \bar{b_1}(\partial_\ell \partial_k a_2) f - \bar{b_1}(\partial_k a_2)\partial_\ell f - \left(\partial_\ell \bar{b_1}\right) a_2 \partial_k f \\ & \quad -\bar{b_1}(\partial_\ell a_2)\partial_k f - \bar{b_1} a_2 \partial_\ell \partial_k f. \end{aligned}$$

Using Lemma 5.17, together with (5.36), yields $\|Af\|^2 \leq K \|f\|_C^2$, and hence for all $f \in \mathcal{D}(C)$, $\|Af\|^2 \leq K \|f\|_C^2$ since $C_c^\infty(\mathbb{R}^d, \mathbb{C})$ is a core for C. Similarly, we obtain $\|A^* f\|^2 \leq K \|f\|_C^2$ for all $f \in \mathcal{D}(C)$. Thus Condition H5.3 holds in this case. In the same manner we can check Condition H5.3 for the other possible choices of B_1 and B_2. Finally, applying Theorems 4.1 and 4.2 we get (4.1) and (4.2), respectively. ☐

5.7. *Proof of Corollary 4.4.*

Proof of Corollary 4.4: Set $P = -id/dx$. Suppose that either $A = P^2$ or $A = \lceil V \rceil$. From $L_1^* = -L_1$ it follows that for all $f \in C_c^\infty(\mathbb{R}, \mathbb{C})$,

$$\langle A^* f, Gf\rangle + \langle Gf, Af\rangle + \left\langle \sqrt{A} L_1 f, \sqrt{A} L_1 f \right\rangle = \left\langle f, \left(-i[A,H] - \frac{1}{2}[[L_1, A], L_1]\right) f \right\rangle.$$

Using $[\lceil V \rceil, P] = i \lceil V' \rceil$ yields

$$-i[A,H] - \frac{1}{2}[[L_1,A],L_1] = \begin{cases} \frac{1}{2M}\left(\lceil V' \rceil P + P \lceil V' \rceil\right), & \text{if } A = \lceil V \rceil \\ -\left(\lceil V' \rceil P + P \lceil V' \rceil\right) + \eta^2, & \text{if } A = P^2 \end{cases}.$$

Since A, G, $\sqrt{A} L_1$, $\lceil V' \rceil P$ and $P \lceil V' \rceil$ are relatively bounded with respect to $C = -d^2/dx^2 + \lceil x^2 \rceil$, for all $f \in \mathcal{D}(C)$ we have

$$\begin{aligned} &\langle A^* f, Gf\rangle + \langle Gf, Af\rangle + \left\langle \sqrt{A} L_1 f, \sqrt{A} L_1 f \right\rangle \\ &\quad = \begin{cases} \frac{1}{2M}\left\langle f, \left(\lceil V' \rceil P + P \lceil V' \rceil\right) f\right\rangle, & if\ A = \lceil V \rceil \\ -\left\langle f, \left(\lceil V' \rceil P + P \lceil V' \rceil\right) f\right\rangle + \left\langle f, \eta^2 f\right\rangle, & if\ A = P^2 \end{cases}, \end{aligned} \tag{5.37}$$

because $C_c^\infty(\mathbb{R}, \mathbb{C})$ is a core for C. Combining (5.37) with Theorem 4.3 we obtain

$$\mathbb{E}\langle X_t, \lceil V \rceil X_t\rangle = \mathbb{E}\langle \xi, \lceil V \rceil \xi\rangle + \frac{1}{2M}\int_0^t \mathbb{E}\left\langle X_s, \left(\lceil V' \rceil P + P \lceil V' \rceil\right) X_s\right\rangle ds \tag{5.38}$$

and

$$\begin{aligned} \mathbb{E}\left\langle X_t, \frac{1}{2M} P^2 X_t \right\rangle &= \mathbb{E}\left\langle \xi, \frac{1}{2M} P^2 \xi \right\rangle - \frac{1}{2M}\int_0^t \mathbb{E}\left\langle X_s, \left(\lceil V' \rceil P + P \lceil V' \rceil\right) X_s\right\rangle ds \\ &\quad + \frac{\eta^2}{2M} t, \end{aligned} \tag{5.39}$$

where we abbreviate $X_t(\xi)$ to X_t. Adding (5.38) and (5.39) gives (4.3). ☐

Appendix A.

A.1. *Proof of Theorem 2.7.*

Proof of Theorem 2.7: Define $\tau_n = \inf\{t \geq 0 : \|X_t(\xi)\| > n\} \wedge T$, where T is a given positive real number and $n \in \mathbb{N}$. Combining Condition H3.1 with Itô's formula we obtain

$$\|X_{t\wedge\tau_n}(\xi)\|^2 = \|\xi\|^2 + \sum_{\ell=1}^{\infty} \int_0^{t\wedge\tau_n} 2\Re\langle X_s(\xi), L_\ell(s) X_s(\xi)\rangle dW_s^\ell. \tag{A.1}$$

Conditions H2.1 and H3.1 yield

$$\sum_{\ell=1}^{\infty} \mathbb{E} \int_0^{\tau_n} \left(\Re \left\langle X_s(\xi), L_\ell(s) X_s(\xi) \right\rangle \right)^2 ds \leq K_{n,T} \left(1 + \mathbb{E} \left\| \xi \right\|_C^2 \right),$$

where $K_{n,T}$ is a constant depending of n and T, hence (A.1) shows that $\left\| X^{\tau_n}(\xi) \right\|^2$ is a martingale. We now use Fatou's lemma to deduce the supermartingale property of $\left(\left\| X_t(\xi) \right\|^2 \right)_{t \in [0,T]}$.

Since $\mathbb{E} \left(\sup_{s \in [0,T]} \left\| X_s(\xi) \right\|^2 \right) < \infty$ (see, e.g., Th. 4.2.5 of Prévôt and Röckner (2007)), applying the dominated convergence theorem gives

$$\mathbb{E} \left\| X_t(\xi) \right\|^2 = \lim_{n \to \infty} \mathbb{E} \left\| X_{t \wedge \tau_n}(\xi) \right\|^2 = \lim_{n \to \infty} \mathbb{E} \left\| \xi \right\|^2.$$

Therefore the supermartingale $\left(\left\| X_t(\xi) \right\|^2 \right)_{t \in [0,T]}$ is in fact a martingale. □

A.2. *Proof of Lemma 5.12.*

Proof of Lemma 5.12: Let $(\chi_j)_{j \in \mathbb{N}}$ be an orthonormal basis of

$$L^2 \left(\left(\Omega, \mathfrak{G}_T^{\xi,W}, \mathbb{P} \right), \mathfrak{h} \right).$$

Combining the Cauchy-Schwarz inequality with (5.8) we obtain the equicontinuity of the family of complex functions $\left(\mathbb{E} \left\langle \chi_j, X^n \right\rangle \right)_{n \in \mathbb{N}}$, with $j \in \mathbb{N}$. Using Lemma 5.8, the Arzelà-Ascoli theorem and diagonalization arguments we deduce that can extract from any subsequence of $(X^n)_{n \in \mathbb{N}}$ a subsequence $(X^{n_k})_{k \in \mathbb{N}}$ such that $\mathbb{E} \left\langle \chi_j, X^{n_k} \right\rangle$ is uniformly convergent in $[0,T]$ for any $j \in \mathbb{N}$. Lemma 5.8 now shows that $X_t^{n_k}$ is weakly convergent in $L^2 \left(\left(\Omega, \mathfrak{G}_T^{\xi,W}, \mathbb{P} \right), \mathfrak{h} \right)$ for any $t \in [0,T]$. Since $X_t^{n_k}$ is $\mathfrak{G}_t^{\xi,W}$-measurable, for any $t \in [0,T]$ there exists a $\mathfrak{G}_t^{\xi,W}$-measurable random variable ψ_t satisfying

$$X_t^{n_k} \longrightarrow_{k \to \infty} \psi_t \qquad \textit{weakly in } L^2 \left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P} \right), \mathfrak{h} \right). \tag{A.2}$$

Assume that $(e_j)_{j \in \mathbb{N}}$ is an orthonormal basis of $\mathfrak{h}$. According to (A.2) we have

$$\left\langle e_j, X_t^{n_k} \right\rangle \longrightarrow_{k \to \infty} \left\langle e_j, \psi_t \right\rangle \quad \textit{weakly in } L^2 \left(\left(\Omega, \mathfrak{G}_t^{\xi,W}, \mathbb{P} \right), \mathbb{C} \right).$$

Thus, from (5.8) it follows that

$$\mathbb{E} \left| \left\langle e_j, \psi_t - \psi_s \right\rangle \right|^2 \leq \liminf_{k \to \infty} \mathbb{E} \left| \left\langle e_j, X_t^{n_k} - X_t^{n_k} \right\rangle \right|^2 \leq K_{T,\xi} (t-s).$$

It follows that $\langle e_j, \psi \rangle$ has a $\left(\mathfrak{G}_{t+}^{\xi,W} \right)_{t \in [0,T]}$-predictable version $\widetilde{\langle e_j, \psi \rangle}$ (see, e.g., Proposition 3.6 of Da Prato and Zabczyk (1992)). We define $\mathfrak{a}$ to be the set of all (t, ω) belonging to $[0,T] \times \Omega$ such that $\sum_{j=1}^n \widetilde{\langle e_j, \psi \rangle}_t (\omega) e_j$ converge as n goes to ∞. The proof is completed by choosing $Z_t(\omega) = \sum_{j=1}^\infty \widetilde{\langle e_j, \psi \rangle}_t (\omega) e_j$ if $(t, \omega) \in \mathfrak{a}$, and $Z_t(\omega) = 0$ provided that $(t, \omega) \notin \mathfrak{a}$. Thus Z becomes a version of ψ. □

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