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Random walks with unbounded jumps among random conductances II: Conditional quenched CLT

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Abstract. We study a one-dimensional random walk among random conductances, with unbounded jumps. Assuming the ergodicity of the collection of conductances and a few other technical conditions (uniform ellipticity and polynomial bounds on the tails of the jumps) we prove a quenched *conditional* invariance principle for the random walk, under the condition that it remains positive until time n. As a corollary of this result, we study the effect of conditioning the random walk to exceed level n before returning to 0 as $n \to \infty$.

1. Introduction and results

In this paper, we study one-dimensional random walks among random conductances, with unbounded jumps. This is the continuation of the paper Gallesco and Popov, 2012, where we proved a *uniform* quenched invariance principle for this model, where "uniform" refers to the starting position of the walk (i.e., one obtains the same estimates on the speed of convergence as long as this position lies in a certain interval around the origin). Here, our main results concern the (quenched) limiting law of the trajectory of the random walk $(X_n, n = 0, 1, 2, \dots)$ starting from the origin up to time n, under condition that it remains positive at the moments $1, \dots, n$. In Theorem 1.1 we prove that, after suitable rescaling, for a.e. environment it converges to the *Brownian meander* process, which is, roughly speaking, a Brownian motion conditioned on staying positive up to some finite time, and the main result of the paper Gallesco and Popov, 2012 will be an important tool for prooving Theorem 1.1.

This kind of problem was extensively studied for the case of space-homogeneous random walk, i.e., when one can write $X_n = \xi_1 + \cdots + \xi_n$, where the ξ_i-s are i.i.d. random variables. These random variables are usually assumed to have expectation

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0, and to possess some (nice) tail properties. Among the first papers on the subject we mention Belkin, 1972 and Iglehart, 1974, where the convergence of the rescaled trajectory to the Brownian meander was proved. Afterwards, finer results (such as local limit theorems, convergence to other processes if the original walk is in the domain of attraction of some stable Lévy process, etc.) for space-homogeneous random walks were obtained, see e.g. Bertoin and Doney, 1994; Caravenna, 2005; Caravenna and Chaumont, 2008; Vatutin and Wachtel, 2009 and references therein. Also, it is worth noting that in the paper Bolthausen, 1976 the approach of Iglehart, 1974 was substantially simplified by taking advantage of the homogeneity of the random walk; however, since in our case the random walk is not space-homogeneous, we rather use methods similar to those of Iglehart, 1974.

Also, as mentioned in Gallesco and Popov, 2012, another motivation for this work came from Knudsen billiards in random tubes, see Comets and Popov, 2012; Comets et al., 2009, 2010a,b. We refer to Section 1 of Gallesco and Popov, 2012 for the discussion on the relationship of the present model to random billiards.

Now, we define the model formally. For $x, y \in \mathbb{Z}$, we denote by $\omega_{x,y} = \omega_{y,x}$ the conductance between x and y. Define $\theta_z \omega_{x,y} = \omega_{x+z,y+z}$, for all $z \in \mathbb{Z}$. Note that, by Condition K below, the vectors $\omega_{x,\cdot}$ are elements of the Polish space $\ell^2(\mathbb{Z})$. We assume that $(\omega_{x,\cdot})_{x\in\mathbb{Z}}$ is a stationary ergodic (with respect to the family of shifts θ) sequence of random vectors; $\mathbb{P}$ stands for the law of this sequence. The collection of all conductances $\omega = (\omega_{x,y}, x, y \in \mathbb{Z})$ is called the *environment*. For all $x \in \mathbb{Z}$, define $C_x = \sum_y \omega_{x,y}$. Given that $C_x < \infty$ for all $x \in \mathbb{Z}$ (which is always so by Condition K below), the random walk X in random environment ω is defined through its transition probabilities

$$p_\omega(x, y) = \frac{\omega_{x,y}}{C_x};$$

that is, if $\mathtt{P}^x_\omega$ is the quenched law of the random walk starting from x, we have

$$\mathtt{P}^x_\omega[X_0 = x] = 1, \quad \mathtt{P}^x_\omega[X_{k+1} = z \mid X_k = y] = p_\omega(y, z).$$

Clearly, this random walk is reversible with the reversible measure $(C_x, x \in \mathbb{Z})$. Also, we denote by $\mathtt{E}^x_\omega$ the quenched expectation for the process starting from x. When the random walk starts from 0, we use shortened notations $\mathtt{P}_\omega, \mathtt{E}_\omega$.

In order to prove our results, we need to make two technical assumptions on the environment:

Condition E. There exists $\kappa > 0$ such that, $\mathbb{P}$-a.s., $\omega_{0,1} \geq \kappa$.

Condition K. There exist constants $K, \beta > 0$ such that $\mathbb{P}$-a.s., $\omega_{0,y} \leq \frac{K}{1+y^{3+\beta}}$, for all $y \geq 0$.

For future reference, note that combining Conditions E and K we have that there exists $\hat\kappa > 0$ such that $\mathbb{P}$-a.s.,

$$\hat\kappa \leq \sum_{y\in\mathbb{Z}} \omega_{0,y} \leq \hat\kappa^{-1}. \tag{1.1}$$

We decided to formulate Condition E this way because, due to the fact that this work was motivated by random billiards, the main challenge was to deal with the long-range jumps. It is plausible that Condition E could be relaxed to some extent; however, for the sake of cleaner presentation of the argument, we prefer not trying to deal with *both* long-range jumps and the lack of nearest-neighbor ellipticity.

Next, for all $n \geq 1$, we define the continuous map $Z^n = (Z^n(t), t \in \mathbb{R}_+)$ as the natural polygonal interpolation of the map $k/n \mapsto \sigma^{-1}n^{-1/2}X_k$ (with σ from Theorem 1.1 in Gallesco and Popov, 2012). In other words,

$$\sigma\sqrt{n}Z^n_t = X_{\lfloor nt\rfloor} + (nt - \lfloor nt\rfloor)X_{\lfloor nt\rfloor+1}$$

with $\lfloor\cdot\rfloor$ the integer part. Also, we denote by W the standard Brownian motion.

Now, let $\hat{\tau} = \inf\{k \geq 1 : X_k \in (-\infty, 0]\}$ and $\Lambda_n = \{\hat{\tau} > n\} = \{X_k > 0 \text{ for all } k = 1, \dots, n\}$. Consider the conditional quenched probability measure $Q^n_\omega[\,\cdot\,] = \mathtt{P}_\omega[\,\cdot \mid \Lambda_n]$, for all $n \geq 1$. For each n, the random map Z^n induces a probability measure μ^n_ω on $(C[0,1], \mathcal{B}_1)$, where $\mathcal{B}_1$ is the Borel σ-algebra on $C[0,1]$ with the supremum norm: for any $A \in \mathcal{B}_1$,

$$\mu^n_\omega(A) = Q^n_\omega[Z^n \in A].$$

Let us next recall the formal definition of the Brownian meander W^+. For this, let W be a standard Brownian motion and define $\tau_1 = \sup\{s \in [0,1] : W(s) = 0\}$ and $\Delta_1 = 1 - \tau_1$. Then,

$$W^+(s) = \Delta_1^{-1/2}|W(\tau_1 + s\Delta_1)|, \qquad 0 \leq s \leq 1.$$

Now, we are ready to formulate the quenched invariance principle for the random walk conditioned to stay positive, which is the main result of this paper:

Theorem 1.1. *Under Conditions E and K, we have that, $\mathbb{P}$-a.s., μ^n_ω tends weakly to P_{W^+} as $n \to \infty$, where P_{W^+} is the law of the Brownian meander W^+ on $C[0,1]$.*

As a corollary of Theorem 1.1, we obtain a limit theorem for the process conditioned on crossing a large interval. Define $\hat{\tau}_n = \inf\{k \geq 0 : X_k \in [n, \infty)\}$ and $\Lambda'_n = \{\hat{\tau}_n < \hat{\tau}\}$. We also define $T_n = \inf\{t > 0 : Z^{n^2}_t = \sigma^{-1}\}$ and the stopped process $Y^n_\cdot = Z^{n^2}_{\cdot\wedge T_n}$. Denoting by B_3 the three-dimensional Bessel process (we recall that B_3 is the radial part of a 3-dimensional Brownian motion, that is, if (W_1, W_2, W_3) is a three-dimensional Brownian motion, we have $B_3(t) = \sqrt{W_1^2(t) + W_2^2(t) + W_3^2(t)}$) and by $\varrho_1 = \inf\{t > 0 : B_3 = \sigma^{-1}\}$, we have

Corollary 1.2. *Assume Conditions E and K. We have that, $\mathbb{P}$-a.s., under the law $\mathtt{P}_\omega[\,\cdot \mid \Lambda'_n]$, the couple (Y^n, T_n) converges in law to $(B_3(\cdot \wedge \varrho_1), \varrho_1)$ as $n \to \infty$.*

In the next section, we prove some auxiliary results which are necessary for the proof of Theorem 1.1. Then, in Section 3, we give the proof of Theorem 1.1. Finally, in Section 4, we give the proof of Corollary 1.2.

We will denote by K_1, K_2, ... the "global" constants, that is, those that are used all along the paper and by γ_1, γ_2, ... the "local" constants, that is, those that are used only in the subsection in which they appear for the first time. For the local constants, we restart the numeration in the beginning of each subsection. Besides, to simplify notations, if x is not integer, $\mathtt{P}^x_\omega$ must be understood as $\mathtt{P}^{\lfloor x\rfloor}_\omega$.

2. Auxiliary results

In this section, we will prove some technical results that will be needed later to prove Theorem 1.1. Let us introduce the following notations. If $A \subset \mathbb{Z}$,

$$\tau_A = \inf\{n \geq 0 : X_n \in A\} \quad \text{and} \quad \tau^+_A = \inf\{n \geq 1 : X_n \in A\}. \tag{2.1}$$

Whenever $A = \{x\}$, $x \in \mathbb{Z}$, we write τ_x (respectively, τ_x^+) instead of $\tau_{\{x\}}$ (respectively, $\tau_{\{x\}}^+$).

2.1. *Auxiliary environments.* From some fixed environment ω, we are going to introduce three derived environments denoted by $\omega^{(1)}$, $\omega^{(2)}$ and $\omega^{(3)}$ which will be important tools for the proofs of the lemmas in the rest of this section.

Fix two disjoint intervals $B = (-\infty, 0]$ and $E = [N, \infty)$ of $\mathbb{Z}$. For some realization ω of the environment, consider the new environment $\omega^{(1)}$ obtained from ω by deleting all the conductances $\omega_{x,y}$ if x and y belong to $(B \setminus \{0\}) \cup E$. The reversible measure (up to a constant factor) on this new environment $\omega^{(1)}$ is given by

$$\begin{aligned} C_0^{(1)} &= C_0, & \\ C_x^{(1)} &= C_x, & \text{if } x \notin B \cup E, \\ C_x^{(1)} &= \sum_{y \notin (B \setminus \{0\}) \cup E} \omega_{x,y}, & \text{otherwise.} \end{aligned}$$

Now, we define $C_B^{(1)} = \sum_{x \in B} C_x^{(1)}$ and for all $x \in B$, $\pi_B(x) = C_x^{(1)}/C_B^{(1)}$. Observe that by Conditions E and K, $C_B^{(1)}$ is positive and finite $\mathbb{P}$-a.s. Hence π_B is $\mathbb{P}$-a.s. a probability measure on B. In the same way we define π_E on E. For the sake of simplicity we denote $\mathsf{P}_{\omega^{(1)}}^{B}$ (respectively, $\mathsf{P}_{\omega^{(1)}}^{E}$) instead of $\mathsf{P}_{\omega^{(1)}}^{\pi_B}$ (respectively, $\mathsf{P}_{\omega^{(1)}}^{\pi_E}$) for the random walk on $\omega^{(1)}$ starting with initial distribution π_B (respectively, π_E). The same convention will be adopted for environments $\omega^{(2)}$ and $\omega^{(3)}$ defined below.

From the environment $\omega^{(1)}$, we now construct a new environment $\omega^{(2)}$ by setting if $x > 0$, $y > 0$,

$$\omega_{x,0}^{(2)} = \sum_{y \in B} \omega_{x,y}^{(1)}, \;\; \omega_{0,0}^{(2)} = \sum_{y \in B} \omega_{y,0}^{(1)}, \;\; \omega_{x,y}^{(2)} = \omega_{x,y}^{(1)}$$

and $\omega_{x,y}^{(2)} = 0$ otherwise. Defining the reversible measure associated to $\omega^{(2)}$ as $C_x^{(2)} = \sum_{y \in \mathbb{Z}} \omega_{x,y}^{(2)}$, for $x \in \mathbb{Z}$, observe in particular that $C_0^{(2)} = C_B^{(1)}$ and $C_x^{(2)} = C_x^{(1)}$ for $x > 0$.

From the environment $\omega^{(1)}$, we finally create a last environment $\omega^{(3)}$ by setting if $x \in (0, N)$,

$$\omega_{x,N}^{(3)} = \sum_{y \in E} \omega_{x,y}^{(1)}, \;\; \omega_{x,0}^{(3)} = \sum_{y \in B} \omega_{x,y}^{(1)}.$$

Then, let

$$\omega_{N,0}^{(3)} = \sum_{y \in E} \omega_{y,0}^{(1)}, \;\; \omega_{0,0}^{(3)} = \sum_{y \in B} \omega_{y,0}^{(1)}.$$

For $x \in (0, N)$ and $y \in (0, N)$ we just set $\omega_{x,y}^{(3)} = \omega_{x,y}^{(1)}$ and $\omega_{x,y}^{(3)} = 0$ in all other cases. We define the reversible measure associated to $\omega^{(3)}$ as $C_x^{(3)} = \sum_{y \in \mathbb{Z}} \omega_{x,y}^{(3)}$, for $x \in \mathbb{Z}$. Observe in particular that $C_0^{(3)} = C_B^{(1)}$, $C_N^{(3)} = C_E^{(1)}$ and $C_x^{(3)} = C_x^{(1)}$ for $x \in (0, N)$.

2.2. *Crossing probabilities and estimates on the conditional exit distribution.* Fix $\varepsilon > 0$, $n \in \mathbb{N}$ such that $\varepsilon\sqrt{n} \geq 1$ and take $N = \lfloor \varepsilon\sqrt{n} \rfloor$ (N is from section 2.1). Then define the event $A_{\varepsilon,n} = \{\tau_E < \tau_B^+\}$ (B and E are from section 2.1). For an arbitrary positive integer M define $I_M = [N, N + M]$.

Lemma 2.1. *For all $\eta > 0$ there exists $M > 0$ such that $\mathbb{P}$-a.s.,*

$$\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}] \geq 1 - \eta, \quad \text{for all } n \text{ such that } N > 1.$$

Proof. The proof of this lemma is very similar to the proof of Proposition 2.3 of Gallesco and Popov, 2012. Here, we just give the first steps of the proof and then indicate the exact place where it matches with the proof of Proposition 2.3 of Gallesco and Popov, 2012. First, we write

$$\begin{aligned}\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}] &= 1 - \mathtt{P}_\omega[X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}] \\ &= 1 - \sum_{y > N+M} \mathtt{P}_\omega[X_{\tau_E} = y \mid A_{\varepsilon,n}]. \end{aligned} \tag{2.2}$$

By definition of $\omega^{(1)}$ (cf. section 2.1), we can couple the random walks in environments ω and $\omega^{(1)}$ to show that $\mathtt{P}_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \mathtt{P}_\omega[X_{\tau_E} = y \mid A_{\varepsilon,n}]$. Then, by construction of $\omega^{(2)}$, we can couple the random walks in environments $\omega^{(1)}$ and $\omega^{(2)}$ to show that $\mathtt{P}_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \mathtt{P}^B_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$. Thus, we obtain

$$\begin{aligned}\mathtt{P}_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \mathtt{P}^B_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] &= \sum_{x \in B} \pi_B(x) \mathtt{P}^x_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] \\ &= \sum_{x \in B} \pi_B(x) \mathtt{P}^x_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] \\ &\geq \pi_B(0) \mathtt{P}_{\omega^{(1)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] \\ &= \frac{C_0}{C_B^{(1)}} \mathtt{P}_\omega[X_{\tau_E} = y \mid A_{\varepsilon,n}].\end{aligned}$$

By (2.2) we obtain

$$\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}] \geq 1 - \frac{C_B^{(1)}}{C_0} \sum_{y > N+M} \mathtt{P}_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}].$$

Note that, by Condition K and (1.1), $C_B^{(1)}/C_0 \leq \gamma_1$ for some constant γ_1. The terms $\mathtt{P}_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}]$ can be treated in the same way as the terms $\mathtt{P}^x_\omega[X_{\tau_E} = y \mid A_E]$ of equation (2.6) in Gallesco and Popov, 2012. In particular, following the reasoning anteceding equation (2.9) in Gallesco and Popov, 2012, we can show that

$$\mathtt{P}_{\omega^{(2)}}[X_{\tau_E} = y \mid A_{\varepsilon,n}] = \frac{C_y^{(1)} \mathtt{P}^y_{\omega^{(2)}}[\tau_0 < \tau_E^+]}{C_E^{(1)} \mathtt{P}^E_{\omega^{(2)}}[\tau_0 < \tau_E^+]}.$$

Then, the numerator and denominator of the above equation can be treated by using the same techniques as those used to treat (2.9) in Gallesco and Popov, 2012. □

Lemma 2.2. *There exists a positive constant K_1 such that, $\mathbb{P}$-a.s., $\mathtt{P}_\omega[A_{\varepsilon,n}] \geq K_1 N^{-1}$ for all n such that $N > 1$.*

Proof. Recall that $\mathtt{P}_\omega[A_{\varepsilon,n}] = \mathtt{P}_\omega[\tau_E < \tau_B^+]$. We can couple the random walks in environments ω and $\omega^{(1)}$ (cf. section 2.1) to show that $\mathtt{P}_\omega[\tau_E < \tau_B^+] = \mathtt{P}_{\omega^{(1)}}[\tau_E < \tau_B^+]$.

Let us denote by $\Gamma_{z',z''}$ the set of finite paths $(z', z_1, \ldots, z_k, z'')$ such that $z_i \notin B \cup E \cup \{z', z''\}$ for all $i = 1, \ldots, k$. Let $\gamma = (z', z_1, \ldots, z_k, z'') \in \Gamma_{z',z''}$ and define

$$\mathtt{P}^{z'}_{\omega^{(1)}}[\gamma] := \mathtt{P}^{z'}_{\omega^{(1)}}[X_1 = z_1, \ldots, X_k = z_k, X_{k+1} = z''].$$

By reversibility we obtain

$$
\begin{aligned}
\mathtt{P}_{\omega^{(1)}}[\tau_E < \tau_B^+] &= \sum_{z\in E}\sum_{\gamma\in\Gamma_{0,z}} \mathtt{P}_{\omega^{(1)}}[\gamma] \\
&= \sum_{z\in E}\sum_{\gamma\in\Gamma_{z,0}} \frac{C_z^{(1)}}{C_0}\mathtt{P}^z_{\omega^{(1)}}[\gamma] \\
&= \frac{C_E^{(1)}}{C_0}\sum_{z\in E}\pi_E(z)\sum_{\gamma\in\Gamma_{z,0}}\mathtt{P}^z_{\omega^{(1)}}[\gamma] \\
&= \frac{C_E^{(1)}}{C_0}\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0]. \qquad (2.3)
\end{aligned}
$$

Now, define $B' = (-\infty, 1]$. We have

$$
\begin{aligned}
&\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0] \\
&= \mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, \tau_{B'} < \tau_E^+, X_{\tau_B} = 0] \\
&= \mathtt{P}^E_{\omega^{(1)}}[\tau_{B'} < \tau_E^+]\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0 \mid \tau_{B'} < \tau_E^+] \\
&\geq \mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+]\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0 \mid \tau_{B'} < \tau_E^+]. \qquad (2.4)
\end{aligned}
$$

Let us treat the term $\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+]$. By definition of $\omega^{(3)}$ (cf. section 2.1), we can couple the random walks in environments $\omega^{(1)}$ and $\omega^{(3)}$ to show that $\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+] = \mathtt{P}^N_{\omega^{(3)}}[\tau_0 < \tau_N^+]$. We obtain

$$
C_E^{(1)}\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+] = C_E^{(1)}\mathtt{P}^N_{\omega^{(3)}}[\tau_0 < \tau_N^+] = C_N^{(3)}\mathtt{P}^N_{\omega^{(3)}}[\tau_0 < \tau_N^+] = C_{\text{eff}}(1,N) \qquad (2.5)
$$

where $C_{\text{eff}}(1,N)$ is the effective conductance between the points 1 and N of the electrical network associated to $\omega^{(3)}$ (cf. Doyle and Snell, 1984, section 3.4). Using Condition E, we obtain

$$
C_{\text{eff}}(1,N) \geq \Big(\sum_{i=1}^{N-1}\omega_{i,i+1}^{-1}\Big)^{-1} \geq \frac{\kappa}{N-1}.
$$

Therefore, there exists a constant γ_1 such that, whenever $N > 1$

$$
C_E^{(1)}\mathtt{P}^E_{\omega^{(1)}}[\tau_{B'} < \tau_E^+] \geq \frac{\gamma_1}{N}. \qquad (2.6)
$$

Let us treat the term $\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0 \mid \tau_{B'} < \tau_E^+]$. We have by the Markov property

$$
\begin{aligned}
&\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0 \mid \tau_{B'} < \tau_E^+] \\
&= \sum_{y\in\{0,1\}}\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0, X_{\tau_{B'}} = y \mid \tau_{B'} < \tau_E^+] \\
&= \sum_{y\in\{0,1\}}\mathtt{P}^E_{\omega^{(1)}}[\tau_B < \tau_E^+, X_{\tau_B} = 0, \mid X_{\tau_{B'}} = y, \tau_{B'} < \tau_E^+]\mathtt{P}^E_{\omega^{(1)}}[X_{\tau_{B'}} = y \mid \tau_{B'} < \tau_E^+] \\
&= \sum_{y\in\{0,1\}}\mathtt{P}^y_{\omega}[\tau_B < \tau_E, X_{\tau_B} = 0]\mathtt{P}^E_{\omega^{(1)}}[X_{\tau_{B'}} = y \mid \tau_{B'} < \tau_E^+] \\
&\geq \min_{y\in\{0,1\}}\mathtt{P}^y_{\omega}[\tau_B < \tau_E, X_{\tau_B} = 0]
\end{aligned}
$$

$\geq \mathtt{P}^1_\omega[X_1 = 0]$.

By Condition E and (1.1), this last probability is bounded from below by the constant $\kappa\hat{\kappa}$. Thus, combining this last result with (2.3), (2.4), (2.6) and, since by (1.1) we have $C_0 \leq \hat{\kappa}^{-1}$, it follows that $\mathbb{P}$-a.s.,

$$\mathtt{P}_\omega[A_{\varepsilon,n}] \geq \frac{\gamma_1 \kappa \hat{\kappa}^2}{N}.$$

This concludes the proof of Lemma 2.2. □

Lemma 2.3. *There exists a positive constant K_2 such that we have, $\mathbb{P}$-a.s.,*

$$\mathtt{E}_\omega[\tau_B^+ \wedge \tau_E] \leq K_2 N$$

for all n such that $N > 1$.

Proof. First notice that by construction of $\omega^{(1)}$ (cf. section 2.1), we can couple the random walks in environments ω and $\omega^{(1)}$ to show that $\mathtt{E}_\omega[\tau_B^+ \wedge \tau_E] = \mathtt{E}_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E]$. Hence, we obtain

$$\begin{aligned} \mathtt{E}^B_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E] &= \sum_{y \in B} \pi_B(y) \mathtt{E}^y_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E] \\ &= \pi_B(0) \mathtt{E}^0_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E] + \sum_{y \in B \setminus \{0\}} \pi_B(y) \mathtt{E}^y_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E] \\ &\geq \pi_B(0) \mathtt{E}_\omega[\tau_B^+ \wedge \tau_E]. \end{aligned}$$

Therefore, we obtain

$$\mathtt{E}_\omega[\tau_B^+ \wedge \tau_E] \leq \frac{\mathtt{E}^B_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E]}{\pi_B(0)}. \tag{2.7}$$

Then, observe that

$$\mathtt{E}^B_{\omega^{(1)}}[\tau_B^+ \wedge \tau_E] = \mathtt{E}_{\omega^{(3)}}[\tau_0^+ \wedge \tau_N]. \tag{2.8}$$

We are going to bound the right-hand side term of (2.8) from above. Before this, we make a brief digression to study the invariant measure of a particular process of interest.

Consider the following particle system in continuous time on the interval $[0, N]$ of $\mathbb{Z}$. Suppose that we have injection (according to some Poisson process) and absorption of particles at states 0 and N. Once injected, particles move according to transition rates given by $q_{x,y} = \omega^{(3)}_{x,y}/C^{(3)}_x$, for $(x, y) \in \{0, \dots, N\}^2$, until they reach 0 or N. We suppose that injections at 0 and N happen accordingly to independent Poisson processes with rates respectively $\lambda_0 = C^{(3)}_0$ and $\lambda_N = C^{(3)}_N$. We are interested in the continuous time Markov process $\big(\eta(t) = ((\eta_0(t), \dots, \eta_N(t)), t \geq 0\big)$ with state space $\Omega = \mathbb{Z}_+^{\{0,\dots,N\}}$ where $\eta_i(t)$ represents the number of particles in i at time t. Hereafter, for $(i, j) \in \{0, \dots, N\}^2$, we will use the symbol $\eta^{i,j}$ to denote the configuration obtained from η by moving a particle from site i to site j, i.e., if for example $i < j$, $\eta^{i,j} = (\eta_0, \dots, \eta_i - 1, \dots, \eta_j + 1, \dots, \eta_N)$. We also define $\eta^{i,+} = (\eta_0, \dots, \eta_i + 1, \dots, \eta_N)$ and $\eta^{i,-} = (\eta_0, \dots, \eta_i - 1, \dots, \eta_N)$ for $i \in \{0, \dots, N\}$. The generator of this process defined by its action on functions $f : \Omega \to \mathbb{R}$ is given by

$$\mathcal{L}f(\eta) = \lambda_0(f(\eta^{0,+}) - f(\eta)) + \sum_{i=0}^{N} \eta_i q_{i,0}(f(\eta^{i,-}) - f(\eta))$$

$$+\sum_{i=1}^{N-1}\sum_{j=1}^{N-1}\eta_j q_{j,i}(f(\eta^{j,i})-f(\eta))$$

$$+\lambda_N(f(\eta^{N,+})-f(\eta))+\sum_{i=0}^{N}\eta_i q_{i,N}(f(\eta^{i,-})-f(\eta)). \tag{2.9}$$

Let $\mu=\bigotimes_{i=1}^{N}\mu_i$ be the product measure of laws μ_i where for each $i\in\{0,\dots,N\}$, μ_i is a Poisson law with parameter $C_i^{(3)}$. We can check that for any configurations η, $\eta'\in\Omega$,

$$L(\eta,\eta')\mu(\eta)=L(\eta',\eta)\mu(\eta') \tag{2.10}$$

where $L(\eta,\eta')$ is the transition rate from the configuration η to η', i.e., $L(\eta,\eta')=\mathcal{L}f(\eta)$ with $f(\eta)=\delta_{\eta,\eta'}$. This implies that the probability measure μ is reversible and invariant for the Markov process η.

Now, consider the model above with injection at rate λ_0 and absorption at 0 and only absorption (without injection) at N. Such a system can be considered as a $M/G/\infty$ queue where the customers arrive according to a Poisson process of rate λ_0 and the service time law is that of the lifetime of a particle in the interval $[0,N]$. Thus, the expected service time of a customer, denoted by $E[T]$, equals $\mathtt{E}_{\omega^{(3)}}[\tau_0^+\wedge\tau_N]$. By Little's formula (see e.g. Section 5.2 of Cooper, 1981) we have

$$E[T]=\frac{E[R]}{\lambda_0}$$

where $E[R]$ is the mean number of particles in the queue in the stationary regime. By a coupling argument, we can see that the distribution of the number of customers in the system in the stationary regime is stochastically dominated by the distribution of the total number of particles in the interval $[0,N]$ in the stationary regime for the particle system with both injection and absorption of particles at states 0 and N. It is not difficult to see that this last distribution is $\mu_0\star\cdots\star\mu_N$ (here $\star$ is the convolution product of measures). Therefore, combining the foregoing observations, we obtain

$$\mathtt{E}_{\omega^{(3)}}[\tau_0^+\wedge\tau_N]=E[T]=\frac{E[R]}{\lambda_0}\le\frac{1}{\lambda_0}\sum_{x\in\mathbb{Z}}x\mu_0\star\cdots\star\mu_N(x)=\frac{1}{C_0^{(3)}}\sum_{x=0}^{N}C_x^{(3)}. \tag{2.11}$$

Finally, by (2.7), (2.8) and (2.11) we obtain

$$\mathtt{E}_\omega[\tau_B^+\wedge\tau_E]\le\frac{1}{C_0^{(3)}\pi_B(0)}\sum_{x=0}^{N}C_x^{(3)}.$$

By Conditions E and K, it holds that there exists a positive constant K_2 such that $\mathbb{P}$-a.s.,

$$\mathtt{E}_\omega[\tau_B^+\wedge\tau_E]\le K_2N.$$

This concludes the proof of Lemma 2.3. □

3. Proof of Theorem 1.1

In this section, we prove Theorem 1.1. To simplify notations, we consider $\sigma=1$. Our strategy to prove Theorem 1.1 is to use Theorems 3.6 and 3.10 of Durrett, 1978 (which are restated here as Theorems 3.1 and 3.2). These theorems give equivalent conditions for the tightness and convergence of finite dimensional distributions of the conditioned processes Z^n that are easier to verify in our case. In Durrett, 1978, these theorems are stated in a quite general form that can be simplified here. Also, since in our problem all the processes considered have continuous trajectories, we will transpose these theorems on $C[0,1]$ (instead of $D[0,1]$, the Skorokhod space):

Theorem 3.1. *The sequence of measures $(\mu_\omega^n, n \geq 1)$ is tight if and only if*

$$\lim_{x\to\infty} \limsup_{n\to\infty} \mathtt{P}_\omega[Z_1^n > x \mid \Lambda_n] = 0 \qquad \textit{and} \tag{3.1}$$

$$\lim_{t\to 0} \limsup_{n\to\infty} \mathtt{P}_\omega[Z_t^n > h \mid \Lambda_n] = 0 \quad \textit{for each } h > 0. \tag{3.2}$$

We recall that the measures μ_ω^n are defined in the introduction. Now, let us define the following conditions:

(i) if $x_n \to x$, then $(\mathtt{P}_\omega^{x_n\sqrt{n}}[Z_\cdot^n \in \cdot], n \geq 1)$ tends weakly to $P^x[W_\cdot \in \cdot]$ in $C[0,1]$,

(ii) let $x_n \geq 0$, for all $n \geq 1$, then $\lim_{n\to\infty} \mathtt{P}_\omega^{x_n\sqrt{n}}[Z_s^n > 0, s \leq t_n] = P^x[W_s > 0, s \leq t]$, whenever $x_n \to x$ and $t_n \to t > 0$.

Theorem 3.2. *Suppose (i)-(ii) hold and $(\mu_\omega^n, n \geq 1)$ is tight. Then, $(\mu_\omega^n, n \geq 1)$ tends wealky to W^+ if and only if*

$$\lim_{h\to 0} \liminf_{n\to\infty} \mathtt{P}_\omega[Z_t^n > h \mid \Lambda_n] = 1 \qquad \textit{for all } t > 0. \tag{3.3}$$

In our case, condition (i) is an immediate consequence of the quenched Uniform CLT (cf. Theorem 1.2 of Gallesco and Popov, 2012) which in the rest of this paper will be referred as UCLT. For condition (ii), let $\varepsilon > 0$, we have for all n large enough

$$\mathtt{P}_\omega^{x_n\sqrt{n}}[Z_s^n > 0, s \leq t+\varepsilon] \leq \mathtt{P}_\omega^{x_n\sqrt{n}}[Z_s^n > 0, s \leq t_n] \leq \mathtt{P}_\omega^{x_n\sqrt{n}}[Z_s^n > 0, s \leq t-\varepsilon].$$

Thus, condition (ii) follows from the UCLT and the continuity in t of $P^x[W_s > 0, s \leq t]$. Our next step is to obtain the weak limit of the sequence $(\mathtt{P}_\omega[Z_1^n \in \cdot \mid \Lambda_n], n \geq 1)$. This is the object of Proposition 3.3. Then, we obtain the weak limit of $(\mathtt{P}_\omega[Z_t^n \in \cdot \mid \Lambda_n], n \geq 1)$ for all $t \in (0,1)$. This is done in Proposition 3.4. In the last step, we check that (3.1), (3.2), and (3.3) hold to end the proof of Theorem 1.1.

At this point, let us recall some notations of Section 2.2. Fix $\varepsilon > 0$ and define $N = \lfloor \varepsilon\sqrt{n} \rfloor$. Let $B = (-\infty, 0]$ and $E = [N, +\infty)$. Then, define the event $A_{\varepsilon,n} = \{\tau_E < \tau_B^+\}$. For an arbitrary positive integer M define $I_M = [N, N+M]$. First, let us prove

Proposition 3.3. *We have $\mathbb{P}$-a.s.,*

$$\lim_{n\to\infty} \mathtt{P}_\omega[Z_1^n > x \mid \Lambda_n] = \exp(-x^2/2), \quad \textit{for all } x \geq 0. \tag{3.4}$$

Proof. For notational convenience, let us only treat the case $x = 1$. The generalization to any $x \geq 0$ is straightforward. Fix $\varepsilon \in (0,1)$, $\delta \in (0,1)$ and write

$$\begin{aligned}
&\mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n] \\
&= \frac{1}{\mathtt{P}_\omega[\Lambda_n]} \mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n] \\
&= \frac{1}{\mathtt{P}_\omega[\Lambda_n]} \Big(\mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E} \in I_M] \\
&\qquad\qquad + \mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E} \notin I_M] \Big)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\mathtt{P}_\omega[\Lambda_n]}\Big(\mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E} \in I_M, \tau_E > \delta n] \\
&\qquad + \mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E} \in I_M, \tau_E \le \delta n] \\
&\qquad + \mathtt{P}_\omega[X_n > \sqrt{n}, A_{\varepsilon,n}, \Lambda_n, X_{\tau_E} \notin I_M]\Big) \\
&= \frac{\mathtt{P}_\omega[A_{\varepsilon,n}]}{\mathtt{P}_\omega[\Lambda_n]}\Big(\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]\mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \\
&\qquad \times \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E > \delta n] \\
&\qquad + \mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]\mathtt{P}_\omega[\tau_E \le \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \\
&\qquad \times \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \le \delta n] \\
&\qquad + \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n, X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}]\Big). \qquad (3.5)
\end{aligned}
$$

Informally, the rest of the proof consists in using the decomposition (3.5) in order to find good lower and upper bounds L_n and U_n for $\mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n]$ such that $U_n/L_n \to 1$ as $n \to \infty$. We start with the upper bound. Let us write

$$
\begin{aligned}
&\mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n] \\
&\le \frac{\mathtt{P}_\omega[A_{\varepsilon,n}]}{\mathtt{P}_\omega[\Lambda_n]}\Big(\mathtt{P}_\omega[X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}] + \mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \\
&\qquad + \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \le \delta n]\Big). \qquad (3.6)
\end{aligned}
$$

Observe that we can bound the term $\mathtt{P}_\omega[X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}]$ from above using Lemma 2.1: let $\eta > 0$, then we can choose M large enough in such a way that

$$
\mathtt{P}_\omega[X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}] \le \eta. \qquad (3.7)
$$

Next, let us bound the other terms of the right-hand side of (3.6) from above. For $\mathtt{P}_\omega[A_{\varepsilon,n}]/\mathtt{P}_\omega[\Lambda_n]$, we write

$$
\begin{aligned}
\mathtt{P}_\omega[\Lambda_n] &\ge \mathtt{P}_\omega[\Lambda_n, A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n] \\
&= \mathtt{P}_\omega[A_{\varepsilon,n}]\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]\mathtt{P}_\omega[\tau_E \le \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \\
&\quad \times \mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n]. \qquad (3.8)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{\mathtt{P}_\omega[\Lambda_n]}{\mathtt{P}_\omega[A_{\varepsilon,n}]} &\ge \mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]\mathtt{P}_\omega[\tau_E \le \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \\
&\quad \times \mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n].
\end{aligned}
$$

Again, we use Lemma 2.1 to bound the term $\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]$ from below. For the term $\mathtt{P}_\omega[\tau_E \le \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}]$ we write

$$
\mathtt{P}_\omega[\tau_E \le \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] = 1 - \mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \qquad (3.9)
$$

and

$$
\begin{aligned}
\mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] &= \frac{\mathtt{P}_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\varepsilon,n}]}{\mathtt{P}_\omega[X_{\tau_E} \in I_M, A_{\varepsilon,n}]} \\
&= \frac{\mathtt{P}_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\varepsilon,n}]}{\mathtt{P}_\omega[X_{\tau_E} \in I_M \mid A_{\varepsilon,n}]\mathtt{P}_\omega[A_{\varepsilon,n}]}. \qquad (3.10)
\end{aligned}
$$

We first treat the numerator of (3.10). By Chebyshev's inequality we obtain

$$\mathtt{P}_\omega[\tau_E > \delta n, X_{\tau_E} \in I_M, A_{\varepsilon,n}] \le \mathtt{P}_\omega[\tau_B^+ \wedge \tau_E > \delta n] \le \frac{\mathtt{E}_\omega[\tau_B^+ \wedge \tau_E]}{\delta n}.$$

Using (3.10) and Lemmas 2.3, 2.1 and 2.2 we obtain

$$\mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] \le \frac{K_2 N^2}{K_1 \delta n(1-\eta)}. \tag{3.11}$$

Then, we deal with the term $\mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n]$. By the Markov property we obtain

$$\begin{aligned}
&\mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n] \\
&= \frac{1}{\mathtt{P}_\omega[A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \le \delta n]} \sum_{x \in I_M} \sum_{u=1}^{\lfloor \delta n \rfloor} \mathtt{P}_\omega[\Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}] \\
&\qquad\qquad \times \mathtt{P}_\omega[X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}] \\
&\ge \min_{x \in I_M} \min_{u \le \lfloor \delta n \rfloor} \mathtt{P}_\omega^x[\Lambda_{n-u}] \\
&\ge \min_{x \in I_M} \mathtt{P}_\omega^x[\Lambda_n].
\end{aligned} \tag{3.12}$$

Thus, by (3.8), (3.9), (3.10), (3.12) and Lemma 2.1, we have

$$\frac{\mathtt{P}_\omega[\Lambda_n]}{\mathtt{P}_\omega[A_{\varepsilon,n}]} \ge (1-\eta)\Big(1 - \frac{K_2 N^2}{K_1 \delta n(1-\eta)}\Big) \min_{x \in I_M} \mathtt{P}_\omega^x[\Lambda_n]. \tag{3.13}$$

To bound the term $\mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \le \delta n]$ from above we do the following. Let us denote by $\mathcal{E}$ the event $\{X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \le \delta n\}$. Since $A_{\varepsilon,n} \in \mathcal{F}_{\tau_E}$ the σ-field generated by X until the stopping time τ_E, we have by the Markov property and the fact that $\delta < 1$,

$$\begin{aligned}
\mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid \mathcal{E}] &= \frac{1}{\mathtt{P}_\omega[\mathcal{E}]} \sum_{x \in I_M} \sum_{u=1}^{\lfloor \delta n \rfloor} \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}] \\
&\qquad\qquad \times \mathtt{P}_\omega[X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}] \\
&\le \max_{x \in I_M} \max_{u \le \lfloor \delta n \rfloor} \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}] \\
&= \max_{x \in I_M} \max_{u \le \lfloor \delta n \rfloor} \mathtt{P}_\omega^x[X_{n-u} > \sqrt{n}, X_k > 0, 1 \le k \le n-u] \\
&= \max_{x \in I_M} \max_{u \le \lfloor \delta n \rfloor} \mathtt{P}_\omega^x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}].
\end{aligned} \tag{3.14}$$

Now, fix $\delta' \in (0,1)$. Then, we use the following estimate for $x \in I_M$ and $u \le \lfloor \delta n \rfloor$,

$$\begin{aligned}
&\mathtt{P}_\omega^x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}] \\
&\le \mathtt{P}_\omega^x\big[(\{X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}\} \cup \{|X_{n-\lfloor \delta n \rfloor} - X_{n-u}| > \delta' \sqrt{n}\}) \cap \Lambda_{n-u}\big] \\
&\le \mathtt{P}_\omega^x\Big[\Big(\Big\{X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}\Big\} \\
&\qquad \cup \Big\{\max_{u \le \lfloor \delta n \rfloor} |X_{n-\lfloor \delta n \rfloor} - X_{n-u}| > \delta' \sqrt{n}\Big\}\Big) \cap \Lambda_{n-\lfloor \delta n \rfloor}\Big].
\end{aligned}$$

Hence, we obtain for all $x \in I_M$ that

$$\max_{u \le \lfloor \delta n \rfloor} \mathtt{P}_\omega^x[X_{n-u} > \sqrt{n}, \Lambda_{n-u}] \le \mathtt{P}_\omega^x[X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}] + \mathtt{P}_\omega^x\Big[\max_{u \le \lfloor \delta n \rfloor} |X_{n-\lfloor \delta n \rfloor} - X_{n-u}| > \delta' \sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}\Big]. \tag{3.15}$$

To sum up, using (3.7), (3.13), (3.11) and (3.15) we obtain that $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n] \\
&\le (1-\eta)^{-1}\Big(1 - \frac{K_2 N^2}{K_1 \delta n (1-\eta)}\Big)^{-1}\Big(\min_{x \in I_M} \mathtt{P}_\omega^x[\Lambda_n]\Big)^{-1} \\
&\quad \times \Big(\frac{K_2 N^2}{K_1 \delta n (1-\eta)} + \eta + \max_{x \in I_M} \mathtt{P}_\omega^x[X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}] \\
&\qquad + \max_{x \in I_M} \mathtt{P}_\omega^x\Big[\max_{u \le \lfloor \delta n \rfloor} |X_{n-\lfloor \delta n \rfloor} - X_{n-u}| > \delta' \sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}\Big]\Big).
\end{aligned} \tag{3.16}$$

Our goal is now to calculate the lim sup as $n \to \infty$ of both sides of (3.16). Let us first compute $\limsup_{n\to\infty} (\mathtt{P}_\omega^x[\Lambda_n])^{-1}$ for $x \in I_M$. We have by definition of Z^n

$$\mathtt{P}_\omega^x[\Lambda_n] = \mathtt{P}_\omega^x[X_m > 0, 0 \le m \le n] = \mathtt{P}_\omega^x\big[Z_t^n > 0, t \in [0,1]\big].$$

Thus, by the UCLT, we have

$$\lim_{n\to\infty} \mathtt{P}_\omega^x\big[Z_t^n > 0, t \in [0,1]\big] = P^\varepsilon\Big[\min_{0 \le t \le 1} W(t) > 0\Big]$$

with W a standard Brownian motion. Using the reflexion principle (see Chap. III, Prop. 3.7 in Revuz and Yor, 1999), we obtain

$$P^\varepsilon\Big[\min_{0 \le t \le 1} W(t) > 0\Big] = P^0[|W(1)| < \varepsilon] = \int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx.$$

So, we obtain

$$\lim_{n\to\infty} \min_{x \in I_M} (\mathtt{P}_\omega^x[\Lambda_n])^{-1} = \Big(\int_{-\varepsilon}^{\varepsilon} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx\Big)^{-1} = \Big(\frac{2\varepsilon}{\sqrt{2\pi}} + o(\varepsilon)\Big)^{-1} \tag{3.17}$$

as $\varepsilon \to 0$.

Now, let us bound $\limsup_{n\to\infty} \mathtt{P}_\omega^x[X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}]$ from above. We have

$$\begin{aligned}
&\mathtt{P}_\omega^x[X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n}, \Lambda_{n-\lfloor \delta n \rfloor}] \\
&\le \mathtt{P}_\omega^x\Big[X_{n-\lfloor \delta n \rfloor} > (1-\delta')\sqrt{n - \lfloor \delta n \rfloor}, \Lambda_{n-\lfloor \delta n \rfloor}\Big] \\
&= \mathtt{P}_\omega^x\Big[Z_1^{n-\lfloor \delta n \rfloor} > (1-\delta'), Z_t^{n-\lfloor \delta n \rfloor} > 0, t \in [0,1]\Big].
\end{aligned}$$

As $\delta < 1$ and $x \in I_M$, we have by the UCLT,

$$\begin{aligned}
&\lim_{n\to\infty} \mathtt{P}_\omega^x\Big[Z_1^{n-\lfloor \delta n \rfloor} > (1-\delta'), Z_t^{n-\lfloor \delta n \rfloor} > 0, t \in [0,1]\Big] \\
&\quad = P^{\frac{\varepsilon}{\sqrt{1-\delta}}}\Big[W(1) > (1-\delta'), \min_{0 \le t \le 1} W(t) > 0\Big].
\end{aligned}$$

Abbreviate $\varepsilon' := \varepsilon(1-\delta)^{-\frac{1}{2}}$ and let us compute $P^{\varepsilon'}\Big[W(1) > (1-\delta'), \min_{0\le t\le 1} W(t) > 0\Big]$ for sufficiently small ε. By the reflexion principle for Brownian motion, we have

$$\begin{aligned}
&P^{\varepsilon'}\Big[W(1) > (1-\delta'), \min_{0\le t\le 1} W(t) > 0\Big]\\
&= P^{\varepsilon'}\Big[W(1) > (1-\delta')\Big] - P^{\varepsilon'}\Big[W(1) < -(1-\delta')\Big]\\
&= P\Big[W(1) > 1-(\delta'+\varepsilon')\Big] - P\Big[W(1) < -1+(\delta'-\varepsilon')\Big]\\
&= \frac{1}{\sqrt{2\pi}}\int_{1-(\delta'+\varepsilon')}^{1-(\delta'-\varepsilon')} e^{-\frac{x^2}{2}}\,dx.
\end{aligned}$$

Therefore, we obtain, as $\varepsilon \to 0$

$$\begin{aligned}
&\limsup_{n\to\infty}\max_{x\in I_M} \mathtt{P}^x_\omega[X_{n-\lfloor\delta n\rfloor} > (1-\delta')\sqrt{n}, \Lambda_{n-\lfloor\delta n\rfloor}]\\
&\le \frac{1}{\sqrt{2\pi}}\int_{1-(\delta'+\varepsilon')}^{1-(\delta'-\varepsilon')} e^{-\frac{x^2}{2}}\,dx = \frac{2\varepsilon}{\sqrt{2\pi(1-\delta)}}e^{-\frac{1}{2}} + o(\varepsilon).
\end{aligned}\tag{3.18}$$

Then, let us bound $\limsup_{n\to\infty}\mathtt{P}^x_\omega\Big[\max_{u\le\lfloor\delta n\rfloor}|X_{n-\lfloor\delta n\rfloor} - X_{n-u}| > \delta'\sqrt{n}, \Lambda_{n-\lfloor\delta n\rfloor}\Big]$ from above in (3.16) for $x \in I_M$. First, observe that

$$\begin{aligned}
&\mathtt{P}^x_\omega\Big[\max_{u\le\lfloor\delta n\rfloor}|X_{n-\lfloor\delta n\rfloor} - X_{n-u}| \ge \delta'\sqrt{n}, \Lambda_{n-\lfloor\delta n\rfloor}\Big]\\
&\le \mathtt{P}^x_\omega\Big[\max_{u\le\lfloor\delta n\rfloor}|X_{n-\lfloor\delta n\rfloor} - X_{n-u}| \ge \delta'\sqrt{n}\Big]
\end{aligned}$$

and

$$\begin{aligned}
&\mathtt{P}^x_\omega\Big[\max_{u\le\lfloor\delta n\rfloor}|X_{n-\lfloor\delta n\rfloor} - X_{n-u}| \ge \delta'\sqrt{n}\Big]\\
&= \mathtt{P}^x_\omega\Big[\max_{n-\lfloor\delta n\rfloor\le k\le n}|X_k - X_{n-\lfloor\delta n\rfloor}| \ge \delta'\sqrt{n}\Big]\\
&\le \mathtt{P}^x_\omega\Big[\max_{n-\lfloor\delta n\rfloor\le k\le n}(X_k - \min_{n-\lfloor\delta n\rfloor\le l\le k} X_l) \ge \delta'\sqrt{n}\Big]\\
&\qquad + \mathtt{P}^x_\omega\Big[\min_{n-\lfloor\delta n\rfloor\le k\le n}(X_k - \max_{n-\lfloor\delta n\rfloor\le l\le k} X_l) \le -\delta'\sqrt{n}\Big]\\
&\le \mathtt{P}^x_\omega\Big[\max_{1-\delta\le t\le 1}(Z^n_t - \min_{1-\delta\le s\le t} Z^n_s) \ge \delta'\Big] + \mathtt{P}^x_\omega\Big[\min_{1-\delta\le t\le 1}(Z^n_t - \max_{1-\delta\le s\le t} Z^n_s) \le -\delta'\Big].
\end{aligned}$$

Using the UCLT, we obtain

$$\begin{aligned}
&\lim_{n\to\infty}\mathtt{P}^x_\omega\Big[\max_{1-\delta\le t\le 1}(Z^n_t - \min_{1-\delta\le s\le t} Z^n_s) \ge \delta'\Big]\\
&\qquad = P^\varepsilon\Big[\max_{1-\delta\le t\le 1}\big(W(t) - \min_{1-\delta\le s\le t} W(s)\big) \ge \delta'\Big]
\end{aligned}\tag{3.19}$$

and

$$\begin{aligned}
&\lim_{n\to\infty}\mathtt{P}^x_\omega\Big[\min_{1-\delta\le t\le 1}(Z^n_t - \max_{1-\delta\le s\le t} Z^n_s) \le -\delta'\Big]\\
&\qquad = P^\varepsilon\Big[\min_{1-\delta\le t\le 1}\big(W(t) - \max_{1-\delta\le s\le t} W(s)\big) \le -\delta'\Big].
\end{aligned}\tag{3.20}$$

Observe that the right-hand sides of (3.19) and (3.20) are equal since $(-W)$ is a Brownian motion. Thus, let us compute for example $P^\varepsilon[\max_{1-\delta\le t\le 1}(W(t) -$

$\min_{1-\delta\leq s\leq t} W(s)) \geq \delta']$. First, by the Markov property and since the event is invariant by space shifts, we have

$$P^{\varepsilon}\Big[\max_{1-\delta\leq t\leq 1}\big(W(t)-\min_{1-\delta\leq s\leq t}W(s)\big)\geq \delta'\Big] = P\Big[\max_{0\leq t\leq \delta}\big(W(t)-\min_{0\leq s\leq t}W(s)\big)\geq \delta'\Big].$$

By Lévy's Theorem (cf. Revuz and Yor, 1999, Chapter VI, Theorem 2.3), we have

$$P\Big[\max_{0\leq t\leq \delta}\big(W(t)-\min_{0\leq s\leq t}W(s)\big)\geq \delta'\Big] = P\Big[\max_{0\leq t\leq \delta}|W(t)|\geq \delta'\Big].$$

Then, by the reflexion principle, we have

$$P\Big[\max_{0\leq t\leq \delta}|W(t)|\geq \delta'\Big] \leq 2P\Big[\max_{0\leq t\leq \delta}W(t)\geq \delta'\Big] = 4P[W(\delta)\geq \delta'].$$

Using an estimate on the tail of the Gaussian law (cf. Mörters and Peres, 2010, Appendix II, Lemma 3.1) we obtain

$$P\Big[\max_{0\leq t\leq \delta}|W(t)|\geq \delta'\Big] \leq \frac{4\sqrt{\delta}}{\delta'\sqrt{2\pi}}\exp\Big\{-\frac{(\delta')^2}{2\delta}\Big\}.$$

Thus, we find

$$\limsup_{n\to\infty}\max_{x\in I_M}\mathtt{P}^x_\omega\Big[\max_{u\leq\lfloor\delta n\rfloor}|X_{n-\lfloor\delta n\rfloor}-X_{n-u}|>\delta'\sqrt{n},\Lambda_{n-\lfloor\delta n\rfloor}\Big] \leq \frac{8\sqrt{\delta}}{\delta'\sqrt{2\pi}}\exp\Big\{-\frac{(\delta')^2}{2\delta}\Big\}. \tag{3.21}$$

Finally, combining (3.16), (3.17), (3.18) and (3.21), we obtain

$$\begin{aligned}&\limsup_{n\to\infty}\mathtt{P}_\omega[X_n>\sqrt{n}\mid\Lambda_n]\\ &\leq (1-\eta)^{-1}\Big(1-\frac{K_2\varepsilon^2}{K_1\delta(1-\eta)}\Big)^{-1}\Big(\frac{2\varepsilon}{\sqrt{2\pi}}+o(\varepsilon)\Big)^{-1}\\ &\times\Big(\frac{K_2\varepsilon^2}{K_1\delta(1-\eta)}+\eta+\frac{2\varepsilon}{\sqrt{2\pi(1-\delta)}}e^{-\frac{1}{2}}+o(\varepsilon)+\frac{8\sqrt{\delta}}{\delta'\sqrt{2\pi}}\exp\Big\{-\frac{(\delta')^2}{2\delta}\Big\}\Big).\end{aligned} \tag{3.22}$$

Next, let us bound the quantity $\mathtt{P}_\omega[X_n>\sqrt{n}\mid\Lambda_n]$ from below. Using (3.5), we write

$$\begin{aligned}\mathtt{P}_\omega[X_n>\sqrt{n}\mid\Lambda_n] \geq \frac{\mathtt{P}_\omega[A_{\varepsilon,n}]}{\mathtt{P}_\omega[\Lambda_n]}&\mathtt{P}_\omega[X_{\tau_E}\in I_M\mid A_{\varepsilon,n}]\mathtt{P}_\omega[\tau_E\leq\delta n\mid X_{\tau_E}\in I_M,A_{\varepsilon,n}]\\ &\times\mathtt{P}_\omega[X_n>\sqrt{n},\Lambda_n\mid X_{\tau_E}\in I_M,A_{\varepsilon,n},\tau_E\leq\delta n].\end{aligned} \tag{3.23}$$

As we have already treated the terms $\mathtt{P}_\omega[\tau_E\leq\delta n\mid X_{\tau_E}\in I_M,A_{\varepsilon,n}]$ and $\mathtt{P}_\omega[X_{\tau_E}\in I_M\mid A_{\varepsilon,n}]$ in (3.9) and Lemma 2.1 respectively, we just need to bound the terms $\mathtt{P}_\omega[A_{\varepsilon,n}]/\mathtt{P}_\omega[\Lambda_n]$ and $\mathtt{P}_\omega[X_n>\sqrt{n},\Lambda_n\mid X_{\tau_E}\in I_M,A_{\varepsilon,n},\tau_E\leq\delta n]$ from below.

Let us start with the term $\mathtt{P}_\omega[A_{\varepsilon,n}]/\mathtt{P}_\omega[\Lambda_n]$. Observe that

$$\begin{aligned}\mathtt{P}_\omega[\Lambda_n] &= \mathtt{P}_\omega[\Lambda_n,\tau_E\leq\delta n]+\mathtt{P}_\omega[\Lambda_n,\tau_E>\delta n]\\ &= \mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E\leq\delta n]+\mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E>\delta n]+\mathtt{P}_\omega[\Lambda_n,A^c_{\varepsilon,n},\tau_E>\delta n]\\ &\leq \mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E\leq\delta n]+\mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E>\delta n]+\mathtt{P}_\omega[\Lambda_n,A^c_{\varepsilon,n}]\\ &\leq \mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E\leq\delta n,X_{\tau_E}\in I_M]+\mathtt{P}_\omega[\Lambda_n,A_{\varepsilon,n},\tau_E>\delta n,X_{\tau_E}\in I_M]\\ &\quad +2\mathtt{P}_\omega[X_{\tau_E}\notin I_M,A_{\varepsilon,n}]+\mathtt{P}_\omega[\Lambda_n,A^c_{\varepsilon,n}]\end{aligned}$$

$$\leq \mathtt{P}_\omega[A_{\varepsilon,n}]\Big[\mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, \tau_E \leq \delta n, X_{\tau_E} \in I_M] + 2\mathtt{P}_\omega[X_{\tau_E} \notin I_M \mid A_{\varepsilon,n}] + \mathtt{P}_\omega[\tau_E > \delta n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}] + \frac{\mathtt{P}_\omega[\Lambda_n, A^c_{\varepsilon,n}]}{\mathtt{P}_\omega[A_{\varepsilon,n}]}\Big]. \tag{3.24}$$

From the first equality in (3.12) we obtain

$$\mathtt{P}_\omega[\Lambda_n \mid A_{\varepsilon,n}, X_{\tau_E} \in I_M, \tau_E \leq \delta n] \leq \max_{x\in I_M} \max_{u\leq\lfloor\delta n\rfloor} \mathtt{P}^x_\omega[\Lambda_{n-u}] \leq \max_{x\in I_M} \mathtt{P}^x_\omega[\Lambda_{n-\lfloor\delta n\rfloor}]. \tag{3.25}$$

Now, let us treat the term $\mathtt{P}_\omega[\Lambda_n, A^c_{\varepsilon,n}]$. First, observe that by definition of $A_{\varepsilon,n}$ we have

$$\mathtt{P}_\omega[\Lambda_n, A^c_{\varepsilon,n}] \leq \mathtt{P}_\omega[\tau^+_B \wedge \tau_E > n].$$

Then, by Chebyshev's inequality we obtain

$$\mathtt{P}_\omega[\tau^+_B \wedge \tau_E > n] \leq \frac{\mathtt{E}_\omega[\tau^+_B \wedge \tau_E]}{n}.$$

By Lemma 2.3, we obtain

$$\mathtt{P}_\omega[\tau^+_B \wedge \tau_E > n] \leq \frac{K_2 N}{n}. \tag{3.26}$$

Thus, by (3.11), (3.24), (3.25), (3.26) and Lemmas 2.1 and 2.2 we obtain

$$\frac{\mathtt{P}_\omega[A_{\varepsilon,n}]}{\mathtt{P}_\omega[\Lambda_n]} \geq \Big(\max_{x\in I_M} \mathtt{P}^x_\omega[\Lambda_{n-\lfloor\delta n\rfloor}] + 2\eta + \frac{K_2N^2}{K_1\delta n(1-\eta)} + \frac{K_2N^2}{K_1 n}\Big)^{-1}. \tag{3.27}$$

Let us find a lower bound for $\mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \leq \delta n]$ in (3.23). Since $A_{\varepsilon,n} \in \mathcal{F}_{\tau_E}$ we have by the Markov property,

$$\begin{aligned}
&\mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} \in I_M, A_{\varepsilon,n}, \tau_E \leq \delta n]\\
&\qquad \geq \min_{x\in I_M} \min_{u\leq\lfloor\delta n\rfloor} \mathtt{P}_\omega[X_n > \sqrt{n}, \Lambda_n \mid X_{\tau_E} = x, \tau_E = u, A_{\varepsilon,n}]\\
&\qquad = \min_{x\in I_M} \min_{u\leq\lfloor\delta n\rfloor} \mathtt{P}^x_\omega[X_{n-u} > \sqrt{n}, X_k > 0, 1 \leq k \leq n-u]\\
&\qquad = \min_{x\in I_M} \min_{u\leq\lfloor\delta n\rfloor} \mathtt{P}^x_\omega[X_{n-u} > \sqrt{n}, \Lambda_{n-u}].
\end{aligned} \tag{3.28}$$

For $x \in I_M$ and $u \leq \lfloor\delta n\rfloor$ we write

$$\begin{aligned}
&\mathtt{P}^x_\omega[X_{n-u} > \sqrt{n}, \Lambda_{n-u}]\\
&\geq \mathtt{P}^x_\omega[X_n > (1+\delta')\sqrt{n}, |X_n - X_{n-u}| \leq \delta'\sqrt{n}, \Lambda_{n-u}]\\
&\geq \mathtt{P}^x_\omega\Big[X_n > (1+\delta')\sqrt{n}, \max_{u\leq\lfloor\delta n\rfloor} |X_n - X_{n-u}| \leq \delta'\sqrt{n}, \Lambda_{n-u}\Big]\\
&\geq \mathtt{P}^x_\omega\Big[X_n > (1+\delta')\sqrt{n}, \max_{u\leq\lfloor\delta n\rfloor} |X_n - X_{n-u}| \leq \delta'\sqrt{n}, \Lambda_n\Big]\\
&\geq \mathtt{P}^x_\omega[X_n > (1+\delta')\sqrt{n}, \Lambda_n] - \mathtt{P}^x_\omega\Big[\max_{u\leq\lfloor\delta n\rfloor} |X_n - X_{n-u}| > \delta'\sqrt{n}\Big].
\end{aligned} \tag{3.29}$$

To sum up, by (3.23), (3.27), (3.29), (3.9), (3.11) and Lemma 2.1 we obtain that $\mathbb{P}$-a.s.,

$$
\begin{aligned}
\mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n] \geq (1-\eta)\Big(1 - \frac{K_2 N^2}{K_1 \delta n(1-\eta)}\Big) \\
\times \Big(\max_{x\in I_M} \mathtt{P}^x_\omega[\Lambda_{n-\lfloor \delta n\rfloor}] + 2\eta + \frac{K_2 N^2}{K_1 \delta n(1-\eta)} + \frac{K_2 N^2}{K_1 n}\Big)^{-1} \\
\times \Big(\min_{x\in I_M} \mathtt{P}^x_\omega[X_n > (1+\delta')\sqrt{n}, \Lambda_n] \\
- \max_{x\in I_M} \mathtt{P}^x_\omega\Big[\max_{u\leq \lfloor \delta n\rfloor} |X_n - X_{n-u}| > \delta'\sqrt{n}\Big]\Big). \quad (3.30)
\end{aligned}
$$

Let us now compute $\liminf_{n\to\infty}$ of both sides of (3.30). First, by (3.17) we have

$$
\lim_{n\to\infty} \max_{x\in I_M} \mathtt{P}^x_\omega[\Lambda_{n-\lfloor \delta n\rfloor}] = \frac{2\varepsilon}{\sqrt{2\pi(1-\delta)}} + o(\varepsilon) \quad (3.31)
$$

as $\varepsilon \to 0$. Then, by the UCLT and after some elementary computations similar to those which led to (3.18) and (3.21) we obtain

$$
\begin{aligned}
\lim_{n\to\infty} \min_{x\in I_M} \mathtt{P}^x_\omega[X_n > (1+\delta')\sqrt{n}, \Lambda_n] &= \frac{1}{\sqrt{2\pi}} \int_{1+(\delta'-\varepsilon)}^{1+(\delta'+\varepsilon)} e^{-\frac{x^2}{2}}\, dx \\
&= \frac{2\varepsilon}{\sqrt{2\pi}} e^{-\frac{1}{2}} + o(\varepsilon) \quad (3.32)
\end{aligned}
$$

as $\varepsilon \to 0$, and

$$
\limsup_{n\to\infty} \max_{x\in I_M} \mathtt{P}^x_\omega\Big[\max_{u\leq \lfloor \delta n\rfloor} |X_n - X_{n-u}| > \delta'\sqrt{n}\Big] \leq \frac{8\sqrt{\delta}}{\delta'\sqrt{2\pi}} \exp\Big\{-\frac{(\delta')^2}{2\delta}\Big\}. \quad (3.33)
$$

Thus, combining (3.30) with (3.31), (3.32) and (3.33) leads to

$$
\begin{aligned}
\liminf_{n\to\infty} \mathtt{P}_\omega[X_n > \sqrt{n} \mid \Lambda_n] \geq (1-\eta)\Big(1 - \frac{K_2\varepsilon^2}{K_1\delta(1-\eta)}\Big) \\
\times \Big(\frac{2\varepsilon}{\sqrt{2\pi(1-\delta)}} + o(\varepsilon) + 2\eta + \frac{K_2\varepsilon^2}{K_1\delta(1-\eta)} + \frac{K_2\varepsilon^2}{K_1}\Big)^{-1} \\
\times \Big(\frac{2\varepsilon}{\sqrt{2\pi}} e^{-\frac{1}{2}} + o(\varepsilon) - \frac{8\sqrt{\delta}}{\delta'\sqrt{2\pi}} \exp\Big\{-\frac{(\delta')^2}{2\delta}\Big\}\Big). \quad (3.34)
\end{aligned}
$$

Now take $\eta = \varepsilon^2$, $\delta = \varepsilon^{\frac{1}{2}}$ and $\delta' = \varepsilon^{\frac{1}{8}}$ and let $\varepsilon \to 0$ in (3.22) and (3.34) to prove (3.4). □

The next step is to show the weak convergence of $(\mathtt{P}_\omega[Z^n_t \in \cdot \mid \Lambda_n], n \geq 1)$ for all $t \in (0,1)$. We start by recalling the transition density function from $(0,0)$ to (t,y) of the Brownian meander (see Iglehart, 1974):

$$
q(t,y) = t^{-\frac{3}{2}} y \exp\Big(-\frac{y^2}{2t}\Big) \tilde{N}(y(1-t)^{-\frac{1}{2}}) \quad (3.35)
$$

for $y > 0$, $0 < t \leq 1$, where

$$
\tilde{N}(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}}\, du
$$

for $x \geq 0$. We will prove the following

Proposition 3.4. *We have $\mathbb{P}$-a.s., for all $x \geq 0$ and $0 < t < 1$,*

$$\lim_{n\to\infty} \mathtt{P}_\omega[Z_t^n \leq x \mid \Lambda_n] = \int_0^x q(t,y)dy. \tag{3.36}$$

Proof. First notice the following. For all $\tilde{\varepsilon} > 0$ we have

$$\begin{aligned}
&\mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x - \tilde{\varepsilon} \mid \Lambda_n\Big] \\
&\leq \mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x - \tilde{\varepsilon}, \Big|Z^n_{\frac{\lfloor nt\rfloor+1}{n}} - Z^n_{\frac{\lfloor nt\rfloor}{n}}\Big| \leq \tilde{\varepsilon} \mid \Lambda_n\Big] + \mathtt{P}_\omega\Big[\Big|Z^n_{\frac{\lfloor nt\rfloor+1}{n}} - Z^n_{\frac{\lfloor nt\rfloor}{n}}\Big| > \tilde{\varepsilon} \mid \Lambda_n\Big] \\
&\leq \mathtt{P}_\omega[Z_t^n \leq x \mid \Lambda_n] + \mathtt{P}_\omega[\Lambda_n]^{-1}\mathtt{P}_\omega[|X_{\lfloor nt\rfloor+1} - X_{\lfloor nt\rfloor}| > \tilde{\varepsilon}\sqrt{n}].
\end{aligned} \tag{3.37}$$

By (3.13), (3.17), Lemma 2.2 and Condition K, the second term of (3.37) tends to 0 as $n \to \infty$. Hence, assuming that the following limits exist, we deduce that

$$\begin{aligned}
\lim_{n\to\infty} \mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x - \tilde{\varepsilon} \mid \Lambda_n\Big] &\leq \lim_{n\to\infty} \mathtt{P}_\omega[Z_t^n \leq x \mid \Lambda_n] \\
&\leq \lim_{n\to\infty} \mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x + \tilde{\varepsilon} \mid \Lambda_n\Big]
\end{aligned} \tag{3.38}$$

for all $\tilde{\varepsilon} > 0$. Now, suppose that we have for all $x \geq 0$ and $0 < t < 1$,

$$\lim_{n\to\infty} \mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x \mid \Lambda_n\Big] = \int_0^x q(t,y)dy. \tag{3.39}$$

Combining (3.38) and (3.39), we obtain (3.36) since the limit distribution $q(t,x)$ is absolutely continuous. Our goal is now to show (3.39). For this, observe that

$$\begin{aligned}
&\mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \leq x \mid \Lambda_n\Big] \\
&= \frac{1}{\mathtt{P}_\omega[\Lambda_n]} \int_0^{\frac{xn^{1/2}}{\lfloor nt\rfloor^{1/2}}} \mathtt{P}_\omega[Z_1^{\lfloor nt\rfloor} \in dy, \Lambda_{\lfloor nt\rfloor}, X_k > 0, \lfloor nt\rfloor < k \leq n] \\
&= \frac{\mathtt{P}_\omega[\Lambda_{\lfloor nt\rfloor}]}{\mathtt{P}_\omega[\Lambda_n]} \int_0^{\frac{xn^{1/2}}{\lfloor nt\rfloor^{1/2}}} \mathtt{P}_\omega^{y\sqrt{\lfloor nt\rfloor}}\Big[Z_s^n > 0, 0 \leq s \leq 1 - \frac{\lfloor nt\rfloor}{n}\Big]\mathtt{P}_\omega[Z_1^{\lfloor nt\rfloor} \in dy \mid \Lambda_{\lfloor nt\rfloor}].
\end{aligned} \tag{3.40}$$

By (3.13), (3.27), (3.17), and (3.31) we have

$$\lim_{n\to\infty} \frac{\mathtt{P}_\omega[\Lambda_{\lfloor nt\rfloor}]}{\mathtt{P}_\omega[\Lambda_n]} = t^{-\frac{1}{2}}. \tag{3.41}$$

Using part (v) of the UCLT and Dini's theorem on uniform convergence of non-decreasing sequences of continuous functions, we obtain

$$\begin{aligned}
\lim_{n\to\infty} \mathtt{P}_\omega^{z\sqrt{\lfloor nt\rfloor}}\Big[Z_s^n > 0, 0 \leq s \leq 1 - \frac{\lfloor nt\rfloor}{n}\Big] &= P^z\Big[\min_{s\in[0,1-t]} W_s > 0\Big] \\
&= P[|W_{1-t}| < z] = \tilde{N}\Big(z\Big(\frac{t}{1-t}\Big)^{\frac{1}{2}}\Big)
\end{aligned}$$

uniformly in z on every compact set of $\mathbb{R}_+$. By Proposition 3.3, we have

$$\lim_{n\to\infty} \mathtt{P}_\omega[Z_1^{\lfloor nt\rfloor} \leq x \mid \Lambda_{\lfloor nt\rfloor}] = \int_0^x y\, e^{-\frac{y^2}{2}} dy.$$

Now, applying Lemma 2.18 of Iglehart, 1974 to (3.40), we obtain

$$\lim_{n\to\infty} \mathtt{P}_\omega[Z_t^n \le x \mid \Lambda_n] = \lim_{n\to\infty} \mathtt{P}_\omega\Big[Z^n_{\frac{\lfloor nt\rfloor}{n}} \le x \mid \Lambda_n\Big]$$
$$= \int_0^{xt^{-\frac{1}{2}}} t^{-\frac{1}{2}} \tilde{N}\Big(y\Big(\frac{t}{1-t}\Big)^{\frac{1}{2}}\Big) y e^{-\frac{y^2}{2}}\, dy.$$

Finally, make the change of variables $u = t^{\frac{1}{2}} y$ to obtain the desired result. □

We can now use Propositions 3.3 and 3.4 to easily check that (3.1), (3.2) and (3.3) of Theorems 3.1 and 3.2 are satisfied. This ends the proof of Theorem 1.1. □

4. Proof of Corollary 1.2

In this last part, for the sake of brevity, we will use the same notation for a real number x and its integer part $\lfloor x \rfloor$. The interpretation of the notation should be clear by the context where it is used. We also suppose without loss of generality that $\sigma = 1$. Let us first introduce some spaces needed in the proof of Corollary 1.2.

For any $l > 0$, let $C_0([0,l])$ the space of continuous functions f from $[0,l]$ into $\mathbb{R}$ such that $f(0) = 0$. We endow this space with the metric

$$d(f,g) = \sup_{x\in[0,l]} |f(x) - g(x)|$$

and the Borel sigma-field on $C_0([0,l])$ corresponding to the metric d.

Then, let $C_0(\mathbb{R}_+)$ the space of continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$ such that $f(0) = 0$. We endow this space with the metric

$$\mathtt{d}(f,g) = \sum_{n=1}^{\infty} 2^{-n+1} \min\{1, \sup_{x\in[0,n]} |f(x) - g(x)|\}$$

and the Borel sigma-field on $C_0(\mathbb{R}_+)$ corresponding to the metric $\mathtt{d}$. Next, let G be the set of functions of $C_0(\mathbb{R}_+)$ for which there exists x_0 (depending on f) such that $f(x_0) = 1$. Let us also define the set H as the set of functions of $C_0(\mathbb{R}_+)$ such that there exists $x_1 = x_1(f) = \min\{s > 0 : f(s) = 1\}$ and $f(x) = 1$ for all $x \ge x_1$; observe that G and H are closed subsets of $C_0(\mathbb{R}_+)$. We define the continuous map $\Psi : G \to H$ by

$$\Psi(f)(x) = \begin{cases} f(x) & \text{for } x \le x_1, \\ 1 & \text{for } x > x_1. \end{cases}$$

Now, Corollary 1.2 can be restated as follows: under the conditions of Theorem 1.1, we have $\mathbb{P}$-a.s., for all measurable $A \subset H$ such that $P[B_3(\cdot \wedge \varrho_1) \in \partial A] = 0$ and all $a \ge 0$,

$$\lim_{n\to\infty} \mathtt{P}_\omega[Y^n \in A, T_n \le a \mid \Lambda'_n] = P[B_3(\cdot\wedge\varrho_1) \in A, \varrho_1 \le a]. \tag{4.1}$$

Before proving this last statement, let us start by denoting $R = \{Y^n \in A\}$. We will bound the term $\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n]$ from above and below, for sufficiently large n.

We start with the upper bound. Let $M > 0$ be an integer and $I_M = [n, n+M]$. We obtain

$$\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n] = \frac{1}{\mathtt{P}_\omega[\Lambda'_n]} \mathtt{P}_\omega[R, T_n \le a, \Lambda'_n]$$

$$= \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\Big(\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \in I_M] \\ + \mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \notin I_M]\Big) \\ \le \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \in I_M] + \mathtt{P}_\omega[X_{\hat\tau_n} \notin I_M \mid \Lambda'_n] \tag{4.2}$$

for all sufficiently large n. The second term of the right-hand side of (4.2) can be treated easily. Indeed, by the same method we used to prove Lemma 2.1, we can show that, $\mathbb{P}$-a.s., for all $\eta > 0$, there exists $M > 0$ such that

$$\mathtt{P}_\omega[X_{\hat\tau_n} \notin I_M \mid \Lambda'_n] \le \eta \tag{4.3}$$

for all $n \ge 1$. Let $c > 2a$ and observe that $R \cap \{T_n \le a\} \in \mathcal{F}_{\hat\tau_n}$, where $\mathcal{F}_{\hat\tau_n}$ is the sigma-field generated by X until time $\hat\tau_n$. For the first term of the right-hand side of (4.2), we have by the Markov property

$$\frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \in I_M] \\ = \sum_{u=0}^{M} \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} = n+u] \\ = \sum_{u=0}^{M} \frac{\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}{\mathtt{P}_\omega[\Lambda'_n]\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} = n+u] \\ \le \sum_{u=0}^{M} \frac{1}{\mathtt{P}_\omega[\Lambda'_n]\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}\mathtt{P}_\omega[R, T_n \le a, \Lambda_{(c-a)n^2}, X_{\hat\tau_n} = n+u].$$

Next, let us define the event $E = \{X_1 > 0, \dots, X_{\hat\tau_n} > 0, \dots, X_{\hat\tau_n+(c-a)n^2} > 0\}$. Using the Markov property, we can write

$$\mathtt{P}_\omega[E] \le \sum_{v=0}^{M} \mathtt{P}_\omega[X_1 > 0, \dots, X_{\hat\tau_n} = n+v, \dots, X_{\hat\tau_n+(c-a)n^2} > 0] + \mathtt{P}_\omega[X_{\hat\tau_n} \notin I_M, \Lambda'_n] \\ = \sum_{v=0}^{M} \mathtt{P}_\omega[\Lambda'_n, X_{\hat\tau_n} = n+v]\mathtt{P}_\omega^{n+v}[\Lambda_{(c-a)n^2}] + \mathtt{P}_\omega[X_{\hat\tau_n} \notin I_M \mid \Lambda'_n]\mathtt{P}_\omega[\Lambda'_n].$$

But, by the UCLT, we have for all $\varepsilon > 0$ that uniformly in $v \in [0, M]$ and $u \in [0, M]$,

$$\Big|\mathtt{P}_\omega^{n+v}[\Lambda_{(c-a)n^2}] - \mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]\Big| \le \varepsilon \tag{4.4}$$

for all n sufficiently large. Therefore, we obtain for all $u \in [0, M]$,

$$\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]\mathtt{P}_\omega[\Lambda'_n] \ge \mathtt{P}_\omega[E] - (\varepsilon + \eta)\mathtt{P}_\omega[\Lambda'_n] \tag{4.5}$$

for all n sufficiently large. Now, let us bound the first term of the right-hand side of (4.5) from below. Fix some $\delta > 0$. We write

$$\mathtt{P}_\omega[E] \ge \mathtt{P}_\omega[E, \hat\tau_n \le (a+\delta)n^2] \\ \ge \mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}, \hat\tau_n \le (a+\delta)n^2] \\ \ge \mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}]\mathtt{P}_\omega[\hat\tau_n \le (a+\delta)n^2 \mid \Lambda_{((c+\delta)n^2+3)}]. \tag{4.6}$$

Finally, by (4.2), (4.3), (4.5) and (4.6) we obtain $\mathbb{P}$-a.s.,

$$\begin{aligned}
&\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n] \\
&\le \frac{(\mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}])^{-1}\mathtt{P}_\omega[\Lambda_{(c-a)n^2}]\mathtt{P}_\omega[R, T_n \le a \mid \Lambda_{(c-a)n^2}]}{\mathtt{P}_\omega[\hat\tau_n \le (a+\delta)n^2 \mid \Lambda_{((c+\delta)n^2+3)}] - (\varepsilon+\eta)\mathtt{P}_\omega[\Lambda'_n](\mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}])^{-1}} + \eta
\end{aligned} \tag{4.7}$$

for all sufficiently large n.

We now estimate the term $\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n]$ from below. Let us write

$$\begin{aligned}
\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n] &= \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n] \\
&\ge \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \in I_M].
\end{aligned} \tag{4.8}$$

Then, we have by the Markov property

$$\begin{aligned}
&\frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} \in I_M] \\
&= \sum_{u=0}^{M} \frac{1}{\mathtt{P}_\omega[\Lambda'_n]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} = n+u] \\
&= \sum_{u=0}^{M} \frac{\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}{\mathtt{P}_\omega[\Lambda'_n]\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}\mathtt{P}_\omega[R, T_n \le a, \Lambda'_n, X_{\hat\tau_n} = n+u] \\
&\ge \sum_{u=0}^{M} \frac{1}{\mathtt{P}_\omega[\Lambda'_n]\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]}\mathtt{P}_\omega[R, T_n \le a, \Lambda_{cn^2}, X_{\hat\tau_n} = n+u].
\end{aligned} \tag{4.9}$$

Again using the Markov property, we can write

$$\begin{aligned}
\mathtt{P}_\omega[E] &\ge \sum_{v=0}^{M} \mathtt{P}_\omega[X_1 > 0, \dots, X_{\hat\tau_n} = n+v, \dots, X_{\hat\tau_n + (c-a)n^2} > 0] \\
&\ge \sum_{v=0}^{M} \mathtt{P}_\omega[\Lambda'_n, X_{\hat\tau_n} = n+v]\mathtt{P}_\omega^{n+v}[\Lambda_{(c-a)n^2}].
\end{aligned}$$

Using (4.4), we obtain for all $u \in [0, M]$,

$$\mathtt{P}_\omega^{n+u}[\Lambda_{(c-a)n^2}]\mathtt{P}_\omega[\Lambda'_n] \le \mathtt{P}_\omega[E] + \varepsilon\mathtt{P}_\omega[\Lambda'_n] \tag{4.10}$$

for all sufficiently large n. Then, as $\hat\tau_n \ge 1$, we have

$$\mathtt{P}_\omega[E] \le \mathtt{P}_\omega[\Lambda_{(c-a)n^2}]. \tag{4.11}$$

Finally, by (4.8), (4.9), (4.10) and (4.11), we obtain $\mathbb{P}$-a.s.,

$$\mathtt{P}_\omega[R, T_n \le a \mid \Lambda'_n] \ge \frac{(\mathtt{P}_\omega[\Lambda_{(c-a)n^2}])^{-1}\mathtt{P}_\omega[\Lambda_{cn^2}]\mathtt{P}_\omega[R, T_n \le a \mid \Lambda_{cn^2}]}{1 + \varepsilon\mathtt{P}_\omega[\Lambda'_n](\mathtt{P}_\omega[\Lambda_{(c-a)n^2}])^{-1}} \tag{4.12}$$

for all sufficiently large n.

Our intention is now to take the lim sup as $n \to \infty$ in (4.7). Before this, observe that by (3.13), (3.27), (3.17) and (3.31) we have for $\varepsilon \le 1$,

$$\lim_{n\to\infty} \frac{\mathtt{P}_\omega[\Lambda_{(c-a)n^2}]}{\mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}]} = \sqrt{\frac{c+\delta}{c-a}}, \tag{4.13}$$

$$\limsup_{n\to\infty} \frac{\mathtt{P}_\omega[\Lambda'_n]}{\mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}]} \leq \limsup_{n\to\infty} \frac{\mathtt{P}_\omega[A_{\varepsilon,n^2}]}{\mathtt{P}_\omega[\Lambda_{((c+\delta)n^2+3)}]} \leq \gamma_1\sqrt{c+\delta} \tag{4.14}$$

for some constant γ_1. By the usual scaling, from the Brownian meander W^+ on $[0,1]$ it is possible to define the Brownian meander W_t^+ on any finite interval $[0,t]$: $W_t^+(\cdot) := \sqrt{t}W^+(\cdot/t)$. Thus, Theorem 1.1 implies that

$$\lim_{n\to\infty} \mathtt{P}_\omega[\hat{\tau}_n \leq (a+\delta)n^2 \mid \Lambda_{((c+\delta)n^2+3)}] = P\Big[\sup_{0\leq s\leq(a+\delta)} W_{c+\delta}^+(s) \geq 1\Big]. \tag{4.15}$$

Denoting by $\mathcal{U}_a$ the measurable set of functions f in H such that $f(a)=1$ and by π_l the projection map from $C_0(\mathbb{R}_+)$ onto $C_0([0,l])$, we have

$$\begin{aligned}\mathtt{P}_\omega[R,T_n \leq a \mid \Lambda_{(c-a)n^2}] &= \mathtt{P}_\omega[Z^{n^2}_{\cdot\wedge T_n} \in A\cap\mathcal{U}_a \mid \Lambda_{(c-a)n^2}]\\ &= \mathtt{P}_\omega[Z^{n^2} \in \Psi^{-1}(A\cap\mathcal{U}_a) \mid \Lambda_{(c-a)n^2}]\\ &= \mathtt{P}_\omega[Z^{n^2}_{\cdot\wedge(c-a)} \in \pi_{c-a}(\Psi^{-1}(A\cap\mathcal{U}_a)) \mid \Lambda_{(c-a)n^2}].\end{aligned}$$

The next step is to show that

$$\begin{aligned}\lim_{n\to\infty} \mathtt{P}_\omega[Z^{n^2}_{\cdot\wedge(c-a)} &\in \pi_{c-a}(\Psi^{-1}(A\cap\mathcal{U}_a)) \mid \Lambda_{(c-a)n^2}]\\ &= P[W^+_{c-a} \in \pi_{c-a}(\Psi^{-1}(A\cap\mathcal{U}_a))],\end{aligned} \tag{4.16}$$

where W^+_{c-a} is the Brownian meander on $[0,c-a]$. As the law of the Brownian meander on $[0,c-a]$ is absolutely continuous with respect to the law of the three dimensional Bessel process B_3 on $[0,c-a]$ (see Imhof, 1984 section 4), to prove (4.16) we will show that

$$P[B_3(\cdot\wedge c-a) \in \partial\{\pi_{c-a}(\Psi^{-1}(A\cap\mathcal{U}_a))\}] = 0. \tag{4.17}$$

Observe that, as π_{c-a} is a projection, we have

$$\begin{aligned}&P[B_3(\cdot\wedge c-a) \in \partial\{\pi_{c-a}(\Psi^{-1}(A\cap\mathcal{U}_a))\}]\\ &\leq P[B_3(\cdot\wedge c-a) \in \pi_{c-a}\partial\{\Psi^{-1}(A\cap\mathcal{U}_a)\}] = P[B_3 \in \partial\{\Psi^{-1}(A\cap\mathcal{U}_a)\}].\end{aligned}$$

Now, as Ψ is a continuous map, we have

$$\begin{aligned}P[B_3 \in \partial\{\Psi^{-1}(A\cap\mathcal{U}_a)\}] &\leq P[B_3 \in \Psi^{-1}(\partial\{A\cap\mathcal{U}_a\})]\\ &\leq P[B_3 \in \Psi^{-1}(\partial A\cup\partial\mathcal{U}_a)]\\ &\leq P[B_3(\cdot\wedge\varrho_1) \in \partial A] + P[\varrho_1 = a].\end{aligned} \tag{4.18}$$

By hypothesis, $P[B_3(\cdot\wedge\varrho_1) \in \partial A] = 0$. As the law of ϱ_1 is absolutely continuous with respect to the Lebesgue measure (see Imhof, 1984 Theorem 4), we also have $P[\varrho_1 = a] = 0$. This proves (4.16).

Then, we want to take the lim inf as $n\to\infty$ in (4.12). Before this, notice that

$$\lim_{n\to\infty} \frac{\mathtt{P}_\omega[\Lambda_{cn^2}]}{\mathtt{P}_\omega[\Lambda_{(c-a)n^2}]} = \sqrt{\frac{c-a}{c}}, \tag{4.19}$$

$$\limsup_{n\to\infty} \frac{\mathtt{P}_\omega[\Lambda'_n]}{\mathtt{P}_\omega[\Lambda_{(c-a)n^2}]} \leq \gamma_2\sqrt{c-a} \tag{4.20}$$

for some constant γ_2. By the same argument we used to prove (4.16), we have

$$\lim_{n\to\infty} \mathtt{P}_\omega[R,T_n \leq a \mid \Lambda_{cn^2}] = P[W_c^+ \in \pi_c(\Psi^{-1}(A\cap\mathcal{U}_a))] \tag{4.21}$$

where W_c^+ is the Brownian meander on $[0,c]$. Then, define $V_l = \{W_l^+ \in \pi_l(\Psi^{-1}(A\cap \mathcal{U}_a))\}$ for $l \in \{c-a, c\}$. Combining (4.13), (4.14), (4.15), (4.16), (4.19), (4.20) and (4.21) we see that

$$\begin{aligned}\frac{P[V_c]\sqrt{\frac{c-a}{c}}}{1+\gamma_2\varepsilon\sqrt{c-a}} &\le \liminf_{n\to\infty} \mathtt{P}_\omega[R, T_n \le a \mid \Lambda_n'] \le \limsup_{n\to\infty} \mathtt{P}_\omega[R, T_n \le a \mid \Lambda_n'] \\ &\le \frac{P[V_{c-a}]\sqrt{\frac{c+\delta}{c-a}}}{P[\sup_{0\le s\le (a+\delta)} W_{c+\delta}^+(s)\ge 1] - \gamma_1(\varepsilon+\eta)\sqrt{c+\delta}} + \eta. \qquad (4.22)\end{aligned}$$

Now, take $\varepsilon = \eta = c^{-1}$ and $\delta = \sqrt{c}$ and let c tend to infinity. Since

$$P[W_l^+ \in \pi_l(\Psi^{-1}(A\cap\mathcal{U}_a))] = P[W_l^+(\cdot\wedge a) \in \pi_a(\Psi^{-1}(A\cap\mathcal{U}_a))],$$

we have by Lemma 11-1 of Biane and Yor, 1988

$$\lim_{c\to\infty} P[V_l] = P[B_3(\cdot\wedge\varrho_1)\in A, \varrho_1 \le a]$$

for $l \in \{c-a, c\}$.

The last thing we have to check to obtain (4.1) is that

$$\lim_{c\to\infty} P\Big[\sup_{0\le s\le (a+\delta)} W_{c+\delta}^+(s) < 1\Big] = 0. \qquad (4.23)$$

First, we start by noting that by scaling property

$$\begin{aligned}P\Big[\sup_{0\le s\le (a+\delta)} W_{c+\delta}^+(s) < 1\Big] &= P\Big[(c+\delta)^{\frac12} \sup_{0\le s\le (a+\delta)} W^+\Big(\frac{s}{c+\delta}\Big) < 1\Big] \\ &= P\Big[\sup_{0\le s\le \frac{a+\delta}{c+\delta}} W^+(s) < (c+\delta)^{-\frac12}\Big] \\ &\le P\Big[W^+\Big(\frac{a+\delta}{c+\delta}\Big) \le (c+\delta)^{-\frac12}\Big]\end{aligned}$$

where W^+ is a Brownian meander on $[0,1]$. This last term is easily computable using the transition density function from $(0,0)$ of W^+ given in (3.35). Let $u = \frac{a+\delta}{c+\delta}$,

$$P\Big[W^+\Big(\frac{a+\delta}{c+\delta}\Big) \le (c+\delta)^{-\frac12}\Big] = \int_0^{(c+\delta)^{-\frac12}} u^{-\frac32} x \exp\Big(-\frac{x^2}{2u}\Big)\tilde{N}(x(1-u)^{-\frac12})dx$$

Let us make the change of variable $y = (c+\delta)^{\frac12}x$ in the right-hand side integral. Then, we obtain

$$P\Big[W^+\Big(\frac{a+\delta}{c+\delta}\Big) \le (c+\delta)^{-\frac12}\Big] = \int_0^1 \frac{(c+\delta)^{\frac12}}{(a+\delta)^{\frac32}} y \exp\Big(-\frac{y^2}{2(a+\delta)}\Big)\tilde{N}(y(c-a)^{-\frac12})dy.$$

Now, making the change of variable $z = (c-a)^{\frac12}u$ in the following integral

$$\tilde{N}(y(c-a)^{-\frac12}) = \sqrt{\frac{2}{\pi}}\int_0^{y(c-a)^{-\frac12}} \exp\Big(-\frac{u^2}{2}\Big)du,$$

we obtain

$$\begin{aligned}&P\Big[W^+\Big(\frac{a+\delta}{c+\delta}\Big) \le (c+\delta)^{-\frac12}\Big] \\ &= \Big(\frac{c+\delta}{c-a}\Big)^{\frac12}\frac{1}{(a+\delta)^{\frac32}}\int_0^1\int_0^y y\exp\Big(-\frac{y^2}{2(a+\delta)}\Big)\exp\Big(-\frac{z^2}{2(c-a)}\Big)dz\,dy\end{aligned}$$

$$\leq \Big(\frac{c+\delta}{c-a}\Big)^{\frac{1}{2}} \frac{1}{(a+\delta)^{\frac{3}{2}}} \int_0^1 \int_0^y y \, dz \, dy$$
$$= \frac{1}{3}\Big(\frac{c+\delta}{c-a}\Big)^{\frac{1}{2}} \frac{1}{(a+\delta)^{\frac{3}{2}}}. \tag{4.24}$$

Taking $\delta = \sqrt{c}$ and letting $c \to \infty$ in (4.24), we obtain (4.23). This concludes the proof of (4.1). □

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References

B. Belkin. An invariance principle for conditioned recurrent random walk attracted to a stable law. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **21**, 45–64 (1972). MR0309200.

J. Bertoin and R. A. Doney. On conditioning a random walk to stay nonnegative. *Ann. Probab.* **22** (4), 2152–2167 (1994). MR1331218.

Ph. Biane and M. Yor. Quelques précisions sur le méandre brownien. *Bull. Sci. Math. (2)* **112** (1), 101–109 (1988). MR942801.

E. Bolthausen. On a functional central limit theorem for random walks conditioned to stay positive. *Ann. Probability* **4** (3), 480–485 (1976). MR0415702.

F. Caravenna. A local limit theorem for random walks conditioned to stay positive. *Probab. Theory Related Fields* **133** (4), 508–530 (2005). MR2197112.

F. Caravenna and L. Chaumont. Invariance principles for random walks conditioned to stay positive. *Ann. Inst. Henri Poincaré Probab. Stat.* **44** (1), 170–190 (2008). MR2451576.

F. Comets and S. Popov. Ballistic regime for random walks in random environment with unbounded jumps and Knudsen billiards. *Ann. Inst. Henri Poincaré Probab. Stat.* **48** (3), 721–744 (2012). MR2976561.

F. Comets, S. Popov, G. M. Schütz and M. Vachkovskaia. Quenched invariance principle for the Knudsen stochastic billiard in a random tube. *Ann. Probab.* **38** (3), 1019–1061 (2010a). MR2674993.

F. Comets, S. Popov, G. M. Schütz and Marina Vachkovskaia. Billiards in a general domain with random reflections. *Arch. Ration. Mech. Anal.* **191** (3), 497–537 (2009). MR2481068.

F. Comets, S. Popov, G. M. Schütz and Marina Vachkovskaia. Knudsen gas in a finite random tube: transport diffusion and first passage properties. *J. Stat. Phys.* **140** (5), 948–984 (2010b). MR2673342.

R. B. Cooper. *Introduction to queueing theory*. North-Holland Publishing Co., New York, second edition (1981). ISBN 0-444-00379-7. MR636094.

P. G. Doyle and J. L. Snell. *Random walks and electric networks*, volume 22 of *Carus Mathematical Monographs*. Mathematical Association of America, Washington, DC (1984). ISBN 0-88385-024-9. MR920811.

R. Durrett. Conditioned limit theorems for some null recurrent Markov processes. *Ann. Probab.* **6** (5), 798–828 (1978). MR503953.

C. Gallesco and S. Popov. Random walks with unbounded jumps among random conductances I: Uniform quenched CLT. *Electron. J. Probab.* **17**, no. 85, 22 (2012). MR2988400.

D. L. Iglehart. Functional central limit theorems for random walks conditioned to stay positive. *Ann. Probability* **2**, 608–619 (1974). MR0362499.

J.-P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.* **21** (3), 500–510 (1984). MR752015.

P. Mörters and Y. Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge (2010). ISBN 978-0-521-76018-8. With an appendix by Oded Schramm and Wendelin Werner. MR2604525.

D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition (1999). ISBN 3-540-64325-7. MR1725357.

V. A. Vatutin and V. Wachtel. Local probabilities for random walks conditioned to stay positive. *Probab. Theory Related Fields* **143** (1-2), 177–217 (2009). MR2449127.