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On a ternary coalescent process

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Abstract. We present a coalescent process where three particles merge at each coagulation step. Using a random walk representation, we prove duality with a fragmentation process, whose fragmentation law we specify explicitly. Furthermore, we give a second construction of the coalescent in terms of random binary forests and study asymptotic properties. Starting from N particles of unit mass, we obtain under an appropriate rescaling when N tends to infinity a well-known binary coalescent, the so-called standard additive coalescent.

1. Introduction

Generally speaking, a stochastic coalescent is a Markov process describing the coagulation of particles characterized by their size only. The rate at which particles merge depends just on the members involved. Conversely, fragmentation processes describe a Markovian evolution of particles which split independently into new particles (branching property). The goal of this paper is to study the stochastic coalescent with ternary coagulation kernel

$$\kappa(r,s,t) = r+s+t+3, \quad r,s,t>0,$$

to which we will simply refer to as ternary coalescent or ternary coalescent process. Here, three particles of sizes (masses) r,s,t coagulate into a new particle of size $r+s+t$ at rate $r+s+t+3$. Although at first glance, the kernel κ may look somewhat arbitrary (for example, it is not scale invariant), the corresponding process enjoys rather interesting properties. Similarly to the additive coalescent, that is the coalescent where two particles with masses s, t merge at rate $\tilde{\kappa}(s,t) = s+t$, the state chain of the ternary coalescent admits different representations. In the spirit of Bertoin (2001), we show how it can be obtained by looking at excursion intervals of a one-dimensional conditioned random walk. As a by-product of our

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representation, we establish duality with a fragmentation process via time-reversal. We stress that this is a unusual feature, because the branching property normally fails when time is reversed in a coalescent process. Section 7 of Bertoin (2006a) gives a brief overview over cases where such a duality relation has been proven. See also Chapter 5.5 in the lecture notes of Pitman (2006) for further discussions.

Using the same construction, we study asymptotic properties of the ternary coalescent starting from N particles of unit mass. Properly rescaled in space and time, we observe in the limit $N \to \infty$ the so-called standard additive coalescent, which has been obtained by Evans and Pitman (1998) as the weak limit $n \to \infty$ of the (binary) additive coalescent, started at time $-(1/2)\ln n$ with n atoms of size $1/n$. Here, the characterization of Bertoin (2000) of the dual fragmentation process connected to the standard additive coalescent by time-reversal plays a pivotal role. We emphasize that even though κ is a ternary coagulation kernel, we end up in the limit with a binary coagulation process.

We also highlight a second construction of the ternary coalescent involving random binary forests, following the ideas of Pitman (1999). In a final remark, we point out that this representation could instead be used to work out our results. Moreover, we outline a possible extension of the results to certain k-ary coalescent processes.

The rest of this paper is organized as follows. In the first section, we describe the semigroup of the ternary coalescent and derive some further properties. We finish this part by computing the one-dimensional statistics for the underlying state chain starting from an odd number of particles of unit mass. Its special form already hints at a connection to hitting times of a one-dimensional nearest neighbor random walk, which we elaborate in the next section. There we prove duality via time-reversal with a fragmentation process, using an explicit construction of the coalescent in terms of ladder epochs. In the third part, we turn our attention to random binary trees and find a second interpretation of the ternary coalescent which is based on random binary forests. Finally we use again the random walk representation to study asymptotic properties of the coalescent in the last section.

2. Some basic properties

Throughout this text, let

$$\mathbb{N} = \{1, 2, \ldots\}, \quad \mathbb{Z} = \{\ldots, -1, 0, 1, \ldots\}, \quad \mathbb{Z}_+ = \mathbb{N} \cup \{0\}.$$

The coalescent process will take values in the space of decreasing numerical sequences with finitely many non-zero terms

$$\mathcal{S}^{\downarrow} = \{\mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, s_k = 0 \text{ for } k \text{ sufficiently large}\}.$$

We may think of elements of a sequence $\mathbf{s} \in \mathcal{S}^{\downarrow}$ as (sizes of) atoms or particles and simply identify $\mathbf{s}$ with its non-zero components. If we write $\mathbf{s} = (s_1, \ldots, s_l)$, the non-zero components of $\mathbf{s}$ are precisely given by $s_1, \ldots, s_l$. If $\mathbf{s} = (s_1, s_2, \ldots) \in \mathcal{S}^{\downarrow}$ and $1 \leq i < j < k$, we use the notation $\mathbf{s}^{i \oplus j \oplus k}$ for the sequence in $\mathcal{S}^{\downarrow}$ obtained from $\mathbf{s}$ by merging its ith, jth and kth terms, that is one removes s_i, s_j, s_k and rearranges the remaining elements together with the sum $s_i + s_j + s_k$ in decreasing order.

Let us define the object of our interest. Recall the kernel κ from the introduction.

Definition 2.1. The ternary coalescent with values in $\mathcal{S}^{\downarrow}$ and kernel κ is a continuous time Markov process $\mathcal{X} = (\mathcal{X}(t),\, t \geq 0)$ with state space $\mathcal{S}^{\downarrow}_{r}$ for an appropriate subset $\mathcal{S}^{\downarrow}_{r}$ of $\mathcal{S}^{\downarrow}$, and jump rates

$$q(\mathbf{s}, \cdot) = \sum_{1 \leq i < j < k,\, s_k > 0} \kappa(s_i, s_j, s_k) \delta_{\mathbf{s}^{i \oplus j \oplus k}}.$$

This definition can be adapted in an obvious way to other coagulation kernels, leading to different stochastic coalescent models, for example the additive coalescent with kernel $\tilde{\kappa}(s,t) = s + t$.

Before looking at concrete realizations, we collect in this section some basic properties which can be read off from the kernel κ and the very definition of jump-hold processes of the above type. Denote by $\mathcal{X} = (\mathcal{X}(t),\, t \geq 0)$ the ternary coalescent, started from a finite configuration $\mathbf{r} = (r_1, \ldots, r_N) \in \mathcal{S}^{\downarrow}$, where $N = 2n + 1$, $n \in \mathbb{Z}_+$. We write $M = r_1 + \ldots + r_N$ for the total mass in the system. For every $k = 0, \ldots, n+1$, let T_k be the instant of the kth coagulation, with the convention $T_0 = 0$, $T_{n+1} = \infty$. The state chain or skeleton chain $\mathcal{X}'$ of the coalescent process is given by $\mathcal{X}'_k = \mathcal{X}(T_k)$, $k = 0, \ldots, n$. We use the expression $\#(t)$ for the number of particles at time t, whereas $J(t) = \max\{k \in \mathbb{Z}_+ : T_k \leq t\}$ stands for the number of jumps up to time t. Note that $\#(t) = N - 2J(t)$.

2.1. *State chain and semigroup.*

Proposition 2.2. *In the preceding notation, the following holds true.*

(1) *The sequence $\Delta_k = T_k - T_{k-1}$, $k = 1, \ldots, n$, of the waiting times between two coagulations is a sequence of independent exponential variables with respective parameters*

$$\alpha(k) = \frac{1}{2}(M + N + 2 - 2k)(N + 1 - 2k)(N - 2k).$$

In particular, the sequences $\{T_k\}_{0 \leq k \leq n}$ and $\{\mathcal{X}'_k\}_{0 \leq k \leq n}$ are independent.

(2) *The sequence $\{\mathcal{X}'_k\}_{0 \leq k \leq n}$ is a Markov chain with transition probabilities*

$$\mathbb{P}\left(\mathcal{X}'_{l+1} = \mathbf{s}^{i \oplus j \oplus k} \mid \mathcal{X}'_l = \mathbf{s}\right) = \frac{s_i + s_j + s_k + 3}{\alpha(l+1)},$$

where $0 \leq l < n$, $1 \leq i < j < k \leq N - 2l$, and $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$ is a generic finite configuration with total mass $s_1 + \ldots + s_{N-2l} = M$ such that $\mathbb{P}(\mathcal{X}'_l = \mathbf{s}) > 0$.

Proof: Let $0 \leq l < n$, and put $L = N - 2l$. By construction, the time Δ_{l+1} between the lth and the $(l+1)$th coagulation given $\mathcal{X}'_l = \mathbf{s} = (s_1, \ldots, s_L)$ is exponentially distributed with parameter

$$\begin{aligned}
&\sum_{1 \leq i < j < k \leq L} (s_i + s_j + s_k + 3) \\
&\quad = \; 3\binom{L}{3} + \frac{1}{6}\left(\sum_{i,j,k=1}^{L} (s_i + s_j + s_k) - 3\sum_{i=1}^{L} s_i - 3 \sum_{\substack{i,j=1,\\ i \neq j}}^{L} (2s_i + s_j) \right) \\
&\quad = \; \frac{1}{2}(M + L)(L - 1)(L - 2) = \alpha(l+1).
\end{aligned}$$

Therefore, the waiting times $\{\Delta_k\}_{1\le k\le n}$ do not depend on the states $\{\mathcal{X}'_k\}_{1\le k\le n}$. The rest follows from the construction of our process. □

We turn to a description of the semigroup. Recall that $\mathcal{X}$ starts from $\mathcal{X}(0) = \mathbf{r} = (r_1, \dots, r_N)$. In the following, Γ denotes the Gamma function.

Proposition 2.3. *In the notation above, consider a partition π of $\{1, \dots, N\}$ into $N - 2l$ (non-empty) blocks $B_1, \dots, B_{N-2l}$, each of odd cardinality. Denote by $\Lambda'_\pi(N-2l)$ the event that the $N-2l$ atoms of $\mathcal{X}'_l$ result from the coagulation of particles $\{r_i : i \in B_j\}$, $j = 1, \dots, N-2l$. Then, with $\mathbf{r}_{B_j} = \sum_{i\in B_j} r_i$,*

$$\mathbb{P}\left(\Lambda'_\pi(N-2l)\right) = \frac{l!}{\alpha(1)\cdots\alpha(l)} \prod_{j=1}^{N-2l} \frac{\Gamma\left((\mathbf{r}_{B_j} + |B_j| + 2)/2\right)(|B_j|-1)!}{\Gamma\left((\mathbf{r}_{B_j}+3)/2\right)((|B_j|-1)/2)!}.$$

Proof: The first coagulation involves three particles with labels in the block B_j with probability

$$\sum_{i<i'<i''\in B_j} \frac{r_i + r_{i'} + r_{i''} + 3}{\alpha(1)} = \frac{(\mathbf{r}_{B_j} + |B_j|)(|B_j|-1)(|B_j|-2)}{2\alpha(1)}.$$

Now consider an arbitrary sequence $(k_1, \dots, k_l)$ taking values in $\{1, \dots, N-2l\}$ such that for every $j = 1, \dots, N-2l$, $|\{i \le l : k_i = j\}| = (|B_j|-1)/2$. Using the Markov property of $\mathcal{X}'$, we see that the probability that for all $i = 1, \dots, l$, the ith coagulation affected only particles formed from initial particles with labels in B_{k_i} equals

$$\frac{1}{\alpha(1)\cdots\alpha(l)} \prod_{j=1}^{N-2l} \frac{\Gamma\left((\mathbf{r}_{B_j} + |B_j| + 2)/2\right)}{\Gamma\left((\mathbf{r}_{B_j}+3)/2\right)} (|B_j|-1)!\,.$$

Observe that the number of such sequences $(k_1, \dots, k_l)$ is

$$\binom{l}{(|B_1|-1)/2, \dots, (|B_{N-2l}|-1)/2} = \frac{l!}{((|B_1|-1)/2)!\cdots((|B_{N-2l}|-1)/2)!}.$$

This proves the statement. □

In the setting of the proposition, denote by $\Lambda_\pi(t)$ the event that $\mathcal{X}(t)$ has $N-2l$ atoms, each resulting from the merging of $\{r_i : i \in B_j\}$, $j = 1, \dots, N-2l$. Since the sequence of coagulation times and the skeleton chain $\mathcal{X}'$ are independent,

$$\mathbb{P}\left(\Lambda_\pi(t)\right) = \mathbb{P}\left(T_l \le t < T_{l+1},\, \Lambda'_\pi(N-2l)\right) = \mathbb{P}\left(\#(t) = N-2l\right)\mathbb{P}\left(\Lambda'_\pi(N-2l)\right).$$

In particular, the semigroup of $\mathcal{X}$ is described by the preceding proposition and the distribution of the number of particles at time t, which is computed in the following lemma.

Lemma 2.4. *In the notation above, for $l = 0, \dots, n$ and $t \ge 0$,*

$$\mathbb{P}\left(\#(t) = N-2l\right) = \sum_{j=1}^{l+1} \frac{\alpha(j)e^{-\alpha(j)t}}{\alpha(l+1)} \prod_{k=1,k\ne j}^{l+1} \frac{\alpha(k)}{\alpha(k)-\alpha(j)}.$$

Proof: We use

$$\mathbb{P}\left(\#(t) = N-2l\right) = \mathbb{P}\left(T_{l+1} > t\right) - \mathbb{P}\left(T_l > t\right).$$

Note that T_k is distributed according to $\sum_{i=1}^{k} \alpha(i)^{-1}\mathbf{e}_i$, where $\alpha(i)$ is as in the statement of Proposition 2.2, and $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a sequence of independent standard exponential variables. As a general fact, a sum of k independent exponential variables with pairwise distinct parameters $\alpha(i) > 0$ follows the hypoexponential distribution, that is the probability distribution with density

$$f(x) = \sum_{i=1}^{k} \alpha(i) e^{-\alpha(i)x} \prod_{j=1, j\neq i}^{k} \frac{\alpha(j)}{\alpha(j) - \alpha(i)}.$$

Integrating the density and regrouping terms result in the statement of the lemma. □

2.2. *The monodisperse case.* We turn to the situation where $\mathcal{X}(0) = \mathbf{r} = (1, \ldots, 1)$, that is the coalescent process is started from the monodisperse configuration consisting of $N = 2n+1$ atoms of unit mass. In this case, the total mass M equals N, so the rates $\alpha(i)$ simplify to

$$\alpha(i) = (N+1-i)(N+1-2i)(N-2i). \tag{2.1}$$

If $\mathbf{s} = (s_1, \ldots, s_m) \in \mathcal{S}^{\downarrow}$ we denote by $\gamma(\mathbf{s})$ the number of different m-tuples that can be built from the elements s_i (recall that by our convention $s_i > 0$). To put it into a formula, if $\{s_{l_i}\}_{1\leq i\leq p}$ is a maximal family of pairwise disjoint non-zero elements from the sequence $\mathbf{s}$, and $k_i = |\{j = 1, \ldots, m : s_j = s_{l_i}\}|$, we define

$$\gamma(\mathbf{s}) = \binom{m}{k_1, \ldots, k_p}.$$

In other words, the ranking map

$$rk : \bigcup_{m=1}^{\infty} \mathbb{N}^m \longrightarrow \mathcal{S}^{\downarrow}$$

which orders $(r_1, \ldots, r_m) \in \mathbb{N}^m$ decreasingly satisfies $|rk^{-1}(\mathbf{s})| = \gamma(\mathbf{s})$ for each $\mathbf{s} \neq (0, \ldots) \in \mathcal{S}^{\downarrow}$. As a corollary of Proposition 2.3, the one-dimensional statistics for $\mathcal{X}'$ look as follows.

Corollary 2.5. *Let $0 \leq l \leq n$ and $\mathbf{s} = (s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$ with $s_i \in \mathbb{N}$ odd for all i, and $s_1 + \ldots + s_{N-2l} = N$. Then, in the situation described above,*

$$\mathbb{P}\left(\mathcal{X}'_l = \mathbf{s}\right) = \gamma(\mathbf{s}) \frac{N}{N-2l} \binom{N}{l}^{-1} \prod_{i=1}^{N-2l} \frac{1}{s_i} \binom{s_i}{\frac{s_i+1}{2}}.$$

Proof: The starting configuration is given by $(r_1, \ldots, r_N)$ with $r_i = 1$ for each i. Thus, if $\mathcal{X}'_l$ has $N-2l$ atoms of the sizes $s_1 \geq \ldots \geq s_{N-2l}$, then there is a partition π of $\{1, \ldots, N\}$ into $N-2l$ blocks $B_1, \ldots, B_{N-2l}$ of cardinality $|B_j| = s_j$, such that the atoms of $\mathcal{X}'_l$ evolved from merging the particles $\{r_i : i \in B_j\}$. Denote this event by $\Lambda'_\pi(N-2l)$. Since

$$\alpha(1) \cdots \alpha(l) = \frac{N!(N-1)!}{(N-l)!(N-2l-1)!},$$

we obtain from Proposition 2.3 (note that here $\mathbf{r}_{B_j} = |B_j| = s_j$)

$$\mathbb{P}\left(\Lambda'_\pi(N-2l)\right) = \frac{(N-1-2l)!}{(N-1)!} \binom{N}{l}^{-1} \prod_{i=1}^{N-2l} (s_i - 1)! \binom{s_i}{\frac{s_i+1}{2}}.$$

The number of such partitions π is given by

$$\frac{\gamma(\mathbf{s})}{(N-2l)!}\binom{N}{s_1,\ldots,s_{N-2l}}.$$

By multiplying the last two expressions together, we arrive at the stated expression. □

As the reader may already check at this stage, $\mathcal{X}_l'$ has the same distribution as the decreasingly ranked sequence of $N-2l$ independent copies ξ_i of the first hitting time of -1 of a simple random walk, conditioned on $\xi_1+\ldots+\xi_{N-2l}=N$ (see Section 3.3 for a definition of these quantities). Indeed, if $\xi_{(k)}$ denotes the kth order statistic of $\xi_1,\ldots,\xi_{N-2l}$, then for $\mathbf{s}=(s_1,\ldots,s_{N-2l})\in\mathcal{S}^{\downarrow}$

$$\begin{aligned}&\mathbb{P}\left((\xi_{(N-2l)},\ldots,\xi_{(1)})=(s_1,\ldots,s_{N-2l})\,|\,\xi_1+\ldots+\xi_{N-2l}=N\right)\\ &\quad=\quad\gamma(\mathbf{s})\,\mathbb{P}\left((\xi_1,\ldots,\xi_{N-2l})=(s_1,\ldots,s_{N-2l})\,|\,\xi_1+\ldots+\xi_{N-2l}=N\right),\end{aligned}$$

and an application of Lemma 3.2 affirms that the last expression coincides with that obtained in the corollary. The connection between random walks and the ternary coalescent will become much clearer in the next section.

3. Duality with fragmentation via random walks

Our intention of this section is to prove duality of the ternary coalescent with a fragmentation process. Let us begin with an informal description of such processes.

Conversely to the phenomenon of coagulation of particles, one often observes in nature or science processes of fragmentation. In these systems, particles are broken into smaller pieces as time passes. As an example, one may think of DNA fragmentation in biology or fractures in geophysics. Just as for coalescent processes, one needs to impose constraints on such systems to make them mathematically tractable. First, one assumes that the process has no memory in the sense that the future does only depend on the present state and not on the past. Second, one supposes that a particle is entirely characterized by its size, that is by a real number, and third, one requires the system to fulfill the branching property, which means that particles split independently.

Naively, one might first guess that a coalescent process can always be turned into a fragmentation process by reversing time. However, even though the memoryless property is preserved under time reversal, the branching property is typically not fulfilled. In fact, there are only few examples known where a duality relation holds (see Bertoin (2006a) Section 7 for an overview).

In view of our informal characterization, it is natural to call a Markov process with values in $\mathcal{S}^{\downarrow}$ a *ternary fragmentation process*, if each particle splits at a certain rate according to some dislocation law into three smaller pieces, where both the rate and the dislocation law depend only on the particle size s, and the sizes of the newly formed elements sum up to s. Ranked in decreasing order, these three particles together with the ones that did not split form the next state of the process. In particular, different particles split independently.

For our ternary coalescent starting from $N=2n+1$ atoms of unit mass, we shall prove

Theorem 3.1. *Reversing the coalescent chain* $\{\mathcal{X}'_k\}_{0\le k\le n}$ *in time results in the state chain of the fragmentation process, whose dynamics are given in Proposition 3.4.*

We will derive our result from an explicit construction of the skeleton chain $\mathcal{X}'$ in terms of (lengths of) excursion intervals of a conditioned random walk. This representation will also be useful for studying asymptotic properties in the last section.

3.1. *From configurations to paths to mass partitions.* We first show how subsets of $\{0,1,\dots,2n\}$ can be identified with certain paths of nearest neighbor walks on $\mathbb{Z}$ of length $2n+1$. The excursion intervals above two consecutive (new) minima of such paths partition the space $\mathbb{Z}/(2n+1)\mathbb{Z}$ into discrete arcs. Taking the ranked sequence of their lengths, we obtain the main object of our interest.

To begin with, define the configuration space $\mathcal{C}_n$ to be the set of all subsets of $\{0,\dots,2n\}$ which have cardinality less or equal to n. We often represent $x\in\mathcal{C}_n$ by the vector $(x(i))_{0\le i\le 2n}$, where

$$x(i)=\begin{cases}1\ , & i\in x\\ 0\ , & i\notin x\end{cases}\ .$$

Under this identification, we may regard x as a mass distribution. We use the terminology that a site i is occupied by a mass if $x(i)=1$ and vacant otherwise. The number of occupied sites (the cardinality of the subset x) is denoted by

$$|x|=|\{i\in\{0,\dots,2n\}:x(i)=1\}|\,.$$

We identify a configuration $x\in\mathcal{C}_n$ with a path of a nearest neighbor walk of length $2n+1$ on $\mathbb{Z}$ in the following way. Starting from the origin at time zero, the walk goes one step up if site 0 is occupied, i.e. $x(0)=1$, and down otherwise, then above if $x(1)=1$, down if $x(1)=0$ and so on, up to time $2n$. More precisely, the corresponding path $S(x)$ is given by $S(x)_0=0$ and for $1\le j\le 2n+1$,

$$S(x)_j=2\left(\sum_{i=0}^{j-1}x(i)\right)-j.$$

Notice that by definition, $S(x)_{2n+1}=2(|x|-n)-1$. Clearly, the mapping $\mathcal{C}_n\ni x\mapsto S(x)$ is one-to-one.

As we show next, the excursion intervals of such a path provide us with an element $\varphi_1(x)$ in the space of cyclically ordered partitions of $\mathbb{Z}/(2n+1)\mathbb{Z}$ into discrete arcs,

$$\begin{aligned}\mathcal{P}^{\circ}_{2n+1}\quad=\quad&\{\mathtt{s}^{\circ}=(\mathtt{s}_1,\dots,\mathtt{s}_m):\text{there exist } a_1<a_2<\ldots<a_m\le 2n+1,\\ &m,a_i\in\mathbb{N},\ \text{such that for } 1\le i\le m-1,\mathtt{s}_i=[a_i,a_{i+1})\cap\mathbb{N},\\ &\mathtt{s}_m=([a_m,2n+1)\cup[0,a_1))\cap\mathbb{Z}_+\}\,.\end{aligned}$$

Take $x\in\mathcal{C}_n$, and let $M=-S(x)_{2n+1}$. With $\underline{m}(x)=\min_{0\le j\le 2n+1}S_j(x)$, define the first time at which $S(x)$ reaches $\underline{m}(x)+k$, $k=0,\dots,M-1$,

$$m_k(S(x))=\inf\{j\ge 0:S_j(x)=\underline{m}(x)+k\}\,.$$

For $i=1,\dots,M$, put $a_i=m_{M-i}(S(x))$.

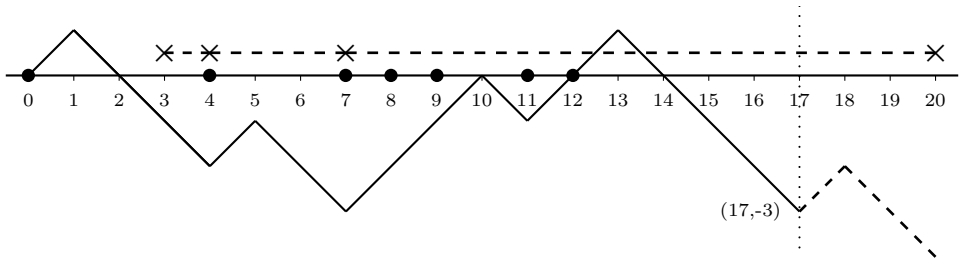

FIGURE 3.1. The black dots represent the configuration $x = \{0, 4, 7, 8, 9, 11, 12\} \subset \mathcal{C}_8$. The corresponding path $S(x)$ starts at zero and ends in -3 at time 17. It is periodically extended up to time 20 to better recognize the excursion intervals $\varphi_1(x)$. They are visualized by the dashed line above the x-axis, where the crosses mark the endpoints of the intervals, i.e. $\varphi_1(x) = (\mathtt{s}_1, \mathtt{s}_2, \mathtt{s}_3)$ with $\mathtt{s}_1 = [3, 4) \cap \mathbb{N}$, $\mathtt{s}_2 = [4, 7) \cap \mathbb{N}$, $\mathtt{s}_3 = ([7, 17) \cup [0, 3)) \cap \mathbb{Z}_+$.

We construct a sequence $\mathtt{s}^\circ = (\mathtt{s}_1, \ldots, \mathtt{s}_M) \in \mathcal{P}^\circ_{2n+1}$ by setting $\mathtt{s}_i = [a_i, a_{i+1}) \cap \mathbb{N}$ for $i = 1, \ldots, M-1$, $\mathtt{s}_M = ([a_M, 2n+1) \cup [0, a_1)) \cap \mathbb{Z}_+$. In other words, if we look for $k = 0, \ldots, 2n$ at the shifted path $\theta_k(S(x))$ defined by

$$\theta_k(S(x))_i = \begin{cases} S(x)_{i+k} - S(x)_k & , \ 0 \le i \le 2n+1-k \\ S(x)_{i+k-(2n+1)} + S(x)_{2n+1} - S(x)_k & , \ 2n+1-k < i \le 2n+1 \end{cases},$$

then the element $\mathtt{s}^\circ$ corresponds to the M successive excursion intervals of $\theta_{m_{M-1}} S(x)$ above two consecutive (new) minima. The length $|\mathtt{s}_i|$ of such an interval is also referred to as a *ladder epoch*. We let $\varphi_1(x) = \mathtt{s}^\circ$ and define φ_2 as the function which sends $\mathtt{s}^\circ = (\mathtt{s}_1, \ldots, \mathtt{s}_m) \in \mathcal{P}^\circ_{2n+1}$ to its arc lengths $\{|\mathtt{s}_i|\}_{1 \le i \le m}$, arranged in decreasing order. In this way, we obtain an element in the space of mass partitions

$$\mathcal{P}^\downarrow_{2n+1} = \left\{ \mathbf{s} = (s_1, \ldots, s_m) : s_1 \ge s_2 \ge \ldots \ge s_m \,, m, s_i \in \mathbb{N}, \sum_{i=1}^m s_i = 2n+1 \right\}.$$

By filling up with an infinite sequence of zeros, we will often identify mass partitions with elements in $\mathcal{S}^\downarrow$. To summarize our construction, the concatenation map φ

$$\varphi = \varphi_2 \circ \varphi_1 : \mathcal{C}_n \xrightarrow{\varphi_1} \mathcal{P}^\circ_{2n+1} \xrightarrow{\varphi_2} \mathcal{P}^\downarrow_{2n+1} \subset \mathcal{S}^\downarrow.$$

sends configurations $x \in \mathcal{C}_n$ via their path representations to partitions of $\mathbb{Z}/(2n+1)\mathbb{Z}$ and then to mass partitions.

3.2. *Random evolution.* Our purpose here is to randomize the input of the map $\varphi : \mathcal{C}_n \to \mathcal{P}^\downarrow_{2n+1}$ to obtain (a sequence of) random mass partitions. More precisely, we construct two Markov chains on $\mathcal{C}_n$ running from time zero up to n as follows. Let $X = \{X_k\}_{0 \le k \le n}$ be the Markov chain with $X_0 = \emptyset$ and transition probabilities

$$p_X(x, y) = \begin{cases} \frac{1}{2n+1-|x|} & , \ x \subset y \text{ and } y \backslash x = \{i\} \text{ for some } i \in \{0, \ldots, 2n\} \backslash x \\ 0 & , \ \text{otherwise} \end{cases}.$$

In words, $(X_0, \dots, X_l)$ is obtained by occupying successively l sites from $\{0, \dots, 2n\}$, chosen uniformly at random. From the point of view of sets, X_l is uniformly distributed on the space of all l-subsets of $\{0, \dots, 2n\}$. By identifying with the random path $S(X_l)$, we will also think of X_l as simple random walk up to time $2n+1$, conditioned to end at position $-2(n-l)-1$.

Let $Y = \{Y_k\}_{0 \le k \le n}$ be the Markov chain with Y_0 being uniformly distributed on the space of all n-subsets of $\{0, \dots, 2n\}$ and transition probabilities

$$p_Y(x,y) = \begin{cases} \frac{1}{|x|} & , \ y \subset x \text{ and } x \backslash y = \{i\} \text{ for some } i \in \{0, \dots, 2n\} \backslash y \\ 0 & , \ \text{otherwise} \end{cases} .$$

In words, $(Y_0, \dots, Y_l)$ is obtained by removing successively l masses chosen uniformly at random from the starting configuration Y_0. In terms of sets, Y_l is uniformly distributed on the space of all $(n-l)$- subsets of $\{0, \dots, 2n\}$. As above, Y_l can be identified with simple random walk up to time $2n+1$, conditioned to end at $-2l-1$. Note that by construction, we have the duality relation

$$(X_0, \dots, X_n) \stackrel{\mathrm{d}}{=} (Y_n, \dots, Y_0). \tag{3.1}$$

3.3. *Realization of the skeleton chains.* We are not interested in X and Y themselves, but rather in $\varphi(X) = \{\varphi(X_k)\}_{0 \le k \le n}$ and $\varphi(Y)$. As we will show in Proposition 3.5, the former is the state chain of the ternary coalescent starting from $N = 2n+1$ atoms of unit mass. The latter is characterized by Proposition 3.4 as the state chain of a fragmentation process starting from a single particle of mass N.

We need some preparation. Recall that simple random walk on $\mathbb{Z}$ is the Markov chain $S = \{S_m\}_{m \ge 0}$ with $S_0 = 0$ and $S_m = \zeta_1 + \ldots + \zeta_m$, where $\zeta_1, \zeta_2, \dots$ are independent random variables with $\mathbb{P}(\zeta_i = \pm 1) = 1/2$. For $k \in \mathbb{Z}$, the first hitting time of k is denoted by

$$H_k = \inf\{m \ge 1 : S_m = k\}.$$

The following result on the distribution of H_k is classical.

Lemma 3.2. *Let $k \in \mathbb{Z}$, $k \neq 0$, and $m \in \mathbb{N}$. Then*

$$\mathbb{P}\left(H_k = m\right) = \begin{cases} \frac{|k|}{m} \binom{m}{(m+|k|)/2} 2^{-m} & , \ k = m [\mathrm{mod}\ 2] \\ 0 & , \ k \neq m [\mathrm{mod}\ 2] \end{cases} .$$

Moreover, if $m = 2n+1$ and k is a fixed odd number, as $n \to \infty$,

$$\mathbb{P}\left(H_k = m\right) \sim \frac{1}{2} \sqrt{\frac{1}{\pi n^3}}.$$

Proof: Clearly, for the probability to be different from zero the numbers k and m must have the same parity. Then, using the hitting time theorem (see for example Kemperman (1961)) in the first equality,

$$\mathbb{P}(H_k = m) = \frac{|k|}{m} \mathbb{P}(S_m = k) = \frac{|k|}{m} \binom{m}{(m+|k|)/2} 2^{-m}.$$

The second statement follows from Stirling's formula for the factorial. □

Before looking at $\varphi(X)$ and $\varphi(Y)$ in detail, let us give an indication that the former is the skeleton chain of the ternary coalescent. Recall Corollary 2.5 and the connection between $\mathcal{X}'$ and hitting times. Let $N = 2n+1$, $0 \le l \le n$, and take

N independent copies ξ_i of the hitting time H_{-1}. Denote by $\xi_{(k)}$ the kth order statistic of $\xi_1, \ldots, \xi_{N-2l}$.

Proposition 3.3. *$\varphi(X_l)$ is distributed according to $(\xi_{(N-2l)}, \ldots, \xi_{(1)})$ conditionally on $\xi_1 + \ldots + \xi_{N-2l} = N$, i.e. the one-dimensional distributions of $\varphi(X)$ and $\mathcal{X}'$ started from N atoms of mass one agree.*

Proof: We identify X_l with simple random walk $S(X_l)$ up to time N, conditioned to end at $-(N-2l)$. For notational simplicity, let us write S instead of $S(X_l)$. Also recall the definitions of $\theta_k(S)$ and $m_k(S)$ from Section 3.1. By Theorem 1 of Bertoin et al. (2003), if ν is a uniform random variable on $\{0, \ldots, N-2l-1\}$ independent of S, then the chain $\theta_{m_\nu}(S)$ has the law of S conditioned on $H_{-(N-2l)} = N$. Moreover, the index m_ν is uniformly distributed on $\{0, \ldots, N-1\}$ and independent of the chain $\theta_{m_\nu}(S)$. Denote by $\theta_k X_l$ the shifted configuration defined by $\theta_k X_l(i) = X_l(i+k[\text{mod } N])$. Clearly, $\varphi(X_l) = \varphi(\theta_k X_l)$ for each k. From Theorem 1 of Bertoin et al. (2003) we thus infer that for $(s_1, \ldots, s_{N-2l}) \in \mathcal{S}^{\downarrow}$,

$$\begin{aligned}\mathbb{P}\left(\varphi(X_l) = (s_1, \ldots, s_{N-2l})\right) &= \mathbb{P}\left(\varphi(\theta_{m_\nu} X_l) = (s_1, \ldots, s_{N-2l})\right) \\ &= \mathbb{P}\left((\xi_{(N-2l)}, \ldots, \xi_{(1)}) = (s_1, \ldots, s_{N-2l}) \,|\, \xi_1 + \ldots + \xi_{N-2l} = N\right).\end{aligned}$$

□

For the moment, we leave $\varphi(X)$ aside and first turn to $\varphi(Y)$. In the sequel it is convenient to use the notion of multisets, which we distinguish from normal sets by using double braces. For example, $\{\{a, b, c, c\}\}$ contains the elements a, b each with multiplicity 1 and the element c with multiplicity 2. The cardinality of this multiset is 4, the order of elements is irrelevant, as for sets.

Let ξ_1, ξ_2, ξ_3 be three independent copies of the hitting time H_{-1}. To state the transition mechanism of $\varphi(Y)$ in a concise way, we define a family $\mu = (\mu_s, s \geq 3 \text{ odd})$ of probability laws, supported on

$$\Omega_s = \{R = \{\{r_1, r_2, r_3\}\} : r_i \in \mathbb{N} \text{ odd}, \, r_1 + r_2 + r_3 = s\},$$

by setting

$$\mu_s(R) = \mathbb{P}\left(\{\{\xi_1, \xi_2, \xi_3\}\} = R \mid \xi_1 + \xi_2 + \xi_3 = s\right). \tag{3.2}$$

More explicitly, applying Lemma 3.2 results in the expression

$$\mu_s(R) = \gamma \frac{s}{3 r_1 r_2 r_3} \binom{r_1}{\frac{r_1+1}{2}} \binom{r_2}{\frac{r_2+1}{2}} \binom{r_3}{\frac{r_3+1}{2}} \left[\binom{s}{\frac{s+3}{2}}\right]^{-1}, \tag{3.3}$$

where γ is the number of triplets (r_i, r_j, r_k) that can be formed from $R = \{\{r_1, r_2, r_3\}\}$

$$\gamma = \begin{cases} 6\,, & |\{r_1, r_2, r_3\}| = 3 \\ 3\,, & |\{r_1, r_2, r_3\}| = 2 \\ 1\,, & |\{r_1, r_2, r_3\}| = 1 \end{cases}.$$

Proposition 3.4. *$\varphi(Y) = \{\varphi(Y_k)\}_{0 \leq k \leq n}$ is a Markov chain. Its transition mechanism from time $l \leq n-1$ to $l+1$ is described as follows.*

(a) *Conditionally on $\varphi(Y_l) = \mathbf{s} = (s_1, \ldots, s_{2l+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$, select an index $\iota \in \{1, \ldots, 2l+1\}$ according to the law*

$$\mathbb{P}\left(\iota = i \mid \varphi(Y_l) = \mathbf{s}\right) = \frac{s_i - 1}{2(n-l)}.$$

(b) *Given $\varphi(Y_l) = \mathbf{s}$ and $\iota = i$, split s_i according to the law μ_{s_i} into three numbers and rank them together with s_m, $m \in \{1, \dots, 2l+1\}\backslash\{i\}$, in decreasing order to obtain a new mass partition.*

Proof: Fix $l \in \{0, \dots, n-1\}$. We write $\varphi(Y)_{0:i}$ for the vector $(\varphi(Y_0), \dots, \varphi(Y_i))$. The Markov property will follow from

(1) $\varphi(Y)_{0:l}$ and $\varphi(Y_{l+1})$ are conditionally independent given $\varphi_1(Y_l)$.
(2) $\varphi_1(Y_l)$ and $\varphi(Y_{l+1})$ are conditionally independent given $\varphi(Y_l)$.

Indeed, assuming (i) and (ii), with $\mathbf{r}_{0:l+1} = (\mathbf{r}_0, \dots, \mathbf{r}_{l+1}) \in \mathcal{P}^{\downarrow}_{2n+1} \times \ldots \times \mathcal{P}^{\downarrow}_{2n+1}$,

$$\begin{aligned}
&\mathbb{P}\left(\varphi(Y)_{0:l+1} = \mathbf{r}_{0:l+1}\right) \\
&\quad = \sum_{\mathbf{u}:\varphi_2(\mathbf{u}^\circ)=\mathbf{r}_l} \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{u}^\circ\right) \mathbb{P}\left(\varphi(Y_{l+1}) = \mathbf{r}_{l+1} \mid \varphi_1(Y_l) = \mathbf{u}^\circ\right) \\
&\qquad \times \mathbb{P}\left(\varphi_1(Y_l) = \mathbf{u}^\circ\right) \\
&\quad = \mathbb{P}\left(\varphi(Y_{l+1}) = \mathbf{r}_{l+1} \mid \varphi(Y_l) = \mathbf{r}_l\right) \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l}\right).
\end{aligned}$$

For (i), the key step is to show that the conditional law of $\varphi(Y)_{0:l}$ given Y_l only depends on $\varphi_1(Y_l)$. In that direction, we work conditionally on $\varphi_1(Y_l) = \mathbf{s}^\circ = (\mathbf{s}_1, \dots, \mathbf{s}_{2l+1})$ and denote by $N_k(i) = |Y_k \cap \mathbf{s}_i|$ the number of sites of the arc $\mathbf{s}_i$ which are occupied by Y_k. Write N_k for the family $\{N_k(i)\}_{1 \le i \le 2l+1}$. Let i_l denote the unique index such that the singleton $Y_{l-1}\backslash Y_l \subset \mathbf{s}_{i_l}$. In other words, i_l is the unique index i such that $N_{l-1}(i) = N_l(i) + 1$. Then $\varphi_1(Y_{l-1})$ results from $\varphi_1(Y_l) = \mathbf{s}$ by merging the arcs $\mathbf{s}_{i_l}$, $\mathbf{s}_{i_l+1}$ and $\mathbf{s}_{i_l+2}$ (with the convention that indices of arcs are taken modulo $2l+1$). By iteration, we realize that the sequence $N_{0:l} = (N_0, \dots, N_l)$ determines $\varphi_1(Y)_{0:l}$ and therefore also $\varphi(Y)_{0:l}$. Hence it now suffices to check that the conditional distribution of $N_{0:l}$ given Y_l only depends on $\varphi_1(Y_l) = \mathbf{s}^\circ$, which is straightforward from the dynamics and the observation that for every $i = 1, \dots, 2l+1$, the arc $\mathbf{s}_i$ has exactly $(|\mathbf{s}_i| + 1)/2$ sites which are not occupied by Y_l.

We are now able to prove (i). Take $\mathbf{t} \in \mathcal{P}^{\downarrow}_{2n+1}$ with $\mathbb{P}\left(\varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) > 0$. Then

$$\begin{aligned}
&\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l}, \varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) \\
&\quad = \sum_{\substack{x \in \varphi_1^{-1}(\mathbf{s}^\circ), \\ y \in \varphi^{-1}(\mathbf{t})}} \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x, Y_{l+1} = y\right) \mathbb{P}\left(Y_l = x, Y_{l+1} = y\right).
\end{aligned}$$

Since Y is a Markov chain, it follows that for $x, y \in \mathcal{C}_n$ with $\mathbb{P}(Y_l = x, Y_{l+1} = y) > 0$,

$$\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x, Y_{l+1} = y\right) = \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x\right).$$

Plugging this into the above formula and using the conditional independence of $\varphi(Y)_{0:l}$ and Y_l given $\varphi_1(Y_l)$, we deduce that for $x \in \varphi_1^{-1}(\mathbf{s}^\circ)$,

$$\mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{s}^\circ, \varphi(Y_{l+1}) = \mathbf{t}\right) = \mathbb{P}\left(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid Y_l = x\right).$$

Similarly, one sees that the right hand side equals $\mathbb{P}(\varphi(Y)_{0:l} = \mathbf{r}_{0:l} \mid \varphi_1(Y_l) = \mathbf{s}^\circ)$, and (i) follows. We turn to (ii) and the description of the transition mechanism. We keep the conditioning on $\varphi_1(Y_l) = \mathbf{s}^\circ$. Note that Y_{l+1} evolves from Y_l by removing uniformly at random one of the $n - l$ masses. By identifying Y_l with $S(Y_l)$, this amounts to switching one of the upward steps chosen uniformly at random into a

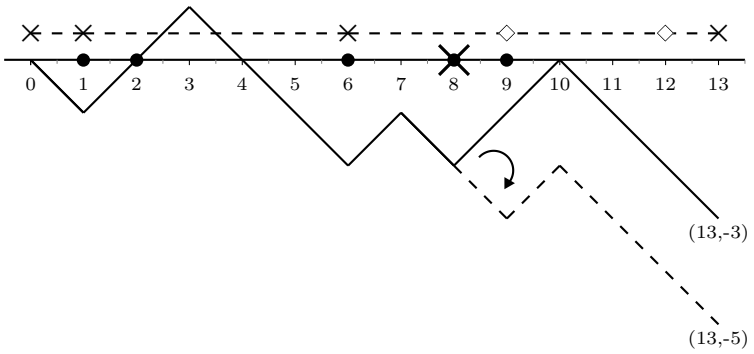

FIGURE 3.2. The transition mechanism from $\varphi(Y_1)$ to $\varphi(Y_2)$, where $n = 6$. Here, at time 1 the chain Y is in the configuration state $Y_1 = \{1, 2, 6, 8, 9\}$. Then the mass at position 8 is removed. For the corresponding path, this means that the upward step at time 8 is changed into a downward step. The new path $S(Y_2)$ coincides up to time 8 with the old path $S(Y_1)$ and is then indicated by the dashed line. The excursion interval $[6, 13)$ is broken into three intervals $[6, 9)$, $[9, 12)$, $[12, 13)$. Therefore, $\varphi_1(Y_2) = (\mathtt{s}_1, \mathtt{s}_2, \mathtt{s}_3, \mathtt{s}_4, \mathtt{s}_5)$ with $\mathtt{s}_1 = [1, 6) \cap \mathbb{N}$, $\mathtt{s}_2 = [6, 9) \cap \mathbb{N}$, $\mathtt{s}_3 = [9, 12) \cap \mathbb{N}$, $\mathtt{s}_4 = [12, 13) \cap \mathbb{N}$, $\mathtt{s}_5 = [0, 1) \cap \mathbb{Z}_+$ and $\varphi(Y_2) = (5, 3, 3, 1, 1)$.

downward step. More precisely, under our conditioning, the ith arc $\mathtt{s}_i$ is picked with probability

$$\frac{\text{number of upward steps over } \mathtt{s}_i}{\text{total number of upward steps}} = \frac{(|\mathtt{s}_i| - 1)/2}{n - l}, \tag{3.4}$$

then one of the upward steps over $\mathtt{s}_i$ is selected with uniform probability and changed into a downward step. Up to a vertical shift in space, $S(Y_l)$ restricted to the arc $\mathtt{s}_i$ obeys the law of simple random walk conditioned on $H_{-1} = |\mathtt{s}_\mathtt{i}|$ (with an obvious modification for the last arc $\mathtt{s}_{2l+1}$). Given an upward step over $\mathtt{s}_\mathtt{i}$ is switched, $S(Y_{l+1})$ restricted to $\mathtt{s}_i$ can therefore be seen as simple random walk conditioned on $H_{-3} = |\mathtt{s}_\mathtt{i}|$. In terms of $\varphi(Y)$, we deduce that $\varphi(Y_{l+1})$ is obtained by first picking the ith arc $\mathtt{s}_i$ with probability given in (3.4), then splitting its length according to $\mu_{|\mathtt{s}_\mathtt{i}|}$ into three numbers r_1, r_2, r_3 corresponding to the first three ladder epochs of simple random walk conditioned on $H_{-3} = |\mathtt{s}_i|$, and finally ranking them together with the numbers $|\mathtt{s}_j|$, $j \neq i$, in decreasing order. In particular, we realize that for predicting $\varphi(Y_{l+1})$ out of $\varphi_1(Y_l)$, the additional information given by $\varphi_1(Y_l)$ compared to $\varphi(Y_l)$, namely the location of the arcs, is irrelevant. Hence also (ii) holds. □

Let us now characterize $\varphi(X)$.

Proposition 3.5. *$\varphi(X) = \{\varphi(X_k)\}_{0 \le k \le n}$ is a Markov chain. Its transition mechanism from time $l \le n - 1$ to $l + 1$ is described as follows.*

(a) *Conditionally on* $\varphi(X_l) = \mathbf{s} = (s_1, \dots, s_{2(n-l)+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$, *select an index* ι *out of the set of all* 3*-subsets of* $\{1, \dots, 2(n-l)+1\}$ *according to the law*

$$\mathbb{P}\left(\iota = \{i,j,k\} \mid \varphi(X_l) = \mathbf{s}\right) = \frac{s_i + s_j + s_k + 3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

(b) *Given* $\varphi(X_l) = \mathbf{s}$ *and* $\iota = \{i,j,k\}$, *rank the sum* $r = s_i + s_j + s_k$ *together with the numbers* s_m, $m \in \{1, \dots, 2(n-l)+1\} \backslash \{i,j,k\}$, *in decreasing order to obtain a new mass partition.*

Proof: From the duality (3.1) it follows that $\varphi(X)$ is obtained by reversing $\varphi(Y)$ in time. In particular, the Markov property carries over from $\varphi(Y)$ to $\varphi(X)$.

It remains to look at the transition mechanism. The step from $l = n-1$ to n is obvious from the construction of X and φ. Now fix $l \in \{0, \dots, n-2\}$, and let $M = 2(n-l)+1$. We work conditionally on $\varphi(X_l) = \mathbf{s} = (s_1, \dots, s_M) \in \mathcal{P}^{\downarrow}_{2n+1}$. By construction, $\varphi(X_{l+1})$ is obtained from $\varphi(X_l)$ by summing up three numbers s_i,s_j,s_k, where i,j,k are pairwise distinct, and rearranging the sum together with s_m, $m \neq i,j,k$, in decreasing order. Write $\mathbf{s}^\circ = (\mathbf{s}_1, \dots, \mathbf{s}_M)$ for the partition $\varphi_1(X_l)$, and let ν be uniformly distributed on $\{0, \dots, M-1\}$, independent of X_l. By the random walk representation and Theorem 1 of Bertoin et al. (2003), the law of the cyclically ordered arc lengths $(|\mathbf{s}_{1+\nu}|, \dots, |\mathbf{s}_{M+\nu}|)$ (indices are taken modulo M) agrees with the law of the M subsequent ladder epochs of simple random walk conditioned on $H_{-M} = 2n+1$. In particular, the law of $(|\mathbf{s}_{1+\nu}|, \dots, |\mathbf{s}_{M+\nu}|)$ is invariant under permutations and therefore equals the law of $\left(s_{\sigma(1)}, \dots, s_{\sigma(M)}\right)$, where σ is a permutation of $\{1, \dots, M\}$, chosen uniformly at random and independently of X_l. Note that this can also be deduced directly from the fact that X_l is uniformly distributed on the space of all l-subsets of $\{0, \dots, 2n\}$. The probability that s_i,s_j,s_k are replaced by their sum is given by the probability that the arcs $\mathbf{s}_{\sigma^{-1}(i)+\nu}$, $\mathbf{s}_{\sigma^{-1}(j)+\nu}$, $\mathbf{s}_{\sigma^{-1}(k)+\nu}$ merge. This is the case if and only if the arcs adjoin each other and the singleton $X_{l+1} \backslash X_l$ is contained in that arc which is followed in clockwise order by the other two. More formally, the arcs merge if and only if there is a permutation ρ of the indices i,j and k such that $X_{l+1} \backslash X_l \subset \mathbf{s}_{\sigma^{-1}(\rho(i))+\nu}$, and $\sigma^{-1}(\rho(j)) = \sigma^{-1}(\rho(i)) + 1$, $\sigma^{-1}(\rho(k)) = \sigma^{-1}(\rho(i)) + 2$ (both equalities are taken modulo M). Given $X_{l+1} \backslash X_l \subset \mathbf{s}_{\sigma^{-1}(i)+\nu}$, the probability that $\mathbf{s}_{\sigma^{-1}(i)+\nu}$, $\mathbf{s}_{\sigma^{-1}(j)+\nu}$, $\mathbf{s}_{\sigma^{-1}(k)+\nu}$ merge is therefore

$$\frac{2}{M-1} \times \frac{1}{M-2}.$$

The probability that $X_{l+1} \backslash X_l \subset \mathbf{s}_{\sigma^{-1}(i)+\nu}$ is

$$\frac{\text{number of vacant sites in } \mathbf{s}_{\sigma^{-1}(i)+\nu} \text{ at time } l}{\text{total number of vacant sites at time } l} = \frac{(s_i+1)/2}{2n+1-l}.$$

Altogether, given $\varphi(X_l) = \mathbf{s}$,

$$\begin{aligned} &\mathbb{P}\left(\mathbf{s}_{\sigma^{-1}(i)+\nu}, \mathbf{s}_{\sigma^{-1}(j)+\nu}, \mathbf{s}_{\sigma^{-1}(k)+\nu} \text{ merge}\right) \\ &\quad = \left(\frac{(s_i+s_j+s_k+3)/2}{2n+1-l}\right) \frac{2}{M-1} \times \frac{1}{M-2}, \end{aligned}$$

which is the probability in (a) in the case $l < n-1$. □

Theorem 3.1 now easily follows. Indeed, from the last proposition we see that $\varphi(X)$ is equal in law to the skeleton chain $\{\mathcal{X}'_k\}_{0 \le k \le n}$ started from N particles of

unit mass. By the duality relation (3.1), reversing $\varphi(X)$ in time yields the process $\varphi(Y)$, which is the state chain of a fragmentation process.

4. Random binary forest representation

In this section, we give a second construction of the skeleton chain of the ternary coalescent in terms of random binary forests. The connection between random forests and coalescent processes was first observed by Pitman (1999). In our description, we are guided by Chapter 5.2.3 of Bertoin (2006b).

4.1. *Basic definitions on graphs.* We first collect some basic notions on graphs which will useful for our purpose.

A (undirected) *graph* is a pair $G = (V, E)$, where V is a finite set and $E \subset \{U \subset V : |U| = 2\}$. The elements of V are called *vertices*, the elements of E *edges*. The *size* of a graph is the number of vertices $|V|$. A *subgraph* of a graph $G = (V, E)$ is a graph $H = (V', E')$ with $V' \subset V$ and $E' \subset E$.

Now let $G = (V, E)$ be a graph. Two vertices v, w are *adjacent*, if $\{v, w\} \in E$. The *degree* of a vertex v is the number of vertices adjacent to v. A sequence $(v_1, e_1, v_2, \ldots, v_m, e_m, v_{m+1})$ such that $m \geq 0$ $v_i \neq v_j$ for $i \neq j$ and $e_i = \{v_i, v_{i+1}\} \in E$ for $1 \leq i \leq m$ is called a *path*, or also a v_1-v_{m+1}-path. A *cycle* is a sequence $(v_1, e_1, \ldots, v_m, e_m, v_1)$ such that $m \geq 2$, $(v_1, e_1, \ldots, v_{m-1}, e_{m-1}, v_m)$ is a path and $e_m = \{v_m, v_1\} \in E$. We say that two vertices v, w are *connected*, if there exists a v-w-path. If there is a v-w-path for any $v, w \in V$, we say that the graph G is connected. The maximal connected subgraphs of G are its *connected components*. A connected graph without a cycle (as a subgraph) is called a *tree*. In a tree, a *leaf* is a vertex of degree equals 1, while the vertices of degree greater than 1 are called *internal* vertices.

We are interested in a special family of trees. A *binary tree* is either a tree consisting of a single vertex only, called the *root* of the tree, or a tree where exactly one vertex has degree 2, which we then call the root of the tree, and all the other vertices have degree 3 or they are leaves. The *height* of a vertex v in a binary tree is the number of edges of the (unique) v-r-path, where r is the root of the tree. If v is not a leaf, then there are exactly two vertices w, w' adjacent to v with height strictly bigger than that of v, the *children* of v. We call the pair $\{\{v, w\}, \{v, w'\}\}$ the *outgoing edges* (from v). Finally, a *binary forest* is a graph such that its connected components are binary trees. The leaves or internal vertices of such a forest are then all those of its tree components.

Observe that a binary forest on N vertices with m tree components has $N - m$ edges, $(N + m)/2$ leaves, and $(N - m)/2$ internal vertices.

Remark 4.1. In the literature, a binary tree in our sense is often called a (rooted) *full labeled* binary tree. The term "full" reflects the fact that every vertex other than the leaves has two children, and "labeled" stresses that the vertices are distinguishable. However, we will use the term "labeled" to indicate a labeling of internal vertices.

4.2. *Dynamics.* Our concern here is to describe the dynamics on the space of binary forests, which will lead to another representation of the ternary coalescent.

As before let $N = 2n + 1$. We consider $V = \{1, 2, \ldots, N\}$ as a set of vertices. Given a binary forest on V, we enumerate its tree components according to the increasing order of their roots.

We will assign additional labels to all internal vertices of such a forest. A *labeling* of a binary forest on V with m tree components is a bijective map from the set of $(N-m)/2$ internal vertices into $\{1,\dots,(N-m)/2\}$. A labeled binary forest on V is then a binary forest together with a labeling. Note that internal vertices are double-labeled, by V and by the labeling just described. The set of all labeled binary forests on V with m tree components is denoted by $\mathcal{F}(m,N)$. Clearly, $\mathcal{F}(m,N)$ is empty if m is an even number.

For every $1 \le k \le n$, we define a map $R : \mathcal{F}(2k-1,N) \longrightarrow \mathcal{F}(2k+1,N)$ as follows. For each $\tau \in \mathcal{F}(2k-1,N)$, select the internal vertex with the highest label and delete both outgoing edges (and the label, since the vertex is now a leaf). We obtain a labeled binary forest with $2k+1$ trees, which we denote by $R(\tau)$.

As the reader might already guess, the map R will be the building block of the fragmentation mechanism - it breaks the tree with the highest label into three (new) trees. The reverse dynamic will correspond to the coagulation mechanism: Out of a binary forest with at least three trees, pick one leaf and connect it by adding edges to two distinct roots from other tree components. Then, three trees have merged into one (new) tree, and the selected leaf has become an internal vertex. Before underlying this procedure with randomness, let us analyze the map R in detail.

Lemma 4.2. *For every $1 \le k \le n$, the map $R : \mathcal{F}(2k-1,N) \longrightarrow \mathcal{F}(2k+1,N)$ is surjective. More precisely, for every $\tau \in \mathcal{F}(2k+1,N)$,*

$$|\{\tilde{\tau} \in \mathcal{F}(2k-1,N) : R(\tilde{\tau}) = \tau\}| = (n+k+1)k(2k-1).$$

Proof: Let $\tau \in \mathcal{F}(2k+1,N)$. In order to construct a generic $\tilde{\tau} \in R^{-1}(\tau)$, pick a leaf i from τ. Write $\rho(i)$ for the root of the tree component containing i. Then select two roots $j \neq j'$ different from $\rho(i)$, add the edges $\{i,j\}$, $\{i,j'\}$ and label the vertex i with the number $n-k+1$. Out of three components, we have obtained a new labeled binary tree with root $\rho(i)$, which is part of a forest with $2k-1$ trees. Clearly, this forest is contained in $R^{-1}(\tau)$. Moreover, different choices of i,j,j' give rise to different forests. To finish the proof, note that there are $n+k+1$ possible choices for a leaf i, and $2k(2k-1)/2$ possible choices for distinct roots $\{j,j'\}$. □

Remark 4.3. Applying the map R at most n times destructs a labeled binary forest into its single vertices. Due to the recursive structure of trees, this method enables one to compute various combinatorial quantities. For example, using $|\mathcal{F}(N,N)| = 1$ and iteratively the identity

$$|\mathcal{F}(2k-1,N)| = (n+k+1)k(2k-1)\,|\mathcal{F}(2k+1,N)|$$

provided by Lemma 4.2, one obtains for $k = 2,\dots,n+1$

$$|\mathcal{F}(2k-1,N)| = \frac{2^{k-(n+1)}n\,(2n+1)!\,(2n-1)!\,(k-2)!}{(n+k)!\,(k-1)!\,(2k-3)!}.$$

In the case $k=1$,

$$|\mathcal{F}(1,N)| = \frac{2^{-n}(2n)!(2n+1)!}{(n+1)!} = 2^{-n}(2n+1)!\,n!\,C_n,$$

where $C_n = (2n)!/((n+1)!\,n!)$ is the nth *Catalan* number. Since there are $n!$ different labelings of internal vertices, we deduce that the number of binary trees on V is given by $2^{-n}(2n+1)!\;C_n$.

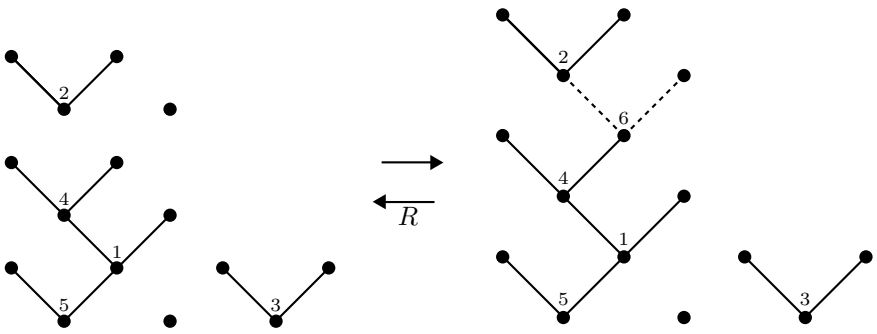

FIGURE 4.3. From left to right (right to left) one step in the coagulation (fragmentation) mechanism is shown. For simplicity, only the labeling of the internal vertices is depicted. On the left, the leaf with number 6 on the right is chosen as well as two roots from different tree components. They are connected by two edges visualized by the dashed lines on the right side.

4.3. *From forests to mass partitions.* Denote by R^k the kth concatenation of R, where R^0 is the identity map. We randomize the input by endowing the space $\mathcal{F}(1,N)$ with the uniform probability measure and interpret the maps R^k as random variables

$$R^k : \mathcal{F}(1,N) \longrightarrow \mathcal{F}(2k+1,N), \quad k=0,\dots,n.$$

In words, $R^k(\tau)$ is the forest with $2k+1$ tree components which arises from $\tau \in \mathcal{F}(1,N)$ by picking the k internal vertices with the highest labels and deleting their outgoing edges. By induction, we deduce from Lemma 4.2 that R^k obeys the uniform law on the space $\mathcal{F}(2k+1)$, for each k. We then consider the random variables

$$|R^k|^{\downarrow} : \mathcal{F}(1,N) \longrightarrow \mathcal{P}^{\downarrow}_{2n+1}, \quad k=0,\dots,n,$$

where for a tree $\tau \in \mathcal{F}(1,N)$, $|R^k|^{\downarrow}(\tau) = \mathbf{s} = (s_1,\dots,s_{2k+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$ is the sequence of the sizes of the tree components, ranked in decreasing order.

Turning back to the ternary coalescent, let $\mathcal{X}'_k$, $k=0,\dots,n$, denote the skeleton chain started from N particles of unit mass. Its connection to the sizes of the tree components is given by

Proposition 4.4. *The sequence of random variables* $\{|R^{n-k}|^{\downarrow}\}_{0\le k\le n}$ *is the state chain of the ternary coalescent, that is*

$$\left(|R^n|^{\downarrow}, |R^{n-1}|^{\downarrow}, \dots, |R^0|^{\downarrow}\right) \stackrel{\mathrm{d}}{=} (\mathcal{X}'_0,\dots,\mathcal{X}'_n)\,.$$

Proof: For each tree $\tau \in \mathcal{F}(1,N)$, the forest $R^n(\tau)$ has no edges, so $|R^n|^{\downarrow} = (1,\dots,1) = \mathcal{X}'_0$. Note that given $|R^l|^{\downarrow} = \mathbf{s} = (s_1,\dots,s_{2l+1})$ for some $1 \le l \le n$, the mass partition $|R^{l-1}|^{\downarrow}$ is obtained from $\mathbf{s}$ by replacing three elements s_i,s_j,s_k, where i,j,k are pairwise distinct, by their sum. Furthermore, observe that the random variables R^k, $l \le k \le n$, are measurable with respect to the sigma-field generated by R^l. In particular, by Proposition 3.5, the claim follows if we show

that for every $0 \leq l < n$, for every $\mathbf{s} = (s_1, \ldots, s_{2(n-l)+1}) \in \mathcal{P}^{\downarrow}_{2n+1}$ and for every 3-subset $\{i, j, k\} \subset \{1, \ldots, 2(n-l)+1\}$,

$$\mathbb{P}\left(|R^{n-l-1}|^{\downarrow} = s^{i\oplus j\oplus k} \mid R^{n-l}, |R^{n-l}|^{\downarrow} = \mathbf{s}\right) = \frac{s_i + s_j + s_k + 3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

Take a forest $\tau \in \mathcal{F}(2(n-l)+1, N)$. We work conditionally on $R^{n-l} = \tau$. By our observation above, R^{n-l-1} is uniformly distributed on the set of $(2n+1-l)(n-l)(2(n-l)-1)$ forests which can be obtained from τ in the way described in Lemma 4.2. We write $\tau_1, \ldots, \tau_{2(n-l)+1}$ for the tree components of τ. For every 3-subset $\{a, b, c\} \subset \{1, \ldots, 2(n-l)+1\}$, the probability that the leaf i is picked in τ_a and the roots are chosen from τ_b and τ_c is therefore

$$\frac{|\tau_a|+1}{2} \times \frac{1}{(2n+1-l)(n-l)(2(n-l)-1)}.$$

Hence the probability that R^{n-l-1} evolves from τ by merging the trees τ_a, τ_b and τ_c, that is the probability that the leaf i is picked in either τ_a, τ_b or τ_c and connected to the roots of the other two components is

$$\frac{|\tau_a| + |\tau_b| + |\tau_c| + 3}{(2n+1-l)2(n-l)(2(n-l)-1)}.$$

□

As a consequence of Theorem 3.1, the time-reversed process $\{|R^k|^{\downarrow}\}_{0\leq k\leq n}$ is a fragmentation chain with dislocation law μ.

Remark 4.5. Adapting the proof of Corollary 5.7 in Bertoin (2006b) to our situation, we find another way to prove Corollary 2.5, based on the binary forest representation. Namely, with $m = 2(n-l)+1$ and $\mathbf{s} = (s_1, \ldots, s_m) \in \mathcal{P}^{\downarrow}_{2n+1}$, there are

$$\frac{1}{m!}\binom{2n+1}{s_1, \ldots, s_m} = \frac{(2n+1)!}{m!\, s_1! \cdots s_m!}$$

possibilities to partition the set of vertices $\{1, \ldots, 2n+1\}$ into non-empty disjoint sets E_i, $i = 1, \ldots, m$, such that $|E_i| = s_i$ and $\min E_i < \min E_j$ for $i < j \leq m$. Without labeling internal vertices, the number of binary tree structures which can be attached to E_i is $|\mathcal{F}(1, s_i)|/((s_i-1)/2)!$. Having chosen a binary tree structure for each E_i, there are $l!$ possible ways to label the l internal vertices. Recall that the tree components of a forest are enumerated in increasing order of their roots. It follows that the number of binary forests $\tau \in \mathcal{F}(m, 2n+1)$ with tree components τ_i such that $|\tau_i| = s_i$ is given by

$$\frac{(2n+1)!\, l!}{m!} \prod_{i=1}^{m} \frac{|\mathcal{F}(1, s_i)|}{s_i!\left(\frac{s_i-1}{2}\right)!}.$$

Since R^{n-l} is uniformly distributed on $\mathcal{F}(m, 2n+1)$, we deduce from Proposition 4.4 that

$$\mathbb{P}\left(\mathcal{X}'_l = (s_1, \ldots, s_m)\right) = \frac{\gamma(\mathbf{s})}{|\mathcal{F}(m, 2n+1)|} \frac{(2n+1)!\, l!}{m!} \prod_{i=1}^{m} \frac{|\mathcal{F}(1, s_i)|}{s_i!\left(\frac{s_i-1}{2}\right)!},$$

where $\gamma(\mathbf{s})$ has been defined in Section 2.2. Plugging in the values for $|\mathcal{F}(m, 2n+1)|$ and $|\mathcal{F}(1, s_i)|$ from Remark 4.3 results in the expression obtained in Corollary 2.5.

4.4. *Encoding forests by paths.* We conclude our discussion of binary forests by illustrating a direct connection to the random walk representation. Here, it is more convenient to consider (rooted unlabeled) plane trees and forests. In a plane forest vertices are regarded as indistinguishable, but the set of children for each vertex is ordered, as well as the set of roots of the different tree components. The ordering induces serveral natural enumerations of the vertices. For example, one of them is provided by the order in which the vertices are visited by a depth-first search, see Figure 4.4. More on this can be found in Chapter 6.2 of Pitman (2006).

We will look at (full) binary plane forests. To relate them to the binary forests considered above, note that the number of binary plane forests on N vertices with k tree components is equal to

$$\frac{2^{(N-k)/2}k!}{N!((N-k)/2)!}|\mathcal{F}(k,N)|,$$

since there are $2^{(N-k)/2}$ possible orderings of the children of the internal vertices of a forest in $\mathcal{F}(k,N)$, $k!$ orderings of the roots, but neither vertices are labeled nor there is an additional identification of internal vertices. Clearly the ternary coalescent with a monodisperse initial configuration can also be realized on the space of binary plane forests, with the same dynamics.

There are various possibilities to code plane trees and forests by discrete functions. For a (finite) plane tree θ on N vertices, one common way is to look at its Lukasiewicz path $\{x_l\}_{0\le l\le N}$. Denoting by $v_0,\dots,v_{N-1}$ the vertices of θ listed in the order of a depth-first search and by $k(v)$ the number of children of vertex v, one defines

$$x_j=\sum_{i=0}^{j-1}(k(v_i)-1),\quad 0\le j\le N.$$

Note that $x_0=0$, $x_N=-1$, and

$$x_j-x_{j-1}=k(v_{j-1})-1,\quad 1\le j\le N. \tag{4.1}$$

It is easy to see that there is a bijection between Lukasiewicz paths and rooted plane trees. A sequence of such trees may then by encoded by gluing together the corresponding Lukasiewicz paths, retaining the relationship (4.1). In other words, the coding of the next tree starts if a new minimum is attained.

Turning to random trees, it follows from Proposition 1.4 in Le Gall (2005) that a Galton-Watson tree with offspring distribution $\eta(k)=1/2(\delta_0(k)+\delta_2(k))$, conditioned to have total progeny size N, is distributed according to a tree chosen uniformly at random among the set of all binary plane trees on N vertices. Further, the corresponding Lukasiewicz path tree is distributed as the path of simple random walk on $\mathbb{Z}$ up to time N, conditioned on $H_{-1}=N$ (see Corollary 1.6 in Le Gall (2005)).

We then realize that for an integer $0\le l\le n$, the path of simple random walk up to time N, conditioned on $H_{-(2l+1)}=N$, encodes a forest distributed uniformly over all binary plane forests on N vertices with $2l+1$ tree components. In particular, the sequence of the sizes of the tree components is distributed as the sequence of the ladder epochs of the conditioned random walk path, if both are put in random uniform order, say. However, the sequence of coding functions induced by the above dynamics on the space of binary forests is not directly related to the sequence of

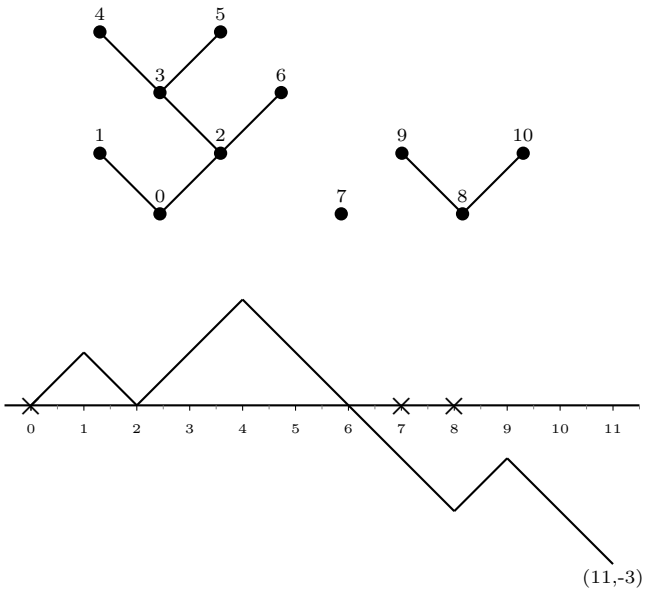

FIGURE 4.4. On the top a binary plane forest on 11 vertices with 3 tree components is shown, where the vertices are enumerated by a depth-first search. The corresponding Lukasiewicz path is depicted below. The crosses indicate where the coding of a new tree starts.

paths of the random walk representation. In this sense, the connection between the two representations is only static.

5. Asymptotics of the ternary coalescent

Having concrete realizations at hand, we are now able to investigate asymptotic properties of the ternary coalescent process. Let us write $\mathcal{X}^{[N]} = (\mathcal{X}^{[N]}(t),\, t \geq 0)$ for the coalescent with kernel κ started from the monodisperse configuration $(1, \ldots, 1)$ consisting of $N = 2n + 1$ atoms of unit mass, and put $\mathcal{X}'^{[N]}_k = \mathcal{X}^{[N]}(T_k)$, $k = 0, \ldots, n$. The number of particles at time $t \geq 0$ is denoted by $\#^{[N]}(t)$, and the number of jumps up to time t by $J^{[N]}(t)$.

We will consider the space of mass partitions with total mass bounded by 1,

$$\mathbb{S}_{\leq 1} = \left\{ \mathbf{s} = (s_1, s_2, \ldots) : s_1 \geq s_2 \geq \ldots \geq 0, \sum_{i=1}^{\infty} s_i \leq 1 \right\},$$

and the subset $\mathbb{S}_1 \subset \mathbb{S}_{\leq 1}$ of sequences with $\sum_{i=1}^{\infty} s_i = 1$. We equip $\mathbb{S}_{\leq 1}$ with the uniform distance. The induced topology coincides with that of pointwise convergence and turns $\mathbb{S}_{\leq 1}$ into a compact space. The l_1-distance induces a finer topology. However, if $(\mathbf{s}_n,\, n \in \mathbb{N})$ is some sequence in $\mathbb{S}_{\leq 1}$ converging pointwise to $\mathbf{s} \in \mathbb{S}_1$, then the convergence does also hold in the l_1-sense, as it can be easily deduced from Scheffé's lemma. Therefore, on $\mathbb{S}_1$ all these types of convergence are equivalent.

We turn to our main result of this section. Recall that the standard additive coalescent $\mathfrak{X} = (\mathfrak{X}(t),\, t \in \mathbb{R})$ is the unique additive coalescent process such that for each $t \in \mathbb{R}$, $\mathfrak{X}(t)$ has the law of the ranked sequence $\mathtt{a}_1 \geq \mathtt{a}_2 \geq \ldots$ of the atoms of a Poisson random measure on $(0, \infty)$ with intensity measure $\Lambda(da) = e^{-t} da/\sqrt{2\pi a^3}$, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_\mathtt{i} = 1$. We refer to Aldous and Pitman (1998) and Evans and Pitman (1998) for background.

Theorem 5.1. *As $n \to \infty$, the $\mathbb{S}_1$-valued process*

$$t \mapsto \frac{1}{N} \mathcal{X}^{[N]}(e^t/N^{3/2}), \quad t \in \mathbb{R},$$

converges in the sense of finite-dimensional distributions towards the standard additive coalescent.

Here, the multiplication with $1/N$ is meant element-wise. At first glance the convergence may look surprising, since the standard additive coalescent is a *binary* coalescent that arises as a limit of additive coalescent processes as follows (Evans and Pitman (1998)). Let $\mathfrak{X}^{[n]} = (\mathfrak{X}^{[n]}(t),\, t \geq 0)$ be the stochastic coalescent with binary coagulation kernel

$$\tilde{\kappa}(r, s) = r + s, \quad r, s > 0,$$

started from the monodisperse configuration with n atoms, each of mass $1/n$. Then, as $n \to \infty$, the time-shifted processes $(\mathfrak{X}^{[n]}(t + (1/2)\ln n), t \geq -(1/2)\ln n)$ converge weakly to $\mathfrak{X}$.

However, our convergence result concerns only the finite-dimensional laws. For the one-dimensional distributions, one might expect a result in this direction if one compares the one-dimensional statistics of the skeleton chains of the ternary and the additive coalescent $\mathfrak{X}^{[n]}$. The states of the additive coalescent can be expressed in terms of independent standard Borel variables (see for example (30) in Evans and Pitman (1998)), which have a similar tail behavior as the hitting time H_k. For the finite-dimensional laws, an analysis of the first hitting time distribution shows that a "true" ternary coagulation step, i.e. the event that three particles merge which are all of a size comparable to n, only occurs with negligible probability. Therefore, under the rescaling, the process looks more like a binary coalescent.

Let us briefly comment on the scaling in the theorem. To obtain a limit for the normalized sequence of masses, the number of atoms must be of order $\sqrt{n}$. We refer to Lemma 5.3 for a better understanding. As Lemma 5.2 shows, if the process $\mathcal{X}^{[N]}$ runs for time $t/N^{3/2}$, then the amount of particles has typically reduced from N to about $\sqrt{N}/t$. Note that when approximating the standard additive coalescent with the processes $\mathfrak{X}^{[n]}$ starting from n atoms of mass $1/n$, the macroscopic picture appears at times $t + (1/2)\ln n$, at which there are about $\sqrt{n}/e^t$ particles. Here, roughly speaking, the standard Borel law plays the role of the hitting time distribution. Precise statements can be found in the books of Pitman (2006), Chapter 10.3, and Bertoin (2006b), Chapter 5.3.

We shall present three different ways to obtain convergence for the rescaled ternary coalescent of which we discuss two in detail. The first more general method will lead to one-dimensional convergence in Proposition 5.5. It relies on the observation that the distribution of the hitting time H_k is in the domain of attraction of a stable(1/2) law. Then a size-biased reordering is used to construct the limiting mass partition. The second method resulting in finite-dimensional convergence

(and therefore in the proof of the theorem) is more specialized to our situation. It is based on the identification of configurations with mass partitions via paths, as described in Section 3.1. Since the two methods do not rely on each other, the reader in a hurry may safely skip Section 5.2. In a closing remark we outline a possible third way to establish finite-dimensional convergence, using the random binary forest representation.

5.1. *Number of particles.* In order to relate the behavior of $\mathcal{X}^{[N]}$ to that of its skeleton chain, we prove a limit theorem for the number of particles. As just remarked, it will become clear later why we choose the spatial scale factor $N^{-1/2}$.

Lemma 5.2. *For every $t > 0$, as $n \to \infty$,*

$$\frac{\#^{[N]}(t/N^{3/2})}{\sqrt{N}} \to \frac{1}{t} \quad \textit{in probability.}$$

Proof: Using the relation $\#^{[N]}(\cdot) = N - 2J^{[N]}(\cdot)$, the claim will follow once we show that

$$\frac{J^{[N]}(t/N^{3/2})}{\sqrt{N}} - \frac{(\sqrt{N} - t^{-1})}{2} \to 0 \quad \text{in probability.} \tag{5.1}$$

Remember that $J^{[N]}(t/N^{3/2}) = \max\{k \in \mathbb{Z}_+ : N^{3/2}T_k \le t\}$, where T_k is the kth coagulation time given by $T_k \stackrel{\mathrm{d}}{=} \sum_{i=1}^{k} \alpha(i)^{-1}\mathbf{e}_i$, the rates $\alpha(i) = \alpha(i, N)$ are as in (2.1) and $\mathbf{e}_1, \mathbf{e}_2, \ldots$ is a sequence of independent standard exponential variables. Heuristically, replacing T_k by its expectation $\sum_{i=1}^{k} \alpha(i)^{-1}$, the number of jumps $J^{[N]}(t/N^{3/2})$ should roughly behave as the maximal k such that $N^{3/2}\sum_{i=1}^{k} \alpha(i)^{-1} \le t$. We will show that with the choice $k_n = n - t^{-1}\sqrt{N}/2$,

$$N^{3/2}\sum_{i=1}^{k_n} \alpha(i)^{-1} = t + o(1), \tag{5.2}$$

where we agree that the sum runs from 1 to the largest integer below k_n. First note that

$$\begin{aligned} N^{3/2}\sum_{i=1}^{k_n} \alpha(i)^{-1} &= N^{3/2}\sum_{i=1}^{k_n} \frac{1}{(N+1-i)(N+1-2i)(N-2i)} \\ &= N^{3/2}\left(\sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2}\right) + O(n^{-1/2}). \end{aligned}$$

Furthermore, some simple computations show that for each $\varepsilon > 0$,

$$\begin{aligned} \sum_{i=1}^{k_n} \frac{1}{(N-i)(N-2i)^2} &= \int_0^{k_n} \frac{\mathrm{d}x}{(N-x)(N-2x)^2} + O(n^{-2}) \\ &= \frac{1}{N(N-2k_n)} + O(n^{-2+\varepsilon}) \\ &= \frac{t}{N^{3/2}} + O(n^{-2+\varepsilon}). \end{aligned}$$

Altogether, we obtain (5.2). Moreover, since

$$\mathrm{Var}\left(N^{3/2}T_{k_n}\right) = N^3\sum_{i=1}^{k_n} \alpha(i)^{-2} = O(n^{-1/2}),$$

we deduce that $N^{3/2}T_{k_n} \to t$ in probability. From this (5.1) readily follows. □

5.2. *Mass partitions induced by Poisson measures.* We now prove one-dimensional convergence of the ternary coalescent process. The reader who is not familiar with Poisson random measures and size-biased reorderings is invited to consult Section 2.2.3 of Bertoin (2006b) first.

Let $\xi_1, \xi_2, \ldots$ be a sequence of independent copies of H_{-1}. Recall that by Lemma 3.2, as $l \to \infty$,

$$\mathbb{P}(\xi_1 = 2l+1) \sim \frac{1}{2}\sqrt{\frac{1}{\pi l^3}}. \tag{5.3}$$

For $k \in \mathbb{Z}_+$, let $\Sigma_{2k+1} = \xi_1 + \ldots + \xi_{2k+1}$, and denote by $S^{(2k+1,N)}$ a random mass partition distributed as the rearrangement in decreasing order of $\xi_1/N, \ldots, \xi_{2k+1}/N$, conditionally on $\Sigma_{2k+1} = N$. As a special case of Corollary 2.2 in Bertoin (2006b) we have

Lemma 5.3. *Fix $b > 0$. Then $S^{(2k+1,N)}$ converges in distribution on $\mathbb{S}_{\le 1}$ as k, $n \to \infty$ with $k \sim bn^{1/2}$ to the ranked sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $\Lambda(da) = b\pi^{-1/2}a^{-3/2}da$, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$.*

For the skeleton chain $\mathcal{X}'^{[N]}$, we derive the following consequence.

Corollary 5.4. *Fix $b > 0$. If $n, k \to \infty$ with $k \sim bn^{1/2}$, then $(1/N)\mathcal{X}'^{[N]}_{n-k}$ converges in distribution on $\mathbb{S}_1$ to the ranked sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ of the atoms of a Poisson random measure on $(0,\infty)$ with intensity $\Lambda(da) = b\pi^{-1/2}a^{-3/2}da$, conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$.*

Proof: This follows from Proposition 3.3 together with the last lemma. □

Combining the corollary with the weak convergence result for the number of particles, we easily obtain one-dimensional convergence.

Proposition 5.5. *Fix $t > 0$. Then*

$$\frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2})$$

converges in distribution on $\mathbb{S}_1$ to the ranked sequence $(\mathtt{a}_1, \mathtt{a}_2, \ldots)$ of the atoms of a Poisson random measure on $(0,\infty)$ with intensity

$$\frac{t^{-1}}{\sqrt{2\pi a^3}}da, \quad a > 0,$$

conditioned on $\sum_{i=1}^{\infty} \mathtt{a}_i = 1$. In particular, the one-dimensional distributions of the process

$$t \mapsto \frac{1}{N}\mathcal{X}^{[N]}(e^t/N^{3/2}), \quad t \in \mathbb{R},$$

converge to those of the standard additive coalescent.

Proof: Let $k_n = n - J^{[N]}(t/N^{3/2})$. Then $\mathcal{X}^{[N]}(t/N^{3/2}) = \mathcal{X}'^{[N]}_{n-k_n}$, so we may show convergence for $(1/N)\mathcal{X}'^{[N]}_{n-k_n}$. From Lemma 5.2 it follows that as $n \to \infty$,

$$\frac{k_n}{\sqrt{n}} \to \frac{t^{-1}}{\sqrt{2}} \quad \text{in probability.} \tag{5.4}$$

Furthermore, we know from Corollary 5.4 that if l_n is a deterministic sequence of integers with $l_n \sim \sqrt{n}t^{-1}/\sqrt{2}$, then we have the asserted convergence for $(1/N)\mathcal{X}'^{[N]}_{n-l_n}$.

It therefore remains to argue that we may replace l_n by the random sequence k_n. To this end, recall that $\mathbb{S}_{\leq 1}$ is a compact metric space, so by Prohorov's theorem (see Billingsley (1968), Section 6) the space of probability measures on $\mathbb{S}_{\leq 1}$ is relatively compact, and we only have to show convergence on $\mathbb{S}_{\leq 1}$ in the sense of finite-dimensional distributions. Since all our random mass partitions lie in $\mathbb{S}_1$ almost surely, this leads to convergence in distribution on $\mathbb{S}_1$. Denote by $x_i^{[N]}$ the ith component of $(1/N)\mathcal{X}'^{[N]}_{n-k_n}$. Finite-dimensional convergence on $\mathbb{S}_{\leq 1}$ is equivalent to say that for each $j \in \mathbb{N}$,

$$\left(x_1^{[N]}, x_1^{[N]} + x_2^{[N]}, \ldots, x_1^{[N]} + \ldots + x_j^{[N]}\right)$$

converges in distribution towards $(\mathtt{a}_1, \mathtt{a}_1 + \mathtt{a}_2, \ldots, \mathtt{a}_1 + \ldots + \mathtt{a}_j)$. This follows if we show that for all $j \in \mathbb{N}$ and $\lambda_i \geq 0$, as $n \to \infty$,

$$\mathbb{E}\left[\exp\left(-\sum_{i=1}^{j} \lambda_i \left(x_1^{[N]} + \ldots + x_i^{[N]}\right)\right)\right] \to \mathbb{E}\left[\exp\left(-\sum_{i=1}^{j} \lambda_i \left(\mathtt{a}_1 + \ldots + \mathtt{a}_i\right)\right)\right]. \tag{5.5}$$

Denote by $f : \mathbb{S}_{\leq 1} \to (0,1]$ the function

$$f(\mathbf{s}) = \exp\left(-\sum_{i=1}^{j} \lambda_i \left(s_1 + \ldots + s_i\right)\right), \quad \mathbf{s} = (s_1, s_2, \ldots) \in \mathbb{S}_{\leq 1}.$$

Note that $f((1/N)\mathcal{X}^{[N]}(t)) \geq f((1/N)\mathcal{X}^{[N]}(s))$ almost surely whenever $t \leq s$. By (5.4) we can find deterministic sequences of integers l_n^- and l_n^+ such that $l_n^- \sim l_n^+ \sim \sqrt{n}t^{-1}/\sqrt{2}$ and the probability of the event $\{l_n^- \leq k_n \leq l_n^+\}$ tends to 1 as $n \to \infty$. But on this event, we have by monotonicity

$$f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-l_n^-}\right) \leq f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-k_n}\right) \leq f\left(\frac{1}{N}\mathcal{X}'^{[N]}_{n-l_n^+}\right).$$

The expectations of the outer quantities converge to the right side of (5.2). This finishes the proof. □

5.3. *Convergence of ladder epochs.* Aldous and Pitman (1998) have shown that the exponential time change

$$F(t) = \mathfrak{X}(-\ln t), \quad t > 0,$$

with $F(0) = (1, 0, \ldots)$ transforms the standard additive coalescent into a fragmentation process which is self-similar with index $\alpha = 1/2$. In Bertoin (2000) one finds an explicit construction of this fragmentation process in terms of ladder epochs of Brownian excursion with drift, and our result on finite-dimensional convergence for the ternary coalescent will be based on this identity.

Let us introduce some notation. We denote by $C[0,1]$ the space of continuous real-valued paths on $[0,1]$, endowed with the uniform topology. For an arbitrary path $\omega \in C[0,1]$, its ladder time set is given by

$$\mathcal{L}(\omega) = \left\{s \in [0,1] : \omega(s) = \inf_{[0,s]} \omega\right\}.$$

Since $\mathcal{L}(\omega)$ is a closed set, there exists a unique decomposition of $[0,1]\backslash\mathcal{L}(\omega)$ into a countable union of disjoint (open) intervals. We denote by $G(\omega)$ the ranked

sequence of their lengths. By filling up with zeros, we may always interpret $G(\omega)$ as a mass partition in $\mathbb{S}_{\leq 1}$. Note that $G(\omega) \in \mathbb{S}_1$ if and only if $\mathcal{L}(\omega)$ has Lebesgue measure zero.

The construction of the dual fragmentation process F in Bertoin (2000) can be summarized as follows. Let $\epsilon = (\epsilon(s),\, 0 \leq s \leq 1)$ be a positive Brownian excursion. For every $t \geq 0$, consider the excursion dragged down with drift t, that is $\epsilon_t(s) = \epsilon(s) - st$, $0 \leq s \leq 1$, and its ladder time set $\mathcal{L}(\epsilon_t)$, which has almost surely Lebesgue measure zero. Then, the law of $(G(\epsilon_t),\, t \geq 0)$ and $(F(t),\, t \geq 0)$ coincide.

In light of our representation of the ternary coalescent in terms of ladder epochs, it seems natural to establish convergence of these objects. In this direction, the main step is to prove convergence of the underlying random paths, with the origin placed at the first instant when their minimum is attained, towards a Brownian excursion with drift.

To begin with, take a process $(J_n(t),\, t \geq 0)$ distributed as $(J^{[N]}(t/N^{3/2}),\, t \geq 0)$, and independently of this a Markov chain $\{X_l\}_{0 \leq l \leq n}$ as defined in Section 3.2. Let us first fix $t > 0$, and write $J_n = J_n(t)$. Remember that given J_n, we may identify X_{J_n} with simple random walk up to time N, conditioned to end at $-(2(n-J_n)+1)$,

$$S(X_{J_n})_j = 2\left(\sum_{i=0}^{j-1} X_{J_n}(i)\right) - j, \quad 0 \leq j \leq 2n+1.$$

By linear interpolation, we define the corresponding continuous random path $S_{n,t}$ on the unit interval,

$$S_{n,t}(s) = 2\left(\sum_{i=0}^{\lfloor Ns \rfloor - 1} X_{J_n}(i) + (Ns - \lfloor Ns \rfloor) X_{J_n}(\lfloor Ns \rfloor)\right) - Ns, \quad 0 \leq s \leq 1.$$

We shall now prove convergence of the finite-dimensional laws of the $C[0,1]$-valued process $(N^{-1/2} S_{n,t},\, t > 0)$. The limiting object $(B^{br}_{t^{-1}},\, t > 0)$ is distributed as

$$(B^{br}_{t^{-1}},\, t > 0) \stackrel{\mathrm{d}}{=} \left((B^{br}(s) - st^{-1},\, 0 \leq s \leq 1),\, t > 0\right), \tag{5.6}$$

where B^{br} is a standard Brownian bridge on the unit interval. In particular, for each fixed t, the distribution of $B^{br}_{t^{-1}}$ on $C[0,1]$ is that of a Brownian bridge from 0 to $-t^{-1}$.

Lemma 5.6. *The $C[0,1]$-valued process $\left(N^{-1/2} S_{n,t},\, t > 0\right)$ converges in the sense of finite-dimensional distributions as $n \to \infty$ to $\left(B^{br}_{t^{-1}},\, t > 0\right)$.*

Proof: Let us fix $t > 0$ as above and first prove one-dimensional convergence. For $0 \leq s \leq 1$, define

$$W_n(s) = 2\left(\sum_{i=0}^{\lfloor Ns \rfloor - 1} X_n(i) + (Ns - \lfloor Ns \rfloor) X_n(\lfloor Ns \rfloor)\right) - Ns,$$

$$D_n(s) = 2\left(\sum_{i=0}^{\lfloor Ns \rfloor - 1} (X_n(i) - X_{J_n}(i)) + (Ns - \lfloor Ns \rfloor)\left(X_n(\lfloor Ns \rfloor) - X_{J_n}(\lfloor Ns \rfloor)\right)\right).$$

We may then express $S_{n,t}$ as $S_{n,t} = W_n - D_n$.

The process $W_n(\cdot)$ is linear interpolation of simple random walk up to time N, conditioned to end at -1. We deduce from a conditioned version of Donsker's

invariance principle (see Dwass and Karlin (1963)) that $(N^{-1/2}W_n(s),\, 0 \le s \le 1)$ converges weakly in $C[0,1]$ to the standard Brownian bridge B^{br}.

Concerning the drift part D_n, we let

$$\begin{aligned} D_n^{(1)}(s) &= \sum_{i=0}^{\lfloor Ns\rfloor -1} \left(X_n(i) - X_{J_n}(i)\right), \\ D_n^{(2)}(s) &= 2(Ns - \lfloor Ns\rfloor)\left(X_n(\lfloor Ns\rfloor) - X_{J_n}(\lfloor Ns\rfloor)\right), \end{aligned}$$

so that $D_n = 2D_n^{(1)} + D_n^{(2)}$. Now fix $s \in [0,1]$. A moment's thought reveals that conditioned on $J_n = n-k$ for some $k \in \{0,\dots,n\}$, the random variable $D_n^{(1)}(s)$ follows the hypergeometric distribution. More precisely,

$$\mathbb{P}\left(D_n^{(1)}(s) = j \mid\ J_n = n-k\right) = \frac{\binom{\lfloor Ns\rfloor}{j}\binom{N-\lfloor Ns\rfloor}{k-j}}{\binom{N}{k}},$$

where $\max\{0, k+\lfloor Ns\rfloor - N\} \le j \le \min\{k, \lfloor Ns\rfloor\}$. As a consequence,

$$\mathbb{E}\left[D_n^{(1)}(s) \mid\ J_n = n-k\right] = k\frac{\lfloor Ns\rfloor}{N}, \quad \mathrm{Var}\left(D_n^{(1)}(s) \mid\ J_n = n-k\right) \le k. \tag{5.7}$$

Let $k_n = n - J_n$. Choosing $\varepsilon > 0$ arbitrarily small, we have for large n by the law of total probability

$$\begin{aligned} &\mathbb{P}\left(N^{-1/2}|D_n(s) - 2k_n s| > \varepsilon\right) \\ &\le \sum_{k=0}^{\lfloor \sqrt{n}t^{-1}\rfloor} \mathbb{P}\left(N^{-1/2}|D_n^{(1)}(s) - \mathbb{E}[D_n^{(1)}(s)| > \varepsilon/3 \mid\ k_n = k\right)\mathbb{P}(k_n = k) \\ &\quad + \mathbb{P}\left(k_n \ge \sqrt{n}t^{-1}\right) \\ &= o(1), \end{aligned}$$

where the last line follows from (5.4), (5.7) and Chebyshev's inequality. Since by (5.1), $N^{-1/2}2k_n s$ converges in probability to $t^{-1}s$, so does $N^{-1/2}D_n(s)$. In particular, the finite-dimensional laws of $(N^{-1/2}D_n(s),\, 0 \le s \le 1)$ converge to those of $(t^{-1}s,\, 0 \le s \le 1)$. Moreover, $D_n(s)$ is increasing in s, and a similar computation entails that for λ large enough, as $n \to \infty$,

$$\mathbb{P}\left(N^{-1/2}D_n(1) \ge \lambda\right) = o(1).$$

Using Theorem 8.4 of Billingsley (1968), we conclude that the distributions of $N^{-1/2}D_n(\cdot)$ form a tight sequence. It follows that $(N^{-1/2}D_n(s),\, 0 \le s \le 1)$ converges in probability to $(t^{-1}s,\, 0 \le s \le 1)$. Applying now Theorem 4.4 from Billingsley (1968) together with the continuous mapping theorem finishes the proof of the one-dimensional convergence.

The arguments obviously extend to finite-dimensional distributions. Indeed, the bridge term W_n is the same for all t, and the drift term D_n converges in probability, for each t. Therefore, finite-dimensional convergence follows again from Theorem 4.4 of Billingsley (1968). □

As for discrete paths, we introduce for $v \in [0,1]$ the shift operator θ on $C[0,1]$,

$$(\theta_v\omega)(s) = \begin{cases} \omega(s+v) - \omega(v) & ,\ 0 \le s \le 1-v \\ \omega(s+v-1) - \omega(v) + \omega(1) - \omega(0) & ,\ 1-v < s \le 1 \end{cases}.$$

Define $H : C[0,1] \to [0,1]$ as the first time when the global minimum is attained,

$$H(\omega) = \inf\left\{ s \in [0,1] : \omega(s) = \inf_{[0,1]} \omega \right\}.$$

Clearly, H is not continuous on the whole space, but it is so restricted to the subset of paths which uniquely attain their minimum. It is well-known and also implied by the subsequent Lemma 5.8 that the distribution of $B^{br}_{t^{-1}}$ is fully supported on this subset. Further, the shift operator is continuous as a map

$$\theta : C[0,1] \times [0,1] \to C[0,1], \quad \theta(\omega, v) = \theta_v \omega.$$

Setting $\theta_H \omega = \theta_{H(\omega)}\omega$, it then follows from the above lemma and the continuous mapping theorem that for $n \to \infty$,

$$\left(N^{-1/2} \theta_H S_{n,t},\, t > 0 \right) \to \left(\theta_H B^{br}_{t^{-1}},\, t > 0 \right)$$

in the sense of finite-dimensional distributions. Recall $(B^{br}_{t^{-1}}(s),\, 0 \le s \le 1) \stackrel{\mathrm{d}}{=} (B^{br}(s) - st^{-1},\, 0 \le s \le 1)$, where B^{br} is a Brownian bridge (the same for all t). Denoting by ϵ a standard Brownian excursion, it has been proven by Vervaat (1979) that

$$\theta_H B^{br} \stackrel{\mathrm{d}}{=} \epsilon.$$

Since $\theta_u \circ \theta_v = \theta_w$ for $w = u + v [\mathrm{mod}\ 1]$, we have $\theta_H = \theta_H \circ \theta_v$ pathwise for every $0 \le v \le 1$. Therefore, if μ denotes the almost surely unique instant when B^{br} attains its minimum,

$$\begin{aligned} \theta_H B^{br}_{t^{-1}} &\stackrel{\mathrm{d}}{=} \theta_H \circ \theta_\mu \left(B^{br} - st^{-1},\, 0 \le s \le 1 \right) \\ &= \theta_H \left(\theta_H B^{br} - st^{-1},\, 0 \le s \le 1 \right) \\ &\stackrel{\mathrm{d}}{=} \theta_H \epsilon_{t^{-1}}. \end{aligned} \tag{5.8}$$

Here, as above, $\epsilon_{t^{-1}}(s) = \epsilon(s) - st^{-1}$ is the Brownian excursion dragged down with drift t^{-1}. Since $\epsilon_{t^{-1}}$ attains its minimal value almost surely at the endpoint, we have proven the following

Corollary 5.7. *In the notation above,* $\left(N^{-1/2} \theta_H S_{n,t},\, t > 0 \right)$ *converges in the sense of finite-dimensional distributions as* $n \to \infty$ *to* $(\epsilon_{t^{-1}},\, t > 0)$.

The convergence of the ternary coalescent is now easy to establish. As last preparation, let us recall a technical result. Call a point $x \in [0,1]$ a *local minimum* of $\omega \in C[0,1]$, if there exists $\delta > 0$ such that for all $y \in [\max\{x-\delta, 0\}, \min\{x+\delta, 1\}]$, $\omega(x) \le \omega(y)$. The following statement is true for all real t.

Lemma 5.8. *With probability one, all local minima of* $(\varepsilon_t(s),\, 0 \le s \le 1)$ *are distinct.*

Proof: By (5.8), we may show the statement for $(B^{br}_t(s),\, 0 \le s \le 1)$ instead. Since for the time-reversed process, it holds that

$$(B^{br}(1-s) - (1-s)t,\, 0 \le s \le 1) \stackrel{\mathrm{d}}{=} (B^{br}(s) + st - t,\, 0 \le s \le 1),$$

it suffices to show that for some $1/2 \le r < 1$, $(B^{br}_t(s),\, 0 \le s \le r)$ has almost surely distinct local minima. However, if $\mathcal{F}_r$ denotes the filtration generated by the canonical process $x.$ on $C[0,1]$ up to time $r < 1$, $\mathbb{Q}$ denotes the law of B^{br}_t

and, for a moment, $\mathbb{P}$ is Wiener measure and p the Gaussian transition kernel, it is well-known that $\mathbb{Q}$ is locally absolute continuous with respect to $\mathbb{P}$,

$$\mathbb{Q}|_{\mathcal{F}_r} = \frac{p_{1-r}(x_r, -t)}{p_1(0, -t)} \cdot \mathbb{P}|_{\mathcal{F}_r}.$$

Since the local minima of Brownian motion on $[0, 1]$ are distinct almost surely (see for example Theorem 2.11 in the book of Mörters and Peres (2010)), the lemma is proven. □

Proof of Theorem 5.1: In view of the result of Bertoin (2000), the claim follows if we show that the finite-dimensional laws of

$$t \mapsto \frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2}), \quad t > 0,$$

converge to those of $(G(\varepsilon_{t^{-1}}),\, t > 0)$. Remember the map φ constructed in Section 3.1 sending configurations to mass partitions. With $J_n(t) = J^{[N]}(t/N^{3/2})$ defined as above, we have already seen that

$$\left(\frac{1}{N}\varphi(X_{J_n(t)}),\, t \geq 0\right) \stackrel{\mathrm{d}}{=} \left(\frac{1}{N}\mathcal{X}^{[N]}(t/N^{3/2}),\, t \geq 0\right).$$

Let $t > 0$, and assume that conditionally on J_n,

$$\frac{1}{N}\varphi(X_{J_n(t)}) = (s_1, \ldots, s_{2(n-J_n(t))+1}),$$

where $Ns_i \in \{1, 3, 5, \ldots, N\}$ with $\sum s_i = 1$. Then by construction of both φ, G and linear interpolation,

$$G(N^{-1/2}\theta_H S_{n,t}) = (g_1, \ldots, g_{2(n-J_n(t))+1}),$$

with $g_i = s_i - 1/N$ for all i. Thus, the theorem follows if we show finite-dimensional convergence of $(G(N^{-1/2}\theta_H S_{n,t}),\, t > 0)$ to $(G(\varepsilon_{t^{-1}}),\, t > 0)$. It is easy to check that $G : C[0, 1] \to \mathbb{S}_{\leq 1}$ is continuous on the subset of those paths which attain their local minima at unique points. By Lemma 5.8, the distribution of $\varepsilon_{t^{-1}}$ assigns mass one to this subset. Therefore, Corollary 5.7 and the continuous mapping theorem yield convergence of the finite-dimensional distributions on $\mathbb{S}_{\leq 1}$, and since $G(\varepsilon_{t^{-1}}) \in \mathbb{S}_1$ with probability one, we obtain finite-dimensional convergence on $\mathbb{S}_1$. □

Remark 5.9. (i) For the proof of Theorem 5.1 we used the random walk representation. Let us point out another possibility to derive convergence, using the random binary forest representation. Following the construction in Section 4, the state chain of the ternary coalescent starting from N particles of unit mass can be realized in reversed time by deleting successively pairs of outgoing edges from a random tree uniformly distributed over all binary plane trees on N vertices. Such a random tree can be seen as a Galton-Watson tree with offspring distribution $\mu(k) = \frac{1}{2}(\delta_0(k) + \delta_2(k))$, conditioned to have total population size N. One finds oneself in the setting of Theorem 23 (in the sublattice case) of Aldous (1993). In particular, if $\tau^{[N]}$ denotes the uniform binary plane tree on N vertices, where mass $1/N$ is assigned to each vertex and the edges are rescaled to have length $1/\sqrt{N}$, then $\tau^{[N]}$ converges weakly as $N \to \infty$ to the Brownian continuum random tree (CRT) introduced in Aldous (1991). By splitting the skeleton of this tree into subtrees according to a Poisson process of cuts with some intensity $t \geq 0$ per unit length, Aldous and Pitman (1998) derived from the CRT an $\mathbb{S}_1$-valued fragmentation process of ranked masses of tree components, indexed by the intensity t.

Further, they showed that the time change $t \mapsto e^{-t}$ turns this process into the standard additive coalescent. As in Aldous and Pitman (1998), it should be possible to approximate the Poisson process of marks on the CRT by the process of deleting edges from the binary plane tree. This would lead to another proof of Theorem 5.1.

(ii) Recall the random walk representation introduced in Section 3. Fix an integer k of size at least 3, and define the configuration space $\mathcal{C}_n^k$ as the set of all subsets of $\{0, \ldots, (k-1)n\}$ with cardinality less or equal to n. Now identify a configuration $x \in \mathcal{C}_n^k$ with a path of a walk that goes up $k-2$ steps if a site is occupied and one step down otherwise, i.e. $S^{(k)}(x)_0 = 0$ and for $1 \leq j \leq (k-1)n+1$,

$$S^{(k)}(x)_j = k \left(\sum_{i=0}^{j-1} x(i) \right) - j.$$

By imposing an analogous dynamics, i.e. by occupying successively n sites chosen uniformly at random from $\{0, \ldots, (k-1)n\}$, the sequence of ladder epochs of the corresponding new paths is now a realization of the state chain of the k-ary coalescent process with kernel $\kappa_k(s_1, \ldots, s_k) = s_1 + \ldots + s_k + k/(k-2)$, started from $(k-1)n+1$ particles of unit mass. As for the case $k = 3$, running this process backwards in time yields a fragmentation process. Moreover, Kemperman's formula applies also to first hitting times of such asymmetric random walks, so that their distributions can easily be computed. With some minor modifications, and under a different rescaling of time, one again obtains convergence of the finite-dimensional laws of this k-ary coalescent process towards those of the standard additive coalescent.

Not surprisingly, there is an analogous random $(k-1)$-ary forest representation of this process. Indeed, when glueing (full) $(k-1)$-ary trees by picking uniformly at random one leaf and $k-1$ roots from different components, in a similar way as described in Section 4 for the case $k = 3$, the ranked sequence of the tree sizes is another realization of the state chain of the k-ary coalescent with kernel κ_k.

This remark shows that our ternary coalescent process is only one particular process out of a family of k-ary coalescents that can be studied by the same means.

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References

D. Aldous. The continuum random tree. I. *Ann. Probab.* **19** (1), 1–28 (1991). MR1085326.

D. Aldous. The continuum random tree. III. *Ann. Probab.* **21** (1), 248–289 (1993). MR1207226.

D. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.* **26** (4), 1703–1726 (1998). MR1675063.

J. Bertoin. A fragmentation process connected to Brownian motion. *Probab. Theory Related Fields* **117** (2), 289–301 (2000). MR1771665.

J. Bertoin. Eternal additive coalescents and certain bridges with exchangeable increments. *Ann. Probab.* **29** (1), 344–360 (2001). MR1825153.

J. Bertoin. Different aspects of a random fragmentation model. *Stochastic Process. Appl.* **116** (3), 345–369 (2006a). MR2199553.

J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2006b). ISBN 978-0-521-86728-3; 0-521-86728-2. MR2253162.

J. Bertoin, L. Chaumont and J. Pitman. Path transformations of first passage bridges. *Electron. Comm. Probab.* **8**, 155–166 (electronic) (2003). MR2042754.

P. Billingsley. *Convergence of probability measures*. John Wiley & Sons Inc., New York (1968). MR0233396.

M. Dwass and S. Karlin. Conditioned limit theorems. *Ann. Math. Statist.* **34**, 1147–1167 (1963). MR0166814.

S. N. Evans and J. Pitman. Construction of Markovian coalescents. *Ann. Inst. H. Poincaré Probab. Statist.* **34** (3), 339–383 (1998). MR1625867.

J. H. B. Kemperman. *The passage problem for a stationary Markov chain*. Statistical Research Monographs, Vol. I. The University of Chicago Press, Chicago, Ill. (1961). MR0119226.

J.-F. Le Gall. Random trees and applications. *Probab. Surv.* **2**, 245–311 (2005). MR2203728.

P. Mörters and Y. Peres. *Brownian motion*. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, Cambridge (2010). ISBN 978-0-521-76018-8. With an appendix by Oded Schramm and Wendelin Werner. MR2604525.

J. Pitman. Coalescent random forests. *J. Combin. Theory Ser. A* **85** (2), 165–193 (1999). MR1673928.

J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2006). ISBN 978-3-540-30990-1; 3-540-30990-X. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard. MR2245368.

W. Vervaat. A relation between Brownian bridge and Brownian excursion. *Ann. Probab.* **7** (1), 143–149 (1979). MR515820.