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High order chaotic limits of wavelet scalograms under long-range dependence

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Abstract. Let G be a non–linear function of a Gaussian process $\{X_t\}_{t\in\mathbb{Z}}$ with long–range dependence. The resulting process $\{G(X_t)\}_{t\in\mathbb{Z}}$ is not Gaussian when G is not linear. We consider random wavelet coefficients associated with $\{G(X_t)\}_{t\in\mathbb{Z}}$ and the corresponding wavelet scalogram which is the average of squares of wavelet coefficients over locations. We obtain the asymptotic behavior of the scalogram as the number of observations and the analyzing scale tend to infinity. It is known that when G is a Hermite polynomial of any order, then the limit is either the Gaussian or the Rosenblatt distribution, that is, the limit can be represented by a multiple Wiener-Itô integral of order one or two. We show, however, that there are large classes of functions G which yield a higher order Hermite distribution, that is, the limit can be represented by a a multiple Wiener-Itô integral of order greater than two. This happens for example if G is a linear combination of a Hermite polynomial of order 1 and a Hermite polynomial of order $q > 3$. The limit in this case can be Gaussian but it can also be a Hermite distribution of order $q - 1 > 2$. This depends not only on the relation between the number of observations and the scale size but also on whether q is larger or smaller than a new critical index q^*. The convergence of the wavelet scalogram is therefore significantly more complex than the usual one.

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1. Introduction

Denote by $X = \{X_t\}_{t\in\mathbb{Z}}$ a centered stationary Gaussian process with unit variance and spectral density $f(\lambda), \lambda \in (-\pi, \pi)$. Such a stochastic process is said to have *short memory* or *short–range dependence* if $f(\lambda)$ is bounded around $\lambda = 0$ and *long memory* or *long–range dependence* if $f(\lambda) \to \infty$ as $\lambda \to 0$. We will suppose that $\{X_t\}_{t\in\mathbb{Z}}$ has long memory with memory parameter $0 < d < 1/2$, that is,

$$f(\lambda) \sim |\lambda|^{-2d} f^*(\lambda) \text{ as } \lambda \to 0 \tag{1.1}$$

where $f^*(\lambda)$ is a bounded spectral density which is continuous and positive at the origin. This hypothesis is semi–parametric in nature because the function f^* plays the role of a "nuisance function". It is convenient to set

$$f(\lambda) = |1 - \mathrm{e}^{-\mathrm{i}\lambda}|^{-2d} f^*(\lambda), \quad \lambda \in (-\pi, \pi] \,. \tag{1.2}$$

Since the process X is defined only if $\int_{-\pi}^{\pi} f(\lambda) d\lambda < \infty$, we need to require $d < \frac{1}{2}$.

Consider now a process $\{Y_t\}_{t\in\mathbb{Z}}$, such that

$$\left(\Delta^K Y\right)_t = G(X_t), \quad t \in \mathbb{Z}\,, \tag{1.3}$$

for $K \geq 0$, where $(\Delta Y)_t = Y_t - Y_{t-1}$, $\{X_t\}_{t\in\mathbb{Z}}$ is Gaussian with spectral density f satisfying (1.2) and where G is a function such that $\mathbb{E}[G(X_t)] = 0$ and $\mathbb{E}[G(X_t)^2] < \infty$. While the process $\{Y_t\}_{t\in\mathbb{Z}}$ is not necessarily stationary, its K–th difference $\Delta^K Y_t$ is stationary and is the output of a non–linear filter G with Gaussian input.

We shall study the asymptotic behavior of the wavelet scalogram of $\{Y_t\}_{t\in\mathbb{Z}}$, that is, the average of squares of its wavelet coefficients. As shown in Flandrin (1992); Abry and Veitch (1998); Veitch and Abry (1999) and Bardet (2000) in a parametric context, the normalized limit of scalogram can be used to estimate the long memory exponent d defined in (1.1).

Empirical studies presented in Abry et al. (2011) consider the problem of estimating d under various types of functions G. The argument, consistent with the one in Clausel et al. (2012), suggests that at large scales the wavelet coefficients behavior only depends on the "Hermite rank", which is defined below, of G. Moreover the authors develop heuristical arguments to deduce the asymptotic behavior of wavelet-based regression estimator of d. We provide here a theoretical analysis in a semi–parametric setting for a large class of functions G. We will show that, as j goes to infinity, there is a delicate interplay between the scale γ_j (typically 2^j) and the number of wavelet coefficients n_j and that the "reduction theorem" (see below) applies only when γ_j is much greater than n_j.

In the semi–parametric context, the case where the function G is linear was firstly considered in Moulines et al. (2007) and the case where G is a Hermite polynomial of arbitrary order was studied in Clausel et al. (2014). The case where $G(X_t)$ is the so–called "Rosenblatt process" was studied by Bardet and Tudor (2010) (see also Tudor (2013)) and is somewhat analogous to the one where G is the second Hermite polynomial. Our goal is to show that for more complicated functions G, one can obtain new types of limits.

We have referred to Hermite polynomials a number of times. This is because they form a basis for the space of functions G and thus appear naturally in our setting. Since the function G satisfies $\mathbb{E}[G(X)] = 0$ and $\mathbb{E}[G(X)^2] < \infty$ for $X \sim \mathcal{N}(0, 1)$, $G(X)$ can be expanded in Hermite polynomials, that is,

$$G(X) = \sum_{q=1}^{\infty} \frac{c_q}{q!} H_q(X)\,. \tag{1.4}$$

One sometimes refer to (1.4) as an expansion in Wiener chaos. The convergence of the infinite sum (1.4) is in $L^2(\Omega)$,

$$c_q = \mathbb{E}[G(X)H_q(X)]\,, \quad q \geq 1\,, \tag{1.5}$$

and

$$H_q(x) = (-1)^q e^{\frac{x^2}{2}} \frac{d^q}{dx^q}\left(e^{-\frac{x^2}{2}}\right)\,,$$

are the Hermite polynomials. These Hermite polynomials satisfy $H_0(x) = 1, H_1(x) = x, H_2(x) = x^2 - 1$ and one has

$$\mathbb{E}[H_q(X)H_{q'}(X)] = \int_{\mathbb{R}} H_q(x)H_{q'}(x)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-x^2/2}\mathrm{d}x = q!\mathbb{1}_{\{q=q'\}}\,.$$

Observe that the expansion (1.4) starts at $q = 1$, since

$$c_0 = \mathbb{E}[G(X)H_0(X)] = \mathbb{E}[G(X)] = 0 , \tag{1.6}$$

by assumption. Denote by $q_0 \geq 1$ the *Hermite rank* of G, namely the index of the first non–zero coefficient in the expansion (1.4). Formally, q_0 is such that

$$q_0 = \min\{q \geq 1,\, c_q \neq 0\} . \tag{1.7}$$

One has then

$$\sum_{q=q_0}^{+\infty} \frac{c_q^2}{q!} = \mathbb{E}[G(X)^2] < \infty . \tag{1.8}$$

We will focus on the wavelet coefficients of the sequence $\{Y_t\}_{t\in\mathbb{Z}}$ in (1.3). Since $\{Y_t\}_{t\in\mathbb{Z}}$ is random so will be its wavelet coefficients which we denote by $\{W_{j,k},\, j \geq 0,\, k \in \mathbb{Z}\}$, where j indicates the scale index and k the location. These wavelet coefficients are defined by

$$W_{j,k} = \sum_{t\in\mathbb{Z}} h_j(\gamma_j k - t) Y_t , \tag{1.9}$$

where $\gamma_j \uparrow \infty$ as $j \uparrow \infty$ is a sequence of non–negative decimation factors applied at scale index j, for example $\gamma_j = 2^j$ and h_j is a filter whose properties are listed in Appendix C. We follow the engineering convention where large values of j correspond to large scales. Our goal is to find the distribution of the empirical quadratic mean of these wavelet coefficients at large scales $j \to \infty$, that is, the asymptotic behavior of the wavelet scalogram

$$S_{n_j,j} = \frac{1}{n_j} \sum_{k=0}^{n_j-1} W_{j,k}^2 , \tag{1.10}$$

adequately centered and normalized as the scale γ_j and the number of wavelets coefficients n_j available at scale index j both tend to infinity.

The reduction theorem of Taqqu (1974/75) states that if $G(X_t)$ is long–range dependent then the limit in the sense of finite–dimensional distributions of $\sum_{k=1}^{[nt]} G(X_k)$ adequately normalized, depends on the first term in the Hermite expansion of G. In other words, there exist normalization factors $a_n \to \infty$ as $n \to \infty$ such that

$$\frac{1}{a_n} \sum_{k=1}^{[nt]} G(X_k) \qquad \text{and} \qquad \frac{1}{a_n} \sum_{k=1}^{[nt]} \frac{c_{q_0}}{q_0!} H_{q_0}(X_k) ,$$

have the same non–degenerate limit as $n \to \infty$.

We are interested here, however, in the asymptotic behavior of the wavelet scalogram $S_{n_j,j}$ in (1.10). We want to find exponents $\alpha > 0$ and $\nu > 0$ such that as the number of wavelet coefficients n_j and the scale γ_j tend to ∞,

$$\{n_j^{\alpha} \gamma_j^{-\nu} S_{n_{j+u},j+u}, u \in \mathbb{Z}\} , \tag{1.11}$$

tends, after centering, to a limit in the sense of the finite–dimensional distributions in the scale increment u. This is a necessary and important step in developing methods for estimating the underlying long memory parameter.

The limit of the sequence $S_{n_j,j}$ will be related to the so–called Hermite process. The Hermite process is a self-similar stochastic process, with stationary increments and long range dependence. The Hermite process of order q lives in the qth Wiener chaos, that is, it can be written as an iterated multiple integral of order q with respect to white noise. We refer to Definition 2.1 below for the precise representation.

We will see that, in the scalogram setting, the reduction theorem mentioned above does not always apply. For example if $G(X_t) = H_1(X_t) + H_{q_1}(X_t)$, $q_1 \geq 3$ then the Hermite rank is $q_0 = 1$. But the limit of the normalized scalogram is not necessarily the same as that of $H_1(X_t) = X_t$. This is essentially due to the fact that the scalogram involves squares and, in addition, depends on two parameters j and n_j which both tend to ∞.

In Clausel et al. (2014), the case

$$G(X_t) = H_q(X_t), \quad q \geq 2 ,$$

was studied and it was shown that in this case the limit is a Rosenblatt process (see Definition 2.1). In the present paper we study other classes of functions G for which different Hermite processes appear in the limit. For example, for the process

$$G(X_t) = H_1(X_t) + H_{q_1}(X_t), \quad q_1 \geq 3 ,$$

considered above, the limit of (1.11) may be either Gaussian, a Hermite process of order $q_1 - 1$ or a Rosenblatt process depending on the specific circumstances. We will show the existence of a *critical index* q_1^* and of critical exponents ν, ν' such that when $q_1 < q_1^*$, then :

- the limit is Gaussian if $n_j \ll \gamma_j^{\nu}$,
- the limit is a Hermite process of order $q_1 - 1$ if $\gamma_j^{\nu} \ll n_j \ll \gamma_j^{\nu'}$,
- the limit is a Rosenblatt process if $\gamma_j^{\nu'} \ll n_j$,

where $a_j \ll b_j$ means that $a_j = o(b_j)$ as $j \to \infty$.

We will also study interesting cases where the function G has a Hermite rank greater than two.

The paper is organized as follows. Long range–dependence and the multidimensional wavelet scalogram are introduced in Section 2. The main results are stated and illustrated in Section 3. The chaos decomposition of the scalogram is given in Section 4. The study of the leading terms is done in Sections 5 and 6. The proofs of the main theorems are given in Section 7 while Section 8 contains some technical lemmas. Basic facts about the Wiener chaos are gathered in Appendix B and Appendix C lists the assumptions on the wavelet filters.

2. Long-range dependence and the multidimensional wavelet scalogram

The Gaussian sequence $X = \{X_t\}_{t\in\mathbb{Z}}$ with spectral density (1.2) is long–range dependent because $d > 0$ and hence its spectrum explodes at $\lambda = 0$. Whether $\{H_q(X_t)\}_{t\in\mathbb{Z}}$ is also long–range dependent depends on the respective values of q and d. We show in Clausel et al. (2012), that the spectral density of $\{H_q(X_t)\}_{t\in\mathbb{Z}}$ behaves like $|\lambda|^{-2\delta_+(q)}$ as $\lambda \to 0$, where

$$\delta_+(q) = \max(\delta(q), 0) \quad \text{and} \quad \delta(q) = qd - (q-1)/2 . \tag{2.1}$$

Hence $\delta_+(q)$ is the memory parameter of $\{H_q(X_t)\}_{t\in\mathbb{Z}}$. Therefore, since $0 < d < 1/2$, $\{H_q(X_t)\}_{t\in\mathbb{Z}}$, $q \geq 1$, is long–range dependent if and only if

$$\delta(q) > 0 \Longleftrightarrow (1/2)(1 - 1/q) < d < 1/2 , \tag{2.2}$$

that is, d must be sufficiently close to $1/2$. Specifically, for long–range dependence,

$$q = 1 \Rightarrow d > 0, \quad q = 2 \Rightarrow d > 1/4, \quad q = 3 \Rightarrow d > 1/3, \quad q = 4 \Rightarrow d > 3/8 . \tag{2.3}$$

From another perspective,

$$\delta(q) > 0 \Longleftrightarrow 1 \leq q < 1/(1 - 2d) , \tag{2.4}$$

and thus $\{H_q(X_t)\}_{t\in\mathbb{Z}}$ is short–range dependent if $q \geq 1/(1 - 2d)$.

We shall suppose that the Hermite rank of G is $q_0 \geq 1$, that is the expansion of $G(X_t)$ starts at q_0. We always assume that $\{H_{q_0}(X_t)\}_{t\in\mathbb{Z}}$ has long memory, that is,

$$q_0 < 1/(1 - 2d) . \tag{2.5}$$

The condition (2.5), with q_0 defined as the Hermite rank (1.7), ensures such that $\{\Delta^K Y\}_{t\in\mathbb{Z}} = \{G(X_t)\}_{t\in\mathbb{Z}}$ is long-range dependent (see Clausel et al. (2012), Lemma 4.1). We are mainly interested in the asymptotic behavior of the scalogram $S_{n_j,j}$, defined by (1.10) as $n_j \to \infty$ (large sample behavior) and $j \to \infty$ (large scale behavior). More precisely, we will study the asymptotic behavior of the sequence

$$\overline{S}_{n_{j+u},j+u} = S_{n_{j+u},j+u} - \mathbb{E}(S_{n_{j+u},j+u}) = \frac{1}{n_{j+u}} \sum_{k=0}^{n_{j+u}-1} \left(W_{j+u,k}^2 - \mathbb{E}(W_{j+u,k}^2)\right) , \tag{2.6}$$

adequately normalized as $j, n_j \to \infty$.

There are two perspectives. One can consider, as in Clausel et al. (2012), that the wavelet coefficients $W_{j+u,k}$ are processes indexed by u taking a finite number of values. A second perspective consists in replacing instead the filter h_j in (1.9) by a multidimensional filter $h_{\ell,j}, \ell = 1, \cdots, m$ and thus replacing $W_{j,k}$ in (1.9) by

$$W_{\ell,j,k} = \sum_{t\in\mathbb{Z}} h_{\ell,j}(\gamma_j k - t) Y_t \; .$$

We adopted this second perspective in Clausel et al. (2014) and we also adopt it here since it allows us to compare our results to those obtained in Roueff and Taqqu (2009) in the Gaussian case.

We use bold faced symbols $\mathbf{W}_{j,k}$ and $\mathbf{h}_j$ to emphasize the multivariate setting and let

$$\mathbf{h}_j = \{h_{\ell,j},\, \ell = 1, \cdots, m\}, \qquad \mathbf{W}_{j,k} = \{W_{\ell,j,k},\, \ell = 1, \cdots, m\} \; ,$$

with

$$\mathbf{W}_{j,k} = \sum_{t\in\mathbb{Z}} \mathbf{h}_j(\gamma_j k - t) Y_t = \sum_{t\in\mathbb{Z}} \mathbf{h}_j(\gamma_j k - t) \Delta^{-K} G(X_t),\; j \geq 0, k \in \mathbb{Z} \; . \tag{2.7}$$

We then will study the asymptotic behavior of the sequence

$$\overline{\mathbf{S}}_{n_j,j} = \frac{1}{n_j} \sum_{k=0}^{n_j-1} \left(\mathbf{W}_{j,k}^2 - \mathbb{E}[\mathbf{W}_{j,k}^2] \right) \; , \tag{2.8}$$

adequately normalized as $j \to \infty$, where, by convention, in this paper,

$$\mathbf{W}_{j,k}^2 = \{W_{\ell,j,k}^2,\, \ell = 1, \cdots, m\} \; . \tag{2.9}$$

The squared Euclidean norm of a vector $\mathbf{x} = [x_1, \ldots, x_m]^T$ will be denoted by $|\mathbf{x}|^2 = x_1^2 + \cdots + x_m^2$.

It turns out that the asymptotic behavior of $\overline{\mathbf{S}}_{n_j,j}$ depends on how the subsequence of Hermite coefficients c_q, $q \geq 1$ which are *non-vanishing* is distributed. We denote this subsequence by $\{c_{q_\ell}\}_{\ell\in\mathcal{L}}$ where $\mathcal{L}$ is a sequence of consecutive integers starting at 0,

$$\mathcal{L} \subseteq \{0, 1, 2, \ldots\} \; , \tag{2.10}$$

with same cardinality as the set of non-vanishing coefficients, and $(q_\ell)_{\ell\in\mathcal{L}}$ is a (finite of infinite) increasing sequence of integers such that

$$\begin{aligned} q_0 &= \text{index of the first non–zero coefficient } c_q, \\ q_\ell &= \text{index of the } (\ell+1)\text{th non–zero coefficient}\,, \quad \ell \geq 1 \; . \end{aligned}$$

Examples

1) If

$$G(X_t) = c_1 H_1(X_t) + \frac{c_3}{3!} H_3(X_t) \; ,$$

where $c_1 \neq 0$, $c_2 = 0$, $c_3 \neq 0$, $c_q = 0$ for $q \geq 4$, then $q_0 = 1$, $q_1 = 3$ and $\mathcal{L} = \{0, 1\}$.

2) If

$$G(X_t) = \frac{c_2}{2!} H_2(X_t) + \frac{c_3}{3!} H_3(X_t) \; + \frac{c_4}{4!} H_4(X_t) \; ,$$

where $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$, $c_4 \neq 0$, $c_q = 0$ for $q \geq 5$, then $q_0 = 2$, $q_1 = 3$, $q_2 = 4$ and $\mathcal{L} = \{0, 1, 2\}$.

3) If

$$G(X_t) = \sum_{q=1}^{\infty} \frac{c_q}{q!} H_q(X_t) \; ,$$

where $c_q \neq 0$ for $q \geq 1$ then $q_0 = 1$, $q_1 = 2, \ldots$, and $\mathcal{L} = \{0, 1, 2, \cdots\}$.

4) If

$$G(X_t) = \frac{c_{q_0}}{q_0!} H_{q_0}(X_t) \; ,$$

where $c_{q_0} \neq 0$ and $c_q = 0$ for $q \neq q_0$, then $\mathcal{L} = \{0\}$.

While c_0 is always equal to 0 (see (1.6)), the assumption (1.7) ensures that $c_{q_0} \neq 0$ and hence that $\mathcal{L}$ always contains the index 0, so that $\mathcal{L}$ is never empty. In particular, we may write

$$(\Delta^K Y)_t = G(X_t) = \sum_{\ell\in\mathcal{L}} \frac{c_{q_\ell}}{q_\ell!} H_{q_\ell}(X_t), \quad t \in \mathbb{Z} \; , \tag{2.11}$$

where, if $\mathcal{L}$ is infinite, the sum converges in the L^2 sense.

We set

$$I = \{\ell \in \mathcal{L} \ : \ell + 1 \in \mathcal{L}, q_{\ell+1} - q_\ell = 1\} \ , \tag{2.12}$$

$$J = \{(\ell_1, \ell_2) \in \mathcal{L}^2 \ : \ \ell_1 < \ell_2, \ q_{\ell_1} \neq 1 \text{ and } q_{\ell_2} - q_{\ell_1} \geq 2\} \ , \tag{2.13}$$

that is, q_ℓ and $q_{\ell+1}$ take consecutive values when $\ell \in I$ and q_{ℓ_1} and q_{ℓ_2} differ by two or more when $(\ell_1, \ell_2) \in J$. The structure of these two sets is particulary important. The set I could be empty (there are no consecutive values of q_ℓ) or not empty. Then we set

$$\ell_0 = \begin{cases} \min(I) \geq 0 \ , & \text{when } I \text{ is not empty} \ , \\ \infty \ , & \text{when } I \text{ is empty} \ . \end{cases} \tag{2.14}$$

When ℓ_0 is finite (that is, I is not empty), q_{ℓ_0} is the smallest index q such that two Hermite coefficients c_q, c_{q+1} are non–zero. It will be involved in the normalization. We define, in addition,

$$m_0 = \min(\{\ell \in \mathcal{L}, \ q_\ell \geq 3\}) \geq 0 \ . \tag{2.15}$$

Thus q_{m_0} is the smallest index q such that c_q is non–zero with $q \geq 3$.

Examples

1) If

$$G(X_t) = c_1 H_1(X_t) + \frac{c_2}{2!} H_2(X_t) + \frac{c_4}{4!} H_4(X_t) \ ,$$

where $c_1 \neq 0$, $c_2 \neq 0$, $c_3 = 0, c_4 \neq 0, c_q = 0$ for $q \geq 5$ then $\mathcal{L} = \{0, 1, 2\}$, $I = \{1\}$, $\ell_0 = 1$, $m_0 = 4$ and $J = \{(2, 4)\}$.

2) If

$$G(X_t) = \frac{c_2}{2!} H_2(X_t) + \frac{c_3}{3!} H_3(X_t) \ + \frac{c_4}{4!} H_4(X_t) \ ,$$

where $c_1 = 0$, $c_2 \neq 0$, $c_3 \neq 0$, $c_4 \neq 0$, $c_q = 0$ for $q \geq 5$, then $\mathcal{L} = \{0, 1, 2\}$, $I = \{2, 3\}$, $\ell_0 = 2$, $m_0 = 3$ and $J = \{(2, 4)\}$.

3) If

$$G(X_t) = c_1 H_1(X_t) = c_1 X_t \ ,$$

where $c_1 \neq 0$ and $c_q = 0$ for $q \geq 2$, then $\mathcal{L} = \{0\}$ and both I and J are empty.

We are interested in the asymptotic behavior of the normalized scalogram $\overline{\mathbf{S}}_{n_j,j}$ defined in (2.8). This behavior depends on the sets J and I. These sets affect both the rate of convergence and the limit distribution of the rescaled sequence. The limit (see Section 3) will be expressed in terms of the Hermite processes which are defined as follows :

Definition 2.1. The Hermite process of order q and index

$$(1/2)(1 - 1/q) < d < 1/2 \ , \tag{2.16}$$

is the continuous time process

$$Z_{q,d}(t) = \int''_{\mathbb{R}^q} \frac{\mathrm{e}^{\mathrm{i}(u_1 + \cdots + u_q)\, t} - 1}{\mathrm{i}(u_1 + \cdots + u_q)} |u_1 \cdots u_q|^{-d} \, \mathrm{d}\widehat{W}(u_1) \cdots \mathrm{d}\widehat{W}(u_q), \ t \in \mathbb{R} \ . \tag{2.17}$$

It is Gaussian and called Fractional Brownian Motion when $q = 1$ and $0 < d < 1/2$. It is non Gaussian and called Rosenblatt process when $q = 2$ and $1/4 < d < 1/2$. The marginal distribution of $Z_{q,d}(t)$ at $t = 1$ is called the Hermite distribution of index q. It is called a Rosenblatt distribution when $q = 2$.

The multiple integral (2.17) is defined in Appendix B. The symbol $\int''_{\mathbb{R}^q}$ indicates that one does not integrate on the diagonal $u_i = u_j$, $j \neq i$. The integral is well-defined when (2.16) holds or equivalently when,

$$1 \leq q < 1/(1 - 2d) \ ,$$

because then it has finite L^2 norm. This process is self–similar with self-similarity parameter

$$H = dq + 1 - q/2 = \delta(q) + 1/2 \in (1/2, 1),$$

that is for all $a > 0$, $\{Z_{q,d}(at)\}_{t \in \mathbb{R}}$ and $\{a^H Z_{q,d}(t)\}_{t \in \mathbb{R}}$ have the same finite dimensional distributions, see Taqqu (1979).

3. Main results

We shall now state the main results and discuss them. They are proved in the following sections. We start with the assumptions

Assumptions A $\{\mathbf{W}_{j,k},\ j \geq 1, k \in \mathbb{Z}\}$ are the multidimensional wavelet coefficients defined by (2.7) , where

(i) $\{X_t\}_{t\in\mathbb{Z}}$ is a stationary Gaussian process with mean 0, variance 1 and spectral density f satisfying (1.2).

(ii) G is a real-valued function whose Hermite expansion (1.4) satisfies condition (2.5), namely $q_0 < 1/(1-2d)$, and whose coefficients in Hermite expansion satisfy the following condition: for any $\lambda > 0$,

$$c_q = O((q!)^d e^{-\lambda q}) \quad \text{as } q \to \infty \ . \tag{3.1}$$

(iii) the wavelet filters $(\mathbf{h}_j)_{j\geq 1}$ and their asymptotic Fourier transform $\widehat{\mathbf{h}}_\infty$ satisfy (W-a)–(W-c) with M vanishing moments. See details in Appendix C.

We shall focus on the asymptotic behavior of the scalogram for two basic classes of functions G.

- The first class involves functions G with Hermite rank greater or equal to 2 and with two consecutive terms in the Hermite expansion, both of which having long–range dependence. The result is stated in Theorem 3.1.
- The second class involves functions G with Hermite rank equal to 1 with no two consecutive terms with long–range dependence. The results are stated in Theorems 3.3 and 3.5.

Other classes are left for future work.

3.1. *G has a Hermite rank greater or equal to* 2. Consider functions G of the form

$$G(x) = \frac{c_2}{2!}H_2(x) + \cdots + \frac{c_{q_{\ell_0}}}{q_{\ell_0}!}H_{q_{\ell_0}}(x) + \frac{c_{q_{\ell_0}+1}}{(q_{\ell_0}+1)!}H_{q_{\ell_0}+1}(x) + \cdots \ .$$

where $c_1 = 0$. Some of the c_q, $q \geq 2$ may be zero as well. More precisely assume that

$$q_0 \geq 2 \ , \tag{3.2}$$

that is, that the Hermite rank of G is 2 or more. Also assume that (a) there exists two *consecutive* terms and that (b) both are long range dependent. Assumption (a) implies that the set I in (2.12) is not empty. Since the index q_{ℓ_0} (see (2.14)) of the first of these two consecutive terms could be $q_0 \geq 2$, we have $q_{\ell_0} \geq 2$. The index of the second of these consecutive terms is $q_{\ell_0}+1 \geq 3$. Assumption (b) will be satisfied if this second term is long-range dependent, that is

$$q_{\ell_0} + 1 < 1/(1-2d) \ , \tag{3.3}$$

by (2.4). We note that this situation implies the following boundaries for the parameter d:

$$1/3 < d < 1/2 \ ,$$

as indicated in (2.16).

Set

$$\nu = 2q_{\ell_0} + 1 - 2q_0 \ . \tag{3.4}$$

The following theorem provides the limit of (2.8) for two different cases, depending on whether the limit of $n_j^{-1}\gamma_j^\nu$ when $j \to +\infty$ is null or infinite. It involves $K \geq 0$ defined in (1.3), q_0 in (1.7), $\delta(q)$ is defined in (2.1), ℓ_0 in (2.14). The integer M is the number of vanishing moments of the wavelet filters and appears in Appendix C.

Theorem 3.1. *Suppose that Assumptions A hold with $M \geq K + \delta(q_0)$. Suppose moreover that the Hermite expansion of G satisfies (3.2) and (3.3).*

Then two limits in distribution of the centered multidimensional scalogram $\overline{\mathbf{S}}_{n,j}$ in (2.8), suitably normalized, are possible. They involve the Hermite processes in Definition 2.1 evaluated at time $t=1$. The coefficients involve ℓ_0 and the multidimensional deterministic vector $\mathbf{L}_q$, whose entries $[L_q(\widehat{h}_{\ell,\infty})]_{\ell=1,\cdots,m}$ are defined as

$$L_q(\widehat{h}_{\ell,\infty}) = \int_{\mathbb{R}^q} \frac{|\widehat{h}_{\ell,\infty}(u_1+\cdots+u_q)|^2}{|u_1+\cdots+u_q|^{2K}} \prod_{i=1}^{q} |u_i|^{-2d}\, \mathrm{d}u_1\cdots\mathrm{d}u_q \ , \tag{3.5}$$

which is finite for any $q<1/(1-2d)$*. Then*

(a) If $n_j \ll \gamma_j^\nu$ *then, as* $j, n_j \to \infty$*,*

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0)+K)}\overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} \frac{c_{q_0}^2}{(q_0-1)!}\left[f^*(0)^{q_0}\mathbf{L}_{q_0-1}\right] Z_{2,d}(1)\ .$$

(b) If $\gamma_j^\nu \ll n_j$ *then, as* $j, n_j \to \infty$*,*

$$n_j^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}\overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} 2\frac{c_{q_{\ell_0}}c_{q_{\ell_0}+1}}{q_{\ell_0}!}\left[f^*(0)^{q_{\ell_0}+1/2}\,\mathbf{L}_{q_{\ell_0}}\right] Z_{1,d}(1)\ .$$

Remarks.

1. Using (C.6) with $M \geq K$ and $\alpha > 1/2$, the integral in (3.5) is finite for any positive integer $q<1/(1-2d)$, see Lemma 5.1 in Clausel et al. (2012). Thus, under Conditions (3.2) and (3.3), the vectors $\mathbf{L}_{q_0-1}$ and $\mathbf{L}_{q_{\ell_0}}$ appearing in the limits of Cases (a) and (b) have finite entries.
2. In case (a), the limit is a deterministic vector times the non-Gaussian Rosenblatt random variable $Z_{2,d}(1)$, that is, the Rosenblatt process $Z_{2,d}(t)$ defined in (2.17) and evaluated at time $t=1$. In case (b), the limit is a deterministic vector times the Gaussian random variable $Z_{1,d}(1)$, that is, Fractional Brownian motion $Z_{1,d}(t)$ defined in (2.17) and evaluated at time $t=1$.
3. In the case where $n_j \sim C_0\gamma_j^\nu$ as $j\to\infty$ for some $C_0>0$, the scalogram is asymptotically a linear combination of a Rosenblatt and a Gaussian variable. Indeed, using the results of Section 6, one can see that the scalogram is the sum of two terms having the same order, both converging in the L^2 sense respectively to a Rosenblatt and a Gaussian variable.

Proof: This theorem is proved in Section 7.1. □

In the framework of wavelet analysis as in Moulines et al. (2007), we have $\gamma_j = 2^j$ and the number $n=n_j$ of wavelet coefficients available at scale index j, is related both to the number N of observations $Y_1,\cdots,Y_N$ of the time series Y and to the length T of the support of the analyzing wavelet. More precisely, one has (see Moulines et al. (2007) for more details),

$$n_j = [2^{-j}(N-T+1)-T+1] = 2^{-j}N + O(1)\ , \tag{3.6}$$

where $[x]$ denotes the integer part of x for any real x. Note that the assumption $n_j\to\infty$ when $j\to\infty$ is equivalent to $N\to\infty$ faster than 2^j. Moreover, for any $\nu>0$,

$$n_j \ll 2^{j\nu} \iff 2^{-j}N \ll 2^{j\nu} \iff N \ll 2^{j(\nu+1)} \text{ when } N\to\infty\ . \tag{3.7}$$

Examples. We now illustrate Theorem 3.1 through three examples:

(i) $G=H_{q_0}$ with $q_0\geq 2$.
(ii) $G=H_{q_0}+H_{q_0+1}$ with $q_0\geq 2$, $q_0+1<1/(1-2d)$.
(iii) $G=H_{q_0}+H_{q_0+1}+H_{q_1}$ with $q_0\geq 2$, $q_0+1<1/(1-2d)$ and with $q_1-(q_0+1)\geq 2$, that is, $J=\{(q_0+1,q_1)\}$.

In all cases, the integer q_0 denotes the Hermite rank of G.

Let us elaborate on the conditions on d and the resulting limits for these examples. For simplicity, we assume that the scalogram $S_{n_j,j}$ is univariate.

Example (i). When $G=H_{q_0}$ with $q_0\geq 2$, I and J are both empty. Since I is empty one can regard ℓ_0 and consequently q_{ℓ_0} and ν as infinity, which suggests that we are in case (a), independently of the growths of n_j versus γ_j as $j\to\infty$. The asymptotic behavior of the scalogram of this example is treated by Theorem 3.1 in Clausel et al. (2014) under the condition $q_0<1/(1-2d)$. Indeed, the obtained rate of convergence is the same as in case (a) of Theorem 3.1 and the limit is also Rosenblatt. This also corresponds to the limit obtained by Bardet and Tudor in the case where Y itself is the Rosenblatt process (see Theorem 4 of Bardet and Tudor (2010)).

Example (ii). Suppose $G=H_{q_0}+H_{q_0+1}$, with $q_0\geq 2$ and $q_0+1<1/(1-2d)$. Then J is empty and $I=\{q_0\}$. The Hermite rank of G is q_0 and thus coincides with q_{ℓ_0}. As a consequence, by (3.4), $\nu=1$. Let us use Eq. (3.7) to relate the asymptotic behavior to the number of observation N and the analyzing scale index j. Since $\nu = 1$, we get that the asymptotic behavior of the scalogram $S_{n_j,j}$ depends on whether, as $j, N \to \infty$,

$$N \ll 2^{2j} \text{ or if } 2^{2j} \ll N .$$

Let us explain how these two regimes show up in the limit. The wavelet coefficients of Y can be expanded as follows

$$W_{j,k} = W_{j,k}^{(q_0)} + W_{j,k}^{(q_0+1)} ,$$

where $W_{j,k}^{(q_0)}, W_{j,k}^{(q_0+1)}$ belong respectively to the chaos of order q_0 and $q_0 + 1$. Then,

$$W_{j,k}^2 = \left([W_{j,k}^{(q_0)}]^2 + [W_{j,k}^{(q_0+1)}]^2\right) + \left(2W_{j,k}^{(q_0)} W_{j,k}^{(q_0+1)}\right) .$$

The term $[W_{j,k}^{(q_0)}]^2$ behaves as in the case $G = H_{q_0}$ and is asymptotically Rosenblatt as proved in Clausel et al. (2014). The term $[W_{j,k}^{(q_0+1)}]^2$ is asymptotically negligible as proved in Proposition 5.1. The term $W_{j,k}^{(q_0)} W_{j,k}^{(q_0+1)}$, on the other hand, turns out to be asymptotically Gaussian. The asymptotic behavior of the scalogram then depends on whether the Rosenblatt term or the Gaussian term is leading. This depends on the limit of $N/2^{2j}$. Hence, both limits stated in Theorem 3.1 may occur:

- If $2^{-2j}N \to 0$, the term corresponding to $[W_{j,k}^{(q_0)}]^2$ is leading and the scalogram $S_{n_j,j}$ of Y is asymptotically Rosenblatt.
- If $2^{-2j}N \to \infty$, the terms corresponding to $W_{j,k}^{(q_0)} W_{j,k}^{(q_0+1)}$ are leading and the scalogram $S_{n_j,j}$ of Y is asymptotically Gaussian.

Example (iii). Suppose $G = H_{q_0} + H_{q_0+1} + H_{q_1}$ with $q_0 \geq 2$, $q_0 + 1 < 1/(1 - 2d)$ and $q_1 - (q_0 + 1) \geq 2$. Then $I = \{q_0\}$, $J = \{(q_0, q_1), (q_0 + 1, q_1)\}$. Observe that in this case, J is not involved in the limit of $S_{n_j,j}$ and the behavior of the scalogram is similar to that of Example (ii). Thus, the two limits of Theorem 3.1 may occur.

3.2. *The Hermite rank of G equals* 1. Here we assume that

$$q_0 = 1,\ q_1 < 1/(1 - 2d) \quad \text{and} \quad \ell_0 = \infty . \tag{3.8}$$

In particular, $\ell_0 = \infty$ implies $q_1 \geq q_0+2 = 3$ and thus this condition implies $d > 1/3$, thus $d \in (1/3, 1/2)$. By definition of ℓ_0 in (2.14), the last condition in (3.8) means that there are no terms with consecutive indices in the Hermite expansion. Thus

$$G = c_1 H_1 + \frac{c_{q_1}}{q_1!} H_{q_1} + \frac{c_{q_2}}{q_2!} H_{q_2} + \cdots$$

where for any $\ell \in \mathcal{L}$, $q_{\ell+1} - q_\ell \geq 2$. In this case the following critical index plays an important role :

$$q_1^* = 2 + \frac{1}{2(1 - 2d)} . \tag{3.9}$$

It will also be useful to relate the number of available wavelet coefficients $n = n_j$ to γ_j^ν where ν takes the following three values :

$$\nu_1 = \frac{(1 - 2d)(q_1 - 1)}{1 - (1 - 2d)(q_1 - 1)},\ \nu_2 = \frac{1 - 2d}{2d - 1/2}(q_1 - 1),\ \nu_3 = \begin{cases} \frac{q_1-1}{q_1-3} & \text{if } q_1 > 3 \\ \infty & \text{if } q_1 = 3 . \end{cases} \tag{3.10}$$

As shown in the following lemma, the relations between ν_1, ν_2 and ν_3 depend on whether $q_1 < q_1^*$ or $q_1 \geq q_1^*$:

Lemma 3.2.

- *If $q_1 < q_1^*$ then $\nu_1 < \nu_2 < \nu_3$.*
- *If $q_1 \geq q_1^*$ then $\nu_3 \leq \nu_2 \leq \nu_1$ (in particular we have $\nu_3 < \infty$).*

Proof: First observe that

$$\nu_1 = \frac{(1 - 2d)(q_1 - 1)}{1 - (1 - 2d)(q_1 - 1)} < \nu_2 = \frac{(1 - 2d)(q_1 - 1)}{2d - 1/2} \iff 2d - \frac{1}{2} < 1 - (1 - 2d)(q_1 - 1)$$

$$\iff q_1 < 1 + \frac{1 - (2d - \frac{1}{2})}{1 - 2d} = q_1^* .$$

Now, if $q_1 > 3$ then

$$\nu_2 = \frac{(1-2d)(q_1-1)}{2d-1/2} < \nu_3 = \frac{q_1-1}{q_1-3} \Longleftrightarrow q_1 - 3 < \frac{2d-\frac{1}{2}}{1-2d} \Longleftrightarrow q_1 < 3 + \frac{2d-\frac{1}{2}}{1-2d} = q_1^* ,$$

and, since $q_1^* > 3$, the case $q_1 = 3$ can only happen for $q_1 < q_1^*$, and yields $\nu_2 < \infty = \nu_3$. □

The next theorems indicate the limits in the various cases. We first consider the case where q_1 is lower than the critical index q_1^*.

Theorem 3.3. *Suppose that Assumptions A hold with $M \geq K + d$. Suppose moreover that the Hermite expansion of G satisfies (3.8) and assume that $q_1 < q_1^*$, where q_1^* is defined in (3.9).*

Then three limits of the multidimensional scalogram $\overline{\mathbf{S}}_{n,j}$ in (2.8), suitably normalized, are possible :

(a) If $n_j \ll \gamma_j^{\nu_1}$ then as $j, n_j \to \infty$,

$$n_j^{1/2} \gamma_j^{-(2d+2K)} \overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} c_1^2\, \mathcal{N}(0, \Gamma) ,$$

where Γ is defined as

$$\Gamma_{i,i'} = 4\pi (f^*(0))^2 \int_{-\pi}^{\pi} \left| \sum_{p\in\mathbb{Z}} |\lambda + 2p\pi|^{-2(K+d)} [\widehat{h}_{i,\infty} \overline{\widehat{h}_{i',\infty}}](\lambda + 2p\pi) \right|^2 \mathrm{d}\lambda, \quad 1 \leq i, i' \leq m . \tag{3.11}$$

(b) If $\gamma_j^{\nu_1} \ll n_j$ and either $\nu_3 = \infty$ or $n_j \ll \gamma_j^{\nu_3}$ then as $j, n_j \to \infty$

$$n_j^{(1-2\delta(q_1-1))/2} \gamma_j^{-(2\delta(\frac{q_1+1}{2})+2K)} \overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} \frac{2c_1 c_{q_1}}{(q_1-1)!} [f^*(0)]^{(q_1+1)/2} \mathbf{L}_1 Z_{q_1-1,d}(1) .$$

(c) If $\nu_3 < \infty$ and $\gamma_j^{\nu_3} \ll n_j$ then as $j, n_j \to \infty$,

$$n_j^{1-2d} \gamma_j^{-2(\delta(q_1)+K)} \overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} \frac{c_{q_1}^2}{(q_1-1)!} [f^*(0)]^{q_1}\ \mathbf{L}_{q_1-1}\ Z_{2,d}(1) .$$

Remark 3.4. In case (c), the limit is a deterministic vector times the non-Gaussian Rosenblatt random variable $Z_{2,d}(1)$. In case (b), the limit is a deterministic vector times a Hermite random variable of order $q_1 - 1 > 3 - 1 = 2$, which can be represented by a multiple Wiener integral of order 3 or more (see Definition 2.1).

In the case where $n_j \sim C_0 \gamma_j^{\nu_3}$ as $j \to \infty$ for some $C_0 > 0$, the scalogram is asymptotically a linear combination of a Rosenblatt and a Hermite random variable. This is because, up to an equality in distribution, it is the sum of two terms both converging in L^2 after normalization (see Section 6). On the other hand if $n_j \sim C_0 \gamma_j^{\nu_1}$ as $j \to \infty$ for some $C_0 > 0$, the situation is complicated. This is because the scalogram is the sum of two terms of same order, one converging in L^2 to a Hermite random variable, the other converging only in law to a Gaussian random variable.

Proof: This theorem is proved in Section 7.2. □

We now consider the case where q_1 is greater than the critical exponent q_1^*.

Theorem 3.5. *Suppose that Assumptions A hold with $M \geq K + d$. Suppose moreover that the Hermite expansion of G satisfies (3.8) and assume that $q_1 \geq q_1^*$, where q_1^* is defined in (3.9).*

Then two limits of the multidimensional scalogram $\overline{\mathbf{S}}_{n,j}$ in (2.8), suitably normalized, are possible :

(a) If $n_j \ll \gamma_j^{\nu_2}$ then as $j, n_j \to \infty$,

$$n_j^{1/2} \gamma_j^{-(2d+2K)} \overline{\mathbf{S}}_{n_j,j} \overset{(\mathcal{L})}{\to} c_1^2 \mathcal{N}(0, \Gamma) ,$$

where Γ is as in Theorem 3.3 (a).

(b) If $\gamma_j^{\nu_2} \ll n_j$ then as $j, n_j \to \infty$,

$$n_j^{1-2d} \gamma_j^{-2(\delta(q_1)+K)} \overline{\mathbf{S}}_{n,j} \overset{(\mathcal{L})}{\to} \frac{c_{q_1}^2}{(q_1-1)!} [f^*(0)]^{q_1} \mathbf{L}_{q_1-1}\ Z_{2,d}(1) .$$

Remark 3.6. As in the case of Theorem 3.3, the case where $n_j \sim C_0 \gamma_j^{\nu_2}$ as $j \to \infty$, seems quite complicated to deal with.

Proof: This theorem is proved in Section 7.2. □

Example. We now illustrate Theorem 3.3 and 3.5. Our setting is still that of Moulines et al. (2007) as above.

The memory parameter d is assumed to belong to $(3/8, 1/2)$. Consider the case where

$$G = H_1 + H_{q_1} \ ,$$

with $3 < q_1 < 1/(1-2d)$. We will prove in the sequel that the wavelet coefficients of Y can be expanded as

$$W_{j,k} = W_{j,k}^{(1)} + W_{j,k}^{(q_1)} \ ,$$

where $W_{j,k}^{(1)}$ is Gaussian and $W_{j,k}^{(q_1)}$ belongs to the chaos of order q_1. Then,

$$W_{j,k}^2 = [W_{j,k}^{(1)}]^2 + [W_{j,k}^{(q_1)}]^2 + 2W_{j,k}^{(1)} W_{j,k}^{(q_1)} \ .$$

The empirical mean of the terms $[W_{j,k}^{(1)}]^2$ behaves as in the Gaussian case and is asymptotically Gaussian. The empirical mean of the terms $[W_{j,k}^{(q_1)}]^2$ behaves as in the case $G = H_{q_1}$ with $q_1 \geq 2$ and is asymptotically Rosenblatt. Finally the empirical mean of the terms $2W_{j,k}^{(1)} W_{j,k}^{(q_1)}$ belongs to the chaos of order $q_1 - 1 > 2$. The asymptotic behavior of the scalogram then depends on which of the three terms is leading.

To see what happens, let N be as before the number of observations and assume that $\gamma_j = 2^j$. Let $n_j \sim N 2^{-j}$ as $j \to \infty$ as in (3.6). Distinguish two cases : $q_1 < q_1^*$ and $q_1 \geq q_1^*$ where q_1^* is defined in (3.9).

If $q_1 < q_1^*$, the three possibilities stated in Theorem 3.3 can occur :

- if $2^{-j(\nu_1+1)} N \to 0$ as $N, j \to \infty$, then the term corresponding to $[W_{j,k}^{(1)}]^2$ is leading and the scalogram $S_{n_j,j}$ of the process $\{Y_t\}_{t\in\mathbb{Z}}$ is asymptotically Gaussian (case (a)).
- if $2^{-j(\nu_1+1)} N \to \infty$ and $2^{-j(\nu_3+1)} N \to 0$ as $N, j \to \infty$, then the term corresponding to $2W_{j,k}^{(1)} W_{j,k}^{(q_1)}$ is leading and the scalogram $S_{n_j,j}$ of $\{Y_t\}$ belongs asymptotically to the chaos of order $q_1 - 1 > 2$ (case (b)).
- if $2^{-j(\nu_3+1)} N \to \infty$ as $N, j \to \infty$, then the term corresponding to $[W_{j,k}^{(q_1)}]^2$ with $q_1 > 3$ is leading and the scalogram $S_{n_j,j}$ of $\{Y_t\}$ is asymptotically Rosenblatt (case (c)).

If we now assume that $q_1 \geq q_1^*$, we are in the setting of Theorem 3.5 and the term corresponding to $2W_{j,k}^{(1)} W_{j,k}^{(q_1)}$ is always negligible. Then only two different situations can occur :

- if $2^{-j(\nu_2+1)} N \to 0$ as $N, j \to \infty$, then the term corresponding to $[W_{j,k}^{(1)}]^2$ is leading and the scalogram $S_{n_j,j}$ of $\{Y_t\}$ is asymptotically Gaussian (case (a)).
- if $2^{-j(\nu_2+1)} N \to \infty$ as $N, j \to \infty$, then the term corresponding to $[W_{j,k}^{(q_1)}]^2$ is leading and the scalogram $S_{n_j,j}$ of $\{Y_t\}$ is asymptotically Rosenblatt (case (b)).

4. The basic decomposition

Our goal is to investigate the asymptotic behavior of $\overline{\mathbf{S}}_{n_j,j}$ as defined in (2.8) when $j \to +\infty$. As in Clausel et al. (2014), our main tool will be the Wiener-Itô chaos expansion of $\overline{\mathbf{S}}_{n_j,j}$ which involves multiple stochastic integrals $\widehat{I}_q$, $q = 1, 2, \ldots$. These are defined in Appendix B. In this case, the situation is more complex than in the case $G = H_{q_0}$ since as proved in Clausel et al. (2012), the wavelet coefficients $\mathbf{W}_{j,k}$, defined in (2.7), admit an expansion into Wiener chaos as follows :

$$\mathbf{W}_{j,k} = \sum_{q=1}^{\infty} \frac{c_q}{q!} \mathbf{W}_{j,k}^{(q)} \ , \tag{4.1}$$

where $\mathbf{W}_{j,k}^{(q)}$ is a multiple integral of order q. Then, using the same convention as in (2.9), we have

$$\mathbf{W}_{j,k}^2 = \sum_{q=1}^{\infty} \left(\frac{c_q}{q!}\right)^2 \left(\mathbf{W}_{j,k}^{(q)}\right)^2 + 2 \sum_{q'=2}^{\infty} \sum_{q=1}^{q'-1} \frac{c_q}{q!} \frac{c_{q'}}{q'!} \mathbf{W}_{j,k}^{(q)} \mathbf{W}_{j,k}^{(q')} \ , \tag{4.2}$$

where the convergence of the infinite sums hold in $L^1(\Omega)$ sense.

Each $\mathbf{W}_{j,k}^{(q)}$ is a multiple integral of order q of some multidimensional kernel $\mathbf{f}_{j,k}^{(q)}$, that is

$$\mathbf{W}_{j,k}^{(q)} = \widehat{I}_q(\mathbf{f}_{j,k}^{(q)}) \; . \tag{4.3}$$

Now, using the product formula for multiple stochastic integrals (B.8), one gets, as shown in Proposition 4.2 that, for any $(n,j) \in \mathbb{N}^2$,

$$\begin{aligned}\overline{\mathbf{S}}_{n,j} &= \frac{1}{n}\sum_{k=0}^{n-1}\mathbf{W}_{j,k}^2 - \mathbb{E}[\mathbf{W}_{j,0}^2] \\ &= \sum_{q=1}^{\infty}\left(\frac{c_q}{q!}\right)^2 \sum_{p=0}^{q-1} p!\binom{q}{p}^2 \mathbf{S}_{n,j}^{(q,q,p)} \\ &\quad + 2\sum_{q'=2}^{\infty}\sum_{q=1}^{q'-1}\frac{c_q}{q!}\frac{c_{q'}}{q'!}\sum_{p=0}^{q} p!\binom{q}{p}\binom{q'}{p}\mathbf{S}_{n,j}^{(q,q',p)} \; ,\end{aligned} \tag{4.4}$$

where, for all $q, q' \geq 1$ and $0 \leq p \leq \min(q,q')$, $\mathbf{S}_{n,j}^{(q,q',p)}$ is of the form

$$\mathbf{S}_{n,j}^{(q,q',p)} = \widehat{I}_{q+q'-2p}(\mathbf{g}_{n,j}^{(q,q',p)}) \; . \tag{4.5}$$

We call $q + q' - 2p$ the *order* of the summand $\mathbf{S}_{n,j}^{(q,q',p)}$. For any n, j, q, q', p, the function $\mathbf{g}_{n,j}^{(q,q',p)}(\xi)$, $\xi = (\xi_1, \ldots, \xi_{q+q'-2p}) \in \mathbb{R}^{q+q'-2p}$ is defined for every p, q, q' as

$$\mathbf{g}_{n,j}^{(q,q',p)}(\xi) = \frac{1}{n}\sum_{k=0}^{n-1}\left(\mathbf{f}_{j,k}^{(q)}\overline{\otimes}_p\mathbf{f}_{j,k}^{(q')}\right) \; , \tag{4.6}$$

where the operation $\overline{\otimes}_p$ is defined in (B.9) for each entry. The expansion in Wiener chaos of $\overline{\mathbf{S}}_{n,j}$ implies that

$$\overline{\mathbf{S}}_{n,j} = c_1^2\mathbf{S}_{n,j}^{(1,1,0)} + \mathbf{\Sigma}_{n,j}^{(0)} + \mathbf{\Sigma}_{n,j}^{(1)} + \mathbf{\Sigma}_{n,j}^{(2)} + \mathbf{\Sigma}_{n,j}^{(3)} \; , \tag{4.7}$$

with

$$\mathbf{\Sigma}_{n,j}^{(0)} = \sum_{\ell\in\mathcal{L},\, q_\ell\neq 1}\frac{c_{q_\ell}^2}{(q_\ell!)^2}\sum_{p=0}^{q_\ell-1} p!\binom{q_\ell}{p}^2\mathbf{S}_{n,j}^{(q_\ell,q_\ell,p)}, \tag{4.8}$$

$$\mathbf{\Sigma}_{n,j}^{(1)} = 2\sum_{(\ell_1,\ell_2)\in J}\frac{c_{q_{\ell_1}}}{q_{\ell_1}!}\frac{c_{q_{\ell_2}}}{q_{\ell_2}!}\sum_{p=0}^{q_{\ell_1}} p!\binom{q_{\ell_1}}{p}\binom{q_{\ell_2}}{p}\mathbf{S}_{n,j}^{(q_{\ell_1},q_{\ell_2},p)}, \tag{4.9}$$

$$\begin{aligned}\mathbf{\Sigma}_{n,j}^{(2)} &= 2\sum_{\ell\in\mathcal{L},\ell\geq m_0}\frac{c_1 c_{q_\ell}}{q_\ell!}\sum_{p=0}^{1} p!\binom{1}{p}\binom{q_\ell}{p}\mathbf{S}_{n,j}^{(1,q_\ell,p)}, \\ &= 2\sum_{\ell\in\mathcal{L},\ell\geq m_0}\left(\frac{c_1 c_{q_\ell}}{q_\ell!}\mathbf{S}_{n,j}^{(1,q_\ell,0)} + \frac{c_1 c_{q_\ell}}{(q_\ell-1)!}\mathbf{S}_{n,j}^{(1,q_\ell,1)}\right) \; ,\end{aligned} \tag{4.10}$$

$$\mathbf{\Sigma}_{n,j}^{(3)} = 2\sum_{\ell\in I}\frac{c_{q_\ell}}{q_\ell!}\frac{c_{q_\ell+1}}{(q_\ell+1)!}\sum_{p=0}^{q_\ell} p!\binom{q_\ell}{p}\binom{q_\ell+1}{p}\mathbf{S}_{n,j}^{(q_\ell,q_\ell+1,p)} \; . \tag{4.11}$$

The sets $\mathcal{L}$, I and J are defined in (2.10), (2.12) and (2.13) respectively and the index m_0, defined in (2.15), is such that $q_{m_0} \geq 3$.

Let us comment on the decomposition (4.7). The sum $\mathbf{\Sigma}_{n,j}^{(0)}$ contains terms of the form $\mathbf{S}_{n,j}^{(q,q,p)}$ that is multiple integrals of order $2(q-p)$. Then this sum, after subtracting its expectation, has only summands of order $2, 4, 6, \ldots$ in the Wiener chaos.

The sum $\mathbf{\Sigma}_{n,j}^{(1)}$ contains multiple integrals of orders $q+q'-2p$ with $q \neq 1$, $q' \neq 1$, $p \leq q \wedge q'$ and $|q-q'| \geq 2$. That means that all the summands in $\mathbf{\Sigma}_{n,j}^{(1)}$ are of order greater than or equal to 2.

The sum $\mathbf{\Sigma}_{n,j}^{(2)}$ contains multiple integrals of orders $q + q' - 2p$ with $q = 1$, $q' \geq q_{m_0} \geq 3$ and $p = 0$ or 1. All the summands in $\mathbf{\Sigma}_{n,j}^{(2)}$ are then of order greater than or equal to $q_{m_0} - 1 \geq 2$.

The last sum $\mathbf{\Sigma}^{(3)}_{n,j}$ contains terms of the form $\mathbf{S}^{(q,q+1,p)}_{n,j}$, that is multiple integrals of order $q+(q+1)-2p = 2q+1-2p$. When $p=q$, $q+1+q-2q=1$, thus one can have components in the first Wiener chaos, that is Gaussian terms.

We will see that $\mathbf{\Sigma}^{(0)}_{n,j}+\mathbf{\Sigma}^{(1)}_{n,j}$ will converge to a non-Gaussian limit, more precisely to a random variable in the second Wiener chaos. The sum $\mathbf{\Sigma}^{(2)}_{n,j}$ will also converge to a non-Gaussian limit, more precisely to a random variable in the Wiener chaos of order $q_{m_0}-1$. Finally $\mathbf{\Sigma}^{(3)}_{n,j}$ will tend to a Gaussian limit.

Remark 4.1. It is the presence of $\mathbf{\Sigma}^{(2)}_{n,j}$ which creates the possibility of having as limit a multiple integral of order greater than 2. Thus, starting with a process

$$G(X_t) = H_1(X_t) + H_{q_1}(X_t) \ ,$$

with $q_1 \geq 4$, then $q_{m_0} = q_1$ and one may obtain as limit of the scalogram a Hermite process of order $q_1 - 1 \geq 3$.

Let us formalize the above decomposition of $\mathbf{S}_{n,j}$ and give a more explicit expression for the function $\mathbf{g}^{(q,q',p)}_{n,j}$ in (4.6). The next proposition is a generalization of Proposition 6.1 of Clausel et al. (2014).

Proposition 4.2. *For all j, $\{\mathbf{W}_{j,k}\}_{k\in\mathbb{Z}}$ is a weakly stationary sequence. Moreover, for any $(n,j)\in\mathbb{N}^2$, $\overline{\mathbf{S}}_{n,j}$ can be expressed as (4.4) where the infinite sums converge in the $L^1(\Omega)$ sense. The function $\mathbf{g}^{(q,q',p)}_{n,j}(\xi)$, $\xi=(\xi_1,\dots,\xi_{q+q'-2p})\in\mathbb{R}^{q+q'-2p}$, in (4.5), equals*

$$\begin{aligned}\mathbf{g}^{(q,q',p)}_{n,j}(\xi) \quad &= \quad D_n(\gamma_j\{\xi_1+\cdots+\xi_{q+q'-2p}\}) \times \textstyle\prod_{i=1}^{q+q'-2p}[\sqrt{f(\xi_i)}\mathbb{1}_{(-\pi,\pi)}(\xi_i)] \\ &\quad \times \widehat{\kappa}^{(p)}_j(\xi_1+\cdots+\xi_{q-p},\xi_{q-p+1}+\cdots+\xi_{q+q'-2p}) \ .\end{aligned} \tag{4.12}$$

Here f denotes the spectral density of the underlying Gaussian process X and

$$D_n(u) = \frac{1}{n}\sum_{k=0}^{n-1} \mathrm{e}^{\mathrm{i}ku} = \frac{1-\mathrm{e}^{\mathrm{i}nu}}{n(1-\mathrm{e}^{\mathrm{i}u})} \ , \tag{4.13}$$

denotes the normalized Dirichlet kernel. Finally, for $\xi_1,\xi_2\in\mathbb{R}$, if $p\neq 0$,

$$\widehat{\boldsymbol{\kappa}}^{(p)}_j(\xi_1,\xi_2) = \int_{(-\pi,\pi)^p}\left(\prod_{i=1}^p f(\lambda_i)\right)\widehat{\mathbf{h}}^{(K)}_j(\lambda_1+\cdots+\lambda_p+\xi_1)\overline{\widehat{\mathbf{h}}^{(K)}_j(\lambda_1+\cdots+\lambda_p-\xi_2)}\,\mathrm{d}^p\lambda \ , \tag{4.14}$$

and, if $p=0$,

$$\widehat{\boldsymbol{\kappa}}^{(p)}_j(\xi_1,\xi_2) = \widehat{\mathbf{h}}^{(K)}_j(\xi_1)\overline{\widehat{\mathbf{h}}^{(K)}_j(\xi_2)} \ . \tag{4.15}$$

Notation. To simplify the notation, for any integer p and $q_1,\dots,q_p\in\mathbb{Z}_+$ we shall denote by $\Sigma_{q_1,\dots,q_p}$, the $\mathbb{C}^{q_1+\cdots+q_p}\to\mathbb{C}^p$ function defined, for all $y=(y_1,\dots,y_{q_1+\cdots+q_p})\in\mathbb{C}^{q_1+\cdots+q_p}$ by

$$\Sigma_{q_1,\dots,q_p}(y) = \left(\sum_{i=1}^{q_1} y_i, \sum_{i=q_1+1}^{q_1+q_2} y_i,\dots,\sum_{i=q_1+\cdots+q_{p-1}+1}^{q_1+\cdots+q_p} y_i\right) \ . \tag{4.16}$$

Note that, for $p=1$, one simply has $\Sigma_q(y)=y_1+\cdots+y_q$.

With this notation, (4.5) and (4.14) become respectively

$$\mathbf{S}^{(q,q',p)}_{n,j} = \widehat{I}_{q+q'-2p}\left(D_n\circ\Sigma_{q+q'-2p}(\gamma_j\times\cdot)\times[\sqrt{f}\mathbb{1}_{(-\pi,\pi)}]^{\otimes(q+q'-2p)}\times\widehat{\boldsymbol{\kappa}}^{(p)}_j\circ\Sigma_{q-p,q'-p}\right) \ , \tag{4.17}$$

$$\widehat{\boldsymbol{\kappa}}^{(p)}_j(\xi_1,\xi_2) = \int_{(-\pi,\pi)^p} f^{\otimes p}(\lambda)\,\widehat{\mathbf{h}}^{(K)}_j(\Sigma_p(\lambda)+\xi_1)\overline{\widehat{\mathbf{h}}^{(K)}_j(\Sigma_p(\lambda)-\xi_2)}\,\mathrm{d}^p\lambda \ , \text{ if } p\neq 0, \tag{4.18}$$

where $\circ$ denotes the composition of functions, $\lambda=(\lambda_1,\cdots,\lambda_p)$ and $f^{\otimes p}(\lambda)=f(\lambda_1)\cdots f(\lambda_p)$ is written as a tensor product.

Proof of Proposition 4.2: For sake of simplicity we can assume that $\mathbf{W}$ is a vector of length $m=1$ since the case $m\geq 2$ can be deduced by applying the case $m=1$ to each entries. We must give an expansion in

Wiener chaos of the one dimensional scalogram $S_{n,j}-\mathbb{E}(S_{n,j})$ in our setting. Using (1.10), (4.2), (4.3) and the product formula (B.8) of Proposition B.1, we have as in Proposition 6.1 of Clausel et al. (2014)

$$S_{n,j}=\sum_{q,q'=1}^{\infty}\frac{c_q c_{q'}}{q!q'!}\sum_{p=0}^{q\wedge q'}p!\binom{q}{p}\binom{q'}{p}\widehat{I}_{q+q'-2p}\left(g_{n,j}^{(q,q',p)}\right)\,, \tag{4.19}$$

where

$$g_{n,j}^{(q,q',p)}=\frac{1}{n}\sum_{k=0}^{n-1}f_{j,k}^{(q)}\overline{\otimes}_p f_{j,k}^{(q')}\,.$$

By (B.7),

$$f_{j,k}^{(q)}(\xi)=\exp\circ\Sigma_q(\mathrm{i}k\gamma_j\xi)\left(\widehat{h}_j^{(K)}\circ\Sigma_q(\xi)\right)\left(f^{\otimes q}(\xi)\right)^{1/2}\mathbb{1}_{(-\pi,\pi)}^{\otimes q}(\xi)\,,\xi\in\mathbb{R}^q\,. \tag{4.20}$$

If $q+q'-2p\neq 0$, let $\xi=(\xi_1,\cdots,\xi_{q+q'-2p})$. As in Clausel et al. (2014) using (B.9), we get that $g_{n,j}^{(q,q',p)}$ is a function with $q+q'-2p$ variables given by

$$g_{n,j}^{(q,q',p)}(\xi)=\frac{1}{n}\sum_{k=0}^{n-1}\exp\circ\Sigma_{q+q'-2p}(\mathrm{i}k\gamma_j\xi)\times[\sqrt{f}\mathbb{1}_{(-\pi,\pi)}]^{\otimes q+q'-2p}(\xi)\times\widehat{\kappa}_j^{(p)}\circ\Sigma_{q-p,q'-p}(\xi)\,.$$

The Dirichlet kernel D_n appears when one computes the sum $\frac{1}{n}\sum_{k=0}^{n-1}\exp\circ\Sigma_{q+q'-2p}(ik\gamma_j\xi)$. This implies the formula (4.12).

In addition, the chaos of order zero appears in the expression (4.19) of $S_{n,j}$ in the terms with $p=q=q'$ since $\widehat{I}_{q+q'-2p}=\widehat{I}_0$. In this case, a similar argument as in Clausel et al. (2014) leads to

$$\frac{1}{n}\sum_{q=1}^{\infty}\frac{c_q^2}{(q!)^2}\sum_{k=1}^{n}\mathbb{E}(|W_{j,k}^{(q)}|^2)=\sum_{q=1}^{\infty}\frac{c_q^2}{(q!)^2}\times\mathbb{E}(|W_{j,0}^{(q)}|^2)=\mathbb{E}(|W_{j,0}|^2)=\mathbb{E}(S_{n,j})\,,$$

by (4.1) and (4.2). Therefore, in the univariate case $m=1$, $\overline{\mathbf{S}}_{n,j}=S_{n,j}-\mathbb{E}(S_{n,j})$ can be expressed as stated in (4.4). The generalization to the case $m\geq 2$ is straightforward. □

We prove in Section 6 that

- The leading term of $\mathbf{\Sigma}_{n,j}^{(0)}+\mathbf{\Sigma}_{n,j}^{(1)}$ is $\frac{c_{q_0^*}^2}{(q_0^*-1)!}\mathbf{S}_{n,j}^{(q_0^*,q_0^*,q_0^*-1)}$ (see Propositions 6.1 and 6.2) where
$$q_0^*=\begin{cases}q_1 \text{ if } q_0=1,\\ q_0 \text{ otherwise.}\end{cases} \tag{4.21}$$
Note that $\mathbf{S}_{n,j}^{(q_0^*,q_0^*,q_0^*-1)}$ always is in the 2nd Wiener chaos.
- The leading term of $\mathbf{\Sigma}_{n,j}^{(2)}$ is $\frac{c_1 c_{q_{m_0}}}{(q_{m_0}-1)!}\mathbf{S}_{n,j}^{(1,q_{m_0},1)}$ (see Propositions 6.3 and 6.4), which is in the $(q_{m_0}-1)$-th Wiener chaos.
- The leading term of $\mathbf{\Sigma}_{n,j}^{(3)}$ is $2\frac{c_{q_{\ell_0}}c_{q_{\ell_0}+1}}{q_{\ell_0}}\mathbf{S}_{n,j}^{(q_{\ell_0},q_{\ell_0}+1,q_{\ell_0})}$ (see Propositions 6.5 and 6.6), which is Gaussian.

Hence, for the two classes of functions considered in Sections 3.1 and 3.2, we have to compare at most four terms: $\mathbf{S}_{n,j}^{(1,1,0)}$, $\mathbf{S}_{n,j}^{(q_0^*,q_0^*,q_0^*-1)}$, $\mathbf{S}_{n,j}^{(q_{\ell_0},q_{\ell_0}+1,p)}$, $\mathbf{S}_{n,j}^{(1,q_{m_0},1)}$ which are respectively asymptotically Gaussian, Rosenblatt, Gaussian and in the chaos of order $q_{m_0}-1$.

Our three theorems are based on the study of the asymptotic behavior of each sum (see Section 6 below). We first establish some preliminary results.

5. Preliminary results

5.1. *L^2 bounds.* To identify the leading terms, using the same approach than in Clausel et al. (2014) we will give an upper bound for the L^2 norm of the multidimensional terms $\mathbf{S}_{n,j}^{(q,q',p)}$, q,q',p defined in (4.5) and (4.17). Here, the main difficulty is that unlike the case where $G=H_{q_0}$, we have to deal with an infinity of terms. We have also to obtain more precise bounds than in Clausel et al. (2014). In the following, for any random vector $\mathbf{Z}$, the $L^2(\Omega)$ norm of $\mathbf{Z}$ is denoted by

$$\|\mathbf{Z}\|_2=\left(\mathbb{E}\left[|\mathbf{Z}|^2\right]\right)^{1/2}\,. \tag{5.1}$$

(Recall that $|\mathbf{Z}|$ denotes the Euclidean norm of $\mathbf{Z}$.) Our goal in this section is to specify how $\|\mathbf{S}_{n,j}^{(q,q',p)}\|_2$ depends on q, q' and p. The difficulty is that the sum $\overline{\mathbf{S}}_{n,j}$ contains long–memory and short–memory terms having then different normalization factors. To recover all the cases, we shall use not only $\delta_+(q)$ and $\delta(q)$ defined in (2.1) but also

$$\delta_-(q) = \max(-\delta(q), 0) \;, \quad q \geq 0 \;, \tag{5.2}$$

so that $\delta = \delta_+ - \delta_-$ and δ_+, δ_- are nonnegative. In particular, $\delta(0) = \delta_+(0) = 1/2$ and $\delta_-(0) = 0$.

As in Clausel et al. (2014), the expression (4.17) of $\mathbf{S}_{n,j}^{(q,q',p)}$ involves the kernel $\widehat{\boldsymbol{\kappa}}_j^{(p)}$ defined in (4.18) and we have to distinguish the two cases $p \neq 0$ and $p = 0$. The following notations will be used in the sequel. For any $s \in \mathbb{Z}_+$ and $d \in (0, 1/2)$, set

$$\Lambda_s(a) = \prod_{i=1}^{s} (a_i!)^{1-2d}, \quad \forall a = (a_1, \cdots, a_s) \in \mathbb{N}^s \;. \tag{5.3}$$

For any $q, q', p \geq 0$, set

$$\alpha(q, q', p) = \begin{cases} \min\left(1 - \delta_+(q-p) - \delta_+(q'-p), 1/2\right) & \text{if } p \neq 0 \;, \\ 1/2 & \text{if } p = 0 \;, \end{cases} \tag{5.4}$$

$$\beta(q, p) = \max\left(\delta_+(p) + \delta_+(q-p) - 1/2, 0\right) \tag{5.5}$$

$$\beta'(q, q', p) = \max\left(2\delta_+(p) + \delta_+(q-p) + \delta_+(q'-p) - 1, -1/2\right) \;. \tag{5.6}$$

Notice that for any $q \geq 0$, $\beta(q, 0) = \delta_+(q)$. Define the function ε on $\mathbb{Z}_+$ as

$$\varepsilon(p) = \begin{cases} 0 & \text{if for any } s \in \{1, \cdots, p\},\ s(1-2d) \neq 1 \;, \\ 1 & \text{if for some } s \in \{1, \cdots, p\},\ s(1-2d) = 1 \;. \end{cases} \tag{5.7}$$

The index K is defined in (1.3) and the index M is defined in (C.3), and, as noted in Appendix C, the filter $\mathbf{h}_j(t)$ has null moments of order $0, 1, \ldots, M-1$.

Proposition 5.1. *Under Assumptions A, the following bounds hold:*

(i) There exists some $C > 0$ such that, for all $n, \gamma_j \geq 2$ and $1 \leq q \leq q'$ and $1 \leq p \leq \min(q, q'-1)$,

$$\begin{aligned} \|\mathbf{S}_{n,j}^{(q,q',p)}\|_2 \quad \leq \quad & C^{\frac{q+q'}{2}} \Lambda_2(q-p, p)^{1/2} \Lambda_2(q'-p, p)^{1/2} \gamma_j^{2K} \\ & \times [n^{-\alpha(q,q',p)} \gamma_j^{\beta'(q,q',p)} + n^{-1/2} \gamma_j^{\beta(q,p)+\beta(q',p)}] \\ & \times (\log n)^{\varepsilon(q+q'-2p)} (\log \gamma_j)^{3\varepsilon(q')} \;. \end{aligned} \tag{5.8}$$

(ii) If $M \geq K + \max(\delta_+(q), \delta_+(q'))$, there exists some $C > 0$ such that $n, \gamma_j \geq 2$ and $1 \leq q \leq q'$,

$$\|\mathbf{S}_{n,j}^{(q,q',0)}\|_2 \leq C^{\frac{q+q'}{2}} \Lambda_1(q)^{1/2} \Lambda_1(q')^{1/2} n^{-1/2} \gamma_j^{2K + \delta_+(q) + \delta_+(q')} (\log \gamma_j)^{\varepsilon(q')} \;. \tag{5.9}$$

Proof: Proposition 5.1 extends Proposition 7.1 in Clausel et al. (2014). Its proof follows the same lines, see Appendix A for details. □

The following result will be sufficient to find the leading term in Section 6.

Corollary 5.2. *Under Assumptions A, if $M \geq K + \max(\delta_+(q), \delta_+(q'))$, then here exists some $C > 0$ whose value depends only on d and f^* such that for all $n, j \geq 2$, $1 \leq q \leq q'$ and $0 \leq p \leq \min(q, q'-1)$,*

$$\begin{aligned} \|\mathbf{S}_{n,j}^{(q,q',p)}\|_2 \leq C^{\frac{q+q'}{2}} \Lambda_2(q-p, p)^{1/2} \Lambda_2(q'-p, p)^{1/2} n^{-\alpha(q,q',p)} (\log n)^{\varepsilon(q+q'-2p)} \\ \times \gamma_j^{2K+\beta(q,p)+\beta(q',p)} (\log \gamma_j)^{3\varepsilon(q')} \;. \end{aligned} \tag{5.10}$$

Proof: We observe that, for all $1 \leq p \leq q \leq q'$, $\beta'(q, q', p) \leq \beta(q, p) + \beta(q', p)$ and $\alpha(q, q', p) \leq 1/2$. Hence the term between brackets in the right-hand side of (5.8) is bounded by $n^{-\alpha(q,q',p)} \times \gamma_j^{2K+\beta(q,p)+\beta(q',p)}$. This gives (5.10) in the case $p \geq 1$. The case $p = 0$ is obtained by using (5.9) and computing $\Lambda_2(q, 0) = \Lambda_1(q)$, $\alpha(q, q', 0) = 1/2$, $\beta(q, 0) = \delta_+(q)$ and $\beta(q', 0) = \delta_+(q')$. □

5.2. *Asymptotic behavior of the leading terms.* We now investigate the exact asymptotic behavior of the terms that will turn out to be leading in the sum (4.4).

Let us first suppose that the bounds in Proposition 5.1 are sharp enough to determine which terms are leading. Since $\gamma_j \to \infty$ and $n = n_j \to \infty$, those for which the bounds have the largest exponents $\beta(q,p)$ and $\beta(q',p)$ and the lowest exponent $\alpha(q,q',p)$ are more likely to dominate, in particular, if $\delta_+(p)$, $\delta_+(q-p)$, $\delta_+(q'-p), \delta_+(p)+\delta_+(q-p)-1/2$, $\delta_+(p)+\delta_+(q'-p)-1/2$ and $1/2-(1-\delta_+(q-p)-\delta_+(q'-p))$ are all positive. Using (2.4), if $p>0$, this happens for $0<p, q-p, q'-p, q, q', q-p+q'-p<1/(1-2d)$, that is (taking $q \leq q'$ without loss of generality),

$$0<p\leq q\leq q'<1/(1-2d) \quad \text{and} \quad 0<q+q'-2p<1/(1-2d)\ . \tag{5.11}$$

In particular, for such a triplet (p,q,q'), we have $\varepsilon(q')=\varepsilon(q+q'-2p)=0$ so that bounds in (5.8) and (5.9) involving logarithms will not appear in these terms. We shall check afterwards (in Section 6) that indeed, in all the cases we consider, either such a term is leading in the sum (4.4), or the leading term is $\mathbf{S}^{(1,1,0)}_{n_j,j}$ ($q=q'=1$ and $p=0$). The bounds established in Proposition 5.1 will be sharp enough for this goal.

This is why, in the following, we shall only determine the asymptotic behaviors of $\mathbf{S}^{(1,1,0)}_{n_j,j}$ and of $\mathbf{S}^{(q,q',p)}_{n_j,j}$ under Condition (5.11), when $j, n_j \to \infty$.

Proposition 5.3. *Suppose that Assumptions A hold with $M \geq K+\delta(1)=K+d$ and that γ_j is even for all j. Let (n_j) be any diverging sequence of integers. Then as $j\to\infty$,*

$$n_j^{1/2}\gamma_j^{-2(d+K)}\mathbf{S}^{(1,1,0)}_{n_j,j} \overset{(\mathcal{L})}{\to} \mathcal{N}(0,\Gamma)\ , \tag{5.12}$$

where Γ is defined by (3.11).

Proof: This is a direct application of Theorem 3.1 case (a) in Clausel et al. (2014). □

We now consider the case where Condition (5.11) is satisfied.

Proposition 5.4. *Let q,q' and p be non-negative integers such that (5.11) holds. Assume that Assumptions A hold with $M\geq K$ and let (n_j) be any diverging sequence of integers. Then, as $j\to\infty$,*

$$(n_j\gamma_j)^{1-\delta(q-p)-\delta(q'-p)}\gamma_j^{-2(K+\delta(p))}\mathbf{S}^{(q,q',p)}_{n_j,j} \overset{(\mathcal{L})}{\to} [f^*(0)]^{(q+q')/2}\,\mathbf{L}_p\ \ Z_{q+q'-2p,d}(1)\ , \tag{5.13}$$

where $Z_{q+q'-2p,d}$ is the Hermite process defined in (2.17) and $\mathbf{L}_p$ is defined in (3.5).

Proof: The proof follows the same line as the proof of Proposition 8.1 in Clausel et al. (2014). Therefore we only explain how to adapt this proof to our setting. Set $r=q+q'-2p$. Using (4.17) and that, for all $g\in L^2(\mathbb{R}^r)$,

$$\widehat{I}_r(g) \overset{d}{=} (n\gamma_j)^{-r/2}\widehat{I}_r(g(\cdot/(n\gamma_j)))\ ,$$

we have

$$\mathbf{S}^{(q,q',p)}_{n,j} \overset{d}{=} (n\gamma_j)^{-r/2}\widehat{I}_r\left(D_n\circ\Sigma_{q+q'-2p}(\cdot/n)\times[\mathbb{1}_{(-\gamma_j\pi,\gamma_j\pi)}]^{\otimes r}(\cdot/n)\times\mathbf{f}_j\right)\ , \tag{5.14}$$

where, for all $\xi\in\mathbb{R}^r$,

$$\mathbf{f}_j(n\gamma_j\xi)=\sqrt{f}^{\otimes r}(\xi)\times\widehat{\boldsymbol{\kappa}}_j^{(p)}\circ\Sigma_{q-p,q'-p}(\xi)\ .$$

The rest of the proof consists in proving the L^2 convergence of the Itô integral in (5.14), adequately normalized. This is done in the proof of Proposition 8.1 in Clausel et al. (2014) with $q-p=q'-p=1$ (hence $r=2$). The same proof applies in our setting but results in a multiple integral of order r with $r\geq 2$. In particular, if $r>2$ the asymptotic limit is not Rosenblatt but an r-order Hermite process. □

6. Leading terms

Recall the decomposition (4.7) of $\overline{\mathbf{S}}_{n,j}$ using sums $\mathbf{\Sigma}^{(0)}_{n,j}$, $\mathbf{\Sigma}^{(1)}_{n,j}$, $\mathbf{\Sigma}^{(2)}_{n,j}$ and $\mathbf{\Sigma}^{(3)}_{n,j}$. The aim of this section is to identify the leading terms of the three following sums : $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$, $\mathbf{\Sigma}^{(2)}_{n_j,j}$, $\mathbf{\Sigma}^{(3)}_{n_j,j}$ under the conditions specified in Sections 3.1 and 3.2.

6.1. *Leading term of* $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$. Recall that the two sums $\mathbf{\Sigma}^{(0)}_{n_j,j}, \mathbf{\Sigma}^{(1)}_{n_j,j}$ are defined in equations (4.8), (4.9) and that q_0^*, defined in (4.21), equals q_1 if $q_0=1$ and equals q_0 otherwise. Therefore $q_0\geq 2$. We shall prove that, if $q_0^*<1/(1-2d)$, the main term in $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$ is $\frac{c^2_{q_0^*}}{(q_0^*-1)!}\mathbf{S}^{(q_0^*,q_0^*,q_0^*-1)}_{n,j}$, and has rate $n_j^{-(1-2d)}\gamma_j^{2(\delta(q_0^*)+K)}$. The following proposition is used to show that the remainder terms are negligible.

Proposition 6.1. *Suppose that Assumptions A hold with $M\geq K+\delta(q_0^*)$ and that*

$$q_0^*<1/(1-2d)\ ,$$

where q_0^* is defined in (4.21). Let (n_j) be a diverging sequence. Then, when $j\to\infty$,

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\left(\sum_{p=0}^{q_0^*-2}\frac{c^2_{q_0^*}}{(q_0^*!)^2}p!\binom{q_0^*}{p}^2\|\mathbf{S}^{(q_0^*,q_0^*,p)}_{n_j,j}\|_2\right)\to 0\ , \tag{6.1}$$

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\left(\sum_{\ell\in\mathcal{L},q_\ell>q_0^*}\frac{c^2_{q_\ell}}{(q_\ell!)^2}\sum_{p=0}^{q_\ell-1}p!\binom{q_\ell}{p}^2\|\mathbf{S}^{(q_\ell,q_\ell,p)}_{n_j,j}\|_2\right)\to 0\ , \tag{6.2}$$

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\left(\sum_{(\ell_1,\ell_2)\in J}\frac{c_{q_{\ell_1}}}{q_{\ell_1}!}\frac{c_{q_{\ell_2}}}{q_{\ell_2}!}\sum_{p=0}^{q_{\ell_1}}p!\binom{q_{\ell_1}}{p}\binom{q_{\ell_2}}{p}\|\mathbf{S}^{(q_{\ell_1},q_{\ell_2},p)}_{n_j,j}\|_2\right)\to 0\ . \tag{6.3}$$

Proof: We first note that, since $q_0^*\geq 2$ by definition and $q_0^*<1/(1-2d)$ by assumption, we have $1/4<d<1/2$.

As in the proofs of Proposition 4.2, we prove the result in the case where $m=1$ without loss of generality. We thus use non bold faced symbols.

We first prove (6.1). Since there is a finite number of terms in the sum appearing on the left-hand side of (6.1), it is sufficient to show that each term converges to 0. Let $p\in\{0,\dots,q_0^*-2\}$. We apply Corollary 5.2 with $q=q'=q_0^*$. Since $q_0^*<1/(1-2d)$ and thus $\varepsilon(q_0^*)=0$, Inequality (5.10) reads

$$\gamma_j^{-2(\delta(q_0^*)+K)}\|S^{(q_0^*,q_0^*,p)}_{n,j}\|_2\leq C^{q_0^*}\Lambda_2(q_0^*-p,p)n^{-\alpha(q_0^*,q_0^*,p)}\left(\log n\right)^{\varepsilon(2(q_0^*-p))}\gamma_j^{2(\beta(q_0^*,p)-\delta(q_0^*))}\ .$$

By (8.8) and (8.11) in Lemma 8.3, we have $\alpha(q_0^*,q_0^*,p)\geq\min(2(1-2d),1/2)$ and $\beta(q_0^*,p)\leq\delta_+(q_0^*)=\delta(q_0^*)$. Hence,

$$\gamma_j^{-2(\delta(q_0^*)+K)}\|S^{(q_0^*,q_0^*,p)}_{n,j}\|_2\leq C^{q_0^*}\Lambda_2(q_0^*-p,p)n^{-\min(2(1-2d),1/2)}\left(\log n\right)^{\varepsilon(2(q_0^*-p))}\ .$$

Since $d\in(1/4,1/2)$, we have $n^{-\min(2(1-2d),1/2)}(\log n)^{\varepsilon(2(q_0^*-p))}=o(n^{2d-1})$. Thus Inequality (6.1) holds.

We now prove (6.2). We apply (5.10), in the cases $q=q'=q_\ell$, $0\leq p\leq q_\ell-2$ and $q=q'=q_\ell$, $p=q_\ell-1$, successively. Then for some $C>0$, for any $\ell\in\mathcal{L}$ such that $q_\ell>q_0^*$ and for any $0\leq p\leq q_\ell-2$ one has,

$$\gamma_j^{-2(\delta(q_0^*)+K)}\|S^{(q_\ell,q_\ell,p)}_{n,j}\|_2\leq C^{q_\ell}\Lambda_2(q_\ell-p,p)n^{-\alpha(q_\ell,q_\ell,p)}\left(\log n\right)^{\varepsilon(2(q_\ell-p))}\gamma_j^{2(\beta(q_\ell,p)-\delta(q_0^*))}(\log\gamma_j)^3\ .$$

On the other hand, since $d>1/4$, $\varepsilon(2)=0$. Thus in the case where $p=q_\ell-1$, the exponent of $\log n$ vanishes. Moreover, in this case, by (8.9) in Lemma 8.3, $\alpha(q_\ell,q_\ell,p)=\min(1-2d,1/2)=1-2d$. In the alternative case $p<q_\ell-1$, we use (8.8) in Lemma 8.3, which gives

$$n^{-\alpha(q_\ell,q_\ell,p)}\left(\log n\right)^{\varepsilon(2(q_\ell-p))}\leq n^{-\min(2(1-2d),1/2)}\left(\log n\right)^{\varepsilon(2(q_\ell-p))}\leq n^{2d-1}\ ,$$

for n large enough, since $2(1-2d)>1-2d$ and $1/2>1-2d$. Hence in all the cases, the terms in n can be bounded by $C'\,n^{2d-1}$. As for the terms in γ_j, we use that, by (8.11) in Lemma 8.3, $\beta(q_\ell,p)\leq\delta_+(q_\ell)\leq\delta_+(q_0^*+1)$, since $q_\ell\geq q_0^*+1$ and δ_+ is non-increasing. Hence we get that

$$\gamma_j^{-2(\delta(q_0^*)+K)}\|S^{(q_\ell,q_\ell,p)}_{n,j}\|_2\leq C^{q_\ell}\Lambda_2(q_\ell-p,p)n^{2d-1}\ \gamma_j^{2(\delta_+(q_0^*+1)-\delta(q_0^*))}(\log\gamma_j)^3\ ,$$

where $C>0$ may have changed from the previous line. Hence, summing over $\ell \in \mathcal{L}$ such that $q_\ell > q_0^*$ and all $p \in \{0,\dots,q_\ell - 1\}$, we get that

$$n^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\left(\sum_{q_\ell>q_0^*}\sum_{p=0}^{q_\ell-1}\frac{c_{q_\ell}^2}{(q_\ell!)^2}p!\binom{q_\ell}{p}^2\|S_{n,j}^{(q_\ell,q_\ell,p)}\|_2\right)$$
$$\le \gamma_j^{2(\delta_+(q_0^*+1)-\delta(q_0^*))}(\log\gamma_j)^3\sum_{\ell=1}^{+\infty}\frac{c_{q_\ell}^2}{(q_\ell!)^2}C^{q_\ell}\sum_{p=0}^{q_\ell-1}p!\binom{q_\ell}{p}^2\Lambda_2(q_\ell-p,p)\ . \quad (6.4)$$

Observe now that

$$\sum_{\ell=1}^{+\infty}\frac{c_{q_\ell}^2}{(q_\ell!)^2}C^{q_\ell}\sum_{p=0}^{q_\ell-1}p!\binom{q_\ell}{p}^2\Lambda_2(q_\ell-p,p)$$
$$\le \sum_{q,q'\ge1}\sum_{p=0}^{q\wedge q'}\frac{|c_q|}{q!}\frac{|c_{q'}|}{q'!}\,p!\,\binom{q}{p}\binom{q'}{p}C^{\frac{q+q'}{2}}\Lambda_2(q-p,p)^{1/2}\Lambda_2(q'-p,p)^{1/2} \quad (6.5)$$

Since Condition (3.1) holds, Lemma 8.6 implies that :

$$\sum_{\ell=1}^{+\infty}\frac{c_{q_\ell}^2}{(q_\ell!)^2}C^{q_\ell}\sum_{p=0}^{q_\ell-1}p!\binom{q_\ell}{p}^2\Lambda_2(q_\ell-p,p)<\infty\ . \quad (6.6)$$

Finally we observe that, since $\delta(q_0^*)>\delta(q_0^*+1)$ and $\delta(q_0^*)>0$, we have

$$\gamma_j^{2(\delta_+(q_0^*+1)-\delta(q_0^*))}(\log\gamma_j)^3\to 0\ , \quad (6.7)$$

as $\gamma_j\to\infty$. Hence (6.6) and (6.7) imply that Inequality (6.2) holds.

We finally prove that (6.3) holds. Inequality (5.10) for $(\ell_1,\ell_2)\in J$ with $q=q_{\ell_1}, q'=q_{\ell_2}$ and $p\le q_{\ell_1}$ implies that

$$\gamma_j^{-2(\delta(q_0^*)+K)}\|S_{n,j}^{(q_{\ell_1},q_{\ell_2},p)}\|_2\le C^{\frac{q_{\ell_1}+q_{\ell_2}}{2}}\Lambda_2(q_{\ell_1}-p,p)^{1/2}\Lambda_2(q_{\ell_2}-p,p)^{1/2}$$
$$\times\, n^{-\alpha(q_{\ell_1},q_{\ell_2},p)}\,(\log n)^{\varepsilon(q_{\ell_1}+q_{\ell_2}-2p)}\,\gamma_j^{\beta(q_{\ell_1},p)+\beta(q_{\ell_2},p)-2\delta(q_0^*)}(\log\gamma_j)^3\ . \quad (6.8)$$

We first bound the terms that depend on n. First suppose that $p=q_{\ell_1}$ and $q_{\ell_2}=q_{\ell_1}+2$. In this case, the exponent of $\log n$ vanishes, since $\varepsilon(2)=0$ for $d>1/4$, and by (8.6) in Lemma 8.3, the exponent of n $\alpha(q_{\ell_1},q_{\ell_2},p)\ge 1-2d$. Hence, in this case, the terms in n are bounded by n^{2d-1}. Otherwise, if $p<q_{\ell_1}$ or $q_{\ell_2}>q_{\ell_1}+2$, we observe that for $(\ell_1,\ell_2)\in J$, we have $p\le q_{\ell_2}-3$ and hence, by definition of α in (5.4) and since δ_+ is non-increasing,

$$\alpha(q_{\ell_1},q_{\ell_2},p)\ge 1/2-\delta^+(3)=\min(1/2,1/2-(3d-1))>1-2d\ ,$$

since $1/4<d<1/2$. Whatever the exponent of $\log n$, we again obtain that the terms in n are bounded by n^{2d-1}, up to a multiplicative constant:

$$\sup_{(\ell_1,\ell_2)\in J,0\le p\le q_{\ell_1}} n^{-\alpha(q_{\ell_1},q_{\ell_2},p)}\,(\log n)^{\varepsilon(q_{\ell_1}+q_{\ell_2}-2p)}=O\left(n^{2d-1}\right)\ . \quad (6.9)$$

We now bound the terms that depend on γ_j in (6.8). By (8.11) in Lemma 8.3, we have $\beta(q,p)\le\delta_+(q)$ for $0\le p\le q$. Thus $\beta(q_{\ell_1},p)+\beta(q_{\ell_2},p)-\delta(q_0^*)\le\delta_+(q_{\ell_1})+\delta_+(q_{\ell_2})-2\delta(q_0^*)$. Since δ is non–increasing, $q_{\ell_1}\ge q_0^*$ and $q_{\ell_2}\ge q_{\ell_1}+2$ we deduce that $\delta_+(q_{\ell_1})\le\delta_+(q_0^*)=\delta(q_0^*)$ and $\delta_+(q_{\ell_2})\le\delta_+(q_0^*+2)<\delta(q_0^*)$. Hence the exponent of γ_j is bounded by a negative constant and

$$\sup_{(\ell_1,\ell_2)\in J,0\le p\le q_{\ell_1}}\gamma_j^{\beta(q_{\ell_1},p)+\beta(q_{\ell_2},p)-2\delta(q_0^*)}(\log\gamma_j)^3\to 0 \quad \text{as } j\to\infty\ . \quad (6.10)$$

In view of (6.8), (6.9) and (6.10), the proof of (6.3) follows from the bound

$$\sum_{(\ell_1,\ell_2)\in J}\frac{|c_{q_{\ell_1}}|}{q_{\ell_1}!}\frac{|c_{q_{\ell_2}}|}{q_{\ell_2}!}C^{\frac{q_{\ell_1}+q_{\ell_2}}{2}}\sum_{p=0}^{q_{\ell_1}}p!\,(2\pi)^p\prod_{i=1}^{2}\binom{q_{\ell_i}}{p}[\Lambda_2(q_{\ell_i}-p,p)]^{1/2}<\infty\ , \quad (6.11)$$

which follows from the inequality

$$\sum_{(\ell_1,\ell_2)\in J} \frac{|c_{q_{\ell_1}}|}{q_{\ell_1}!}\frac{|c_{q_{\ell_2}}|}{q_{\ell_2}!} C^{\frac{q_{\ell_1}+q_{\ell_2}}{2}} \sum_{p=0}^{q_{\ell_1}} p!\,(2\pi)^p \prod_{i=1}^{2}\binom{q_{\ell_i}}{p}[\Lambda_2(q_{\ell_i}-p,p)]^{1/2}$$

$$\leq \sum_{q,q'\geq 1}\sum_{p=0}^{q\wedge q'} \frac{|c_q|}{q!}\frac{|c_{q'}|}{q'!}\, p!\binom{q}{p}\binom{q'}{p}(2\pi C)^{\frac{q+q'}{2}}\Lambda_2(q-p,p)^{1/2}\Lambda_2(q'-p,p)^{1/2}$$

and from Lemma 8.6 with Condition (3.1). This concludes the proof. □

We now focus on the leading term of the sum $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$.

Proposition 6.2. *Under the same assumptions as Proposition 6.1, we have, as $j\to\infty$,*

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\left(\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}\right)\xrightarrow{(\mathcal{L})}\left[\frac{c_{q_0^*}^2}{(q_0^*-1)!}f^*(0)^{q_0^*}\mathbf{L}_{q_0^*-1}\right]Z_{2,d}(1)\ . \tag{6.12}$$

Proof: We apply Proposition 5.4 with $q=q'=q_0^*$ and $p=q_0^*-1$. Since

$$2\delta(1)+2\delta(q_0^*-1)-1=2\delta(q_0^*)\ ,$$

we get that

$$n_j^{1-2d}\gamma_j^{-2(\delta(q_0^*)+K)}\frac{c_{q_0^*}^2\,(2\pi)^{q_0^*-1}}{(q_0^*-1)!}\mathbf{S}^{(q_0^*,q_0^*,q_0^*-1)}_{n,j}\xrightarrow{(\mathcal{L})}\left[\frac{c_{q_0^*}^2}{(q_0^*-1)!}f^*(0)^{q_0^*}\mathbf{L}_{q_0^*-1}\right]Z_{2,d}(1)\ . \tag{6.13}$$

The left-hand side in (6.13) corresponds to the term $q_\ell=q_0^*$ and $p=q_0^*-1$ of $\mathbf{\Sigma}^{(0)}_{n_j,j}$ in (4.8). The terms of $\mathbf{\Sigma}^{(0)}_{n_j,j}$ with $q_\ell=q_0^*$ and $p<q_0^*-1$ are gathered in the left-hand side of (6.1). The terms of $\mathbf{\Sigma}^{(0)}_{n_j,j}$ with $q_\ell>q_0^*$ are gathered in the left-hand side of (6.2). Finally the left-hand side of (6.3) corresponds to $\mathbf{\Sigma}^{(1)}_{n_j,j}$ in (4.8). Hence, by Proposition 6.1, all these terms are negligible and (6.12) holds. □

6.2. *Leading term of $\mathbf{\Sigma}^{(2)}_{n_j,j}$.* In this section, we investigate the asymptotical behavior of the sum $\mathbf{\Sigma}^{(2)}_{n_j,j}$ defined in (4.10). We shall prove that, if $q_{m_0}<1/(1-2d)$, the leading term of this sum is $\frac{c_1c_{q_{m_0}}}{(q_{m_0}-1)!}\mathbf{S}^{(1,q_{m_0},1)}_{n,j}$ and has rate $n_j^{-(1-2\delta(q_{m_0}-1))/2}\gamma_j^{\delta(q_{m_0})+d+2K}$. To this end we first show that the remainder terms are negligible.

Proposition 6.3. *Assume that Assumptions A hold with $M\geq K+d$ and that*

$$q_{m_0}<1/(1-2d)\ ,$$

where q_{m_0} is defined by (2.15).

Let (n_j) be a diverging sequence. Then, as $j\to\infty$,

$$n_j^{1/2-\delta(q_{m_0}-1)}\gamma_j^{-\delta(q_{m_0})-d-2K}\left(\sum_{\ell\geq m_0}\frac{c_1c_{q_\ell}}{q_\ell!}\|\mathbf{S}^{(1,q_\ell,0)}_{n_j,j}\|_2\right)\to 0\ , \tag{6.14}$$

$$n_j^{1/2-\delta(q_{m_0}-1)}\gamma_j^{-\delta(q_{m_0})-d-2K}\left(\sum_{\ell>m_0}\frac{c_1c_{q_\ell}}{(q_\ell-1)!}\|\mathbf{S}^{(1,q_\ell,1)}_{n_j,j}\|_2\right)\to 0\ . \tag{6.15}$$

Proof: Observe that $\delta_+(1)=d$. We apply (5.10) in Corollary 5.1 with $q=1$ and $q'=q_\ell$. Thus there exists some $C>0$ such that for any $\ell\geq m_0$

$$\gamma_j^{-\delta(q_{m_0})-d-2K}\|S^{(1,q_\ell,0)}_{n,j}\|_2\leq C^{\frac{q_\ell+1}{2}}(q_\ell!)^{1/2-d}\,n^{-1/2}\gamma_j^{\delta_+(q_\ell)-\delta(q_{m_0})}(\log\gamma_j)^{3\varepsilon(q_\ell)}\ . \tag{6.16}$$

Since by assumption $q_{m_0}<1/(1-2d)$, we have $\varepsilon(q_{m_0})=0$ and $\delta_+(q_{m_0})=\delta(q_{m_0})$. Thus, if $\ell=m_0$, the terms involving γ_j vanish in the right-hand side of (6.16). If $\ell>m_0$, we have $\delta_+(q_\ell)<\delta(q_{m_0})$ and these terms are $o(1)$ as $j\to\infty$. Hence, for j large enough, and for any $\ell\geq m_0$,

$$n_j^{1/2-\delta(q_{m_0}-1)}\gamma_j^{-\delta(q_{m_0})-d-2K}\|S^{(1,q_\ell,0)}_{n_j,j}\|_2\leq C^{\frac{q_\ell+1}{2}}(q_\ell!)^{1/2-d}\,n_j^{-\delta(q_{m_0}-1)}\ .$$

Using that $\delta(q_{m_0}-1)>\delta(q_{m_0})>0$, and that, by Condition (3.1),

$$\sum_{\ell=m_0}^{+\infty} C^{\frac{q_\ell+1}{2}}\,|c_{q_\ell}|\,(q_\ell!)^{-1/2-d}<+\infty\ ,$$

we obtain the limit (6.14).

We now show that (6.15) holds. Applying (5.10) with $q=1$, $q'=q_\ell$ and $p=1$, we get that there exists some $C>0$ such that for any $\ell>m_0$,

$$\|\mathbf{S}_{n,j}^{(1,q_\ell,1)}\|_2\le C^{\frac{q_\ell+1}{2}}\{(q_\ell-1)!\}^{1/2-d}\,n^{-\alpha(1,q_\ell,1)}(\log n)^{\varepsilon(q_\ell-1)}\gamma_j^{\beta(1,1)+\beta(q_\ell,1)+2K}(\log\gamma_j)^3\ . \tag{6.17}$$

The definition of α and β by Equations (5.4) and (5.5), implies that

$$\alpha(1,q_\ell,1)=1/2-\delta_+(q_\ell-1),\ \beta(1,1)=d,\ \beta(q_\ell,1)=\max(d+\delta_+(q_\ell-1)-1/2,0)\ .$$

Since $\ell>m_0$, one has $\delta_+(q_\ell-1)\le\delta_+(q_{m_0+1}-1)$. Thus

$$n_j^{-\alpha(1,q_\ell,1)}(\log n_j)^{\varepsilon(q_\ell-1)}\le n_j^{1/2-\delta_+(q_{m_0+1}-1)}\log n_j=o\left(n_j^{1/2-\delta(q_{m_0}-1)}\right)\ .$$

Observe now that for $\ell>m_0$, we have $q_\ell-1\ge q_{m_0}$ and thus

$$\gamma_j^{\beta(1,1)+\beta(q_\ell,1)+2K}(\log\gamma_j)^3\le\gamma_j^{d+2K+\max(d+\delta_+(q_{m_0})-1/2,0)}(\log\gamma_j)^3=o\left(\gamma_j^{d+2K+\delta(q_{m_0})}\right)\ .$$

Now, using the last two displayed equations, (6.17) and Condition (3.1), we obtain the limit (6.15), which concludes the proof. □

We now deduce the asymptotic behavior of $\mathbf{\Sigma}^{(2)}_{n_j,j}$.

Proposition 6.4. *Under the same assumptions as Proposition 6.3, we have as $j\to\infty$*

$$n_j^{(1-2\delta(q_{m_0}-1))/2}\gamma_j^{-(\delta(q_{m_0})+d+2K)}\mathbf{\Sigma}^{(2)}_{n_j,j}\xrightarrow{(\mathcal{L})}\frac{2c_1c_{q_{m_0}}}{(q_{m_0}-1)!}[f^*(0)]^{(q_{m_0}+1)/2}\mathbf{L}_1Z_{q_{m_0}-1,d}(1)\ , \tag{6.18}$$

where $\mathbf{L}_1$ is defined in (3.5) and $Z_{q-1,d}$ is the Hermite process defined in (2.17).

Proof: We apply Proposition 5.4 with $q=1$, $q'=q_{m_0}$ and $p=1$. For these values, since $q_{m_0}<1/(1-2d)$, Condition (5.11) is satisfied. The exponents of n and γ_j in the left-hand side of (5.13) respectively read

$$1-\delta(q-p)-\delta(q'-p)=1-\delta(0)-\delta(q_{m_0}-1)=1/2-\delta(q_{m_0}-1)$$

and

$$1-\delta(q-p)-\delta(q'-p)-2K-2\delta(p)=-\delta(q_{m_0})-d-2K\ .$$

Hence we get that

$$n_j^{(1-2\delta(q_{m_0}-1))/2}\gamma_j^{-(\delta(q_{m_0})+d+2K)}\mathbf{S}^{(1,q_{m_0},1)}_{n_j,j}\xrightarrow{(\mathcal{L})}[f^*(0)]^{(q_{m_0}+1)/2}\mathbf{L}_1Z_{q_{m_0}-1,d}(1)\ . \tag{6.19}$$

Finally we observe that this term corresponds to the second term of the summand in (4.10) with index $\ell=q_{m_0}$, up to the multiplicative constant $4\pi c_1c_{q_{m_0}}/(q_{m_0}-1)!$. All the other terms are negligible by Proposition 6.3. Thus the limit (6.18) holds. □

6.3. *Leading term of* $\mathbf{\Sigma}^{(3)}_{n_j,j}$. In this section we investigate the asymptotic behavior of $\mathbf{\Sigma}^{(3)}_{n,j}$ defined in (4.11). We first bound the sum over indices $\ell=\ell_0$ and $p\ne q_{\ell_0}$ and the one over indices $\ell>\ell_0$ and $p\in\{0,\dots,q_\ell\}$. The two sums will turn out to be negligible.

Proposition 6.5. *Assume that Assumptions A hold with $M\ge K+\delta(q_{\ell_0})$ and*

$$q_{\ell_0}+1<1/(1-2d)\ . \tag{6.20}$$

Let (n_j) be a diverging sequence. Then, as $j\to\infty$,

$$n_j^{\frac{1-2d}{2}}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}\left(\sum_{p=0}^{q_{\ell_0}-1}\frac{c_{q_{\ell_0}}}{q_{\ell_0}!}\frac{c_{q_{\ell_0}+1}}{(q_{\ell_0}+1)!}p!\binom{q_{\ell_0}}{p}\binom{q_{\ell_0}+1}{p}\|\mathbf{S}^{(q_{\ell_0},q_{\ell_0}+1,p)}_{n_j,j}\|_2\right)\to0\ , \tag{6.21}$$

$$n_j^{\frac{1-2d}{2}}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}\left(\sum_{\ell\in I\setminus\{\ell_0\}}\sum_{p=0}^{q_\ell}\frac{c_{q_\ell}}{q_\ell!}\frac{c_{q_\ell+1}}{(q_\ell+1)!}p!\binom{q_\ell}{p}\binom{q_\ell+1}{p}\|\mathbf{S}_{n_j,j}^{(q_\ell,q_\ell+1,p)}\|_2\right)\to 0\ . \tag{6.22}$$

Proof: Observe that, since $q_{\ell_0}\geq 1$, the assumption $q_{\ell_0}+1<1/(1-2d)$ implies that $d\in(1/4,1/2)$.

We first prove Inequality (6.21). Since there is only a finite number of terms in the left hand side of Inequality (6.21), we only have to prove that each term tends to 0. We apply Corollary 5.2 with $q=q_{\ell_0}$, $q'=q_{\ell_0}+1$ and $p\leq q_{\ell_0}-1$. For these values of q,q' and p, under Condition (6.20), we have $\varepsilon(q')=0$, and by (8.10) and (8.11), we have $\alpha(q,q',p)\geq\min(3(1/2-d),1/2)$, $\beta(q,p)\leq\delta_+(q_{\ell_0})=\delta(q_{\ell_0})$ and $\beta(q',p)\leq\delta_+(q_{\ell_0}+1)=\delta(q_{\ell_0}+1)$. Thus Equation (5.10) yields

$$n_j^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}\|S_{n_j,j}^{(q_{\ell_0},q_{\ell_0}+1,p)}\|_2=O\left(n_j^{-\min(1-2d,d)}\log(n_j)\right)\ .$$

Since $d\in(1/4,1/2)$, we obtain (6.21).

We now prove (6.22). We apply Corollary 5.2 with $q=q_\ell$, $q'=q_\ell+1$ and $p\leq q_\ell$ for some $\ell\in I\setminus\{\ell_0\}$. In this case Inequality (5.10) reads

$$\begin{aligned}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}&\|S_{n,j}^{(q_\ell,q_\ell+1,p)}\|_2\\ &\leq C^{q_\ell+\frac12}\Lambda_2(q_\ell-p,p)^{\frac12}\Lambda_2(q_\ell+1-p,p)^{\frac12}n^{-\alpha(q_\ell,q_\ell+1,p)}\log(n)^{\varepsilon(2q_\ell+1-2p)}\\ &\qquad\times\gamma_j^{(\beta(q_\ell,p)-\delta(q_{\ell_0}))+(\beta(q_\ell+1,p)-\delta(q_{\ell_0}+1))}(\log\gamma_j)^3\ .\end{aligned} \tag{6.23}$$

We observe that for n large enough,

$$n^{-\alpha(q_\ell,q_\ell+1,p)}\log(n)^{\varepsilon(2q_\ell+1-2p)}\leq n^{-(1-2d)/2}\ . \tag{6.24}$$

Indeed, on the one hand, if $p=q_\ell$, then $\varepsilon(2q_\ell+1-2p)=\varepsilon(q_\ell+q_\ell+1-2q_\ell)=\varepsilon(1)=0$ and $\alpha(q_\ell,q_\ell+1,q_\ell)\geq(1-2d)/2$ (8.7). On the other hand, if $p<q_\ell$, since $d>1/4$, (8.6) implies that $\alpha(q_\ell,q_\ell+1,p)\geq 1-2d$.

In addition, by (8.11) one has for any $p\leq q_\ell$, $\beta(q_\ell,p)\leq\delta_+(q_\ell)$. Thus, for any $\ell>\ell_0$ and any $p\leq q_\ell$,

$$\begin{aligned}\gamma_j^{(\beta(q_\ell,p)-\delta(q_{\ell_0}))+(\beta(q_\ell+1,p)-\delta(q_{\ell_0}+1))}(\log\gamma_j)^3&\leq\gamma_j^{(\delta_+(q_\ell)-\delta(q_{\ell_0}))+(\delta_+(q_\ell+1)-\delta(q_{\ell_0}+1))}(\log\gamma_j)^3\\ &\leq\gamma_j^{(\delta_+(q_{\ell_0+1})-\delta(q_{\ell_0}))+(\delta_+(q_{\ell_0+1}+1)-\delta(q_{\ell_0}+1))}(\log\gamma_j)^3=o(1)\ .\end{aligned} \tag{6.25}$$

As in the proof of Proposition 6.1, applying Lemma 8.6 with Condition (3.1), we have

$$\sum_{\ell\geq 0}\frac{C^{q_\ell+1/2}|c_{q_\ell}||c_{q_\ell+1}|}{q_\ell!(q_\ell+1)!}\sum_{p=0}^{q_\ell}p!\binom{q_\ell}{p}\binom{q_\ell+1}{p}\Lambda_2(q_\ell-p,p)^{\frac12}\Lambda_2(q_\ell+1-p,p)^{\frac12}<\infty\ .$$

Applying this, (6.23), (6.24) and (6.25), we obtain (6.22). □

The following result can now be established.

Proposition 6.6. *Under the same assumptions as Proposition 6.5, we have as $j\to\infty$*

$$n_j^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0}+1)+\delta(q_{\ell_0})+2K)}\mathbf{\Sigma}_{n_j,j}^{(3)}\overset{(\mathcal{L})}{\to}2\frac{c_{q_{\ell_0}}c_{q_{\ell_0}+1}}{q_{\ell_0}!}[f^*(0)^{q_{\ell_0}+1/2}\mathbf{L}_{q_{\ell_0}}]Z_{1,d}(1)\ .$$

Proof: We apply Proposition 5.4 with $q=q'-1=q_{\ell_0}$ and $p=q_{\ell_0}$. Indeed we have, under Condition (6.20), $0<q=q_{\ell_0}<q'=q_{\ell_0}+1<1/(1-2d)$ and $q+q'-2p=q_{\ell_0}+q_{\ell_0}+1-2q_{\ell_0}=1<1/(1-2d)$. Thus Condition (5.11) holds. We obtain that, as $j\to\infty$,

$$n_j^{(1-2d)/2}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+2K)}S_{n_j,j}^{(q_{\ell_0},q_{\ell_0}+1,q_{\ell_0})}\overset{(\mathcal{L})}{\to}[f^*(0)^{q_{\ell_0}+1/2}\mathbf{L}_{q_{\ell_0}}]Z_{1,d}(1)\ . \tag{6.26}$$

Using this limit, Proposition 6.5 and the definition of $\mathbf{\Sigma}_{n_j,j}^{(3)}$ in (4.11), we conclude the proof. □

7. Proofs of Theorems 3.1, 3.3 and 3.5

7.1. *Proof of Theorem 3.1.* In the setting of Theorem 3.1, one has $q_0 \geq 2$ and thus $c_1 = 0$ and $q_0^* = q_0 \geq 2$. Thus $\mathbf{\Sigma}^{(2)}_{n_j,j}$ and $\mathbf{S}^{(1,1,0)}_{n_j,j}$, vanish in (4.7) and the asymptotic behavior of $\overline{S}_{n_j,j}$ results from $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$ and $\mathbf{\Sigma}^{(3)}_{n_j,j}$ given in Proposition 6.2 and 6.6, respectively. These propositions apply because we assume (3.3) and $M \geq K+\delta(q_0)$ in Theorem 3.1. Now the ratio of the convergence rates appearing in these propositions reads

$$n_j^{1/2-d}\gamma_j^{-(\delta(q_{\ell_0})+\delta(q_{\ell_0}+1)+K)} = \left(n_j\gamma_j^{-\nu}\right)^{1/2-d} \ .$$

Hence Case (a) of Theorem 3.1 corresponds to

$$\mathbf{\Sigma}^{(3)}_{n_j,j} = o_P\left(\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}\right)$$

and Case (b) to

$$\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j} = o_P\left(\mathbf{\Sigma}^{(3)}_{n_j,j}\right) \ .$$

The proof of Theorem 3.1 follows. □

7.2. *Proof of Theorems 3.3 and 3.5.* Here Condition (3.8) holds, so that $q_0 = 1$, $q_1 < 1/(1-2d)$ and $\ell_0 = \infty$ (or equivalently I is an empty set). In particular $\mathbf{\Sigma}^{(3)}_{n_j,j}$ vanishes in (4.7) and the asymptotic behavior of $\overline{S}_{n_j,j}$ is obtained from those of $\mathbf{S}^{(1,1,0)}_{n_j,j}$, $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$ and $\mathbf{\Sigma}^{(2)}_{n_j,j}$. Since moreover $M > K+d$, Proposition 5.3 applies. Using the definition of q_0^* in (4.21) we have $q_0^* = q_1$, and since $M > K+d \geq K+\delta(q_0^*)$ Propositions 6.2 also applies. Finally, observing that here m_0 defined in (2.15) equals 1 and that $M > K+d$, Proposition 6.4 applies. Thus, using (4.7), it only remains to compare the convergence rates in these propositions.

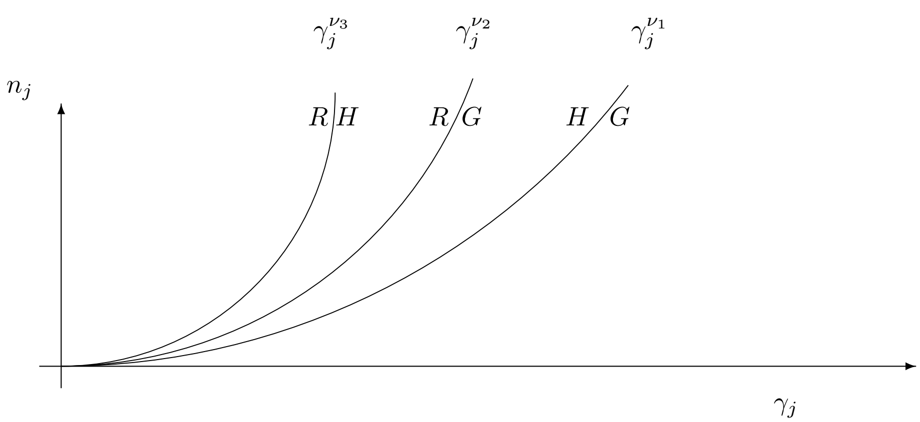

FIGURE 7.1. Pairwise comparisons of the rates of convergence of $\mathbf{S}^{(1,1,0)}_{n_j,j}$ (G), $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$ (R) and $\mathbf{\Sigma}^{(2)}_{n_j,j}$ (H) in the plane γ_j versus n_j.

We first prove Theorem 3.3. Recall that, by Lemma 3.2, since $q_1 < q_1^*$, one has

$$\nu_1 < \nu_2 < \nu_3 \ ,$$

where these three indices are defined in (3.10). In Figure 7.1, we provide pairwise comparisons of the rates of convergence of $\mathbf{S}^{(1,1,0)}_{n_j,j}$, $\mathbf{\Sigma}^{(0)}_{n_j,j}+\mathbf{\Sigma}^{(1)}_{n_j,j}$ and $\mathbf{\Sigma}^{(2)}_{n_j,j}$. We obtain domains separated by the three curves $n_j = \gamma_j^{\nu_1}$, $n_j = \gamma_j^{\nu_2}$ and $n_j = \gamma_j^{\nu_3}$. Each curve is concerned with a pair of two terms among the three and separates the plane (γ_j, n_j) in two domains, where one of the two terms dominates the other. We indicated the dominating term by G for the asymptotically Gaussian term $S^{(1,1,0)}_{n_j,j}$, R for the asymptotically Rosenblatt term $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$ and H for the term $\mathbf{\Sigma}^{(3)}_{n_j,j}$ belonging asymptotically to a chaos of order greater than 2.

We begin with the case $q_1 = 3$. In this case, one has $\nu_3 = \infty$. Further Propositions 6.2 and 6.4 imply that $\Sigma^{(0)}_{n_j,j} + \Sigma^{(1)}_{n_j,j} = o_P(\Sigma^{(2)}_{n_j,j})$. One has then to compare the rates of convergence of $\mathbf{S}^{(1,1,0)}_{n_j,j}$ and $\Sigma^{(2)}_{n_j,j}$. Using the diagram, we then deduce that

- if $n_j \ll \gamma_j^{\nu_1}$, G dominates H and then we obtain Case (a) of Theorem 3.3 for $q_1 = 3$.
- if $\gamma_j^{\nu_1} \ll n_j$, H dominates G and then we obtain Case (b) of Theorem 3.3 for $q_1 = 3$.

If $q_1 > 3$, one has $\nu_3 < \infty$ and the term $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$ is no more always negligible with respect to $\Sigma^{(2)}_{n_j,j}$. We then get three domains where one term dominates over the other two:

- $n_j \ll \gamma_j^{\nu_1}$: G dominates H and R, that is, the two terms $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$ and $\mathbf{\Sigma}^{(2)}_{n_j,j}$ are both negligible with respect to $S^{(1,1,0)}_{n_j,j}$. By Proposition 5.3, we obtain Case (a) of Theorem 3.3.
- $\gamma_j^{\nu_1} \ll n_j \ll \gamma_j^{\nu_3}$: since the domain lies both on the right-hand side of the curve $n_j = \gamma_j^{\nu_3}$ and on the left-hand side of the curve $n_j = \gamma_j^{\nu_1}$, H dominates R and H dominates G, hence R dominates R and G. That is, the two terms $S^{(1,1,0)}_{n_j,j}$ and $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$ are both negligible with respect to $\mathbf{\Sigma}^{(2)}_{n_j,j}$. By Proposition 6.4, we obtain Case (b) of Theorem 3.3.
- $\gamma_j^{\nu_3} \ll n_j$: since the domain lies both on the left-hand side of the curve $n_j = \gamma_j^{\nu_3}$ and on the left-hand side of the curve $n_j = \gamma_j^{\nu_2}$, R dominates H and R dominates G, hence R dominates H and G. That is, the two terms $S^{(1,1,0)}_{n_j,j}$ and $\mathbf{\Sigma}^{(2)}_{n_j,j}$ are both negligible with respect to $\mathbf{\Sigma}^{(0)}_{n_j,j} + \mathbf{\Sigma}^{(1)}_{n_j,j}$. By Proposition 6.2, we obtain Case (c) of Theorem 3.3.

This completes the proof of Theorem 3.3.

The proof of Theorem 3.5 is similar except that the assumption $q_1 \geq q_1^*$ implies that

$$\nu_3 \leq \nu_2 \leq \nu_1 \ .$$

The domains of convergence are now obtained from Figure 7.2. $\square$

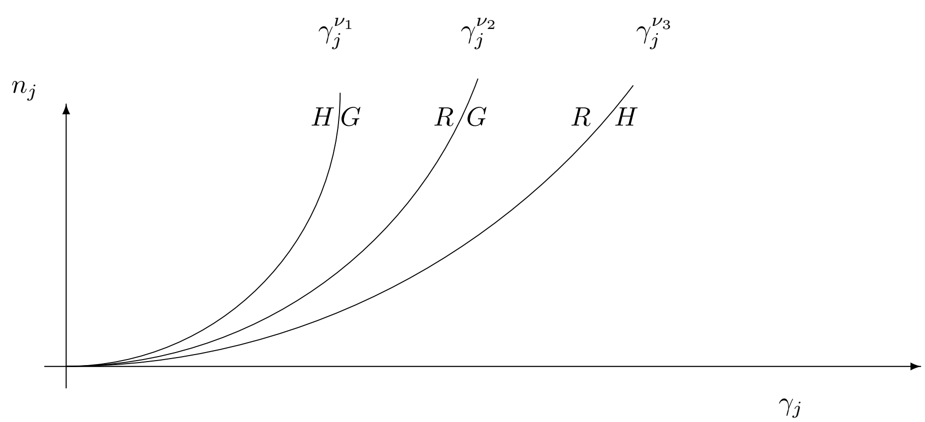

FIGURE 7.2. Domains of convergence for Theorem 3.5

8. Technical lemmas

The following lemma is used in the proof of Proposition 5.1 and in that of Lemma 8.4.

Lemma 8.1. *Define, for all $a > 0$ and $\beta_1 \in (0,1)$,*

$$J_{1,a}(s_1;\beta_1) = |s_1|^{-\beta_1} , \quad s_1 \in \mathbb{R} , \tag{8.1}$$

and, for any integer $p \geq 2$ and $\beta = (\beta_1, \cdots, \beta_p) \in (0,1)^p$,

$$J_{p,a}(s_1;\beta) = \int_{s_2=-(p-1)a}^{(p-1)a} \cdots \int_{s_p=-a}^{a} \prod_{i=2}^{p} |s_{i-1}-s_i|^{-\beta_{i-1}} \, |s_p|^{-\beta_p} \, \mathrm{d}s_p \ldots \mathrm{d}s_2 , \quad s_1 \in \mathbb{R} . \tag{8.2}$$

Then

(i) if $\beta_1 + \cdots + \beta_p > p-1$, we have

$$C_p(\beta) := \sup_{a>0} \sup_{|s_1| \leq pa} \left(|s_1|^{-(p-1-(\beta_1+\cdots+\beta_p))} J_{p,a}(s_1;\beta) \right) < \infty ,$$

(ii) if $\beta_1 + \cdots + \beta_p = p-1$, we have

$$C_p(\beta) := \sup_{a>0} \sup_{|s_1| \leq pa} \left(\frac{1}{1+\log(pa/|s_1|)} J_{p,a}(s_1;\beta) \right) < \infty ,$$

(iii) if there exists $q \in \{2, \ldots, p-1\}$ such that $\beta_q + \cdots + \beta_p = p-q$, we have

$$C_p(\beta) := \sup_{a>0} \sup_{|s_1| \leq pa} \left(\frac{1}{1+\log(pa/|s_1|)} a^{-(q-1-(\beta_1+\cdots+\beta_{q-1}))} J_{p,a}(s_1;\beta) \right) < \infty ,$$

(iv) if $\beta_1 + \cdots + \beta_p < p-1$ and for all $q \in \{1, \ldots, p-1\}$, we have $\beta_q + \cdots + \beta_p \neq p-q$, we have

$$C_p(\beta) := \sup_{a>0} \sup_{|s_1| \leq pa} \left(a^{-(p-1-(\beta_1+\cdots+\beta_p))} J_{p,a}(s_1;\beta) \right) < \infty .$$

Moreover, in the case where all the components of β are equal to $b \in (0,1)$, there exists a constant $c > 0$ depending only on b such that

$$\sup_{p \geq 1} c^{-p} (p!)^{b-1} \, C_p(b 1_p) < \infty , \tag{8.3}$$

where 1_p denotes the p–dimensional vector with all entries equal to 1.

Remark 8.2. As in Clausel et al. (2014), all the cases can be compactly written as

$$C_p(\beta) = \sup_{a>0} \sup_{|s_1| \leq pa} \left(\frac{a^{-(p-1-(\beta_1+\cdots+\beta_p))_+} |s_1|^{(p-1-(\beta_1+\cdots+\beta_p))_-}}{(1+\log(pa/|s_1|))^{\varepsilon}} J_{p,a}(s_1;\beta) \right)$$

where $\varepsilon = 1$ if there exists $q \in \{1, \cdots, p\}$ such that $\beta_q + \cdots + \beta_p = p - q$ and $\varepsilon = 0$ otherwise, and $x_+ = \max(x,0)$, $x_- = \max(-x,0)$. Now, observing that

$$(p-1-2pd)_+ = (p(1-2d)-1)_+ = (-2\delta(p))_+ = 2\delta_-(p) ,$$

and, similarly, $(p-1-2pd)_- = 2\delta_+(p)$, Inequality (8.3) with $b = 2d \in (0,1)$ implies there exists a constant $c > 0$ depending only on d such that for any $a > 0$, $|s_1| \leq pa$

$$J_{p,a}(s_1; 2d1_p) \leq c^p (p!)^{1-2d} a^{2\delta_-(p)} |s_1|^{-2\delta_+(p)} (1+\log(pa/|s_1|))^{\varepsilon(p)} , \tag{8.4}$$

where $\varepsilon(p)$ is here defined by (5.7), which corresponds to the ε above in the case $\beta_1 = \cdots = \beta_p = 2d$.

Proof: Observe first that for all $p \geq 1$,

$$J_{p,a}(s_1;\beta) = \int_{s_2=-(p-1)a}^{(p-1)a} |s_2 - s_1|^{-\beta_1} \, J_{p-1,a}(s_2;\beta') \, \mathrm{d}s_2 , \tag{8.5}$$

where $\beta' = (\beta_2, \ldots, \beta_p)$. The finiteness of the bounds $C_p(\beta)$ for any integer p and any $\beta \in (0,1)^p$ is then proved by induction on p in the different cases in Lemma 9.3 of Clausel et al. (2014).

Finally we show the uniform bound (8.3), that is, that $C_p(b, \ldots, b) = O(c_1^p (p!)^{1-b})$ as $p \to \infty$ for any fixed $b \in (0,1)$. We provide a proof only in the case where $1/(1-b)$ is not an integer (to avoid cases (ii) and (iii)). The proof is similar in the other case. Hence we use the induction step described in Case 1 above. Observe that there exists some integer p_0 depending only on b, such that for any $p \geq p_0$ we have $(p-1)b < p-2$, which corresponds above to $\beta_2 + \cdots + \beta_p < p-2$ (case (iv)). Hence using the induction assumption (8.5), the finiteness of C_p in case (iv) and the fact that $|s_1| \leq pa$, we get that there exists some positive constant c depending only on b such that,

$$\begin{aligned} J_{p,a}(s_1; b, \cdots, b) &\leq C_{p-1}(b, \cdots, b)\, a^{p-2-(p-1)b} \left(\int_{-(p-1)a}^{(p-1)a} |s_2 - s_1|^{-b} \mathrm{d}s_2 \right) \\ &\leq C_{p-1}(b, \cdots, b)\, a^{p-2-(p-1)b} \times \left(c(b)((2p-1)a)^{1-b} \right) \\ &= \left(c(b) p^{1-b}\, C_{p-1}(b, \ldots, b) \right) \, a^{p-1-pb} \,. \end{aligned}$$

This yields that for any $p \geq p_0(b)$, $C_p(b, \ldots, b) \leq c(b) p^{1-b}\, C_{p-1}(b, \ldots, b)$. Since this holds for any $p \geq p_0(b)$, the bound (8.3) follows by induction. □

The following lemma provides bounds of α and β defined in (5.4) and (5.5). It is used in the proofs of Propositions 6.1, 6.3 and 6.5.

Lemma 8.3. *One has*

(1) Assume that $d > 1/4$. Then for any $(q, q') \in \mathbb{N}^2$

$$\inf_{0 \leq p \leq \min(q \vee q' - 2, q \wedge q')} (\alpha(q, q', p)) \geq 1 - 2d \,, \tag{8.6}$$

In any case,

$$\alpha(q, q', \min(q \vee q' - 1, q \wedge q')) \geq 1/2 - d \,. \tag{8.7}$$

(2) For any $q \in \mathbb{N}$

$$\inf_{0 \leq p \leq q-2} (\alpha(q, q, p)) \geq \min(2(1 - 2d), 1/2) \,. \tag{8.8}$$

Further,

$$\alpha(q, q, q-1) = \min(1 - 2d, 1/2) \,. \tag{8.9}$$

(3) For any $q \in \mathbb{N}$

$$\inf_{0 \leq p \leq q-1} (\alpha(q+1, q, p)) \geq \min(3/2(1 - 2d), 1/2) \,. \tag{8.10}$$

(4) For any $q \in \mathbb{N}$

$$\sup_{0 \leq p \leq q} (\beta(q, p)) \leq \delta_+(q) \,. \tag{8.11}$$

Proof: (1) Let us fix $(q, q') \in \mathbb{N}^2$ and assume that $q' \leq q$. Since the map

$$m \mapsto \delta_+(m) = \max(dm - (m-1)/2, 0) \,,$$

is non–increasing with range in $[0, 1/2]$, one has for $0 \leq p \leq \min(q - 2, q')$

$$\alpha(q, q', p) = \min(1 - \delta_+(q - p) - \delta_+(q' - p), 1/2) \geq \min(1 - \delta_+(2) - 1/2, 1/2) \,.$$

If $d > 1/4$, $\delta_+(2) = 2d - 1/2$ and thus

$$\alpha(q, q', p) \geq \min(1 - 2d, 1/2) = 1 - 2d \,,$$

which proves (8.6). Finally, if $p = q - 1$ and $p \leq q'$,

$$\begin{aligned} \alpha(q, q', p) = \min(1 - \delta_+(q - p) - \delta_+(q' - p), 1/2) &\geq \min(1 - \delta_+(1) - 1/2, 1/2) \\ &= \min(1/2 - d, 1/2) = 1/2 - d \,, \end{aligned}$$

which proves (8.7).

(2) Let us fix $q \in \mathbb{N}$, then for any $p \leq q - 2$,

$$\alpha(q, q, p) = \min(1 - \delta_+(q - p) - \delta_+(q - p), 1/2) \geq \min(1 - 2\delta_+(2), 1/2) \,.$$

If $d \leq 1/4$, $\delta_+(2) = 0$ and we get $\alpha(q, q, p) \geq 1/2 \geq \min(2(1 - 2d), 1/2)$. If $d > 1/4$, $2\delta_+(2) = 2\delta(2) = 4d - 1$ and

$$\alpha(q, q, p) \geq \min(1 - (4d - 1), 1/2) = \min(2(1 - 2d), 1/2) \,,$$

which gives (8.8). To prove (8.9), we observe that if $p = q - 1$,

$$\alpha(q, q, p) = \min(1 - 2\delta_+(1), 1/2) = \min(1 - 2d, 1/2) \,.$$

(3) Let us fix $q \in \mathbb{N}$, then for any $p \leq q-1$,

$$\alpha(q+1,q,p) = \min(1-\delta_+(q+1-p)-\delta_+(q-p),1/2) \geq \min(1-\delta_+(2)-\delta_+(1),1/2)$$
$$= \min(1-d-\delta_+(2),1/2)\ .$$

If $d \leq 1/4$, $\delta_+(2)=0$ and $\alpha(q+1,q,p) \geq \min(1-d,1/2)=1/2$. If $d>1/4$, $\delta_+(2)=2d-1/2$ and (8.10) follows from

$$\alpha(q+1,q,p) \geq \min(1-d-(2d-1/2),1/2) = \min(3(1-2d)/2,1/2)\ .$$

(4) If $\beta(q,p)=0$, then $\beta(q,p) \leq \delta_+(q)$. Now consider the case where

$$\beta(q,p) = \max(\delta_+(p)+\delta_+(q-p)-1/2,0) > 0\ ,$$

that is, $\delta_+(p)+\delta_+(q-p)-1/2>0$. In this case, $\delta_+(p)$ and $\delta_+(q-p)$ are both positive (since $0 \leq \delta_+(\cdot) < 1/2$) and they respectively equal $\delta(p)$ and $\delta(q-p)$. Then we obtain

$$\max(\delta_+(p)+\delta_+(q-p)-1/2,0) = \delta(p)+\delta(q-p)-1/2 = \delta(q)\ ,$$

which again implies (8.11).

□

The following result provides a bound of $\widehat{\kappa}_j^{(p)}$ defined in (4.14), in the case where $p>0$. It is a refinement of Lemma 10.1 of Clausel et al. (2014). It is used in the proof of Proposition 5.4.

Lemma 8.4. *Suppose that Assumptions A hold and let p be a positive integer. Then there exists some $C>0$ neither depending on p nor j such that for any $(\xi_1,\xi_2) \in \mathbb{R}^2$,*

(i) if for any $s \in \{1,\cdots,p\}$, $s(1-2d) \neq 1$ then,

$$|\widehat{\kappa}_j^{(p)}(\xi_1,\xi_2)| \leq C^p (p!)^{1-2d}\ \frac{\gamma_j^{2(\delta_+(p)+K)}}{(1+\gamma_j|\{\xi_1\}|)^{\delta_+(p)}(1+\gamma_j|\{\xi_2\}|)^{\delta_+(p)}}\ . \tag{8.12}$$

(ii) if there exists $s \in \{1,\cdots,p\}$ such that $s(1-2d)=1$, then,

$$|\widehat{\kappa}_j^{(p)}(\xi_1,\xi_2)| \leq C^p (p!)^{1-2d}\ \gamma_j^{2K}\log(\gamma_j)\ . \tag{8.13}$$

Remark 8.5. In Case (ii) of Lemma 8.4, we have $p>1/(1-2d)$, hence $\delta_+(p)=0$. Equations (8.13) and (8.12) can thus be written as a single bound, namely,

$$|\widehat{\kappa}_j^{(p)}(\xi_1,\xi_2)| \leq C^p (p!)^{1-2d}\ \frac{\gamma_j^{2(\delta_+(p)+K)}}{(1+\gamma_j|\{\xi_1\}|)^{\delta_+(p)}(1+\gamma_j|\{\xi_2\}|)^{\delta_+(p)}}\ (\log\gamma_j)^{\varepsilon(p)}\ , \tag{8.14}$$

where $\varepsilon(p)$ is defined by (5.7).

Proof: By (2π)-periodicity of $\widehat{\kappa}_j^{(p)}(\xi_1,\xi_2)$ along both variables ξ_1 and ξ_2, we may take $\xi_1,\xi_2 \in [-\pi,\pi]$. The remainder of the proof shows that (8.14) holds for such (ξ_1,ξ_2).

Note that by assumption,

$$f(\lambda) \leq C\,|\lambda|^{-2d}\ ,$$

where $C>0$ only depends on f^*. Using (C.11), (4.14) and (8.2) with

$$\mu_i = \gamma_j(\lambda_i+\cdots+\lambda_p)\ ,$$

we get

$$|\widehat{\kappa}_j^{(p)}(\xi_1,-\xi_2)| \leq C^p \gamma_j^{2(K+\delta(p))} \int_{-p\gamma_j\pi}^{p\gamma_j\pi} \frac{J_{p,\gamma_j\pi}(\mu_1;2d\,1_p)\mathrm{d}\mu_1}{\prod_{i=1}^2 (1+\gamma_j\,|\{\mu_1/\gamma_j+\xi_i\}|)^{K+\alpha}}\ .$$

Then, by (8.4), there exists $C>0$ not depending on j,p such that, for all $(\xi_1,\xi_2) \in [-\pi,\pi]^2$,

$$|\widehat{\kappa}_j^{(p)}(\xi_1,-\xi_2)| \leq C^p (p!)^{1-2d} \gamma_j^{2K+2(\delta(p)+\delta_-(p))} \int_{-p\gamma_j\pi}^{p\gamma_j\pi} \frac{|\mu_1|^{-2\delta_+(p)}(1+\log(p\gamma_j\pi/|\mu_1|))^{\varepsilon(p)}\mathrm{d}\mu_1}{\prod_{i=1}^2 (1+\gamma_j\,|\{\mu_1/\gamma_j+\xi_i\}|)^{K+\alpha}}\ .$$

Using that $\delta(p)=\delta_+(p)-\delta_-(p)$ and the Cauchy–Schwarz inequality, to obtain (8.14), it is sufficient to show that, for all $\xi \in (-\pi,\pi]$,

$$\int_{-p\gamma_j\pi}^{p\gamma_j\pi} \frac{|\mu_1|^{-2\delta_+(p)}(1+\log(p\gamma_j\pi/|\mu_1|))^{\varepsilon(p)}\mathrm{d}\mu_1}{(1+|\gamma_j\,\{\mu_1/\gamma_j+\xi\}|)^{2(K+\alpha)}} \leq C\ p\log p\ (1+\gamma_j|\xi|)^{-2\delta_+(p)}\,(\log\gamma_j)^{\varepsilon(p)}\ , \tag{8.15}$$

where C is a positive constant.

If $\delta(p) > 0$ the rest of the proof is similar to that of Lemma 10.1 in Clausel et al. (2014) and is thus omitted.

We now take $\delta_+(p) = 0$, so that (8.15) becomes

$$\int_{-p\gamma_j\pi}^{p\gamma_j\pi} \frac{(1+\log(p\gamma_j\pi/|\mu_1|))^{\varepsilon(p)}\mathrm{d}\mu_1}{(1+|\gamma_j\{\mu_1/\gamma_j+\xi\}|)^{2(K+\alpha)}} \leq C\, p\log p\, (\log\gamma_j)^{\varepsilon(p)}\,, \tag{8.16}$$

The denominator in the integral is a $(2\pi\gamma_j)$-periodic function of μ_1, hence the integral over $[-p\gamma_j\pi, p\gamma_j\pi]$ is bounded by the sum of at most $p+1$ integral of the form

$$I(-\gamma_j\xi+2k\gamma_j\pi) \quad \text{with} \quad I(y) = \int_{A(y)} \frac{(1+\log(p\gamma_j\pi/|\mu_1|))^{\varepsilon(p)}\mathrm{d}\mu_1}{(1+|\mu_1-y|)^{2(K+\alpha)}}\,,$$

where $k \in \mathbb{Z}$ and $A(y) = [-p\gamma_j\pi, p\gamma_j\pi] \cap (y-\gamma_j\pi, y+\gamma_j\pi]$. We observe that $I(y)$ is maximal at $y=0$ where it takes value

$$I(0) = \int_{-\gamma_j\pi}^{\gamma_j\pi} \frac{(1+\log(p\gamma_j\pi/|\mu_1|))^{\varepsilon(p)}\mathrm{d}\mu_1}{(1+|\mu_1|)^{2(K+\alpha)}} \leq (1+\log(p\gamma_j\pi))^{\varepsilon(p)} \int_{-\infty}^{\infty} \frac{(1+|\log(|\mu_1|)|)^{\varepsilon(p)}\mathrm{d}\mu_1}{(1+|\mu_1|)^{2(K+\alpha)}}\,.$$

Since the last integral in the previous display is finite for $\varepsilon(p) = 0, 1$, we finally obtain (8.16). $\square$

The following Lemma will be used when identifying the leading terms of the three sums $\Sigma_{n,j}^{(0)} + \Sigma_{n,j}^{(1)}$, $\Sigma_{n,j}^{(2)}$ and $\Sigma_{n,j}^{(3)}$:

Lemma 8.6. *Condition (3.1) implies that for any $C > 0$,*

$$\sum_{q,q'\geq 1}\sum_{p=0}^{q\wedge q'} \frac{|c_q|}{q!}\frac{|c_{q'}|}{q'!}\, p! \binom{q}{p}\binom{q'}{p} C^{\frac{q+q'}{2}} \Lambda_2(q-p,p)^{\frac12}\Lambda_2(q'-p,p)^{\frac12} < \infty\,, \tag{8.17}$$

where Λ_2 is defined by (5.3).

Proof: Let $C > 0$. By definition of Λ_2 in (5.3), we have

$$\begin{aligned}
&\sum_{(q,q')}\sum_{p=0}^{q\wedge q'} \frac{|c_q|}{q!}\frac{|c_{q'}|}{q'!}\, p! \binom{q}{p}\binom{q'}{p} C^{\frac{q+q'}{2}} \Lambda_2(q-p,p)^{1/2}\Lambda_2(q'-p,p)^{1/2} \\
&= \sum_{p\geq 0}\sum_{q,q'\geq p\vee 1} |c_q c_{q'}|\, C^{\frac{q+q'}{2}}\, [p!]^{-2d}\, [(q-p)!(q'-p)!]^{-1/2-d} \\
&= \sum_{p\geq 0}[p!]^{-2d}\left(\sum_{q\geq p\vee 1}|c_q|\, C^{q/2}\, [(q-p)!]^{-1/2-d}\right)^2 \\
&\leq \left(\sum_{q\geq p\geq 0}[p!]^{-d}|c_q|\, C^{q/2}\, [(q-p)!]^{-1/2-d}\right)^2
\end{aligned} \tag{8.18}$$

Using that $x \mapsto x^d$ is concave, we have, for any $q \geq 0$,

$$\sum_{p=0}^{q}(q!)^d[p!\,(q-p)!]^{-d} = q\cdot\left(\frac1q\sum_{p=0}^{q}\binom{q}{p}^d\right) \leq q\left(\frac1q\sum_{p=0}^{q}\binom{q}{p}\right)^d = q^{1-d}2^{dq}\,.$$

We deduce that, for any $q \geq 0$,

$$\sum_{p=0}^{q}[p!]^{-d}\,[(q-p)!]^{-1/2-d} \leq \sum_{p=0}^{q}[p!\,(q-p)!]^{-d} \leq (q!)^{-d}\, q^{1-d}2^{dq}\,.$$

Using this to bound the sum in p in (8.18) and then Condition (3.1) , we get (8.17), which concludes the proof of Lemma 8.6. $\square$

Appendix A. Proof of Proposition 5.1

As in the proof of Proposition 4.2, we can take $m = 1$ without loss of generality. In what follows, $C_1, C_2, \cdots$ denote positive constants which do not depend on n, j, q, q', p. The following function ε' defined on $\mathbb{R}_+$ is used in the sequel,

$$\varepsilon'(a) = \mathbb{1}_{\{1\}}(a) = \begin{cases} \varepsilon'(a) = 1 & \text{if } a = 1 \,, \\ \varepsilon'(a) = 0 & \text{otherwise.} \end{cases} \tag{A.1}$$

We shall prove (i) and (ii), successively.

Proof of (i). Set $r = q-p$ and $r' = q'-p$. The starting point of the proof is the integral expression of $S_{n,j}^{(q,q',p)}$ given by (4.17). Thereafter we follow the same approach as in the proof of Proposition 7.1 of Clausel et al. (2014), using Lemma 8.4 to bound the kernel $\widehat{\kappa}_j^{(p)}$ involved in the integral expression of $S_{n,j}^{(q,q',p)}$ instead of Lemma 10.1 of Clausel et al. (2014), replacing $2r$, (r,r), $\delta(p)$ with $r+r'$, (r,r'), $\delta_+(p)$ and adding if necessary a logarithmic correction.

We obtain the following inequality, similar to (7.2) and (7.3) in Clausel et al. (2014),

$$\mathbb{E}\left[\left|S_{n,j}^{(q,q',p)}\right|^2\right] \leq C_1^{2p}(p!)^{2(1-2d)}\gamma_j^{-2+2\delta(r)+2\delta(r')+4\delta_+(p)}\gamma_j^{4K}(\log\gamma_j)^{2\varepsilon(p)}I_{n,j} \,, \tag{A.2}$$

where, for any j, n,

$$I_{n,j} = \int_{-\gamma_j\pi r}^{\gamma_j\pi r}\int_{-\gamma_j\pi r'}^{\gamma_j\pi r'} \frac{J_{r,\gamma_j\pi}(u_1;2d1_r)J_{r',\gamma_j\pi}(v_1;2d1_{r'})\mathrm{d}u_1\mathrm{d}v_1}{(1+n\left|\{u_1+v_1\}\right|)^2\left(1+\gamma_j\left|\{\frac{u_1}{\gamma_j}\}\right|\right)^{2\delta_+(p)}\left(1+\gamma_j\left|\{\frac{v_1}{\gamma_j}\}\right|\right)^{2\delta_+(p)}} \,,$$

and where $J_{r,\gamma_j\pi}(u_1;2d1_r)$ and $J_{r',\gamma_j\pi}(v_1;2d1_{r'})$ are defined in Lemma 8.1.

We now use (8.4) of Lemma 8.1 successively with $p = r$, $a = \gamma_j\pi$, $s_1 = u_1$ and $p = r'$, $a = \gamma_j\pi$, $s_1 = v_1$. We get that

$$\begin{aligned} I_{n,j} \quad &\leq \quad C_2^{r+r'}(r!r'!)^{1-2d}\gamma_j^{2\delta_-(r)+2\delta_-(r')}(\log\gamma_j)^{2\varepsilon'_0} \\ &\times \int_{\mathbb{R}^2} \frac{\mathbb{1}_{(-\pi r,\pi r)}(\frac{u_1}{\gamma_j})\mathbb{1}_{(-\pi r',\pi r')}(\frac{v_1}{\gamma_j})|u_1|^{-2\delta_+(r)}|v_1|^{-2\delta_+(r')}\mathrm{d}u_1\,\mathrm{d}v_1}{(1+n\left|\{u_1+v_1\}\right|)^2\left(1+\gamma_j\left|\{\frac{u_1}{\gamma_j}\}\right|\right)^{2\delta_+(p)}\left(1+\gamma_j\left|\{\frac{v_1}{\gamma_j}\}\right|\right)^{2\delta_+(p)}} \,, \end{aligned}$$

where $\varepsilon'_0 = \frac{1}{2}[\varepsilon(r)+\varepsilon(r')+2\varepsilon(p)]$.

The next step relies on the inequality $|\{x\}| \leq |x|$ on $\mathbb{R}$ and on the 2π–periodicity of $x \mapsto \{x\}$. We then get that

$$I_{n,j} \leq C_3^{r+r'}(r!r'!)^{1-2d}\gamma_j^{2\delta_-(r)+2\delta_-(r')}(\log\gamma_j)^{2\varepsilon'_0}\tilde{I}_{n,j} \,, \tag{A.3}$$

with

$$\tilde{I}_{n,j} = \int_{(-\gamma_j\pi,\gamma_j\pi)^2} \frac{|u_1|^{-2\delta_+(r)}|v_1|^{-2\delta_+(r')}\mathrm{d}u_1\,\mathrm{d}v_1}{(1+n\left|\{u_1+v_1\}\right|)^2(1+|u_1|)^{2\delta_+(p)}(1+|v_1|)^{2\delta_+(p)}} \,.$$

The bound of $\tilde{I}_{n,j}$ is obtained using the decomposition $\tilde{I}_{n,j} = A + 2B$ with

$$A = \int_{\Delta_j^{(0)}} \frac{|u_1|^{-2\delta_+(r)}|v_1|^{-2\delta_+(r')}\mathrm{d}u_1\,\mathrm{d}v_1}{(1+n\left|\{u_1+v_1\}\right|)^2(1+|u_1|)^{2\delta_+(p)}(1+|v_1|)^{2\delta_+(p)}} \,,$$

and

$$B = \sum_{s=1}^{\gamma_j}\int_{\Delta_j^{(s)}} \frac{|u_1|^{-2\delta_+(r)}|v_1|^{-2\delta_+(r')}\mathrm{d}u_1\,\mathrm{d}v_1}{(1+n\left|\{u_1+v_1\}\right|)^2(1+|u_1|)^{2\delta_+(p)}(1+|v_1|)^{2\delta_+(p)}} \,,$$

where

$$\Delta_j^{(s)} = \{(u_1,v_1) \in (-\gamma_j\pi,\gamma_j\pi)^2, |u_1+v_1-2\pi s| \leq \pi\} \,,$$

with $s \in \{-\gamma_j, \cdots, \gamma_j\}$. This decomposition is similar to the one used in the proof of Proposition 7.1 in Clausel et al. (2014) and is obtained by partitioning $(-\gamma_j\pi,\gamma_j\pi)^2$ using the domains $\Delta_j^{(s)}$.

In the proof of Clausel et al. (2014, Proposition 7.1), bounds of A and B are provided in the case where $r = r'$ and $\delta(r) > 0$. It turns out that the same arguments apply in the present case and yield

$$A \leq \begin{cases} Cn^{-2+2\delta(r)+2\delta(r')} & \text{if } 2\delta_+(r) + 2\delta_+(r') > 1 , \\ Cn^{-1}(\log n)^{2\varepsilon'_1}\gamma_j^{\max(1-2\delta_+(r)-2\delta_+(r')-4\delta_+(p),0)}(\log\gamma_j)^{2\epsilon'_2} & \text{otherwise,} \end{cases}$$
$$B \leq Cn^{-1}\gamma_j^{\max(1-2\delta_+(r)-2\delta_+(p),0)+\max(1-2\delta_+(r')-2\delta_+(p),0)}(\log\gamma_j)^{2\varepsilon'_3} ,$$

where

$$\varepsilon'_1 = \frac{1}{2}\varepsilon'(2\delta_+(q-p) + 2\delta_+(q'-p)),\ \varepsilon'_2 = \frac{1}{2}\varepsilon'(2(\delta_+(r) + \delta_+(r') + 2\delta_+(p))) ,$$
$$\varepsilon'_3 = \frac{1}{2}\varepsilon'(2\delta_+(r) + 2\delta_+(p)) + \varepsilon'(2\delta_+(r') + 2\delta_+(p)) .$$

Hence we obtain that there exists some $C > 0$ depending only on $\delta_+(r), \delta_+(r'), \delta_+(p), d$ such that

$$\tilde{I}_{n,j} \leq C\Big(n^{-\min(2(1-\delta_+(r)-\delta_+(r')),1)}(\log n)^{2\varepsilon'_1}\gamma_j^{\max(1-2\delta_+(r)-2\delta_+(r')-4\delta_+(p),0)}(\log\gamma_j)^{2\varepsilon'_2}$$
$$+ n^{-1}\gamma_j^{\max(1-2(\delta_+(r)+\delta_+(p)),0)+\max(1-2(\delta_+(r')+\delta_+(p)),0)}(\log\gamma_j)^{2\varepsilon'_3}\Big) .$$

Observe now that for any fixed d, there exists only a finite number of possible values for $\delta_+(r)$, $\delta_+(r')$, $\delta_+(p)$ and then a finite number of possible values for C. Then, provided we replace C by its maximum possible value, we can assume that C only depends on d.

The bound on $\tilde{I}_{n,j}$ and (A.2),(A.3) then yields

$$\|S_{n,j}^{(q,q',p)}\|_2 \leq C_4^{\frac{q+q'}{2}}\Lambda_2(q-p,p)^{1/2}\Lambda_2(q'-p,p)^{1/2}\,\gamma_j^{2K}\gamma_j^{-1+\delta_+(q-p)+\delta+(q'-p)+2\delta_+(p)}(\log\gamma_j)^{\varepsilon'_0}$$
$$\times\,[n^{-\min(1-\delta_+(q-p)-\delta_+(q'-p),1/2)}(\log n)^{\varepsilon'_1}\gamma_j^{\max(\frac{1}{2}-\delta_+(q-p)-\delta_+(q'-p)-2\delta_+(p),0)}(\log\gamma_j)^{\varepsilon'_2}$$
$$+\,n^{-1/2}\gamma_j^{\max(1/2-\delta_+(q-p)-\delta_+(p),0)+\max(1/2-\delta_+(q'-p)-\delta_+(p),0)}(\log\gamma_j)^{\varepsilon'_3}] .$$

Inequality (5.8) corresponds to this bound with exponents of γ_j, $\log n$ and $\log\gamma_j$ simplified as follows.

The exponent of γ_j is obtained by observing that $-1+\delta_+(q-p)+\delta_+(q'-p)+2\delta_+(p) = (-1/2+\delta_+(q-p)+\delta_+(p))+(-1/2+\delta_+(q'-p)+\delta_+(p))$ and using $\max(-a,0)+a = \max(a,0)$ with $a = -1/2+\delta_+(q-p)+\delta_+(p)$ and $a = -1/2+\delta_+(q'-p)+\delta_+(p)$ successively.

The log exponents are obtained by observing that, since $r \leq r'$, $\varepsilon(r)+\varepsilon(r')+2\varepsilon(p) \leq 2(\varepsilon(r')+\varepsilon(p)) \leq 4\varepsilon(r' \vee p)$. In addition $\varepsilon'(2\delta_+(m)+2\delta_+(m')) = 0$ iff $m+m' \neq 1/(1-2d)$ and equals 1 otherwise. Thus $\varepsilon'(2\delta_+(m)+2\delta_+(m')) \leq \varepsilon(m+m')$ and we get

$$\varepsilon'(2\delta_+(r)+2\delta_+(p)) \leq \varepsilon(q), \quad \text{and} \quad \varepsilon'(2\delta_+(r')+2\delta_+(p)) \leq \varepsilon(q') .$$

Finally, since ε is non–decreasing and $q \leq q'$, $r' \vee p \leq q'$,

$$\varepsilon'(2\delta_+(r)+2\delta_+(p)) + \varepsilon'(2\delta_+(r')+2\delta_+(p)) + 4\varepsilon(r' \vee p) \leq \varepsilon(q)+\varepsilon(q')+4\varepsilon(q') \leq 6\varepsilon(q') .$$

Proof of (ii). Here, $p = 0$ and thus $\widehat{\kappa}_j^{(p)} = \widehat{h}_j^{(K)\otimes 2}$. The same approach as in the proof of Proposition 7.2 in Clausel et al. (2014) leads to the following inequality which corresponds to (7.12) in Clausel et al. (2014) :

$$\mathbb{E}[|S_{n,j}^{(q,q',0)}|^2] \leq C_5\,\gamma_j^{-(q+q')(1-2d)}\gamma_j^{2(2K+1)}I_{n,j} = C_5\gamma_j^{2(\delta(q)+\delta(q')+2K)}I_{n,j} , \tag{A.4}$$

where

$$I_{n,j} = \int_{u=-q\gamma_j\pi}^{q\gamma_j\pi}\int_{v=-q'\gamma_j\pi}^{q'\gamma_j\pi} g(u,v)J_{q,\gamma_j\pi}(u;d,\cdots,d)J_{q',\gamma_j\pi}(v;d,\cdots,d)\mathrm{d}u\mathrm{d}v_1 ,$$

with $J_{m,a}$ defined as in Lemma 8.1 and with $g(u,v)$ defined for all $(u,v) \in \mathbb{R}^2$ by,

$$g(u,v) = (1+|n\{u+v\}|)^{-2}\frac{|\gamma_j\{u/\gamma_j\}|^{2(M-K)}\cdot|\gamma_j\{v/\gamma_j\}|^{2(M-K)}}{[(1+|\gamma_j\{u/\gamma_j\}|)(1+|\gamma_j\{v/\gamma_j\}|)]^{2(M+\alpha)}} . \tag{A.5}$$

As in the case $p \neq 0$, we can use the bound (8.4) of $J_{m,a}$ and the inequality $|\{u\}| \leq |u|$. We get that

$$\begin{aligned} I_{n,j} &\leq C_6^{q+q'}(q!q'!)^{1-2d}\gamma_j^{2\delta_-(q)-\delta_-(q')} \\ &\times \int_{u=-q\gamma_j\pi}^{q\gamma_j\pi}\int_{v=-q'\gamma_j\pi}^{q'\gamma_j\pi} \frac{\left|\gamma_j\{\frac{u}{\gamma_j}\}\right|^{2M-2K-2\delta_+(q)}\left|\gamma_j\{\frac{v}{\gamma_j}\}\right|^{2M-2K-2\delta_+(q')}\mathrm{d}u\mathrm{d}v}{(1+n|\{u+v\}|)^2\left(1+\left|\gamma_j\{\frac{u}{\gamma_j}\}\right|\right)^{2(M+\alpha)}\left(1+\left|\gamma_j\{\frac{v}{\gamma_j}\}\right|\right)^{2(M+\alpha)}} . \end{aligned}$$

As in the proof of Proposition 7.2 of Clausel et al. (2014), we then obtain that

$$I_{n,j} \leq C_7^{q+q'}(q!q'!)^{1-2d}n^{-1}\gamma_j^{-2(\delta_-(q)+\delta_-(q'))} . \tag{A.6}$$

The conclusion follows from (A.4) and (A.6).

Appendix B. Integral representations

It is convenient to use an integral representation in the spectral domain to represent the random processes (see for example Major (1981); Nualart (2006)). The stationary Gaussian process $\{X_k, k \in \mathbb{Z}\}$ with spectral density (1.2) can be written as

$$X_\ell = \int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\lambda\ell} f^{1/2}(\lambda)\mathrm{d}\widehat{W}(\lambda) = \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}\lambda\ell} f^{*1/2}(\lambda)}{|1-\mathrm{e}^{-\mathrm{i}\lambda}|^d}\mathrm{d}\widehat{W}(\lambda), \quad \ell \in \mathbb{N} . \tag{B.1}$$

This is a special case of

$$\widehat{I}(g) = \int_{\mathbb{R}} g(x)\mathrm{d}\widehat{W}(x), \tag{B.2}$$

where $\widehat{W}(\cdot)$ is a complex–valued Gaussian random measure satisfying, for any Borel sets A and B in $\mathbb{R}$, $\mathbb{E}(\widehat{W}(A)) = 0$, $\mathbb{E}(\widehat{W}(A)\overline{\widehat{W}(B)}) = |A \cap B|$ and

$$\widehat{W}(A) = \overline{\widehat{W}(-A)} .$$

The integral (B.2) is defined for any function $g \in L^2(\mathbb{R})$ and one has the isometry

$$\mathbb{E}(|\widehat{I}(g)|^2) = \int_{\mathbb{R}} |g(x)|^2\mathrm{d}x .$$

The integral $\widehat{I}(g)$, moreover, is real–valued if

$$g(x) = \overline{g(-x)} .$$

We shall also consider multiple Itô–Wiener integrals

$$\widehat{I}_q(g) = \int_{\mathbb{R}^q}^{''} g(\lambda_1, \cdots, \lambda_q)\mathrm{d}\widehat{W}(\lambda_1)\cdots\mathrm{d}\widehat{W}(\lambda_q)$$

where the double prime indicates that one does not integrate on hyperdiagonals $\lambda_i = \pm\lambda_j, i \neq j$. The integrals $\widehat{I}_q(g)$ are handy because we will be able to expand our non–linear functions $G(X_k)$ introduced in Section 1 in multiple integrals of this type.

These multiples integrals are defined for $g \in \overline{L^2}(\mathbb{R}^q, \mathbb{C})$, the space of complex valued functions defined on $\mathbb{R}^q$ satisfying

$$g(-x_1, \cdots, -x_q) = \overline{g(x_1, \cdots, x_q)} \text{ for } (x_1, \cdots, x_q) \in \mathbb{R}^q , \tag{B.3}$$

$$\|g\|_{L^2}^2 := \int_{\mathbb{R}^q} |g(x_1, \cdots, x_q)|^2 \,\mathrm{d}x_1\cdots\mathrm{d}x_q < \infty . \tag{B.4}$$

Hermite polynomials are related to multiple integrals as follows: if $X = \int_{\mathbb{R}} g(x)\mathrm{d}\widehat{W}(x)$ with $\mathbb{E}(X^2) = \int_{\mathbb{R}} |g(x)|^2\mathrm{d}x = 1$ and $g(x) = \overline{g(-x)}$ so that X has unit variance and is real–valued, then

$$H_q(X) = \widehat{I}_q(g^{\otimes q}) = \int_{\mathbb{R}^q}^{''} g(x_1)\cdots g(x_q)\mathrm{d}\widehat{W}(x_1)\cdots\mathrm{d}\widehat{W}(x_q) . \tag{B.5}$$

Since X has unit variance, one has for any $\ell \in \mathbb{Z}$,

$$H_q(X_\ell) = H_q\left(\int_{-\pi}^{\pi} \mathrm{e}^{\mathrm{i}\xi\ell} f^{1/2}(\xi)\mathrm{d}\widehat{W}(\xi)\right)$$
$$= \int_{(-\pi,\pi]^q}'' \mathrm{e}^{\mathrm{i}\ell(\xi_1+\cdots+\xi_q)} \times \left(f^{1/2}(\xi_1)\times\cdots\times f^{1/2}(\xi_q)\right) \mathrm{d}\widehat{W}(\xi_1)\cdots\mathrm{d}\widehat{W}(\xi_q)\ .$$

Then by (4.1), we have

$$\mathbf{W}_{j,k}^{(q)} = \sum_{\ell\in\mathbb{Z}} \mathbf{h}_j^{(K)}(\gamma_j k-\ell) H_q(X_\ell) = \widehat{I}_q(\mathbf{f}_{j,k}^{(q)}) \tag{B.6}$$

with

$$\mathbf{f}_{j,k}^{(q)}(\xi_1,\cdots,\xi_q) = \mathrm{e}^{\mathrm{i}k\gamma_j(\xi_1+\cdots+\xi_q)} \times \widehat{\mathbf{h}}_j^{(K)}(\xi_1+\cdots+\xi_q) f^{1/2}(\xi_1)\cdots f^{1/2}(\xi_q)\mathbb{1}_{(-\pi,\pi)}^{\otimes q}(\xi)\ , \tag{B.7}$$

because

$$\begin{aligned}\sum_{\ell\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}\ell(\xi_1+\cdots+\xi_q)}\mathbf{h}_j^{(K)}(\gamma_j k-\ell) &= \mathrm{e}^{\mathrm{i}\gamma_j k(\xi_1+\cdots+\xi_q)}\sum_{u\in\mathbb{Z}} \mathrm{e}^{-\mathrm{i}u(\xi_1+\cdots+\xi_q)}\mathbf{h}_j^{(K)}(u)\\ &= \mathrm{e}^{\mathrm{i}\gamma_j k(\xi_1+\cdots+\xi_q)}\widehat{\mathbf{h}}_j^{(K)}(\xi_1+\cdots+\xi_q)\ ,\end{aligned}$$

by (C.1).

The following proposition can be found in Peccati and Taqqu (2011), Formula (9.7.32). It is an extension to our complex–valued setting of a corresponding result in Nualart (2006) for multiple integrals in a real–valued setting.

Proposition B.1. *Let $(q,q') \in \mathbb{N}^2$. Assume that f, g are two symmetric functions belonging respectively to $\overline{L^2}(\mathbb{R}^q)$ and $\overline{L^2}(\mathbb{R}^{q'})$ then the following product formula holds:*

$$\widehat{I}_q(f)\widehat{I_{q'}}(g) = \sum_{p=0}^{q\wedge q'} p!\binom{q}{p}\binom{q'}{p}\widehat{I_{q+q'-2p}}(f\overline{\otimes}_p g), \tag{B.8}$$

where for any $p \in \{1,\cdots,q\wedge q'\}$

$$(f\overline{\otimes}_p g)(t_1,\cdots,t_{q+q'-2p}) = \int_{\mathbb{R}^p} f(t_1,\cdots,t_{q-p},s)g(t_{q-p+1},\cdots,t_{q+q'-2p},-s)\mathrm{d}^p s\ . \tag{B.9}$$

Appendix C. The wavelet filters

The sequence $\{Y_t\}_{t\in\mathbb{Z}}$ can be formally expressed as

$$Y_t = \Delta^{-K}G(X_t), \quad t\in\mathbb{Z}\ .$$

The study of the asymptotic behavior of the scalogram of $\{Y_t\}_{t\in\mathbb{Z}}$ at different scales involve multidimensional wavelets coefficients of $\{G(X_t)\}_{t\in\mathbb{Z}}$ and of $\{Y_t\}_{t\in\mathbb{Z}}$. To obtain them, one applies a multidimensional linear filter $\mathbf{h}_j(\tau), \tau\in\mathbb{Z} = (h_{j,\ell}(\tau))$, at each scale index $j\geq 0$. We shall characterize below the multidimensional filters $\mathbf{h}_j(\tau)$ by their discrete Fourier transform :

$$\widehat{\mathbf{h}}_j(\lambda) = \sum_{\tau\in\mathbb{Z}} \mathbf{h}_j(\tau)\mathrm{e}^{-\mathrm{i}\lambda\tau},\ \lambda\in[-\pi,\pi]\ , \quad \mathbf{h}_j(\tau) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\widehat{\mathbf{h}}_j(\lambda)\mathrm{e}^{\mathrm{i}\lambda\tau}\mathrm{d}\lambda, \tau\in\mathbb{Z}\ . \tag{C.1}$$

The resulting wavelet coefficients $\mathbf{W}_{j,k}$, where j is the scale index and k the location are defined as

$$\mathbf{W}_{j,k} = \sum_{t\in\mathbb{Z}}\mathbf{h}_j(\gamma_j k-t)Y_t = \sum_{t\in\mathbb{Z}}\mathbf{h}_j(\gamma_j k-t)\Delta^{-K}G(X_t),\ j\geq 0, k\in\mathbb{Z}, \tag{C.2}$$

where $\gamma_j \uparrow \infty$ as $j\uparrow\infty$ is a sequence of non–negative scale factors applied at scale index j, for example $\gamma_j = 2^j$. We do not assume that the wavelet coefficients are orthogonal nor that they are generated by a multiresolution analysis. Our assumption on the filters $\mathbf{h}_j = (h_{j,\ell})$ are as follows:

(W-a) Finite support: For each ℓ and j, $\{h_{j,\ell}(\tau)\}_{\tau\in\mathbb{Z}}$ has finite support.

(W-b) Uniform smoothness: There exists $M \geq K$, $\alpha > 1/2$ and $C > 0$ such that for all $j \geq 0$ and $\lambda \in [-\pi, \pi]$,

$$|\widehat{\mathbf{h}}_j(\lambda)| \leq \frac{C\gamma_j^{1/2}|\gamma_j \lambda|^M}{(1+\gamma_j|\lambda|)^{\alpha+M}} . \tag{C.3}$$

By 2π-periodicity of $\widehat{h}_j$ this inequality can be extended to $\lambda \in \mathbb{R}$ as

$$|\widehat{\mathbf{h}}_j(\lambda)| \leq C\frac{\gamma_j^{1/2}|\gamma_j\{\lambda\}|^M}{(1+\gamma_j|\{\lambda\}|)^{\alpha+M}} . \tag{C.4}$$

where $\{\lambda\}$ denotes the element of $(-\pi, \pi]$ such that $\lambda - \{\lambda\} \in 2\pi\mathbb{Z}$.

(W-c) Asymptotic behavior: There exists a sequence of phase functions $\Phi_j : \mathbb{R} \to (-\pi, \pi]$ and some non identically zero function $\widehat{\mathbf{h}}_\infty$ such that

$$\lim_{j\to+\infty}(\gamma_j^{-1/2}\widehat{\mathbf{h}}_j(\gamma_j^{-1}\lambda)) = \widehat{\mathbf{h}}_\infty(\lambda) , \tag{C.5}$$

locally uniformly on $\lambda \in \mathbb{R}$.

In (W-c) *locally uniformly* means that for all compact $K \subset \mathbb{R}$,

$$\sup_{\lambda\in K}\left|\gamma_j^{-1/2}\widehat{\mathbf{h}}_j(\gamma_j^{-1}\lambda)\mathrm{e}^{\mathrm{i}\Phi_j(\lambda)} - \widehat{\mathbf{h}}_\infty(\lambda)\right| \to 0 .$$

Assumptions (C.3) and (C.5) imply that for any $\lambda \in \mathbb{R}$,

$$|\widehat{\mathbf{h}}_\infty(\lambda)| \leq C\frac{|\lambda|^M}{(1+|\lambda|)^{\alpha+M}} . \tag{C.6}$$

Hence $\widehat{\mathbf{h}}_\infty$ has entries in $L^2(\mathbb{R})$. We let $\mathbf{h}_\infty$ be the vector of $L^2(\mathbb{R})$ inverse Fourier transforms of $\widehat{h}_{\ell,\infty}$, that is

$$\widehat{\mathbf{h}}_\infty(\xi) = \mathfrak{F}(\mathbf{h}_\infty)(\xi) = \int_{\mathbb{R}^q}\mathbf{h}_\infty(t)\mathrm{e}^{-\mathrm{i}t^T\xi}\,\mathrm{d}^q t, \quad \xi \in \mathbb{R}^q , \tag{C.7}$$

is defined for any $f \in L^2(\mathbb{R}^q, \mathbb{C})$.

Observe that while $\widehat{\mathbf{h}}_j$ is 2π–periodic, the function $\widehat{\mathbf{h}}_\infty$ has non–periodic entries on $\mathbb{R}$. For the connection between these assumptions on h_j and corresponding assumptions on the scaling function φ and the mother wavelet ψ in the classical wavelet setting see Moulines et al. (2007). In particular, in that case, one has $\widehat{h}_\infty = \widehat{\varphi}(0)\overline{\widehat{\psi}}$.

A more convenient way to express the wavelet coefficients $\mathbf{W}_{j,k}$ defined in (C.2) is to incorporate the linear filter Δ^{-K} in (C.2) into the filter $\mathbf{h}_j$ and denote the resulting filter $\mathbf{h}_j^{(K)}$. Then

$$\mathbf{W}_{j,k} = \sum_{t\in\mathbb{Z}}\mathbf{h}_j^{(K)}(\gamma_j k - t)G(X_t) , \tag{C.8}$$

where

$$\widehat{\mathbf{h}}_j^{(K)}(\lambda) = (1-\mathrm{e}^{-\mathrm{i}\lambda})^{-K}\widehat{\mathbf{h}}_j(\lambda) \tag{C.9}$$

is the discrete Fourier transform of $\mathbf{h}_j^{(K)}$. Using (C.4) we get,

$$\left|\widehat{\mathbf{h}}_j^{(K)}(\lambda)\right| \leq C\gamma_j^{1/2+K}\,\frac{|\gamma_j\{\lambda\}|^{M-K}}{(1+\gamma_j|\{\lambda\}|)^{\alpha+M}}, \quad \lambda \in \mathbb{R},\, j \geq 1 . \tag{C.10}$$

In particular, since we assume if $M \geq K$, we get

$$\left|\widehat{\mathbf{h}}_j^{(K)}(\lambda)\right| \leq C\gamma_j^{1/2+K}\,(1+\gamma_j|\{\lambda\}|)^{-\alpha-K}, \quad \lambda \in \mathbb{R},\, j \geq 1 . \tag{C.11}$$

By Assumption (C.3), $\mathbf{h}_j$ has null moments up to order $M-1$, that is, for any $m \in \{0, \cdots, M-1\}$,

$$\sum_{t\in\mathbb{Z}}\mathbf{h}_j(t)t^m = 0 . \tag{C.12}$$

Observe that $\Delta^K Y$ is centered by definition. However, by (C.12), the definition of $\mathbf{W}_{j,k}$ only depends on $\Delta^M Y$. In particular, provided that $M \geq K+1$, its value is not modified if a constant is added to $\Delta^K Y$, whenever $M \geq K+1$.

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