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Lévy Mixing

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Abstract. This paper studies Lévy mixing of multivariate infinitely divisible distributions μ, where the parametrisation is in the form of a rescaling of the Lévy measure and of the cumulant transform of μ by a matrix mapping. Particular examples appear in the study of multivariate operator self-decomposable distributions and the construction of multivariate superpositions of Ornstein-Uhlenbeck processes. Under mild conditions the associated transformations preserve infinite divisibility and have smoothening effects, such as guarantying absolute continuity of the resulting Lévy measures and a decreasing effect on the Blumenthal–Getoor index. Their domains, behavior under convolution and composition, continuity with respect to weak convergence and other basic properties are systematically considered. Moreover, a representation of these transformations as distributions of integrals of a deterministic function with respect to a Lévy basis is established. We present a review and a self-contained treatment of the relevant random integrals with respect to Lévy bases in a unified approach closely connected to Lévy mixing.

1. Introduction

In order to introduce the concept of Lévy mixing we note initially that probability mixing, i.e. the generation of a new probability law by randomising a parameter, does not generally preserve infinite divisibility. Thus, if $(P_\theta)_{\theta\in\Theta}$ is a parametrised family of infinitely divisible probability measures, and if Θ is endowed with a probability law Q, the resulting measure $R\,(\mathrm{d}x) = \int P_\theta\,(\mathrm{d}x)\,Q\,(\mathrm{d}\theta)$ does as a rule not

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determine an infinitely divisible distribution, even if also Q is infinitely divisible. This aspect of probability mixing has motivated a major amount of research in the theory of infinite divisibility for several decades; see for example Steutel and van Harn (2004), Chapter VI.

On the other hand, if ν_θ is the parametrised class of Lévy measures corresponding to the class P_θ, then a measure of the form $\varpi(\mathrm{d}x) = \int_\Theta \nu_\theta(\mathrm{d}x)\,\gamma(\mathrm{d}\theta)$, for some σ-finite measure γ, will, under mild conditions, again be a Lévy measure, hence determining an infinitely divisible law. We illustrate this by a simple example.

Example 1.1 (supOU processes). Let X be a non-Gaussian OU process

$$X_t = \int_{-\infty}^t \mathrm{e}^{-\beta(t-s)}\mathrm{d}L_s, \tag{1.1}$$

where L is an increasing Lévy process on $\mathbb{R}$ (with no drift), such that the Lévy measure ν of L_1 (necessarily concentrated on $(0,\infty)$) satisfies the conditions

$$\int_0^1 x\,\nu(\mathrm{d}x) < \infty, \quad \text{and} \quad \int_1^\infty \ln(x)\,\nu(\mathrm{d}x) < \infty. \tag{1.2}$$

The second condition ensures that the integral in (1.1) exists (as a limit in probability). Now the cumulant transform[1] of X_t may be calculated as

$$\begin{aligned}
\mathrm{C}\{\zeta \ddagger X_t\} &= \int_{-\infty}^t \mathrm{C}\{\zeta \mathrm{e}^{-\beta(t-s)} \ddagger L_1\}\,\mathrm{d}s = \int_0^\infty \mathrm{C}\{\zeta\mathrm{e}^{-\beta u} \ddagger L_1\}\,\mathrm{d}u \\
&= \int_0^\infty \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\zeta\mathrm{e}^{-\beta u}x} - 1\right)\nu(\mathrm{d}x)\,\mathrm{d}u = \int_0^\infty \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\zeta y} - 1\right)\nu(\mathrm{e}^{\beta u}\mathrm{d}y)\,\mathrm{d}u \\
&= \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\zeta y} - 1\right)\varpi(\mathrm{d}y;\beta)
\end{aligned}$$

for all ζ in $\mathbb{R}$, and where

$$\varpi(\mathrm{d}y;\beta) = \int_0^\infty \nu(\mathrm{e}^{\beta u}\mathrm{d}y)\,\mathrm{d}u, \qquad (\beta > 0); \tag{1.3}$$

that is, $\varpi(\mathrm{d}y;\beta)$ is a mixture of scalings of the Lévy measure ν, where the mixing measure is Lebesgue measure. The condition (1.2) then means exactly that

$$\int_0^\infty \min\{1,y\}\,\varpi(\mathrm{d}y,\beta) < \infty,$$

so that ϖ is again the Lévy measure of an increasing Lévy process.

We may proceed to mix $\varpi(\mathrm{d}y;\beta)$ with respect to the parameter β, devising a new measure ω on $(0,\infty)$ by letting

$$\omega(\mathrm{d}y) = \int_0^\infty \varpi(\mathrm{d}y;\beta)\,\chi(\mathrm{d}\beta) \tag{1.4}$$

for some measure χ on $(0,\infty)$. Assuming that $\int_0^\infty \beta^{-1}\chi(\mathrm{d}\beta) < \infty$, one may check that $\int_0^\infty \min\{1,y\}\,\omega(\mathrm{d}y) < \infty$, so that ω is again the Lévy measure of an increasing Lévy process.

In continuation of the above considerations, we now introduce a Lévy basis $\tilde{L}$ on $\mathbb{R}\times\mathbb{R}_+$ with characteristic quadruplet $(\int_0^1 x\,\nu(\mathrm{d}x), 0, \nu, \lambda\otimes\chi)$, and where λ

[1] the log of the Fourier transform of X_t.

denotes Lebesgue measure on $\mathbb{R}$.[2] For any bounded subset F of $\mathbb{R}\times\mathbb{R}_+$, it follows, in particular, that $\tilde{L}(F)$ is an infinitely divisible random variable with characteristic triplet $\lambda\otimes\chi(F)(\int_0^1 x\,\nu(\mathrm{d}x),0,\nu)$. The conditions imposed on ν and χ above ensure the existence of the stochastic integral

$$\tilde{X}_t = \int_0^\infty \int_{-\infty}^t \mathrm{e}^{-\beta(t-s)}\,\tilde{L}\,(\mathrm{d}s,\mathrm{d}\beta) \tag{1.5}$$

(see e.g. Barndorff-Nielsen and Stelzer (2011)). The process $\tilde{X}$ is a supOU process, i.e. a superposition of OU processes, and its cumulant transform may be calculated as (see e.g. Barndorff-Nielsen and Stelzer (2011))

$$\begin{aligned}
\mathrm{C}\{\zeta \ddagger \tilde{X}_t\} &= \int_0^\infty \int_{-\infty}^t \mathrm{C}\{\zeta \mathrm{e}^{-\beta(t-s)} \ddagger L_1\}\,\mathrm{d}s\,\chi(\mathrm{d}\beta)\\
&= \int_0^\infty \int_0^\infty \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\zeta \mathrm{e}^{-\beta u}x}-1\right)\nu_1(\mathrm{d}x)\,\mathrm{d}u\,\chi(\mathrm{d}\beta)\\
&= \int_0^\infty \left(\mathrm{e}^{\mathrm{i}\zeta y}-1\right)\omega(\mathrm{d}y).
\end{aligned}$$

In particular, taking $\chi\,(\mathrm{d}\beta) = \beta\pi\,(\mathrm{d}\beta)$ where π is the gamma law $\Gamma\,(\kappa,1)$, the process $\tilde{X}_t$ will exhibit long range dependence provided $\kappa\in(0,1)$; see Barndorff-Nielsen (2001).

The above example illustrates both how Lévy mixing occurs naturally from stochastic integral representations of random variables or processes (cf. formulae (1.3) and (1.1)), and the construction of a new infinitely divisible process - the supOU process (1.5) - by Lévy mixing of a given infinitely divisible process, the OU process (1.1). The constructions and assertions made in the example will appear as special cases of the theory developed in the following sections. In addition we shall return to this example when multivariate situations are considered (see Examples 3.8 and 6.10 below).

The present paper discusses in a systematic way various types of Lévy mixing, focussing on cases where the parametrisation is in the form of rescaling of a given Lévy measure ν. More specifically, we consider mixtures

$$[\Upsilon_T^0(\nu)](B) = \int_S \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)y)\,\nu(\mathrm{d}y)\,\gamma(\mathrm{d}s) \tag{1.6}$$

for a given Lévy measure ν on $\mathbb{R}^d$, and where $T(s)$ is a $d\times d$ matrix with real entries, while s varies in a measure space $(S,\mathcal{S},\gamma)$. In case $T(s)$ is invertible for all s, formula (1.6) may be written as:

$$[\Upsilon_T^0(\nu)](\mathrm{d}x) = \int_S \nu(T(s)^{-1}\mathrm{d}x)\,\gamma(\mathrm{d}s).$$

The specific mixtures (1.6) are referred to as (matrix) Upsilon transformations of type $\Upsilon^0 := \Upsilon_T^0$, and they map Lévy measures on $\mathbb{R}^d$ into - typically more regular - Lévy measures. Associated to these mappings are another type of transformations

[2]See Section 5.1 for background on Lévy bases (also known as infinitely divisible independently scattered random measures).

denoted by $\Upsilon := \Upsilon_T$. These map multivariate infinitely divisible d-dimensional laws μ into infinitely divisible d-dimensional laws $\Upsilon(\mu)$ via the formula

$$C_{\Upsilon(\mu)}(u) = \int_S C_\mu(T(s)u)\,\gamma(\mathrm{d}s), \qquad (u \in \mathbb{R}^d) \tag{1.7}$$

where the letter C stands for the cumulant transform. In the case $S = \mathbb{R}$ and $T = s\mathrm{I}_d$ (with I_d the $d \times d$ unit matrix) Upsilon transformations of these types have from different points of view and levels of generality been the subject of investigation in numerous papers over the last couple of decades; see Section 2 for a brief summary of some of the previously established results in this direction. The present paper provides a unified and generalized approach to much of the theory developed in those papers. Specifically we study transformations of the types Υ_T^0 and Υ_T corresponding to general classes of $d \times d$ matrices $T(s)$. Our framework and results are relevant in several situations and directions. Indeed, as the introductory example shows, there is a need for considering a more general space S than $\mathbb{R}$. Secondly, matrix Upsilon mappings appear naturally in studies of multidimensional operator-selfdecomposable distributions and multivariate superpositions of Ornstein-Uhlenbeck-type processes, where $T(Q, r) = \mathrm{e}^{rQ}$ with Q in an appropriate space of matrices and $r \in \mathbb{R}$; see Jurek and Mason (1993) and Barndorff-Nielsen and Stelzer (2011), respectively. Thirdly, in our set-up the transformation T is also a variable in the sense that the Upsilon transformations Υ_T^0 and Υ_T are functions of T in addition to being functions of a Lévy measure, respectively an infinitely divisible law on $\mathbb{R}^d$. Accordingly, we investigate interesting properties for $\Upsilon_T(\mu)$ for a given T while μ varies, and reciprocally. Finally, the associated random integral representations are with respect to a general Lévy basis in S rather than to a Lévy process, the relevance of which is also illustrated by Example 1.1. Recently, Lévy bases have been extensively used in Lévy type modelling and several aspects of the theory and applications of Ambit processes; see Barndorff-Nielsen (2011); Barndorff-Nielsen et al. (2011); Jonsdottir et al. (2011) and references therein.

The main results and organization of this paper are as follows. Section 2 provides a brief account on previously established results on transformations of upsilon type. Section 3 starts by introducing the notation used in the paper and proceeds with the definitions of the mappings Υ_T^0 and Υ_T. A number of basic properties of these mappings are subsequently established. This includes the study of their domains as well as their behavior under basic probabilistic operations. In case γ is a probability measure, T may be thought of as a random matrix, and a number of examples of Upsilon transformations corresponding to fundamental classes of random matrices (diagonal exponentials, symmetric Gaussians and Wishart matrices) are specifically considered. Section 4 considers continuity with respect to weak convergence of $\Upsilon_T(\cdot)$ for fixed T and of $\Upsilon_\cdot(\mu)$ for fixed μ. Section 5 deals with regularising properties of Υ_T^0 and Υ_T. Conditions on $\gamma \circ T^{-1}$ ensuring absolute continuity of $\Upsilon_T^0(\nu)$ with respect to Lebesgue measure on $\mathbb{R}^d$ are given. It is also shown that Υ_T has, in general, a decreasing effect on the Blumenthal-Getoor index of an infinitely divisible distribution on $\mathbb{R}^d$, but in the case of the stable distributions the index is preserved. Section 6 deals with the representation of Υ_T as a random integral with respect to a Lévy basis L. Specifically, it is shown that given a measurable family of linear mappings $T(s) : \mathbb{R}^d \to \mathbb{R}^d$, $s \in S$, there exists (under suitable conditions) for any μ in the domain of Υ_T, an $\mathbb{R}^d$ -valued Lévy basis L on $(S, \mathcal{S}, \gamma)$, such that

$$\mathcal{L}\Big\{ \int_S T(s)\, L(\mathrm{d}s) \Big\} = \Upsilon_T(\mu).$$

Furthermore, there exists a Lévy basis $\tilde{L}$ on the product space

$$(S \times [0,\infty), \mathcal{S} \otimes \mathcal{B}([0,\infty)), \gamma \otimes \lambda),$$

(where λ denotes Lebesgue measure) such that the process $(Z_t)_{t\geq 0}$ defined by

$$Z_t = \int_{S\times[0,\infty)} T(s) 1_{(0,t]}(u)\, \tilde{L}(\mathrm{d}s, \mathrm{d}u), \qquad (t \geq 0),$$

realizes the Lévy process (in law) associated to the infinitely divisible measure $\Upsilon_T(\mu)$.

We present a review of the relevant random integrals with respect to Lévy bases, giving a self-contained treatment. This is in line with the study of random integrals with respect to Lévy bases initiated by Urbanik and Woyczyński (1967) and further developed by Rajput and Rosiński (1989) and by Sato in Sato (2004). Our presentation, however, provides a unified approach to random integrals closely related to the properties of Upsilon transformations established in the earlier sections (in particular the continuity results of Section 4).

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2. Previously established results

Over the last decade or so much attention has been given to Upsilon transformations of the types Υ^0 and Υ in the case $S = \mathbb{R}$ and $T(s) = s\mathrm{I}_d$ in (1.6) and (1.7), where I_d is the $d \times d$ unit matrix. The measure γ may then be an arbitrary σ-finite measure on $\mathbb{R}$. This case was methodically considered recently in Maejima et al. (2013), where it is shown in particular that not all mixtures of Lévy measures correspond to such Upsilon transformations. Cases where γ is concentrated on $(0,\infty)$ were previously studied in Barndorff-Nielsen and Thorbjørnsen (2004, 2006); Barndorff-Nielsen et al. (2006b, 2008); Barndorff-Nielsen and Maejima (2008), and related questions regarding stochastic integral representations of infinitely divisible distributions with respect to Lévy processes are investigated in Jurek (1985, 1990); Jurek and Mason (1993); Barndorff-Nielsen et al. (2006b); Maejima et al. (2012); Rosiński (1984); Sato (2006a,b, 2007), among others.

Some important subclasses of the class $\mathcal{ID}(\mathbb{R}^d)$ of infinitely divisible distributions on $\mathbb{R}^d$ are describable as the range of a one-dimensional Upsilon transformation of type Υ. This is the case, for example, for the class of self-decomposable distributions $L(\mathbb{R}^d)$, the Goldie-Steutel-Bondesson class $B(\mathbb{R}^d)$, the Thorin class $\mathcal{T}(\mathbb{R}^d)$, the class $G(\mathbb{R}^d)$ of generalized type G distributions, the Jurek class $U(\mathbb{R}^d)$, and the class $A(\mathbb{R}^d)$ obtained by Lévy mixing with arcsine Lévy measure, among others; see for example Barndorff-Nielsen and Thorbjørnsen (2006); Barndorff-Nielsen et al. (2006b, 2008); Maejima et al. (2012, 2013).

Some further interesting applications of one-dimensional Upsilon transformations were highlighted in the lecture Rosiński (2007) at the Conference on Lévy Processes in Copenhagen in 2007; see also Rosiński (1990, 2007).

Assuming that γ is a probability measure on $\mathbb{R}$ (and still that $T(s) = s\mathrm{I}_d$ for all s), the formulae (1.6)-(1.7) may be written conveniently as

$$\Upsilon_T^0(\nu)(B) = \mathbb{E}\Big\{ \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(Xy)\,\nu(\mathrm{d}y)\Big\}, \qquad (B \in \mathcal{B}(\mathbb{R}^d)) \tag{2.1}$$

and

$$C_{\Upsilon_T(\mu)}(u) = \mathbb{E}\left\{C_\mu(Xu)\right\}, \qquad (u \in \mathbb{R}^d) \tag{2.2}$$

where X is a one-dimensional random variable carrying the distribution γ. Two important one-dimensional Upsilon transformations correspond to the cases when X has the standard Gaussian distribution or the exponential distribution.

The Gaussian case corresponds to the Lévy measures of the type G distributions introduced in Marcus (1987) and systematically studied in Rosiński (1991). These are the infinitely divisible variance mixtures (considered in Kelker (1971)) of the form VZ, where Z is Gaussian and V^2 is infinitely divisible. When $T = Z\mathrm{I}_d$ and Z is a one-dimensional random variable with standard Gaussian distribution, the measure $\Upsilon_T^0(\nu)$ is, for any Lévy measure ν, the Lévy measure of a type G distribution on $\mathbb{R}^d$ as considered in Maejima and Rosiński (2002). Integral representations of type G distributions were established in Aoyama and Maejima (2007). Moreover, the class $G(\mathbb{R}^d)$ of generalized type G distributions considered in Maejima and Sato (2009) correspond to the image of Υ_T. In addition, it was shown in Maejima et al. (2012) that the class $G(\mathbb{R}^d)$ is the image of the class of distributions $A(\mathbb{R}^d)$ under an Upsilon transformation where γ is the arcsine measure.

The exponential case, i.e. where $\gamma(\mathrm{d}x) = \mathrm{e}^{-x}1_{(0,\infty)}\,\mathrm{d}x$, was studied extensively in Barndorff-Nielsen and Thorbjørnsen (2004, 2006). In this case the formula (1.7) is related to the relationship between classical and free probability theory embodied in the formula (see Barndorff-Nielsen and Thorbjørnsen (2004))

$$\mathcal{C}_{\Lambda(\mu)}(\mathrm{i}z) = \int_0^\infty C_\mu(zx)\mathrm{e}^{-x}\,\mathrm{d}x, \qquad (z \in (-\infty, 0)). \tag{2.3}$$

Here Λ is the Bercovici-Pata bijection between the classes of one-dimensional infinitely divisible probability distributions in classical and free probability, respectively (see Bercovici and Pata (1999) or Barndorff-Nielsen et al. (2006a)). Moreover, $\mathcal{C}_\rho$ denotes the free cumulant transform of a probability measure ρ on $\mathbb{R}$.

We mention finally that a class of matrix Upsilon transformations has been briefly considered in Barndorff-Nielsen and Pérez-Abreu (2007) in the case of linear transformations $T : \mathbb{S}_d^+ \to \mathbb{S}_d^+$, where $\mathbb{S}_d^+$ is the open cone of positive definite matrices.

3. Upsilon transformations associated to matrix valued mappings

3.1. *Notation.* In the following we denote by $\|u\|$ the Euclidean norm of a vector $u = (u_1, ..., u_d)^*$ in $\mathbb{R}^d$, i.e.

$$\|u\| = (u_1^2 + \cdots + u_d^2)^{1/2}. \tag{3.1}$$

By B_1 we denote the corresponding closed unit ball in $\mathbb{R}^d$:

$$B_1 = \big\{(u_1, \ldots, u_d)^* \in \mathbb{R}^d \mid \|u\| \le 1\big\}. \tag{3.2}$$

We denote by $\mathbb{M}_{d,m}(\mathbb{R})$ the linear space of $d\times m$ matrices with real entries, by $\mathbb{M}_d(\mathbb{R})$ the space $\mathbb{M}_{d,d}(\mathbb{R})$ and by $\mathbb{S}_d$ the subspace of symmetric matrices in $\mathbb{M}_d(\mathbb{R})$. Furthermore we let $\overline{\mathbb{S}}_d^+$ denote the closed cone of $d\times d$ nonnegative definite symmetric matrices in $\mathbb{M}_d(\mathbb{R})$ while $\mathbb{S}_d^+$ is the open cone of $d\times d$ positive definite matrices. For T_1 and T_2 in $\mathbb{M}_{d,m}(\mathbb{R})$ we denote by $\langle T_1, T_2\rangle$ the inner product $\langle T_1,T_2\rangle = \mathrm{tr}(T_1T_2^*)$ where T^* denotes the transpose of T, and tr is the (un-normalized) trace on $\mathbb{M}_d(\mathbb{R})$.

For a matrix T in $\mathbb{M}_d(\mathbb{R})$ we denote by $\|T\|$ the operator norm of T, i.e.

$$\|T\| = \sup\{\|Tu\| \mid u\in\mathbb{R}^d,\ \|u\|\le 1\}. \tag{3.3}$$

Recall then that

$$\|T^*T\| = \|T\|^2, \quad \|Tu\|\le\|T\|\|u\|, \quad\text{and}\quad \|TV\|\le\|T\|\|V\| \tag{3.4}$$

for any $d\times d$ matrix V and any u in $\mathbb{R}^d$. We shall also consider another norm on $\mathbb{M}_d(\mathbb{R})$: For any T in $\mathbb{M}_d(\mathbb{R})$ the Hilbert-Schmidt norm $\|T\|_{\mathrm{HS}}$ is the euclidean norm of T considered as a vector in $\mathbb{R}^{d^2}$. In other words:

$$\|T\|_{\mathrm{HS}}^2 = \sum_{j=1}^d \|Te_j\|^2 = \sum_{j=1}^d \langle T^*Te_j, e_j\rangle = \mathrm{tr}(T^*T) = \langle T,T\rangle, \tag{3.5}$$

where $\{e_1,\dots,e_d\}$ is the standard orthonormal basis for $\mathbb{R}^d$.

Since all norms on $\mathbb{M}_d(\mathbb{R})$ are equivalent, it follows that there exist strictly positive constants c_d, C_d (depending only on d), such that

$$c_d\|T\| \le \|T\|_{\mathrm{HS}} \le C_d\|T\| \quad \text{for all } T \text{ in } \mathbb{M}_d(\mathbb{R}). \tag{3.6}$$

By $\mathfrak{M}(\mathbb{R}^d)$ we denote the class of all Borel measures on $\mathbb{R}^d$ and by $\mathcal{P}(\mathbb{R}^d)$ the subclass of Borel probability measures. By $\mathcal{ID}(\mathbb{R}^d)$ we denote the class of infinitely divisible measures in $\mathcal{P}(\mathbb{R}^d)$ and by $\mathfrak{M}_L(\mathbb{R}^d)$ the class of Lévy measures on $\mathbb{R}^d$. That is, $\nu\in\mathfrak{M}_L(\mathbb{R}^d)$ if and only if $\nu\in\mathfrak{M}(\mathbb{R}^d)$, such that $\nu(\{0\})=0$ and

$$\int_{\mathbb{R}^d} \min\{1, \|y\|^2\}\,\nu(\mathrm{d}y) < \infty.$$

For μ in $\mathcal{ID}(\mathbb{R}^d)$ we denote by C_μ the cumulant transform of μ (i.e. the logarithm of the characteristic function of μ). We recall that C_μ has the Lévy-Khintchine representation:

$$C_\mu(u) = \mathrm{i}\,\langle\eta,u\rangle - \tfrac{1}{2}\,\langle Au,u\rangle + \int_{\mathbb{R}^d}\big(\mathrm{e}^{\mathrm{i}\langle y,u\rangle} - 1 - \mathrm{i}\,\langle y,u\rangle\,1_{B_1}(y)\big)\,\nu(\mathrm{d}y), \qquad (u\in\mathbb{R}^d),$$

where η is a d-dimensional vector, A is a non-negative definite $d\times d$ matrix and ν is in $\mathfrak{M}_L(\mathbb{R}^d)$. The triplet (η, A, ν) is uniquely determined and is termed the characteristic triplet for μ.

3.2. *Definitions of Υ_T^0 and Υ_T.* Let $(S,\mathcal{S},\gamma)$ be a σ-finite measure space, and for each s in S let $T(s)\colon\mathbb{R}^d\to\mathbb{R}^d$ be a linear transformation depending on s in a measurable way. In other words we consider a Borel-measurable mapping $T\colon S\to\mathbb{M}_d(\mathbb{R})$. In this subsection we define the rescaling mappings $\Upsilon_T^0\colon\mathfrak{M}_L(\mathbb{R}^d)\to\mathfrak{M}_L(\mathbb{R}^d)$ and $\Upsilon_T\colon\mathcal{ID}(\mathbb{R}^d)\to\mathcal{ID}(\mathbb{R}^d)$, such that the following formulae hold:

$$[\Upsilon_T^0(\nu)](B) = \int_S\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)y)\,\nu(\mathrm{d}y)\,\gamma(\mathrm{d}s) \tag{3.7}$$

for any ν in $\mathfrak{M}_L(\mathbb{R}^d)$ and any Borel set B in $\mathbb{R}^d$, and

$$C_{\Upsilon_T(\mu)}(y) = \int_S C_\mu(T(s)^* y)\,\gamma(\mathrm{d}s) \tag{3.8}$$

for any μ in $\mathcal{ID}(\mathbb{R}^d)$ and any y in $\mathbb{R}^d$. In order to ensure that Υ_T^0 maps any Lévy measure into a new Lévy measure, it is, as we shall see in Theorem 3.3 below, necessary and sufficient that

$$\gamma(\{s \in S \mid T(s) \neq 0\}) < \infty, \quad \text{and} \quad \int_S \|T(s)\|^2\,\gamma(\mathrm{d}s) < \infty. \tag{3.9}$$

In this case the right hand side of (3.8) is, for any μ in $\mathcal{ID}(\mathbb{R}^d)$, the cumulant transform for an infinitely divisible probability measure (cf. Corollary 3.6). If (3.9) is not satisfied, then one needs to restrict the domains of Υ_T^0 and Υ_T (see Definitions 3.2 and 3.4 below).

Proposition 3.1. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a (Borel-) measurable mapping.*

Then for any σ-finite measure ν on $\mathbb{R}^d$ the formula:

$$\tilde{\nu}(B) = \int_S \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)y)\,\nu(\mathrm{d}y)\,\gamma(\mathrm{d}s), \qquad (B \in \mathcal{B}(\mathbb{R}^d)), \tag{3.10}$$

defines a new measure $\tilde{\nu}$ on $\mathbb{R}^d$ such that $\tilde{\nu}(\{0\}) = 0$. In addition

$$\int_{\mathbb{R}^d} f(y)\,\tilde{\nu}(\mathrm{d}y) = \int_S \int_{\mathbb{R}^d} f(T(s)y) 1_{\mathbb{R}^d\setminus\{0\}}(T(s)y)\,\nu(\mathrm{d}y)\,\gamma(\mathrm{d}s). \tag{3.11}$$

for any positive measurable or $\tilde{\nu}$-integrable function f.

Proof: We note first that the function $(s, u) \mapsto 1_{B\setminus\{0\}}(T(s)u)$ is measurable with respect to the product σ-algebra $\mathcal{S}\otimes\mathcal{B}(\mathbb{R}^d)$, and therefore the double integral on the right hand side of (3.10) is well-defined. Subsequently it is straightforward to check that $\tilde{\nu}$ defined by (3.10) is indeed a measure on $\mathcal{B}(\mathbb{R}^d)$, and clearly $\tilde{\nu}(\{0\}) = 0$. Formula (3.10) shows that (3.11) holds for indicator functions for Borel subsets of $\mathbb{R}^d$ and a standard extension argument then establishes (3.11) for general f (either positive measurable or in $\mathcal{L}^1(\tilde{\nu})$). □

Definition 3.2. Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. Then the associated Upsilon transform of $\mathfrak{M}_L(\mathbb{R}^d)$ is the mapping $\Upsilon_T^0\colon \mathfrak{M}_L(\mathbb{R}^d) \to \mathfrak{M}(\mathbb{R}^d)$ defined by

$$\Upsilon_T^0(\nu) = \tilde{\nu}, \qquad (\nu \in \mathfrak{M}_L(\mathbb{R}^d)),$$

where, for any ν in $\mathfrak{M}_L(\mathbb{R}^d)$, $\tilde{\nu}$ is the measure described in Proposition 3.1.

In addition we define the Lévy domain $\mathsf{dom}_L(\Upsilon_T^0)$ of Υ_T^0 by the formula

$$\mathsf{dom}_L(\Upsilon_T^0) = \{\nu \in \mathfrak{M}_L(\mathbb{R}^d) \mid \Upsilon_T^0(\nu) \in \mathfrak{M}_L(\mathbb{R}^d)\}.$$

That is, $\nu \in \mathsf{dom}_L(\Upsilon_T^0)$ if and only if

$$\int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)y\|^2\}\,\nu(\mathrm{d}y)\,\gamma(\mathrm{d}s) < \infty. \tag{3.12}$$

Theorem 3.3. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a (Borel-) measurable mapping. Then $\mathsf{dom}_L(\Upsilon_T^0) = \mathfrak{M}_L(\mathbb{R}^d)$, if and only if*

$$\gamma(\{s \in S \mid T(s) \neq 0\}) < \infty, \quad \textit{and} \quad \int_S \|T(s)\|^2\,\gamma(\mathrm{d}s) < \infty. \tag{3.13}$$

Proof: Assume first that (3.13) is satisfied, and put

$$S_T = \{s \in S \mid T(s) \neq 0\}.$$

Consider further a Lévy measure ν on $\mathbb{R}^d$. On account of Proposition 3.1, it suffices to show that $\int_{\mathbb{R}^d} \min\{1, \|u\|^2\}\,\tilde{\nu}(\mathrm{d}u) < \infty$. Using (3.11) we find that

$$\begin{aligned}
\int_{\mathbb{R}^d} \min\{1, \|u\|^2\}\,\tilde{\nu}(\mathrm{d}u) &= \int_{S_T} \Big(\int_{\mathbb{R}^d} \min\{1, \|T(s)u\|^2\} 1_{\mathbb{R}^d \setminus \{0\}}(T(s)u)\,\nu(\mathrm{d}u) \Big)\,\gamma(\mathrm{d}s) \\
&\leq \int_{S_T} \Big(\int_{\mathbb{R}^d} \min\{1, \|T(s)\|^2 \|u\|^2\}\,\nu(\mathrm{d}u) \Big)\,\gamma(\mathrm{d}s) \\
&\leq \int_{S_T} \Big(\int_{\mathbb{R}^d} \min\{1, \|u\|^2\} \max\{1, \|T(s)\|^2\}\,\nu(\mathrm{d}u) \Big)\,\gamma(\mathrm{d}s) \\
&= \int_{S_T} \max\{1, \|T(s)\|^2\}\,\gamma(\mathrm{d}s) \int_{\mathbb{R}^d} \min\{1, \|u\|^2\}\,\nu(\mathrm{d}u),
\end{aligned} \tag{3.14}$$

and by the assumptions the resulting expression is finite.

Assume conversely that $\mathsf{dom}_L(\Upsilon_T^0) = \mathfrak{M}_L(\mathbb{R}^d)$. We establish first that

$$\exists R \in (0, \infty)\ \forall y \in \mathbb{R}^d\colon \int_S \min\{1, \|T(s)y\|^2\}\,\gamma(\mathrm{d}s) \leq R \min\{1, \|y\|^2\}. \tag{3.15}$$

Indeed, if (3.15) is not satisfied, then for any positive integer n we may choose a (non-zero) vector y_n in $\mathbb{R}^d$, such that

$$R_n := \int_S \min\{1, \|T(s)y_n\|^2\}\,\gamma(\mathrm{d}t) > n \min\{1, \|y_n\|^2\}.$$

We consider then the measure ν on $\mathbb{R}^d$ given by

$$\nu = \sum_{n=1}^{\infty} \frac{1}{nR_n} \delta_{y_n},$$

and we note that $\nu(\{0\}) = 0$, and that

$$\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,\nu(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{\min\{1, \|y_n\|^2\}}{nR_n} \leq \sum_{n=1}^{\infty} \frac{n^{-1}R_n}{nR_n} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus, by assumption, $\nu \in \mathfrak{M}_L(\mathbb{R}^d) = \mathsf{dom}_L(\Upsilon_T^0)$, and therefore by Proposition 3.1

$$\begin{aligned}
\infty > \int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,[\Upsilon_T^0(\nu)](\mathrm{d}x) &= \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|T(s)x\|^2\}\,\nu(\mathrm{d}x) \Big)\,\gamma(\mathrm{d}s) \\
= \int_S \Big(\sum_{n=1}^{\infty} \frac{\min\{1, \|T(s)y_n\|^2\}}{nR_n} \Big)\,\gamma(\mathrm{d}s) &= \sum_{n=1}^{\infty} \frac{1}{nR_n} \int_S \min\{1, \|T(s)y_n\|^2\}\,\gamma(\mathrm{d}s) \\
= \sum_{n=1}^{\infty} \frac{1}{n},
\end{aligned}$$

and thus we obtain the desired contradiction.

Assume in the following that R is a positive constant satisfying (3.15). Then for any unit vector u in $\mathbb{R}^d$ and any number a in $(0,\infty)$, we have that

$$\int_S \min\{1, a^2\|T(s)u\|^2\}\,\gamma(\mathrm{d}s) \le R\min\{1,a^2\}, \tag{3.16}$$

and letting then $a \uparrow \infty$, it follows by Monotone Convergence that

$$R \ge \int_S 1_{\{Tu\neq 0\}}(s)\,\gamma(\mathrm{d}s) = \gamma(\{s\in S \mid T(s)u \neq 0\}).$$

Considering then e.g. the standard basis $\{e_j \mid j = 1,\dots,d\}$ for $\mathbb{R}^d$, we conclude that

$$\gamma\big(\{s\in S \mid T(s)\neq 0\}\big) \le \sum_{j=1}^d \gamma\big(\{s\in S \mid T(s)e_j \neq 0\}\big) \le dR < \infty,$$

which proves the first statement in (3.13). Regarding the second statement, division with a^2 in (3.16) leads to the estimate

$$\int_S \min\{a^{-2}, \|T(s)u\|^2\}\,\gamma(\mathrm{d}s) \le R\min\{a^{-2},1\},$$

for any unit vector u in $\mathbb{R}^d$ and any a in $(0,\infty)$. Letting then $a\downarrow 0$, it follows by Monotone Convergence that

$$\int_S \|T(s)u\|^2\,\gamma(\mathrm{d}s) \le R.$$

Considering again the standard basis for $\mathbb{R}^d$, we find by application of (3.5) that

$$\begin{aligned}\infty > dR &\ge \sum_{j=1}^d \int_S \|T(s)e_j\|^2\,\gamma(\mathrm{d}s) = \int_S \Big(\sum_{j=1}^d \|T(s)e_j\|^2\Big)\,\gamma(\mathrm{d}s)\\ &= \int_S \|T(s)\|_{\mathrm{HS}}^2\,\gamma(\mathrm{d}s) \ge c_d^2 \int_S \|T(s)\|^2\,\gamma(\mathrm{d}s),\end{aligned}$$

where c_d is the (strictly positive) constant from (3.6). Thus, the second condition in (3.15) is also satisfied, and this completes the proof. □

We proceed to "extend" the mapping Υ_T^0 to a mapping Υ_T of infinitely divisible laws, essentially by applying Υ_T^0 to the Lévy measure of an infinitely divisible distribution followed by some adjustments of the remaining parameters in the characteristic triplet. We note initially (see Proposition 11.10 in Sato (1999)) that for any measure μ in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν) and any fixed $d\times d$ matrix R, the transformation $\mu\circ R^{-1}$ of μ by the linear mapping associated to R has characteristic triplet (η_R, A_R, ν_R), where

$$\begin{aligned} A_R &= RAR^*,\\ \nu_R(B) &= \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(Ry)\,\nu(\mathrm{d}y), \qquad (B\in\mathcal{B}(\mathbb{R}^d)),\\ \eta_R &= R\eta + \int_{\mathbb{R}^d}\big[1_{B_1}(Ry) - 1_{B_1}(y)\big]Ry\,\nu(\mathrm{d}y).\end{aligned} \tag{3.17}$$

In particular the latter integral (taken coordinate-wise) is well-defined for any Lévy measure ν and any $d \times d$ matrix R. This may be seen e.g. as a consequence of the following useful estimate:

$$\big|1_{B_1}(Ry) - 1_{B_1}(y)\big|\big\|Ry\big\| \le \min\{1, \|y\|^2\} \max\{1, \|R\|^2\}, \quad (y \in \mathbb{R}^d,\ R \in \mathbb{M}_d(\mathbb{R})). \tag{3.18}$$

The passage from μ to $\mu \circ R^{-1}$ corresponds to the definition of Υ_T given below in the case where γ is a one-point measure.

Definition 3.4. Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. We then define a mapping $\Upsilon_T\colon \mathsf{dom}_{ID}(\Upsilon_T) \to \mathcal{ID}(\mathbb{R}^d)$ in the following way:

(a) The domain $\mathsf{dom}_{ID}(\Upsilon_T)$ of Υ_T consists of those measures μ in $\mathcal{ID}(\mathbb{R}^d)$ for which the characteristic triplet (η, A, ν) satisfies the following three conditions:

$$\nu \in \mathsf{dom}_L(\Upsilon_T^0),$$

$$\int_S \|T(s)AT(s)^*\|\,\gamma(\mathrm{d}s) < \infty,$$

$$\int_S \Big\|T(s)\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(T(s)y) - 1_{B_1}(y)\big]T(s)y\,\nu(\mathrm{d}y)\Big\|\,\gamma(\mathrm{d}s) < \infty. \tag{3.19}$$

In (3.19) the "inner" integral in (3.19) is taken coordinatewise.[3]

(b) For any measure μ in $\mathsf{dom}_{ID}(\Upsilon_T)$ we define $\Upsilon_T(\mu)$ as the measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet $(\tilde{\eta}, \tilde{A}, \tilde{\nu})$, where

$$\tilde{A} = \int_S T(s)AT(s)^*\,\gamma(\mathrm{d}s),$$

$$\tilde{\nu} = \Upsilon_T^0(\nu), \tag{3.20}$$

$$\tilde{\eta} = \int_S \Big[T(s)\eta + \Big(\int_{\mathbb{R}^d} \big[1_{B_1}(T(s)y) - 1_{B_1}(y)\big]T(s)y\,\nu(\mathrm{d}y)\Big)\Big]\,\gamma(\mathrm{d}s),$$

and where the integrals of vectors and matrices are taken coordinate-wise.

Regarding condition (a) of the definition above, we note that if A is positive definite (in particular invertible), then $\int_S \|TAT^*\|\,\gamma(\mathrm{d}s) < \infty$, if and only if $\int_S \|T(s)\|^2\,\gamma(\mathrm{d}s) < \infty$.

Proposition 3.5. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. Consider further a measure μ in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν). Then $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, if and only if*

$$\nu \in \mathsf{dom}_L(\Upsilon_T^0), \ \text{and} \int_S |C_\mu(T(s)^*u)|\,\gamma(\mathrm{d}s) < \infty \ \text{for all } u \text{ in } \mathbb{R}^d.$$

In that case it holds furthermore that

$$C_{\Upsilon_T(\mu)}(u) = \int_S C_\mu(T(s)^*u)\,\gamma(\mathrm{d}s), \qquad (u \in \mathbb{R}^d).$$

[3] It follows by from (3.18) and standard arguments that the inner integral in (3.19) is a measurable $\mathbb{R}^d$-valued function of s.

Proof: For any fixed s in S the function $u \mapsto \exp(C_\mu(T^*(s)u))$ is the characteristic function for $\mu \circ T(s)^{-1}$, so according to (3.17) we have, writing T for $T(s)$, that

$$\begin{aligned} C_\mu(T^*u) = \mathrm{i}\Big\langle T\eta + \int_{\mathbb{R}^d} Ty\big[1_{B_1}(Ty) - 1_{B_1}(y)\big]\nu(\mathrm{d}y), u\Big\rangle - \tfrac{1}{2}\langle TAT^*u, u\rangle \\ + \int_{\mathbb{R}^d} \big(\mathrm{e}^{\,\mathrm{i}\langle Ty,u\rangle} - 1 - \mathrm{i}\,\langle Ty, u\rangle\, 1_{B_1}(Ty)\big) 1_{\mathbb{R}^d\setminus\{0\}}(Ty)\,\nu(\mathrm{d}y) \end{aligned} \tag{3.21}$$

for any u in $\mathbb{R}^d$. Now, if $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, then by the Cauchy-Schwarz inequality the first two terms on the right hand side of (3.21) are in $\mathcal{L}^1(\gamma)$, and regarding the last term we find by application of Proposition 3.1 that

$$\begin{aligned} &\int_S \Big| \int_{\mathbb{R}^d} \big(\mathrm{e}^{\,\mathrm{i}\langle Ty,u\rangle} - 1 - \mathrm{i}\,\langle Ty, u\rangle\, 1_{B_1}(Ty)\big) 1_{\mathbb{R}^d\setminus\{0\}}(Ty)\,\nu(\mathrm{d}y)\Big| \,\mathrm{d}\gamma \\ &\quad\le \int_S \Big(\int_{\mathbb{R}^d} \big|\mathrm{e}^{\,\mathrm{i}\langle Ty,u\rangle} - 1 - \mathrm{i}\,\langle Ty, u\rangle\, 1_{B_1}(Ty)\big| 1_{\mathbb{R}^d\setminus\{0\}}(Ty)\,\nu(\mathrm{d}y)\Big) \,\mathrm{d}\gamma \\ &\quad= \int_{\mathbb{R}^d} \big|\mathrm{e}^{\mathrm{i}\langle z,u\rangle} - 1 - \mathrm{i}\,\langle z, u\rangle\, 1_{B_1}(z)\big|\, [\Upsilon_T^0(\nu)](\mathrm{d}z) < \infty, \end{aligned}$$

where the strict inequality is due to the fact that $\Upsilon_T^0(\nu)$ is a Lévy measure. Thus, also the last term on the right hand side of (3.21) is in $\mathcal{L}^1(\gamma)$ and hence so is $C_\mu(T^*u)$.

Conversely, if $\nu \in \mathsf{dom}_L(\Upsilon_T^0)$, and $\int_S |C_\mu(T(s)^*u)|\,\gamma(\mathrm{d}s) < \infty$ for all u, then, as we just saw, the last term on the right hand side of (3.21) is in $\mathcal{L}^1(\gamma)$, and hence the sum

$$\mathrm{i}\Big\langle T\eta + \int_{\mathbb{R}^d} Ty\big[1_{B_1}(Ty) - 1_{B_1}(y)\big]\nu(\mathrm{d}y), u\Big\rangle - \tfrac{1}{2}\langle TAT^*u, u\rangle \tag{3.22}$$

must also be in $\mathcal{L}^1(\gamma)$. Since one term is positive (real) and the other purely imaginary, both terms in (3.22) must themselves be in $\mathcal{L}^1(\gamma)$. Letting u vary throughout the standard basis $\{e_1, \ldots, e_d\}$ for $\mathbb{R}^d$, it follows that each coordinate in the vector

$$T\eta + \int_{\mathbb{R}^d} Ty\big[1_{B_1}(Ty) - 1_{B_1}(y)\big]\nu(\mathrm{d}y)$$

belongs to $\mathcal{L}^1(\gamma)$, and hence so does its norm. Regarding $\langle TAT^*u, u\rangle$, we find by application of (3.5) that

$$\begin{aligned} \infty > \int_S \Big(\sum_{j=1}^d \langle TAT^*e_j, e_j\rangle\Big)\,\mathrm{d}\gamma &= \int_S \|A^{1/2}T^*\|_{\mathrm{HS}}^2\,\mathrm{d}\gamma \\ &\ge c_d^2 \int_S \|A^{1/2}T^*\|^2\,\mathrm{d}\gamma = c_d^2 \int_S \|TAT^*\|\,\mathrm{d}\gamma, \end{aligned}$$

where c_d is the strictly positive constant appearing in (3.6), and where the last equality is due to (3.4). Altogether, $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$.

Assuming now that $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, it follows from the above considerations that we may integrate term by term on the right hand side of (3.21). Together with linearity of the integral (with respect to γ) and Proposition 3.1 this yields for any

u in $\mathbb{R}^d$ that

$$\int_S C_\mu(T^*u)\,\mathrm{d}\gamma = \mathrm{i}\Big\langle \int_S \Big(T\eta + \int_{\mathbb{R}^d} Ty\big[1_{B_1}(Ty) - 1_{B_1}(y)\big]\nu(\mathrm{d}y)\Big)\,\mathrm{d}\gamma, u\Big\rangle$$
$$- \tfrac{1}{2}\Big\langle\Big(\int_S TAT^*\,\mathrm{d}\gamma\Big)u, u\Big\rangle$$
$$+ \int_{\mathbb{R}^d}\big(\mathrm{e}^{\mathrm{i}\langle z,u\rangle} - 1 - \mathrm{i}\,\langle z,u\rangle\,1_{B_1}(z)\big)\,[\Upsilon_T^0(\nu)](\mathrm{d}z),$$

and by definition of $\Upsilon_T(\mu)$, the right hand side equals $C_{\Upsilon_T(\mu)}(u)$. This completes the proof. □

Corollary 3.6. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. Then $\mathsf{dom}_{ID}(\Upsilon_T) = \mathcal{ID}(\mathbb{R}^d)$, if and only if*

$$\gamma(\{s \in S \mid T(s) \neq 0\}) < \infty, \quad \textit{and} \quad \int_S \|T(s)\|^2\,\gamma(\mathrm{d}s) < \infty. \tag{3.23}$$

Proof: If $\mathsf{dom}_{ID}(\Upsilon_T) = \mathcal{ID}(\mathbb{R}^d)$, then in particular (cf. Definition 3.4) $\mathsf{dom}_L(\Upsilon_T^0) = \mathfrak{M}_L(\mathbb{R}^d)$, which according to Theorem 3.3 means that (3.23) is satisfied.

Assume conversely that (3.23) is satisfied. According to Proposition 3.5 and Theorem 3.3 it suffices to verify that

$$\int_S |C_\mu(T^*(s)z)|\,\gamma(\mathrm{d}s) < \infty \quad \text{for all } \mu \text{ in } \mathcal{ID}(\mathbb{R}^d) \text{ and } z \text{ in } \mathbb{R}^d. \tag{3.24}$$

So let z in $\mathbb{R}^d$ and μ in $\mathcal{ID}(\mathbb{R}^d)$ be given, and let (η, A, ν) be the characteristic triplet for μ. Recall then (see e.g. Lemma 7.2 in Barndorff-Nielsen et al. (2008)) that

$$\int_{\mathbb{R}^d}\big|\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y,x\rangle 1_{[0,1]}(\|x\|)\big|\,\nu(\mathrm{d}x) \le \big(2 + \tfrac{1}{2}\|y\|^2\big)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x) \tag{3.25}$$

for any y in $\mathbb{R}^d$. It follows from this and the Cauchy-Schwarz inequality, that

$$|C_\mu(y)| \le \|\eta\|\|y\| + \tfrac{1}{2}\|A\|\|y\|^2 + \big(2 + \tfrac{1}{2}\|y\|^2\big)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x)$$

for all y in $\mathbb{R}^d$, and hence that

$$|C_\mu(T^*z)| \le \|\eta\|\|T\|\|z\| + \tfrac{1}{2}\|A\|\|T\|^2\|z\|^2 + \big(2 + \tfrac{1}{2}\|T\|^2\|z\|^2\big)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x). \tag{3.26}$$

Since $\int_S |C_\mu(T^*(s)z)|\,\gamma(\mathrm{d}s) = \int_{\{T\neq 0\}} |C_\mu(T^*(s)z)|\,\gamma(\mathrm{d}s)$, the assumption (3.23) in conjunction with (3.26) imply that (3.24) is satisfied, which completes the proof. □

In order to calculate (the cumulant transform of) $\Upsilon_T(\mu)$ for concrete examples of T, the following result proves very useful (see Subsection 3.4).

Proposition 3.7. *Let $(S, \mathcal{S}, \gamma)$ be a probability space and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping such that*

$$\int_S \|T(s)\|^2\,\gamma(\mathrm{d}s) < \infty.$$

Let further μ be an infinitely divisible probability measure on $\mathbb{R}^d$ with characteristic triplet (η, A, ν). Denoting by $\mathbb{E}_\gamma$ expectation with respect to γ, it follows that the cumulant transform of $\Upsilon_T(\mu)$ is given by

$$\begin{aligned} C_{\Upsilon_T(\mu)}(u) = \mathrm{i}\big\langle \mathbb{E}_\gamma\{T\}\eta, u\big\rangle - \frac{1}{2}\big\langle \mathbb{E}_\gamma\{TAT^*\}u, u\big\rangle \\ + \int_{\mathbb{R}^d} \Big(\mathbb{E}_\gamma\big\{\mathrm{e}^{\mathrm{i}\langle y, T^* u\rangle}\big\} - 1 - \mathrm{i}\,\langle y, \mathbb{E}_\gamma\{T^*\}u\rangle\, 1_{B_1}(y)\Big)\, \nu(\mathrm{d}y), \end{aligned} \tag{3.27}$$

for all u in $\mathbb{R}^d$.

Proof: Consider the characteristic triplet $(\tilde{\eta}, \tilde{A}, \tilde{\nu})$ for $\Upsilon_T(\mu)$ described in Definition 3.4. Using Proposition 3.1, formula (3.18) and Fubini's Theorem, we find that

$$\begin{aligned} &\int_{\mathbb{R}^d} \Big(\mathrm{e}^{\mathrm{i}\langle y, u\rangle} - 1 - \mathrm{i}\,\langle y, u\rangle\, 1_{B_1}(y)\Big)\, \tilde{\nu}(\mathrm{d}y) \\ &= \mathbb{E}_\gamma\Big\{ \int_{\mathbb{R}^d} \Big(\mathrm{e}^{\mathrm{i}\langle Ty, u\rangle} - 1 - \mathrm{i}\,\langle Ty, u\rangle\, 1_{B_1}(Ty)\Big)\, \nu(\mathrm{d}y)\Big\} \\ &= \int_{\mathbb{R}^d} \Big(\mathbb{E}_\gamma\big\{\mathrm{e}^{\mathrm{i}\langle Ty, u\rangle}\big\} - 1 - \mathrm{i}\,\langle \mathbb{E}_\gamma\{T\}y, u\rangle\, 1_{B_1}(y)\Big)\, \nu(\mathrm{d}y) \\ &\qquad + \mathrm{i}\int_{\mathbb{R}^d} \mathbb{E}_\gamma\big\{\langle Ty, u\rangle (1_{B_1}(y) - 1_{B_1}(Ty))\big\}\, \nu(\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} \big(\mathbb{E}_\gamma\big\{\mathrm{e}^{\mathrm{i}\langle y, T^* u\rangle}\big\} - 1 - \mathrm{i}\,\langle y, \mathbb{E}_\gamma\{T^*\}u\rangle\, 1_{B_1}(y)\big)\, \nu(\mathrm{d}y) + \mathrm{i}\big\langle \mathbb{E}_\gamma\{T\}\eta - \tilde{\eta}, u\big\rangle, \end{aligned}$$

where the last equality uses the expression for $\tilde{\eta}$ in (3.20). Since $\tilde{A} = \mathbb{E}_\gamma\{TAT^*\}$, the above calculation combined with Definition 3.4 proves (3.27). ☐

The following is a multivariate generalization of the illustrative example in the Introduction.

Examples 3.8 (Distributions of multivariate supOU processes). (a) Let $\mathbb{M}_d^-$ be the cone of matrices in $\mathbb{M}_d(\mathbb{R})$ with eigenvalues having negative real part, $S = \mathbb{M}_d^- \times \mathbb{R}$ and $\mathcal{S} = \mathcal{B}(\mathbb{M}_d^- \times \mathbb{R})$. Let π be a probability measure on $\mathcal{B}(\mathbb{M}_d^-)$, λ be the Lebesgue measure on $\mathbb{R}$ and $\gamma = \pi \otimes \lambda$ the corresponding product measure on $\mathcal{S}$. For each $s = (Q, r) \in S$, consider the $d \times d$ matrix $T(s) := T(Q, r) = \mathrm{e}^{rQ}$. In this case, if μ with characteristic triplet (η, A, ν) belongs to the domain of Υ_T, we obtain from (3.10) and (3.20) that the characteristic triplet $(\tilde{\eta}, \tilde{A}, \tilde{\nu})$ of $\Upsilon_T(\mu)$ is given by

$$\begin{aligned} \tilde{A} &= \int_{\mathbb{M}_d^-} \int_{\mathbb{R}} \mathrm{e}^{rQ} A \mathrm{e}^{rQ^*}\, \mathrm{d}r\, \pi(\mathrm{d}Q), \\ \tilde{\nu}(B) &= \int_{\mathbb{M}_d^-} \int_{\mathbb{R}} \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(\mathrm{e}^{rQ} y)\, \nu(\mathrm{d}y)\, \mathrm{d}r\, \pi(\mathrm{d}Q), \qquad (B \in \mathcal{B}(\mathbb{R}^d)), \\ \tilde{\eta} &= \int_{\mathbb{M}_d^-} \int_{\mathbb{R}} \Big[\mathrm{e}^{rQ}\eta + \Big(\int_{\mathbb{R}^d} \big[1_{B_1}(\mathrm{e}^{rQ} y) - 1_{B_1}(y)\big] \mathrm{e}^{rQ} y\, \nu(\mathrm{d}y)\Big)\Big]\, \mathrm{d}r\, \pi(\mathrm{d}Q). \end{aligned} \tag{3.28}$$

This is also the characteristic triplet of the distribution of a stationary d-dimensional supOU process; see (3.5)-(3.7) in Barndorff-Nielsen and Stelzer (2011).

(b) A particular case of this example, where π is degenerate at a matrix Q appears in the theory of multidimensional operator-selfdecomposable distributions; see the book Jurek and Mason (1993).

(c) The measure $\gamma = \pi \otimes \lambda$ does not in general satisfy the first condition in (3.23). Hence $\mathsf{dom}_{ID}(\Upsilon_T)$ is not the whole class $\mathcal{ID}(\mathbb{R}^d)$. A sufficient condition is presented in Barndorff-Nielsen and Stelzer (2011) in connection to the existence of superpositions of Ornstein-Uhlenbeck processes as random integrals with respect to Lévy bases. Let us further assume that there exist measurable functions $\rho : \mathbb{M}_d^- \to (0, \infty)$ and $\kappa : \mathbb{M}_d^- \to [1, \infty)$ such that

$$\left\| \mathrm{e}^{rQ} \right\| \leq \kappa(Q)\mathrm{e}^{-r\rho(Q)} \quad \text{for all } r \text{ in } \mathbb{R}_+ \text{ and } \pi\text{-almost all } Q. \tag{3.29}$$

and

$$\int_{\mathbb{M}_d^-} \frac{\kappa(Q)^2}{\rho(Q)} \pi(\mathrm{d}Q) < \infty. \tag{3.30}$$

Let $\mu \in \mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν) be such that

$$\int_{\{\|x\| \geq 1\}} \ln(\|x\|)\, \nu(\mathrm{d}x) < \infty. \tag{3.31}$$

It is shown in Barndorff-Nielsen and Stelzer (2011) - in the context of existence of random integrals with respect to Lévy bases - that conditions (3.29) - (3.31) imply (3.19) and hence we have $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$ (see also Example 6.10 below).

Remark 3.9. One-dimensional Upsilon transformations Υ^0 and Υ for infinitely divisible distributions on Banach spaces have been considered in Jurek (1985, 1990). Our set-up for the multivariate mixing transformations Υ_T^0 and Υ_T could also be extended to Banach spaces. More specifically, let $\mathbf{X}$ be a Banach space and denote by $\mathsf{B}(\mathbf{X})$ the Banach algebra of all bounded (or continuous) linear operators on $\mathbf{X}$. Equipping $\mathsf{B}(\mathbf{X})$ with its Borel-σ-algebra, we may consider a measurable family of operators with values in $\mathsf{B}(\mathbf{X})$, i.e. a measurable mapping

$$T \colon S \to \mathsf{B}(\mathbf{X}),$$

where $(S, \mathcal{S}, \gamma)$ is a σ-finite measure space. Assuming (3.23), one next considers the formula

$$\Upsilon_T^0(\nu) = \int_S \int_{\mathbf{X}} 1_{B\setminus\{0\}}(T(s)y)\, \nu(\mathrm{d}y)\, \gamma(\mathrm{d}s), \qquad (B \in \mathcal{B}(\mathsf{B}(\mathbf{X})), \tag{3.32}$$

for ν in $\mathfrak{M}_L(\mathbf{X})$, the set of Lévy measures on $\mathsf{B}(\mathbf{X})$. Recall here that for general infinite dimensional Banach spaces, Levy measures are not determined by an integrability condition like: $\int_{\mathbf{X}} \min\{1, \|x\|^2\}\, \nu(\mathrm{d}x) < \infty$. In fact, a σ-finite measure ν on $\mathbf{X}$ is said to be a Lévy measure if there exists a probability measure μ on $\mathbf{X}$ such that

$$\widehat{\mu}(f) = \exp\Big(\int_{\mathbf{X}} \big(\mathrm{e}^{\mathrm{i}f(y)} - 1 - \mathrm{i}f(y)1_{B_1}(y)\big)\, \nu(\mathrm{d}y) \Big), \qquad (f \in \mathbf{X}^*),$$

where $\widehat{\mu}(f) = \int_{\mathbf{X}} \mathrm{e}^{\mathrm{i}f(y)}\mu(\mathrm{d}y)$, $f \in \mathbf{X}^*$, is the characteristic functional of μ; see Araujo and Giné (1980).

However, if we specialize to the case of a Hilbert space $\mathbf{H}$ (with inner product $\langle\cdot,\cdot\rangle_{\mathbf{H}}$ and norm $\|\cdot\|_{\mathbf{H}}$), then the Lévy measures ν on $\mathbf{H}$ are indeed characterized by the condition:

$$\int_{\mathbf{H}} \min\{1, \|h\|_{\mathbf{H}}^2\}\, \nu(\mathrm{d}h) < \infty \tag{3.33}$$

(see Araujo and Giné (1980)). Moreover, the class $\mathcal{ID}(\mathbf{H})$ of infinitely divisible probability measures on $\mathbf{H}$ may be characterized as consisting of those measures μ on $\mathbf{H}$ for which the characteristic functional has the Lévy-Khintchine representation:

$$\log \int_{\mathbf{H}} \mathrm{e}^{\mathrm{i}\langle h,x\rangle} \mu(\mathrm{d}x) = \mathrm{i}\,\langle \eta, h\rangle_{\mathbf{H}} - \tfrac{1}{2}\,\langle Ah, h\rangle_{\mathbf{H}} + \int_{\mathbf{H}} \big(\mathrm{e}^{\mathrm{i}\langle y,h\rangle_{\mathbf{H}}} - 1 - \mathrm{i}\frac{\langle y,h\rangle_{\mathbf{H}}}{1+\|y\|_{\mathbf{H}}}\big)\, \nu(\mathrm{d}y)$$

for all h in $\mathbf{H}$. Here $\eta \in \mathbf{H}$, A is a trace class positive operator in $\mathbf{H}$ and ν is Lévy measure on $\mathbf{H}$ (see e.g. Laha and Rohatgi (1979); Samorodnitsky and Taqqu (1994)).

For a family of linear operators $T\colon S \to \mathcal{B}(\mathbf{H})$, such that (3.23) is satisfied, the mapping Υ_T^0 (defined as in (3.32)) can then be extended to a mapping $\Upsilon_T \colon \mathcal{ID}(H) \to \mathcal{ID}(H)$ by copying the corresponding arguments for matrix mappings given in the proof of Proposition 3.5.

3.3. *Basic properties.* Since the mapping $\Upsilon_T(\mu)$ is a function of two variables, the linear mapping T and the infinitely distribution μ, there are a number of useful properties that can be established keeping one variable fixed while varying the other. In this section we establish a number of basic properties in this respect. We start by considering a fixed T and study the behavior of Υ_T under convolution of probability measures and composition.

Proposition 3.10. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping.*

(i) *For any vector c in $\mathbb{R}^d$, we have that*

$$\delta_c \in \mathsf{dom}_{ID}(\Upsilon_T) \iff \int_S \|T(s)c\|\, \gamma(\mathrm{d}s) < \infty, \tag{3.34}$$

and in that case $\Upsilon_T(\delta_c) = \delta_{\int_S Tc\,\mathrm{d}\gamma}$.

(ii) *If $\mu_1, \mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$, then also $\mu_1 * \mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$, and $\Upsilon_T(\mu_1 * \mu_2) = \Upsilon_T(\mu_1) * \Upsilon_T(\mu_2)$.*

(iii) *If $V \in \mathbb{M}_d(\mathbb{R})$ and $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, we have that*

$$\mu \circ V^{-1} \in \mathsf{dom}_{ID}(\Upsilon_T) \iff \mu \in \mathsf{dom}_{ID}(\Upsilon_{TV}), \tag{3.35}$$

and in that case $\Upsilon_T(\mu \circ V^{-1}) = \Upsilon_{TV}(\mu)$.

(iv) *If $V \in \mathbb{M}_d(\mathbb{R})$ and $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, then also $\mu \in \mathsf{dom}_{ID}(\Upsilon_{VT})$ and $\Upsilon_{VT}(\mu) = \Upsilon_T(\mu) \circ V^{-1}$.*

(v) *If $V \in \mathbb{M}_d(\mathbb{R})$, $VT = TV$ and $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, then also $\mu \circ V^{-1} \in \mathsf{dom}_{ID}(\Upsilon_T)$, and*

$$\Upsilon_T(\mu \circ V^{-1}) = \Upsilon_T(\mu) \circ V^{-1}.$$

(vi) *If $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$ and $c \in \mathbb{R}$, then also $D_c\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, and $\Upsilon_T(D_c\mu) = D_c\Upsilon_T(\mu)$. Here $D_c\mu$ denotes the transformation of μ by the mapping $u \mapsto cu\colon \mathbb{R}^d \to \mathbb{R}^d$.*

Proof: (i) Let c be a vector in $\mathbb{R}^d$, and note then that

$$\int_S |C_{\delta_c}(T(s)^* z)|\,\gamma(\mathrm{d}s) = \int_S |\langle c, T(s)^* z\rangle|\,\gamma(\mathrm{d}s) = \int_S |\langle T(s)c, z\rangle|\,\gamma(\mathrm{d}s)$$

for any z in $\mathbb{R}^d$. Now, if $\delta_c \in \mathsf{dom}_{ID}(\Upsilon_T)$, it follows that

$$\infty > \int_S \Big(\sum_{j=1}^d |\langle T(s)c, e_j\rangle|\Big)\,\gamma(\mathrm{d}s) \geq \int_S \|T(s)c\|\,\gamma(\mathrm{d}s),$$

where $\{e_1, \dots, e_d\}$ denotes the standard basis for $\mathbb{R}^d$.

If, conversely, $\int_S \|Tc\|\,\mathrm{d}\gamma < \infty$, then by the Cauchy-Schwarz Inequality

$$\int_S |C_{\delta_c}(T(s)^* z)|\,\gamma(\mathrm{d}s) \leq \|z\| \int_S \|T(s)c\|\,\gamma(\mathrm{d}s) < \infty,$$

and since the Lévy measure for δ_c is 0, it follows from Proposition 3.5, that $\delta_c \in \mathsf{dom}_{ID}(\Upsilon_T)$.

Having established (3.34) we note finally, that if $\int_S \|Tc\|\,\mathrm{d}\gamma < \infty$, then for any z in $\mathbb{R}^d$

$$C_{\Upsilon_T(\delta_c)}(z) = \int_S C_{\delta_c}(T^*(s)z)\,\gamma(\mathrm{d}s) = \mathrm{i}\int_S \langle T(s)c, z\rangle\,\gamma(\mathrm{d}s) = \mathrm{i}\Big\langle \textstyle\int_S T(s)c\,\gamma(\mathrm{d}s), z\Big\rangle,$$

and the resulting expression is exactly the cumulant transform for $\delta_{\int_S Tc\,\mathrm{d}\gamma}$ at z.

(ii) Assume that $\mu_1, \mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$ with Lévy measures ν_1 and ν_2, respectively. For any z in $\mathbb{R}^d$ we note then that

$$\begin{aligned}\int_S |C_{\mu_1 * \mu_2}(Tu)|\,\mathrm{d}\gamma &= \int_S \big|C_{\mu_1}(Tu) + C_{\mu_2}(Tu)\big|\,\mathrm{d}\gamma \\ &\leq \int_S \big|C_{\mu_1}(Tu)\big|\,\mathrm{d}\gamma + \int_S \big|C_{\mu_2}(Tu)\big|\,\mathrm{d}\gamma < \infty.\end{aligned}$$

In addition,

$$\begin{aligned}&\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,[\Upsilon_T^0(\nu_1 + \nu_2)](\mathrm{d}x) \\ &= \int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)x\|^2\}\,\nu_1(\mathrm{d}x)\,\gamma(\mathrm{d}s) + \int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)x\|^2\}\,\nu_2(\mathrm{d}x)\,\gamma(\mathrm{d}s) \\ &< \infty,\end{aligned}$$

and since $\nu_1 + \nu_2$ is the Lévy measure for $\mu_1 * \mu_2$, it follows from Proposition 3.5 that $\mu_1 * \mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$. For any z in $\mathbb{R}^d$ we note subsequently that

$$\begin{aligned}C_{\Upsilon_T(\mu_1 * \mu_2)}(z) &= \int_S C_{\mu_1 * \mu_2}(Tz)\,\mathrm{d}\gamma = \int_S \big(C_\mu(Tz) + C_\mu(Tz)\big)\,\mathrm{d}\gamma \\ &= C_{\Upsilon_T(\mu_1)}(z) + C_{\Upsilon_T(\mu_2)}(z) = C_{\Upsilon_T(\mu_1) * \Upsilon_T(\mu_2)}(z),\end{aligned}$$

and this completes the proof of (ii).

(iii) Recall first that $C_{\mu \circ V^{-1}}(z) = C_\mu(V^* z)$, and therefore

$$\int_S \big|C_{\mu \circ V^{-1}}(T^* z)\big|\,\mathrm{d}\gamma = \int_S \big|C_\mu(V^* T^* z)\big|\,\mathrm{d}\gamma = \int_S \big|C_\mu((TV)^* z)\big|\,\mathrm{d}\gamma$$

for all z in $\mathbb{R}^d$. Note next that the Lévy measure $\nu^{\#}$ for $\mu \circ V^{-1}$ is given by:

$$\nu^{\#}(B) = \int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(Vx)\,\nu(\mathrm{d}x)$$

(cf. Sato (1999, Proposition 11.10)), and hence

$$\begin{aligned}\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,[\Upsilon_T^0(\nu^{\#})](\mathrm{d}x) &= \int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)x\|^2\}\,\nu^{\#}(\mathrm{d}x)\,\gamma(\mathrm{d}s) \\ &= \int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)Vx\|^2\}\,\nu(\mathrm{d}x)\,\gamma(\mathrm{d}s) \\ &= \int_S \int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\,[\Upsilon_{TV}^0(\nu)](\mathrm{d}x).\end{aligned}$$

The above considerations together with Proposition 3.5 establish (3.35). Assuming now that $\mu \circ V^{-1} \in \mathsf{dom}_{ID}(\Upsilon_T)$, we find for any z in $\mathbb{R}^d$ that

$$C_{\Upsilon_T(\mu\circ V^{-1})}(z) = \int_S C_{\mu\circ V^{-1}}(T^*z)\,\mathrm{d}\gamma = \int_S C_\mu(V^*T^*z)\,\mathrm{d}\gamma = C_{\Upsilon_{TV}(\mu)}(z),$$

which completes the proof of (iii).

(iv) Assume that $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$. For any z in $\mathbb{R}^d$ we note then that

$$\int_S |C_\mu((VT)^*z)|\,\mathrm{d}\gamma = \int_S |C_\mu(T^*(V^*z))|\,\mathrm{d}\gamma < \infty.$$

Letting ν denote the Lévy measure for μ we note next that

$$\begin{aligned}\int_{\mathbb{R}^d} \min\{1,\|x\|^2\}\,[\Upsilon_{VT}^0(\nu)](\mathrm{d}x) &= \int_S \int_{\mathbb{R}^d} \min\{1, \|VT(s)x\|^2\}\,\nu(\mathrm{d}x)\,\gamma(\mathrm{d}s) \\ &\le \max\{1, \|V\|^2\} \int_S \int_{\mathbb{R}^d} \min\{1, \|T(s)x\|^2\}\,\nu(\mathrm{d}x)\,\gamma(\mathrm{d}s) < \infty,\end{aligned}$$

and it follows from Proposition 3.5 that $\mu \in \mathsf{dom}_{ID}(\Upsilon_{VT})$. For any z in $\mathbb{R}^d$ we find finally that

$$C_{\Upsilon_{VT}(\mu)}(z) = \int_S C_\mu(T^*V^*z)\,\mathrm{d}\gamma = C_{\Upsilon_T(\mu)}(V^*z) = C_{\Upsilon_T(\mu)\circ V^{-1}}(z),$$

and this completes the proof of (iv).

(v) This follows immediately by combining (iii) and (iv).

(vi) This follows immediately by applying (v) in the case $V = c\mathrm{I}_d$. □

The following result complements Proposition 3.10(ii) and is important in connection with the study of the action of Upsilon transformations on stable and selfdecomposable measures (cf. Corollary 3.12 below).

Lemma 3.11. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. Suppose further that μ_1, μ_2 are measures from $\mathcal{ID}(\mathbb{R}^d)$ such that $\mu_1, \mu_1 * \mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$. Then also $\mu_2 \in \mathsf{dom}_{ID}(\Upsilon_T)$.*

Proof: Let (η_1, A_1, ν_1) and (η_2, A_2, ν_2) denote, respectively, the characteristic triplets for μ_1 and μ_2, so that $(\eta_1 + \eta_2, A_1 + A_2, \nu_1 + \nu_2)$ is the characteristic triplet for $\mu_1 * \mu_2$.

We note first that by Proposition 3.1

$$\int_{\mathbb{R}^d} \min\{1, \|x\|^2\}\, [\Upsilon_T^0(\nu_2)] = \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|Tx\|^2\}\, \nu_2(\mathrm{d}x) \Big)\, \mathrm{d}\gamma$$
$$\le \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|Tx\|^2\}\, [\nu_1 + \nu_2](\mathrm{d}x) \Big)\, \mathrm{d}\gamma < \infty,$$

so that $\nu_2 \in \mathsf{dom}_L(\Upsilon_T^0)$. Since $0 \le TA_2T^* \le T(A_1 + A_2)T^*$, we also have that $\|TA_2T^*\| \le \|T(A_1 + A_2)T^*\|$, and hence

$$\int_S \|TA_2T^*\|\, \mathrm{d}\gamma \le \int_S \|T(A_1 + A_2)T^*\|\, \mathrm{d}\gamma < \infty.$$

Note finally that

$$\int_S \Big\| T\eta_2 + \int_{\mathbb{R}^d} \big[1_{B_1}(Ty) - 1_{B_1}(y)\big] Ty\, \nu_2(\mathrm{d}y) \Big\|\, \mathrm{d}\gamma$$
$$\le \int_S \Big\| T(\eta_1 + \eta_2) + \int_{\mathbb{R}^d} \big[1_{B_1}(Ty) - 1_{B_1}(y)\big] Ty\, [\nu_1 + \nu_2](\mathrm{d}y) \Big\|\, \mathrm{d}\gamma$$
$$+ \int_S \Big\| T\eta_1 + \int_{\mathbb{R}^d} \big[1_{B_1}(Ty) - 1_{B_1}(y)\big] Ty\, \nu_1(\mathrm{d}y) \Big\|\, \mathrm{d}\gamma < \infty,$$

and this completes the proof. □

As a consequence of this lemma we have that Υ_T preserves the classes of Gaussian, stable and selfdecomposable distributions (within its domain).

Corollary 3.12. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping. Let further $\mathcal{G}(\mathbb{R}^d)$, $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{L}(\mathbb{R}^d)$ denote, respectively, the classes of d-dimensional Gaussian, stable and selfdecomposable laws. We then have that*

(i) $\Upsilon_T\big(\mathsf{dom}_{ID}(\Upsilon_T) \cap \mathcal{G}(\mathbb{R}^d)\big) \subseteq \mathcal{G}(\mathbb{R}^d)$.
(ii) $\Upsilon_T\big(\mathsf{dom}_{ID}(\Upsilon_T) \cap \mathcal{S}(\mathbb{R}^d)\big) \subseteq \mathcal{S}(\mathbb{R}^d)$.
(iii) $\Upsilon_T\big(\mathsf{dom}_{ID}(\Upsilon_T) \cap \mathcal{L}(\mathbb{R}^d)\big) \subseteq \mathcal{L}(\mathbb{R}^d)$.

Proof: (i) A measure μ from $\mathcal{G}(\mathbb{R}^d)$ has characteristic triplet in the form $(\eta, A, 0)$, and hence, if additionally $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, the measure $\Upsilon_T(\mu)$ has characteristic triplet $(\int_S T\eta\, \mathrm{d}\gamma, \int_S TAT^*\, \mathrm{d}\gamma, 0)$. Thus, $\Upsilon_T(\mu)$ is again a measure from $\mathcal{G}(\mathbb{R}^d)$.

(ii) Recall that $\mathcal{S}(\mathbb{R}^d)$ is the class of probability measures μ on $\mathbb{R}^d$ satisfying that (cf. Samorodnitsky and Taqqu (1994, Definition 2.1.1))

$$\forall \alpha, \alpha' > 0\ \exists \alpha'' > 0\ \exists b \in \mathbb{R}^d\colon D_\alpha\mu * D_{\alpha'}\mu = D_{\alpha''}\mu * \delta_b.$$

Assume then that $\mu \in \mathcal{S}(\mathbb{R}^d) \cap \mathsf{dom}_{ID}(\Upsilon_T)$ and that $\alpha, \alpha' \in (0, \infty)$. Then we may choose α'' in $(0, \infty)$ and b in $\mathbb{R}^d$, such that $D_\alpha\mu * D_{\alpha'}\mu = D_{\alpha''}\mu * \delta_b$. According to (ii) and (vi) of Proposition 3.10 we have here that $D_\alpha\mu * D_{\alpha'}\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$ and that $D_{\alpha''}\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, and hence Lemma 3.11 ensures that also $\delta_b \in \mathsf{dom}_{ID}(\Upsilon_T)$. It follows then by application of (i),(ii) and (vi) in Proposition 3.10 that

$$D_\alpha\Upsilon_T(\mu) * D_{\alpha'}\Upsilon_T(\mu) = \Upsilon_T\big((D_\alpha\mu) * (D_{\alpha'}\mu)\big) = \Upsilon_T\big(D_{\alpha''}\mu * \delta_b\big)$$
$$= D_{\alpha''}\Upsilon_T(\mu) * \delta_{\int_S Tb\, \mathrm{d}\gamma},$$

and this shows that $\Upsilon_T(\mu) \in \mathcal{S}(\mathbb{R}^d)$ too.

(iii) Recall that $\mathcal{L}(\mathbb{R}^d)$ may be characterised as the class of probability measures in $\mathcal{ID}(\mathbb{R}^d)$ satisfying that

$$\forall c \in (0,1)\ \exists \mu_c \in \mathcal{P}(\mathbb{R}^d)\colon \mu = D_c\mu * \mu_c. \tag{3.36}$$

In that case μ_c is necessarily in $\mathcal{ID}(\mathbb{R}^d)$ (cf. e.g. Sato (1999, Proposition 15.5)). Assume then that $\mu \in \mathsf{dom}_{ID}(\Upsilon_T) \cap \mathcal{L}(\mathbb{R}^d)$ and choose for c in $(0,1)$ a measure μ_c in $\mathcal{ID}(\mathbb{R}^d)$ such that (3.36) is satisfied. Since $\mu, D_c\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, Lemma 3.11 ensures that $\mu_c \in \mathsf{dom}_{ID}(\Upsilon_T)$, and applications of (ii) and (vi) in Proposition 3.10 then yield that

$$\Upsilon_T(\mu) = \Upsilon_T(D_c\mu * \mu_c) = D_c\Upsilon_T(\mu) * \Upsilon_T(\mu_c).$$

This shows that $\Upsilon_T(\mu) \in \mathcal{L}(*)$. □

Proposition 3.13. *Let $(S, \mathcal{S}, \gamma)$ be a probability space, and let $T, V\colon S \to \mathbb{M}_d(\mathbb{R})$ be measurable mappings such that $\int_S \|T\|^2\,\mathrm{d}\gamma < \infty$. Assume further that T and V are independent with respect to γ. We then have that*

(i) $\mathsf{dom}_{ID}(\Upsilon_V) \subseteq \mathsf{dom}_{ID}(\Upsilon_{TV})$.
(ii) $\Upsilon_{TV}(\mu) = \Upsilon_T(\Upsilon_V(\mu))$ *for all μ in* $\mathsf{dom}_{ID}(\Upsilon_V)$.

Proof: We note first that the independence assumption implies that

$$\int_S f(T)g(V)\,\mathrm{d}\gamma = \int_S f(T)\,\mathrm{d}\gamma \int_S g(V)\,\mathrm{d}\gamma \tag{3.37}$$

for any measurable functions $f, g\colon \mathbb{M}_d(\mathbb{R}) \to [0,\infty)$.

(i) Assume that $\mu \in \mathsf{dom}_{ID}(\Upsilon_V)$. We show directly that the three conditions in Definition 3.4(b) are satisfied (with T replaced by TV). So let (η, A, ν) be the characteristic triplet for μ, and note then that

$$\begin{aligned}\int_S \min\{1,\|x\|^2\}\,[\Upsilon^0_{TV}(\nu)](\mathrm{d}x) &= \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|TVx\|^2\}\,\nu(\mathrm{d}x)\Big)\,\mathrm{d}\gamma \\ &\le \int_S \max\{1, \|T\|^2\}\Big(\int_{\mathbb{R}^d} \min\{1, \|Vx\|^2\}\,\nu(\mathrm{d}x)\Big)\,\mathrm{d}\gamma \\ &= \int_S \max\{1, \|T\|^2\}\,\mathrm{d}\gamma \int_S \int_{\mathbb{R}^d} \min\{1, \|Vx\|^2\}\,\nu(\mathrm{d}x)\Big)\,\mathrm{d}\gamma < \infty,\end{aligned}$$

which verifies that $\nu \in \mathsf{dom}_L(\Upsilon^0_{TV})$. Note next that by (3.4) we have that

$$\begin{aligned}\int_S \|TVAV^*T^*\|\,\mathrm{d}\gamma = \int_S \|TVA^{1/2}\|^2\,\mathrm{d}\gamma &\le \int_S \|T\|^2\|VA^{1/2}\|^2\,\mathrm{d}\gamma \\ &= \int_S \|T\|^2\,\mathrm{d}\gamma \int_S \|VAV^*\|\,\mathrm{d}\gamma < \infty.\end{aligned}$$

Note finally that for fixed s in S (suppressed in the notation) we have that

$$\begin{aligned}
&\Big\|TV\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(TVy) - 1_{B_1}(y)\big] TVy\,\nu(\mathrm{d}y)\Big\| \\
&\qquad = \Big\| \int_{\mathbb{R}^d} \big[1_{B_1}(TVy) - 1_{B_1}(Vy)\big] TVy\,\nu(\mathrm{d}y) + TV\eta \\
&\qquad\qquad + T\int_{\mathbb{R}^d} \big[1_{B_1}(Vy) - 1_{B_1}(y)\big] Vy\,\nu(\mathrm{d}y)\Big\| \\
&\qquad \le \sqrt{d}\int_{\mathbb{R}^d} \big|1_{B_1}(TVy) - 1_{B_1}(Vy)\big| \|TVy\|\,\nu(\mathrm{d}y) \\
&\qquad\qquad + \|T\|\Big\|V\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(Vy) - 1_{B_1}(y)\big] Vy\,\nu(\mathrm{d}y)\Big\|,
\end{aligned}$$

where we have used linearity of the integral as well as the fact that $\| \int_{\mathbb{R}^d} f\,\mathrm{d}\nu\| \le \sqrt{d}\int_{\mathbb{R}^d} \|f\|\,\mathrm{d}\nu$ for any ν-integrable function $f\colon \mathbb{R}^d \to \mathbb{R}^d$. Applying now (3.18) and (3.37), it follows that

$$\begin{aligned}
&\int_S \Big(\int_{\mathbb{R}^d} \big|1_{B_1}(TVy) - 1_{B_1}(Vy)\big| \|TVy\|\,\nu(\mathrm{d}y)\Big)\,\mathrm{d}\gamma \\
&\qquad \le \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|Vy\|^2\} \max\{1, \|T\|^2\}\,\nu(\mathrm{d}y)\Big)\,\mathrm{d}\gamma \\
&\qquad = \int_{\mathbb{R}^d} \Big(\int_S \min\{1, \|Vy\|^2\}\,\mathrm{d}\gamma\Big)\Big(\int_S \max\{1, \|T\|^2\}\,\mathrm{d}\gamma\Big)\,\nu(\mathrm{d}y) \\
&\qquad = \int_S \max\{1, \|T\|^2\}\,\mathrm{d}\gamma \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|Vy\|^2\}\,\nu(\mathrm{d}y)\Big)\,\mathrm{d}\gamma < \infty.
\end{aligned}$$

In addition,

$$\begin{aligned}
&\int_S \|T\|\Big\|V\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(Vy) - 1_{B_1}(y)\big] Vy\,\nu(\mathrm{d}y)\Big\|\,\mathrm{d}\gamma \\
&\qquad = \int_S \|T\|\,\mathrm{d}\gamma \int_S \Big\|V\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(Vy) - 1_{B_1}(y)\big] Vy\,\nu(\mathrm{d}y)\Big\|\,\mathrm{d}\gamma < \infty.
\end{aligned}$$

Combining the preceding three calculations, we conclude that

$$\int_S \Big\|TV\eta + \int_{\mathbb{R}^d} \big[1_{B_1}(TVy) - 1_{B_1}(y)\big] TVy\,\nu(\mathrm{d}y)\Big\|\,\mathrm{d}\gamma < \infty,$$

and this completes the proof of (i).

(ii) Assume that $\mu \in \mathsf{dom}_{ID}(\Upsilon_V)$, and let γ_T and γ_V denote, respectively, the distributions of T and V with respect to γ. For any z in $\mathbb{R}^d$ we find then by application of Proposition 3.5, the independence assumption and Fubini's theorem that

$$\begin{aligned}
C_{\Upsilon_{TV}(\mu)}(z) &= \int_S C_\mu\big((TV)^* z\big)\,\mathrm{d}\gamma = \int_S C_\mu\big(V^* T^* z\big)\,\mathrm{d}\gamma \\
&= \int_{\mathbb{M}_d(\mathbb{R})\times\mathbb{M}_d(\mathbb{R})} C_\mu\big(B^* A^* z\big)\,(\gamma_T \otimes \gamma_V)(\mathrm{d}A, \mathrm{d}B) \\
&= \int_{\mathbb{M}_d(\mathbb{R})} \Big(\int_{\mathbb{M}_d(\mathbb{R})} C_\mu\big(B^* A^* z\big)\,\gamma_V(\mathrm{d}B)\Big)\,\gamma_T(\mathrm{d}A) \\
&= \int_{\mathbb{M}_d(\mathbb{R})} C_{\Upsilon_V(\mu)}\big(A^* z\big)\,\gamma_T(\mathrm{d}A) = C_{\Upsilon_T(\Upsilon_V(\mu))}(z),
\end{aligned}$$

and this verifies (ii). □

3.4. *Examples.* Throughout this subsection we specialize to the case where $(S, \mathcal{S}, \gamma)$ is a probability space. In this set-up we consider concrete examples of Upsilon transformations corresponding to three naturally considered classes of random matrices; namely a) diagonal matrices with i.i.d. exponentially distributed diagonal elements, b) selfadjoint Gaussian random matrices, and c) Wishart matrices. In each case we calculate the cumulant transform of $\Upsilon_T(\mu)$.

Example 3.14 (Diagonal matrices with exponentially distributed diagonals). The one-dimensional Upsilon transform corresponding to the exponential distribution $\mathrm{e}^{-x} 1_{(0,\infty)}(x)\,\mathrm{d}x$ was studied intensively in e.g. Barndorff-Nielsen and Thorbjørnsen (2004, 2006); Barndorff-Nielsen et al. (2006b). A natural way to generalize this particular transform to higher dimensions is to consider a random matrix $T\colon S \to \mathbb{M}_d(\mathbb{R})$ in the form

$$T = \begin{bmatrix} T_1 & & & 0 \\ & T_2 & & \\ & & \ddots & \\ 0 & & & T_d \end{bmatrix}, \tag{3.38}$$

where $T_1, \dots, T_d$ are i.i.d. random variables on S with distribution $\mathrm{e}^{-x} 1_{(0,\infty)}(x)\,\mathrm{d}x$. Since $\int_S \|T\|^2\,\mathrm{d}\gamma < \infty$, we may, for any measure μ in $\mathcal{ID}(\mathbb{R}^d)$, calculate the cumulant transform of $\Upsilon_T(\mu)$ conveniently using Proposition 3.7.

Denoting by (η, A, ν) the characteristic triplet for μ, we note first that $\mathbb{E}\{T\} = \mathrm{I}_d$, and secondly that the entry at position (j,k) in $\mathbb{E}\{TAT\}$ is given by

$$\mathbb{E}\{\langle ATe_k, Te_j\rangle\} = \mathbb{E}\{\langle AT_k e_k, T_j e_j\rangle\} = \langle Ae_k, e_j\rangle \mathbb{E}\{T_k T_j\} = \begin{cases} 2a_{jk}, & \text{if } j = k \\ a_{jk}, & \text{if } j \neq k, \end{cases}$$

where a_{jk} is the corresponding entry of A, and $\{e_1, \dots, e_d\}$ is the standard basis for $\mathbb{R}^d$. It follows that $\mathbb{E}\{TAT\} = A + \Delta_A$, where Δ_A is the diagonal matrix with the same diagonal entries as A. Note finally for $y = (y_1, \dots, y_d)$ and $u = (u_1, \dots, u_d)$ in $\mathbb{R}^d$ that

$$\mathbb{E}\left\{\mathrm{e}^{\mathrm{i}\langle y, Tu\rangle}\right\} = \prod_{j=1}^d \mathbb{E}\left\{\mathrm{e}^{\mathrm{i}T_j y_j u_j}\right\} = \prod_{j=1}^d (1 - \mathrm{i}y_j u_j)^{-1},$$

and therefore

$$\int_{\mathbb{R}^d} \left(\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle y,Tu\rangle}\} - 1 - \mathrm{i}\,\langle y, \mathbb{E}\{T\}u\rangle\, 1_{B_1}(y)\right)\,\nu(\mathrm{d}y)$$

$$= \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \frac{1}{1-\mathrm{i}y_j u_j} - 1 - \mathrm{i}\,\langle y, u\rangle\, 1_{B_1}(y)\right)\nu(\mathrm{d}y).$$

Combining the calculations above with Proposition 3.7 we conclude that

$$\begin{aligned} C_{\Upsilon_T(\mu)}(u) &= \mathrm{i}\,\langle \eta, u\rangle - \tfrac{1}{2}\,\langle (A+\Delta_A)u, u\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(\prod_{j=1}^{d} \frac{1}{1-\mathrm{i}y_j u_j} - 1 - \mathrm{i}\,\langle y, u\rangle\, 1_{B_1}(y)\right)\nu(\mathrm{d}y), \end{aligned} \tag{3.39}$$

for all u in $\mathbb{R}^d$.

Example 3.15 (Gaussian random matrices). Let T be a $d\times d$ Gaussian random matrix with mean $M \in \mathbb{M}_d(\mathbb{R})$ and covariance $\Sigma \in \mathbb{S}^d$. That is,

$$\mathbb{E}(\mathrm{e}^{\mathrm{itr}(T\Theta^*)}) = \mathrm{e}^{\mathrm{itr}(M\Theta^*)-\frac{1}{2}\mathrm{tr}(\Theta\Sigma\Theta^*)} \qquad (\Theta \in \mathbb{M}_d(\mathbb{R})). \tag{3.40}$$

Then it is well-known (see Gupta and Nagar (2000); Muirhead (1982)) that

$$\mathbb{E}\{T\} = M, \quad \text{and} \quad \mathrm{cov}(T) = \mathrm{I}_d \otimes \Sigma, \tag{3.41}$$

and we say that T has the matrix Gaussian distribution $\mathrm{N}_d(M, \mathrm{I}_d \otimes \Sigma)$. In this case $\Upsilon_T(\mathcal{ID}(\mathbb{R}^d))$ forms a new class of multivariate type G distributions (cf. Marcus (1987) and Rosiński (1991)). Since $\mathbb{E}\{\|T\|^2\} < \infty$ (see Gupta and Nagar (2000); Muirhead (1982)), we may apply Proposition 3.7 to calculate $C_{\Upsilon_T}(\mu)$ for any μ in $\mathcal{ID}(\mathbb{R}^d)$.

Denoting by (η, A, ν) the characteristic triplet of μ, we note first that $\mathbb{E}\{TAT\} = \mathbb{E}\{ZZ^*\}$, where $Z = TA^{1/2}$. Using the characteristic function (3.40), one may show that Z has the Gaussian matrix distribution $\mathrm{N}_d(MA^{1/2}, \mathrm{I}_d \otimes (A^{1/2}\Sigma A^{1/2}))$ (see e.g. Gupta and Nagar (2000)). Assuming that A is invertible, this implies that ZZ^* has the matrix noncentral Wishart distribution $\mathrm{W}_d(d, A^{1/2}\Sigma A^{1/2}, \Omega)$, where $\Omega = A^{-1/2}\Sigma^{-1}M^*MA^{1/2}$. Therefore

$$\mathbb{E}\{TAT\} = \mathbb{E}\{ZZ^*\} = dA^{1/2}\Sigma A^{1/2} + A^{1/2}M^*MA^{1/2}$$

(see e.g. Muirhead (1982, 441-442)). Finally, for any u, z in $\mathbb{R}^d$ note that uz^* is a $d\times d$ matrix of rank 1, and $\langle z, Tu\rangle = \mathrm{tr}(Tuz^*)$. Using (3.40) we find thus that

$$\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle z,Tu\rangle}\} = \mathbb{E}\{\mathrm{e}^{\mathrm{itr}(Tuz^*)}\} = \mathrm{e}^{\mathrm{itr}(Mzu^*)-\frac{1}{2}\mathrm{tr}(uz^*\Sigma zu^*)}, \tag{3.42}$$

and therefore

$$\int_{\mathbb{R}^d} \left(\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle z,Tu\rangle}\} - 1 - \mathrm{i}\,\langle z, \mathbb{E}\{T\}u\rangle\, 1_{B_1}(z)\right)\,\nu(\mathrm{d}z)$$

$$= \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{itr}(Mzu^*)-\frac{1}{2}\mathrm{tr}(uz^*\Sigma zu^*)} - 1 - \mathrm{i}\,\langle z, Mu\rangle\, 1_{B_1}(z)\right)\nu(\mathrm{d}z) \tag{3.43}$$

$$= \int_{\mathbb{R}^d} \left(\mathrm{e}^{\mathrm{i}\langle z,Mu\rangle-\frac{1}{2}\|u\|^2\langle z,\Sigma z\rangle} - 1 - \mathrm{i}\,\langle z, Mu\rangle\, 1_{B_1}(z)\right)\nu(\mathrm{d}z) \tag{3.44}$$

where, to obtain (3.44) from (3.43), we have used the fact that $uz^*\Sigma zu^* = z^*\Sigma zuu^*$, since $z^*\Sigma z$ is a real number. Combining the calculations above with Proposition 3.7 we conclude that

$$\begin{aligned} C_{\Upsilon_T(\mu)}(u) = \mathrm{i}\,\langle M\eta, u\rangle - \frac{1}{2}\left\langle \left[dA^{1/2}\Sigma A^{1/2} + A^{1/2}M^*MA^{1/2}\right]u, u\right\rangle \\ + \int_{\mathbb{R}^d}\left(\mathrm{e}^{\mathrm{i}\langle z, Mu\rangle - \frac{1}{2}\|u\|^2\langle z,\Sigma z\rangle} - 1 - \mathrm{i}\,\langle z, Mu\rangle\, 1_{B_1}(z)\right)\nu(\mathrm{d}z) \end{aligned} \tag{3.45}$$

for all u in $\mathbb{R}^d$. In the case $M = 0$, we have in particular that

$$C_{\Upsilon_T(\mu)}(u) = -\frac{d}{2}\left\langle \Sigma A^{1/2}u, A^{1/2}u\right\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{-\frac{1}{2}\|u\|^2\langle z,\Sigma z\rangle} - 1\right)\nu(\mathrm{d}z), \qquad \left(u \in \mathbb{R}^d\right).$$

If additionally $\Sigma = \mathrm{I}_d$, then the distribution of T is invariant under orthogonal transformations, and we have that

$$C_{\Upsilon_T(\mu)}(u) = -\frac{d}{2}\,\langle Au, u\rangle + \int_{\mathbb{R}^d}\left(\mathrm{e}^{-\frac{1}{2}\|u\|^2\|z\|^2} - 1\right)\nu(\mathrm{d}z)$$

for all u in $\mathbb{R}^d$.

Example 3.16 (Wishart matrices I). In Example 3.14 we considered one possible d-dimensional generalization of the one-dimensional exponential case. Another possibility is to consider the case when T is a $d \times d$ symmetric random matrix with the matrix exponential distribution, i.e. T has the following density with respect to Lebesgue measure on $\mathbb{S}_d$:

$$f_T(X) = c_d \mathrm{e}^{-\mathrm{tr}(X)} 1_{\mathbb{S}_d^+}(X), \qquad (X \in \mathbb{S}_d). \tag{3.46}$$

Here $c_d = 1/\Gamma_d((d+1)/2)$, where Γ_d is the multivariate Gamma function defined by

$$\Gamma_d(\alpha) = \int_{\mathbb{S}_d^+} \mathrm{e}^{-\mathrm{tr}(X)} \det(X)^{\alpha-(d+1)/2}\mathrm{d}X \tag{3.47}$$

for all complex numbers α, such that $\mathsf{Re}(\alpha) > (d-1)/2$. The matrix exponential distribution is equal to the Wishart distribution $\mathrm{W}_d(d+1, \frac{1}{2}\mathrm{I}_d)$. For any $\alpha > 0$ we consider more generally a symmetric, positive semi-definite $d \times d$ random matrix T_α carrying the Wishart distribution $\mathrm{W}_d(\alpha, \frac{1}{2}\mathrm{I}_d)$. We refer to the books Eaton (1983), Gupta and Nagar (2000) or Muirhead (1982) for an introduction to the Wishart distribution. Here we will use the following three properties:

(i) Let $\mathrm{G}_d = \{0, 1, \dots, d-1\} \cup (d-1, \infty)$ be the Gindikin set. For $\alpha \in \mathrm{G}_d$, the Fourier transform of T_α is given by

$$\mathbb{E}(\mathrm{e}^{\mathrm{i}\mathrm{tr}(T_\alpha\Theta)}) = \det(\mathrm{I}_d - \mathrm{i}\Theta)^{-\alpha/2}, \qquad (\Theta \in \mathbb{M}_d(\mathbb{R})). \tag{3.48}$$

(ii) The mean and covariance of T_α are given by

$$\mathbb{E}\{T_\alpha\} = \frac{\alpha}{2}\mathrm{I}_d, \quad \text{and} \quad \mathrm{cov}(T_\alpha) = \frac{\alpha}{2}\mathrm{I}_d \otimes \mathrm{I}_d. \tag{3.49}$$

(iii) The matrix T_α is nonsingular with probability one, if and only if $\alpha \geq d$. In this case it has a density with respect to Lebesgue measure on $\mathbb{S}_d$ given by

$$f_{T_\alpha}(X) = \mathrm{c}_{d,\alpha}\det(X)^{(\alpha-d-1)/2}\mathrm{e}^{-\mathrm{tr}(X)}1_{\mathbb{S}_d^+}(X), \qquad (X \in \mathbb{S}_d), \tag{3.50}$$

where $c_{d,\alpha} = 1/\Gamma_d(\frac{\alpha}{2})$.

Since $\mathbb{E}\{\|T_\alpha\|^2\} < \infty$ (see Muirhead (1982, 87-90)), we may once again apply Proposition 3.7 to calculate the cumulant transform of $\Upsilon_{T_\alpha}(\mu)$ for any μ in $\mathcal{ID}(\mathbb{R}^d)$.

According to (ii) we have that $\mathbb{E}\{T_\alpha\} = \frac{\alpha}{2}\mathrm{I}_d$. Denoting as usual the characteristic triplet for μ by (η, A, ν), we put $\widetilde{A}_\alpha = \mathbb{E}\{T_\alpha A T_\alpha\}$ and we consider the matrix $W_\alpha = A^{1/2}T_\alpha A^{1/2}$. Then W_α has the Wishart distribution $\mathrm{W}_d(\alpha, A/2)$, and furthermore

$$\mathbb{E}\{W_\alpha^2\} = \mathbb{E}\{A^{1/2}T_\alpha A T_\alpha A^{1/2}\} = A^{1/2}\mathbb{E}\{T_\alpha A T_\alpha\}A^{1/2} = A^{1/2}\widetilde{A}_\alpha A^{1/2}.$$

From Gupta and Nagar (2000, pp 120) we have

$$\widetilde{A} = \frac{\alpha}{4}\left(2\mathrm{tr}(A)\mathrm{I}_d + (\alpha+1)A\right)$$

and hence

$$\mathbb{E}\{W_\alpha^2\} = \frac{\alpha}{4}\left(2\mathrm{tr}(A)A + (\alpha+1)A^2\right).$$

Finally, for any u, z in $\mathbb{R}^d$ we recall that uz^* is a $d \times d$ matrix of rank 1 and that $\langle z, T_\alpha u\rangle = \mathrm{tr}(T_\alpha u z^*)$. Using (3.48) and that the eigenvalues for $\mathrm{I}_d - \mathrm{i}uz^*$ are $1 - \mathrm{i}\langle z, u\rangle$ and 1 (with multiplicity $d-1$) we find that Gupta and Nagar (2000, pp 120)

$$\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle z, T_\alpha u\rangle}\} = \det(\mathrm{I}_d - \mathrm{i}uz^*)^{-\alpha/2} = (1 - \mathrm{i}\langle z, u\rangle)^{-\alpha/2}, \qquad (u \in \mathbb{R}^d).$$

Therefore

$$\begin{aligned}&\int_{\mathbb{R}^d}\Big(\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle z, T_\alpha u\rangle}\} - 1 - \mathrm{i}\langle z, \mathbb{E}\{T_\alpha\}u\rangle 1_{B_1}(z)\Big)\,\nu(\mathrm{d}z)\\ &= \int_{\mathbb{R}^d}\Big((1 - \mathrm{i}\langle z, u\rangle)^{-\alpha/2} - 1 - \frac{\alpha}{2}\mathrm{i}\langle z, u\rangle 1_{B_1}(z)\Big)\,\nu(\mathrm{d}z)\end{aligned}$$

for any u in $\mathbb{R}^d$. Combining the calculations above with Proposition 3.7 we conclude that

$$\begin{aligned}C_{\Upsilon_{T_\alpha}(\mu)}(u) &= \mathrm{i}\frac{\alpha}{2}\langle \eta, u\rangle - \frac{1}{2}\langle \widetilde{A}_\alpha u, u\rangle\\ &\quad + \int_{\mathbb{R}^d}\Big(\frac{1}{(1 - \mathrm{i}\langle y, u\rangle)^{\alpha/2}} - 1 - \mathrm{i}\frac{\alpha}{2}\langle y, u\rangle 1_{B_1(y)}\Big)\nu(\mathrm{d}y), \qquad \left(u \in \mathbb{R}^d\right),\end{aligned}$$

where $\widetilde{A}_\alpha$ satisfies that

$$A^{1/2}\widetilde{A}_\alpha A^{1/2} = \mathbb{E}\{W_\alpha^2\}, \quad \text{with} \quad W_\alpha \sim \mathrm{W}_d(\alpha, A/2). \tag{3.51}$$

If $A = 0$, then $\widetilde{A}_\alpha = \mathbb{E}\{T_\alpha A T_\alpha\} = 0$ too, while for A non-singular we have that $\widetilde{A}_\alpha = A^{-1/2}\mathbb{E}\{W_\alpha^2\}A^{-1/2}$.

4. Continuity

In this section we study continuity of $\Upsilon_T(\mu)$ in μ (for fixed T) and in T (for fixed μ). We start by phrasing the following well-known lemma (see e.g. the proof of Barndorff-Nielsen et al. (2006b, Proposition 2.4(v))).

Lemma 4.1. *Let (μ_n) be a sequence of measures from $\mathcal{ID}(\mathbb{R}^d)$, and for each n let (η_n, A_n, ν_n) be the characteristic triplet for μ_n. Then (μ_n) is precompact, if and only if the following four conditions are satisfied:*

(a) $\sup_{n\in\mathbb{N}}\|A_n\| < \infty$.
(b) $\sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\nu_n(\mathrm{d}x) < \infty$.

(c) $\forall \epsilon > 0 \; \exists K > 0\colon \; \sup_{n\in\mathbb{N}} \nu_n\big(\{\|x\| > K\}\big) < \epsilon$.
(d) $\sup_{n\in\mathbb{N}} \|\eta_n\| < \infty$.

Proposition 4.2. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping, such that*

$$\gamma(\{T \neq 0\}) < \infty, \quad \textit{and} \quad \int_S \|T\|^2 \, \mathrm{d}\gamma < \infty. \tag{4.1}$$

Then the mapping $\Upsilon_T\colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ is continuous with respect to weak convergence: If (μ_n) is a sequence of measures from $\mathcal{ID}(\mathbb{R}^d)$ converging weakly to some measure μ (necessarily) from $\mathcal{ID}(\mathbb{R}^d)$, then $\Upsilon_T(\mu_n)$ converges weakly to $\Upsilon_T(\mu)$.

Proof: Consider a sequence (μ_n) of measures from $\mathcal{ID}(\mathbb{R}^d)$ converging weakly to a measure μ from $\mathcal{ID}(\mathbb{R}^d)$, and for each n in $\mathbb{N}$ let (η_n, A_n, ν_n) be the characteristic triplet for μ_n. According to Lemma 4.1 we then have that

$$H := \sup_{n\in\mathbb{N}} \|\eta_n\| < \infty, \;\; A := \sup_{n\in\mathbb{N}} \|A_n\| < \infty, \;\; R := \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \min\{1, \|x\|^2\} \, \nu_n(\mathrm{d}x) < \infty,$$

and by Lemma 7.7 in Sato (1999) also that

$$\lim_{n\to\infty} C_{\mu_n}(y) = C_\mu(y) \quad \text{for all } y \text{ in } \mathbb{R}^d. \tag{4.2}$$

Using now formula (3.26) (on μ_n rather than μ), it follows that

$$\begin{aligned} &|C_{\mu_n}(T^*z)| \\ &\le \|\eta_n\|\|T\|\|z\| + \tfrac{1}{2}\|A_n\|\|T\|^2\|z\|^2 + \big(2 + \tfrac{1}{2}\|T\|^2\|z\|^2\big) \int_{\mathbb{R}^d} \min\{1, \|x\|^2\} \, \nu_n(\mathrm{d}x) \\ &\le H\|T\|\|z\| + \tfrac{1}{2}A\|T\|^2\|z\|^2 + \big(2 + \tfrac{1}{2}\|T\|^2\|z\|^2\big)R \end{aligned} \tag{4.3}$$

for all z in $\mathbb{R}^d$. Fixing z in $\mathbb{R}^d$ and denoting by G_z the resulting expression in (4.3), it follows for all n in $\mathbb{N}$ that

$$|C_{\mu_n}(T^*z)| = |C_{\mu_n}(T^*z)| 1_{\{T\neq 0\}} \le G_z 1_{\{T \neq 0\}},$$

and the assumptions (4.1) imply that $\int_{\{T\neq 0\}} G_z \, \mathrm{d}\gamma < \infty$. Hence by (4.2) and dominated convergence, we may conclude that

$$\lim_{n\to\infty} C_{\Upsilon_T(\mu_n)}(z) = \lim_{n\to\infty} \int_S C_{\mu_n}(T^*z) \, \mathrm{d}\gamma = \int_S C_\mu(T^*z) \, \mathrm{d}\gamma = C_{\Upsilon_T(\mu)}(z).$$

Clearly this implies that the characteristic function of $\Upsilon_T(\mu_n)$ converges pointwise, as $n \to \infty$, to that of $\Upsilon_T(\mu)$, which yields the desired conclusion. □

Proposition 4.3. *Let $(S, \mathcal{S}, \gamma)$ be a finite measure space, and let $T, T_1, T_2, T_3, \ldots$ be measurable mappings from $(S, \mathcal{S})$ into $\mathbb{M}_d(\mathbb{R})$ such that*

(a) $\lim_{n\to\infty} \|T_n(s) - T(s)\| = 0$ *for γ-almost all s in S.*
(b) *There exists a measurable function $g\colon S \to [0, \infty)$ such that*

$$\int_S g(s) \, \gamma(\mathrm{d}s) < \infty, \quad \textit{and} \quad \|T_n\|^2 \le g \textit{ almost everywhere for all } n.$$

Then $\int_S \|T(s)\|^2 \, \gamma(\mathrm{d}s) < \infty$, $\int_S \|T_n(s)\|^2 \, \gamma(\mathrm{d}s) < \infty$ for all n and

(i) $\Upsilon_{T_n}(\mu) \xrightarrow{\mathrm{w}} \Upsilon_T(\mu)$ *as $n \to \infty$ for all μ in $\mathcal{ID}(\mathbb{R}^d)$.*
(ii) $\Upsilon_{T_n - T_m}(\mu) \xrightarrow{\mathrm{w}} \delta_0$ *as $n, m \to \infty$ for all μ in $\mathcal{ID}(\mathbb{R}^d)$.*

Proof: The assumptions (a) and (b) clearly imply that the integrals $\int_S \|T(s)\|^2\gamma(\mathrm{d}s)$ and $\int_S \|T_n(s)\|^2\,\gamma(\mathrm{d}s)$ are finite.

(i) Let μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν). It follows from (a) and the continuity of C_μ that for all s outside a γ null-set we have that $C_\mu(T_n(s)^*z) \to C_\mu(T(s)^*z)$ as $n \to \infty$ for all z in $\mathbb{R}^d$. Moreover, we find by application of (3.26) that

$$|C_\mu(T_n(s)^*z)| \le$$

$$\|\eta\|\|T_n(s)\|\|z\| + \tfrac{1}{2}\|A\|\|T_n(s)\|^2\|z\|^2 + (2 + \tfrac{1}{2}\|T_n(s)\|^2\|z\|^2)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x)$$

$$\le \|\eta\|\|z\|g(s)^{1/2} + \tfrac{1}{2}\|A\|\|z\|^2 g(s) + (2 + \tfrac{1}{2}\|z\|^2 g(s))\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x)$$

for almost all s and for all z in $\mathbb{R}^d$. Since γ is a finite measure, it follows thus by dominated convergence that

$$C_{\Upsilon_{T_n}(\mu)}(z) = \int_S C_\mu(T_n(s)^*z)\,\gamma(\mathrm{d}s) \underset{n\to\infty}{\longrightarrow} \int_S C_\mu(T(s)^*z)\,\gamma(\mathrm{d}s) = C_{\Upsilon_T(\mu)}(z)$$

for all z in $\mathbb{R}^d$, and this implies (i).

(ii) Using again (3.26) we find for any n, m in $\mathbb{N}$ (suppressing s in the notation) that

$$\begin{aligned}
&|C_\mu((T_n - T_m)^*z)| \\
&\le \|\eta\|\|T_n - T_m\|\|z\| + \tfrac{1}{2}\|A\|\|T_n - T_m\|^2\|z\|^2 \\
&\qquad + (2 + \tfrac{1}{2}\|T_n - T_m\|^2\|z\|^2)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x) \\
&\le 2\|\eta\|\|z\|g^{1/2} + 2\|A\|\|z\|^2 g + 2(1 + g\|z\|^2)\int_{\mathbb{R}^d}\min\{1, \|x\|^2\}\,\nu(\mathrm{d}x).
\end{aligned} \tag{4.4}$$

If we assume that $C_{\Upsilon_{T_n - T_m}(\mu)}(z)$ does not converge to 0 as $n, m \to \infty$ for some z in $\mathbb{R}^d$, then we may choose a positive number ϵ and a sequence $n_1 < n_2 < n_3 < \cdots$ of positive integers, such that

$$\left|C_{\Upsilon_{T_{n_{2k}} - T_{n_{2k-1}}}(\mu)}(z)\right| \ge \epsilon \quad \text{for all } k \text{ in } \mathbb{N}. \tag{4.5}$$

However, (a) implies that $(T^*_{n_{2k}} - T^*_{n_{2k-1}})z \to 0$ almost everywhere, as $k \to \infty$, and together with (4.4) and dominated convergence this implies that

$$C_{\Upsilon_{T_{n_{2k}} - T_{n_{2k-1}}}(\mu)}(z) = \int_S C_\mu((T_{n_{2k}} - T_{n_{2k-1}})^*z)\,\mathrm{d}\gamma \underset{k\to\infty}{\longrightarrow} 0,$$

which contradicts (4.5). Thus, $C_{\Upsilon_{T_n - T_m}(\mu)}(z) \to 0$ as $n, m \to \infty$ for all z in $\mathbb{R}^d$, and this implies (ii). □

5. Regularisation

Consider a finite measure space $(S, \mathcal{S}, \gamma)$. In this section we establish that many of the measurable mappings $T\colon S \to \mathbb{M}_d(\mathbb{R})$ considered in the foregoing give rise to Upsilon transforms which have a regularising effect. Thus, if e.g. the measure $\gamma \circ T^{-1}$ on $\mathbb{M}_d(\mathbb{R})$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{M}_d(\mathbb{R})$, then all Lévy measures in the range of Υ_T^0 are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. This is a special case of the results established in Subsection 5.2 (see Corollary 5.8). We start however in Subsection 5.1 by considering the simple but instructive situation, where $S = \mathbb{R}^d$, and T has the form

$$T(s_1, \dots, s_d) = \begin{bmatrix} s_1 & & & 0 \\ & s_2 & & \\ & & \ddots & \\ 0 & & & s_n \end{bmatrix}, \qquad ((s_1, \dots, s_d) \in \mathbb{R}^d). \tag{5.1}$$

In this case we obtain rather specific expressions for the achieved densities. The considerations in Subsection 5.1-5.2 are as such unrelated to the question of whether $\Upsilon_T^0(\nu)$ is a Lévy measure or not. The results in these subsections are thus generally freed from the assumption that $\nu \in \mathsf{dom}_L(\Upsilon_T^0)$. In Subsection 5.3 we establish another regularising feature of many Upsilon transformations, namely that Υ_T has a decreasing effect on the Blumenthal-Getoor index of an infinitely divisible distribution on $\mathbb{R}^d$. For stable distributions, however, the index is preserved.

5.1. *The case of Diagonal Matrices.* Throughout this subsection we consider a finite measure γ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$ and the mapping $T\colon \mathbb{R}^d \to \mathbb{M}_d(\mathbb{R})$ given by (5.1).

Proposition 5.1. *Let $(S, \mathcal{S}, \gamma)$ and T be as described above, and assume that γ has a density $g_\gamma\colon \mathbb{R}^d \to [0, \infty)$ with respect to Lebesgue measure on $\mathbb{R}^d$. Then for any Lévy measure ν on $\mathbb{R}^d$, satisfying the condition:*

$$\nu\big(\mathbb{R}^d \setminus (\mathbb{R} \setminus \{0\} \times \cdots \times \mathbb{R} \setminus \{0\})\big) = 0, \tag{5.2}$$

the measure $\tilde{\nu} = \Upsilon_\gamma^0(\nu)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ with a density given by

$$r_{\tilde{\nu}}(y_1, \dots, y_d) = \int_{(\mathbb{R}\setminus\{0\})^d} |u_1 \cdots u_d|^{-1} g_\gamma(y_1/u_1, \dots, y_d/u_d)\, \nu(\mathrm{d}u_1, \dots, \mathrm{d}u_d) \tag{5.3}$$

for any $(y_1, \dots, y_d)$ in $\mathbb{R}^d$.

Proof: Let ν be a Lévy measure on $\mathbb{R}^d$ satisfying condition (5.2), and put $\tilde{\nu} = \Upsilon_T^0(\nu)$. For any $s = (s_1, \dots, s_d)$ in $\mathbb{R}^d$, we identify $T(s)$ with the linear mapping $T(s)\colon \mathbb{R}^d \to \mathbb{R}^d$ defined by

$$T(s)(y_1, \dots, y_d) = (s_1 y_1, \dots, s_d y_d), \qquad ((y_1, \dots, y_d) \in \mathbb{R}^d).$$

Note then that $T(s)u = T(u)s$ for any s, u in $\mathbb{R}^d$, and also that $\det(T(s)^{-1}) = |s_1 \cdots s_d|^{-1}$ for any s in $(\mathbb{R} \setminus \{0\})^d$.

Now, let B be an arbitrary Borel set in $\mathbb{R}^d$. Using Tonelli's Theorem, we then find that

$$\begin{aligned} &\tilde{\nu}(B) \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)u)\, \nu(\mathrm{d}u) \Big)\, \gamma(\mathrm{d}s) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(u)s)\, \gamma(\mathrm{d}s) \Big)\, \nu(\mathrm{d}u) \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(u)s) g_\gamma(s)\, \mathrm{d}s_1 \cdots \mathrm{d}s_d \Big)\, \nu(\mathrm{d}u). \end{aligned}$$

Note here that for u in $(\mathbb{R} \setminus \{0\})^d$ we have that $T(u)s \neq 0$, whenever $s \neq 0$. Hence by the assumption (5.2), and the transformation theorem for Lebesgue measure, it follows that

$$\begin{aligned}
\tilde{\nu}(B) &= \int_{(\mathbb{R}\setminus\{0\})^d} \Big(\int_{\mathbb{R}^d} 1_B(T(u)s) g_\gamma(s)\, \mathrm{d}s_1 \cdots \mathrm{d}s_d \Big)\, \nu(\mathrm{d}u) \\
&= \int_{(\mathbb{R}\setminus\{0\})^d} \Big(\int_{\mathbb{R}^d} 1_B(y) g_\gamma(T(u)^{-1}(y)) |\det(T(u)^{-1})|\, \mathrm{d}y_1 \cdots \mathrm{d}y_d \Big)\, \nu(\mathrm{d}u) \\
&= \int_{\mathbb{R}^d} 1_B(y) \Big(\int_{(\mathbb{R}\setminus\{0\})^d} g_\gamma(y_1/u_1, \dots, y_d/u_d) |u_1 \cdots u_d|^{-1}\, \nu(\mathrm{d}u) \Big)\, \mathrm{d}y_1 \cdots \mathrm{d}y_d,
\end{aligned}$$

and the proposition follows. □

Remarks 5.2. Consider the setting of Proposition 5.1.

(1) The conclusion of Proposition 5.1 does not hold without the assumption (5.2). Indeed, assume that γ is a non-zero, finite measure on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$ which is absolutely continuous with respect to two-dimensional Lebesgue measure, and consider further a Lévy measure ν on $\mathbb{R}^2$, such that e.g. $\nu(\{0\} \times \mathbb{R}) > 0$. Note then that for any s in $(\mathbb{R} \setminus \{0\})^2$ and u in $\mathbb{R}^2$ we have that

$$T(s)u \in \{0\} \times (\mathbb{R} \setminus \{0\}) \iff u \in \{0\} \times (\mathbb{R} \setminus \{0\}),$$

and therefore

$$\begin{aligned}
\tilde{\nu}(\{0\} \times \mathbb{R}) &= \int_{\mathbb{R}^2} \Big(\int_{\mathbb{R}^2} 1_{\{0\}\times(\mathbb{R}\setminus\{0\})}(T(s)u)\, \nu(\mathrm{d}u) \Big)\, \gamma(\mathrm{d}s) \\
&\geq \int_{(\mathbb{R}\setminus\{0\})^2} \Big(\int_{\mathbb{R}^2} 1_{\{0\}\times(\mathbb{R}\setminus\{0\})}(u)\, \nu(\mathrm{d}u) \Big)\, \gamma(\mathrm{d}s) \\
&= \nu(\{0\} \times \mathbb{R}) \gamma((\mathbb{R} \setminus \{0\})^2) = \nu(\{0\} \times \mathbb{R}) \gamma(\mathbb{R}^2),
\end{aligned}$$

where the last equality uses the absolute continuity of γ. Since $\gamma \neq 0$, we conclude that $\tilde{\nu}(\{0\} \times \mathbb{R}) > 0$, which clearly implies that $\tilde{\nu}$ is not absolutely continuous with respect to 2-dimensional Lebesgue measure.

(2) Maintaining assumption (5.2), we may consider the transformation β of ν under the mapping

$$\varphi(u_1, \dots, u_d) = (u_1^{-1}, \dots, u_d^{-1}), \quad ((u_1, \dots, u_d) \in (\mathbb{R} \setminus \{0\})^d). \tag{5.4}$$

Then formula (5.3) may be re-written to the form:

$$r_{\tilde{\nu}}(y_1, \dots, y_d) = \int_{(\mathbb{R}\setminus\{0\})^d} |v_1 \cdots v_d| g_\gamma(y_1 v_1, \dots, y_d v_d)\, \beta(\mathrm{d}v_1, \dots, \mathrm{d}v_d). \tag{5.5}$$

In particular, if ν has a density r_ν with respect to d-dimensional Lebesgue measure, then, by the transformation theorem for Lebesgue measure, β has density

$$(v_1, \dots, v_d) \mapsto r_\nu(v_1^{-1}, \dots, v_d^{-1}) v_1^{-2} \cdots v_d^{-2}, \qquad ((v_1, \dots, v_d) \in (\mathbb{R} \setminus \{0\})^d),$$

with respect to d-dimensional Lebesgue measure. Hence (5.3) becomes

$$r_{\tilde{\nu}}(y_1, \dots, y_d) = \int_{(\mathbb{R}\setminus\{0\})^d} |v_1 \cdots v_d|^{-1} r_\nu(v_1^{-1}, \dots, v_d^{-1}) g_\gamma(y_1 v_1, \dots, y_d v_d)\, \mathrm{d}v_1 \cdots \mathrm{d}v_d.$$

(3) From the key formula (5.5) we have that the integral of $r_{\tilde{\nu}}(y)$ equals $\nu(\mathbb{R}^d)$ in the case where γ is a probability measure. It follows that if ν is a probability measure too, then the Lévy process generated by $r_{\tilde{\nu}}$ is a compound Poisson process. In particular, if β is the delta measure at a point $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d_+$ then the innovations of the compound Poisson are distributed according to the probability density $\lambda_1 \cdots \lambda_d g(\lambda_1 y_1, \ldots, \lambda_d y_d)$.

Proposition 5.3. *Let $T_1, \ldots, T_d$ be i.i.d. random variables with common distribution $e^{-x} 1_{[0,\infty)}(x)\,dx$, and let T denote the corresponding random $d \times d$ diagonal matrix. Then for any Lévy measure ν on $\mathbb{R}^d$, satisfying condition (5.2), the Lévy measure $\tilde{\nu} = \Upsilon^0_T(\nu)$ is absolutely continuous with respect to Lebesgue measure, and the density $r_{\tilde{\nu}}$ is given by*

$$r_{\tilde{\nu}}(y_1, \ldots, y_d) = \int_{(\mathbb{R}\setminus\{0\})^d} |v_1 \cdots v_d| e^{-(y_1 v_1 + \cdots + y_d v_d)} \prod_{j=1}^{d} 1_{[0,\infty)}(v_j y_j)\, \beta(dv_1, \ldots, dv_d), \tag{5.6}$$

for any $(y_1, \ldots, y_d)$ in $\mathbb{R}^d$, and where β is the transformation of ν under the mapping φ given in (5.4). In particular Υ^0_T is injective on the class of Lévy measures satisfying (5.2).

Proof: Formula (5.6) follows immediately from the general formula (5.5). To establish the injectivity statement we consider, for any tuple $\ell = (\ell_1, \ldots, \ell_d)$ in $\{1, 2\}^d$ the set

$$V_\ell := \{(v_1, \ldots, v_d) \in (\mathbb{R} \setminus \{0\})^d \mid \mathsf{sign}(v_j) = (-1)^{\ell_j},\ j = 1, \ldots, d\}.$$

Then let $T_\ell \colon \mathbb{R}^d \to \mathbb{R}^d$ be the unique orthogonal linear transformation that maps V_ℓ onto $(0,\infty)^d$ (and vice versa). For any $z = (z_1, \ldots, z_d)$ in $(0,\infty)^d$, it follows then from (5.6) that

$$r_{\tilde{\nu}}(T_\ell(z)) = \int_{(\mathbb{R}\setminus\{0\})^d} |v_1 \cdots v_d| e^{-\langle z, T_\ell(v)\rangle} 1_{V_\ell}(v)\, \beta(dv) = \int_{(0,\infty)^d} e^{-\langle z, w\rangle} \varpi_\ell(dw)$$

where ϖ_ℓ is the transformation by T_ℓ of the measure $\omega_\ell(dv) = |v_1 \cdots v_d| 1_{V_\ell}(v)\, \beta(dv)$. In particular ϖ_ℓ is concentrated on $(0,\infty)^d$, and the above calculation identifies $r_{\tilde{\nu}} \circ T_\ell$ with the Laplace transform of ϖ_ℓ. Since ν is a Lévy measure, this Laplace transform is finite for all z in $(0,\infty)^d$, and hence it determines ϖ_ℓ uniquely. Thus, $r_{\tilde{\nu}}$ determines uniquely ω_l for all ℓ; whence β and therefore also ν (recall that β and ν have no mass outside $\bigcup_{\ell \in \{1,2\}^d} V_\ell$ by assumption). □

5.2. *General random matrices.* Throughout this subsection we consider a finite measure space $(S, \mathcal{S}, \gamma)$, and for convenience (and without loss of generality) we shall assume that γ is a probability measure. Then $T \colon S \to \mathbb{M}_d(\mathbb{R})$ may be referred to as a random matrix, and we can freely apply the usual convenient probability terminology.

Proposition 5.4. *Let T be a random $d \times d$-matrix on the probability space $(S, \mathcal{S}, \gamma)$, and let $\mathcal{U}$ be a k-dimensional subspace of $\mathbb{M}_d(\mathbb{R})$ such that $\gamma(T \in \mathcal{U}) = 1$. Assume in addition that the distribution of T has a density with respect to Lebesgue measure on $\mathcal{U}$.*

Consider further a non-zero Lévy measure ν on $\mathbb{R}^d$ such that

$$\nu\big(\{y \in \mathbb{R}^d \mid \dim(\mathcal{U}y) < d\}\big) = 0, \tag{5.7}$$

where, for any y in $\mathbb{R}^d$, we use the notation: $\mathcal{U}y = \{Uy \mid U \in \mathcal{U}\} \subseteq \mathbb{R}^d$.

Then the measure $\Upsilon_T^0(\nu)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$ with a density given by

$$\tilde{r}(z) = \int_G g_y(z)\,\nu(\mathrm{d}y), \qquad (z \in \mathbb{R}^d),$$

where $G = \{y \in \mathbb{R}^d \mid \dim(\mathcal{U}y) = d\}$, and, for each y in G, g_y is a density of the d-dimensional random vector Ty with respect to d-dimensional Lebesgue measure (cf. Lemmas 5.5-5.6 below).

To prove Proposition 5.4 we need a few preliminary results.

Lemma 5.5. *Let k and m be positive integers, and let $L\colon \mathbb{R}^{k+m} \to \mathbb{R}^k$ be a surjective linear transformation. Let further Y be a $(k+m)$-dimensional random vector, and assume that the distribution of Y has a density $f_Y\colon \mathbb{R}^{k+m} \to [0,\infty)$ with respect to Lebesgue measure on $\mathbb{R}^{k+m}$.*

Then the k-dimensional random vector LY is again absolutely continuous with respect to k-dimensional Lebesgue measure with density

$$f_{LY}(z) = \int_{\mathbb{R}^m} \frac{1}{|\det(A)|} f_Y\left(A^{-1}\begin{bmatrix} z \\ y \end{bmatrix}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_m, \qquad (z \in \mathbb{R}^k), \tag{5.8}$$

where $y = (y_1, \dots, y_m)$, and A is an invertible $(k+m)\times(k+m)$-matrix whose first k rows equal the rows of the $k \times (k+m)$ matrix corresponding to L.

Proof: This is well-known, but we include a proof for convenience: We identify L with the corresponding $k \times (k+m)$-matrix. Since L is assumed surjective, the k rows of L are linearly independent, so we may extend them to a basis for $\mathbb{R}^{k+m}$ by adding suitable vectors $h_1, \dots, h_m$ from $\mathbb{R}^{k+m}$. Then let A be the $(k+m)\times(k+m)$-matrix whose first k rows are those of L, and whose last m rows are $h_1, \dots, h_m$. Then A is invertible, so by the usual (linear) transformation theorem for Lebesgue measure, the random vector AY has the density

$$f_{AY}(y) = \frac{1}{|\det(A)|} f_Y(A^{-1}y), \qquad (y \in \mathbb{R}^{k+m}),$$

with respect to Lebesgue measure on $\mathbb{R}^{k+m}$. As the first k-rows of AY form the random vector LY, it follows that LY has the density (5.8). $\square$

Lemma 5.6. *Let $(W, \mathcal{E})$ be a measurable space, let k, m be positive integers, and let $\Phi\colon W \to \mathbb{M}_{k,k+m}(\mathbb{R})$ be an $\mathcal{E}$-$\mathcal{B}(\mathbb{M}_{k,k+m}(\mathbb{R}))$-measurable mapping such that $\Phi(w)$ has rank k for all w. Let further Y be a $(k+m)$-dimensional random vector, and assume that the distribution of Y has a density $f_Y\colon \mathbb{R}^{k+m} \to [0,\infty)$ with respect to Lebesgue measure on $\mathbb{R}^{k+m}$. For each w in W, let g_w denote the density of the k -dimensional random vector $\Phi(w)Y$ given in Lemma 5.5.*

Then g_w may be chosen such that the mapping $(w,z) \mapsto g_w(z)\colon W \times \mathbb{R}^k \to [0,\infty)$ is measurable with respect to the product σ-algebra $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^k)$.

Proof: For each w in W we have that

$$g_w(z) = \int_{\mathbb{R}^m} \frac{1}{|\det(A_w)|} f_Y\left(A_w^{-1}\begin{bmatrix} z \\ y \end{bmatrix}\right) \mathrm{d}y_1 \cdots \mathrm{d}y_m, \qquad (z \in \mathbb{R}^k), \tag{5.9}$$

where A_w is obtained by extending $\Phi(w)$ to an invertible $(k+m)\times(k+m)$-matrix by adding suitable rows. According to Lemma 5.7 below (applied to $\Phi(w)^*$) the mapping $w \mapsto A_w$ can be chosen $\mathcal{E}$-measurable on W. Since the mapping $A \mapsto A^{-1}$ is a homeomorphism on the group of invertible $(k+m) \times (k+m)$-matrices, this implies that the mapping

$$(w, z, y) \mapsto \frac{1}{|\det(A_w)|} f_Y\left(A_w^{-1}\begin{bmatrix} z \\ y \end{bmatrix}\right) : W \times \mathbb{R}^{k+m} \to [0, \infty)$$

is measurable with respect to the product σ-algebra $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^{k+m})$. It follows subsequently from Tonelli's Theorem that the right hand side of (5.9) is an $\mathcal{E} \otimes \mathcal{B}(\mathbb{R}^k)$-measurable function of (w, z). □

Lemma 5.7. *Let $(W, \mathcal{E})$ be a measurable space, let k, m be positive integers, and let $\Phi\colon W \to \mathbb{M}_{k+m,k}(\mathbb{R})$ be an $\mathcal{E}$-$\mathcal{B}(\mathbb{M}_{k+m,k}(\mathbb{R}))$-measurable mapping such that $\Phi(w)$ has rank k for all w in W. Then there exists an $\mathcal{E}$-$\mathcal{B}(\mathbb{M}_{k+m,m}(\mathbb{R}))$ -measurable mapping $\Psi\colon W \to \mathbb{M}_{k+m,m}(\mathbb{R})$ such that the $(k+m)\times(k+m)$-matrix $[\Phi(w) \mid \Psi(w)]$ is invertible for all w.*

Proof: For each w in W, let $\mathcal{U}_w$ denote the k-dimensional subspace of $\mathbb{R}^{k+m}$ spanned by the columns of $\Phi(w)$. The orthogonal projection onto $\mathcal{U}_w$ is then given by

$$E(w) = \Phi(w)[\Phi(w)^*\Phi(w)]^{-1}\Phi(w)^*,$$

which is $\mathcal{E}$-measurable in w. The columns of $F(w) := \mathbf{I}_n - E(w)$ will then span $\mathcal{U}_w^\perp$. Let $F_1(w), \dots, F_{k+m}(w)$ denote the columns of $F(w)$, which are clearly $\mathcal{E}$-measurable in w.

Consider next the function $r\colon \mathbb{R}^d \to \mathbb{R}^d$ given by

$$r(x) = \begin{cases} \|x\|^{-2}x, & \text{if } x \neq 0, \\ 0, & \text{if } x = 0, \end{cases}$$

and note that r is a Borel-function. Then define column-vectors $Q_1(w), \dots, Q_{k+m}(w)$ recursively as follows:

$$Q_1(w) = F_1(w), \quad \text{and} \quad Q_j(w) = F_j(w) - \sum_{k=1}^{j-1}\langle F_j(w), Q_k(w)\rangle r(Q_k(w)), \quad (j \geq 2),$$

and note that these vectors are orthogonal and that they span $\mathcal{U}_w^\perp$ (in particular k of them must equal 0). Moreover it follows by induction that $Q_1, \dots, Q_{k+m}$ are $\mathcal{E}$-measurable functions of w. Next define $\tau_1, \dots, \tau_m\colon W \to \{1, 2, \dots, k+m\}$ recursively as follows:

$$\tau_1(w) = \min\{j \in \{1, \dots, k+m\} \mid Q_j(w) \neq 0\},$$

and

$$\tau_j(w) = \min\{\tau_{j-1} < j \leq k+m \mid Q_j(w) \neq 0\}, \quad (j \geq 2).$$

It follows then by standard "stopping-time arguments" that $\tau_1, \dots, \tau_m$ are $\mathcal{E}$-measurable functions of w. We finally define

$$R_j(w) = Q_{\tau_j(w)}(w), \qquad (w \in \mathbb{R}^d),$$

for any j in $\{1, \dots, m\}$, and we note that each R_j is an $\mathcal{E}$-measurable function of w, since

$$\{R_j \in B\} = \bigcup_{k=1}^{k+m}\{\tau_j = k\} \cap \{Q_k \in B\}$$

for any Borel-set B in $\mathbb{R}^{k+m}$. In addition $R_1(w), \dots, R_m(w)$ span $\mathcal{U}_w^{\perp}$, so it follows that if we define $\Psi\colon W \to \mathbb{M}_{k+m,m}(\mathbb{R})$ by

$$\Psi(w) = \big[R_1(w) \mid \cdots \mid R_m(w)\big], \qquad (w \in W),$$

then Ψ has the desired properties. $\square$

Proof of Proposition 5.4. Choose an orthonormal basis $\mathfrak{B} = \{h_1, \dots, h_k\}$ for $\mathcal{U}$, and let Y be the random k-dimensional coordinate vector for T with respect to $\mathfrak{B}$. Then, since the distribution of T has a density with respect to Lebesgue measure on $\mathcal{U}$, the distribution of Y has a density with respect to Lebesgue measure on $\mathbb{R}^k$.

For each y in $\mathbb{R}^d$, let $\Phi_0(y)\colon \mathcal{U} \to \mathbb{R}^d$ be the linear mapping defined by

$$[\Phi_0(y)](U) = Uy, \qquad (U \in \mathcal{U}).$$

From the assumption (5.7) it follows that $\Phi_0(y)$ is surjective for ν-almost all y in $\mathbb{R}^d$, and in particular we must have that $k \geq d$. For each y in $\mathbb{R}^d$, we let $\Phi(y)$ denote the $d \times k$ matrix for $\Phi_0(y)$ with respect to $\mathfrak{B}$ and the standard basis for $\mathbb{R}^d$. Then for any y in $\mathbb{R}^d$ we have that

$$Ty = \Phi(y)Y.$$

Note also that $\Phi\colon \mathbb{R}^d \to \mathbb{M}_{d,k}(\mathbb{R})$ is clearly a continuous mapping, and in particular the set

$$\begin{aligned} G &:= \{y \in \mathbb{R}^d \mid \dim(\mathcal{U}y) = d\} = \{y \in \mathbb{R}^d \mid \Phi_0(y) \text{ is surjective}\} \\ &= \{y \in \mathbb{R}^d \mid \det(\Phi(y)\Phi^*(y)) > 0\} \end{aligned}$$

is an open subset of $\mathbb{R}^d$.

For each y in G it follows from Lemma 5.5 that the distribution of Ty $(= \Phi(y)Y)$ is absolutely continuous with respect to k-dimensional Lebesgue measure, and by Lemma 5.6 we may choose a Lebesgue density g_y for Ty in such a way that the mapping

$$(y, z) \mapsto g_y(z)\colon G \times \mathbb{R}^d \to [0, \infty)$$

is Borel-measurable on $G \times \mathbb{R}^d$.

Now, for any Borel-set B in $\mathbb{R}^d$ we find by application of Tonelli's Theorem and the fact that $\nu(G^c) = 0$ that

$$\begin{aligned} [\Upsilon_T^0(\nu)](B) &= \int_S \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)y)\, \nu(\mathrm{d}y) \Big)\, \gamma(\mathrm{d}s) \\ &= \int_{\mathbb{R}^d} \Big(\int_S 1_{B\setminus\{0\}}(T(s)y)\, \gamma(\mathrm{d}s) \Big)\, \nu(\mathrm{d}y) \\ &= \int_G \Big(\int_S 1_{B\setminus\{0\}}(T(s)y)\, \gamma(\mathrm{d}s) \Big)\, \nu(\mathrm{d}y) \\ &= \int_G \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(z) g_y(z)\, \mathrm{d}z_1 \cdots \mathrm{d}z_d \Big)\, \nu(\mathrm{d}y) \\ &= \int_{\mathbb{R}^d} 1_B(z) \Big(\int_G g_y(z)\, \nu(\mathrm{d}y) \Big)\, \mathrm{d}z_1 \cdots \mathrm{d}z_d, \end{aligned}$$

which completes the proof. $\square$

Corollary 5.8. *Let T be a random $d \times d$-matrix on the probability space $(S, \mathcal{S}, \gamma)$, and assume that one of the following two conditions is satisfied:*

(a) *The distribution of T is absolutely continuous with respect to Lebesgue measure on $\mathbb{M}_d(\mathbb{R})$.*
(b) *$T = T^*$, and the distribution of T is absolutely continuous with respect to Lebesgue measure on the space $\mathbb{S}_d$ of symmetric $d \times d$-matrices.*

Then for any non-zero Lévy measure ν on $\mathbb{R}^d$ the measure $\Upsilon_T^0(\nu)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$, and the density is given by

$$\tilde{r}(z) = \int_{\mathbb{R}^d \setminus \{0\}} g_y(z)\, \nu(\mathrm{d}y), \qquad (z \in \mathbb{R}^d),$$

where, for each non-zero vector y in $\mathbb{R}^d$, g_y is a density of the d-dimensional random vector Ty with respect to d-dimensional Lebesgue measure (cf. Lemma 5.6).

Proof: The corollary follows by application of Proposition 5.4 in the cases (a) $\mathcal{U} = \mathbb{M}_d(\mathbb{R})$, and (b) $\mathcal{U} = \mathbb{S}_d$. Condition (5.7) is then satisfied for any Lévy measure ν on $\mathbb{R}^d$, since

$$\mathbb{M}_d(\mathbb{R})y = \mathbb{R}^d, \quad \text{and} \quad \mathbb{S}_d y = \mathbb{R}^d \qquad \text{for any } y \text{ in } \mathbb{R}^d \setminus \{0\}. \tag{5.10}$$

To see this, it clearly suffices to establish the second equation in (5.10) for any unit vector y in $\mathbb{R}^d$. If $y = e_1$ (the first vector in the standard basis for $\mathbb{R}^d$), then this equality follows from the fact that any vector in $\mathbb{R}^d$ can obviously be placed as the first column of a symmetric $d \times d$ matrix. For a general unit vector y in $\mathbb{R}^d$, we may choose an orthogonal $d \times d$-matrix U, such that $Uy = e_1$. Then, for any vector z in $\mathbb{R}^d$, it follows from the previous argument that we may choose a matrix A in $\mathbb{S}_d$, such that $Ae_1 = Uz$. Now $U^*AU \in \mathbb{S}_d$, and $U^*AUy = U^*Ae_1 = U^*Uz = z$. This completes the proof of (5.10) and hence that of the corollary. □

Example 5.9 (Wishart matrices II). Let d be a positive integer and let α be a positive number, such that $\alpha \geq d$. Consider further as in Example 3.16 a symmetric, positive semi-definite $d \times d$ random matrix T_α carrying the Wishart distribution $\mathrm{W}_d(\alpha, \frac{1}{2}\mathrm{I}_d)$. Since $\alpha \geq d$, $P(T_\alpha \in \overline{\mathbb{S}}_d^+) = 1$, and the distribution of T_α has the density (3.50) with respect to Lebesgue measure on $\mathbb{S}_d$. Moreover, $\mathbb{E}\{\|T_\alpha\|^2\} < \infty$, so that $\mathsf{dom}_L(\Upsilon_{T_\alpha}^0) = \mathfrak{M}_L(\mathbb{R}^d)$. For any non-zero Lévy measure ν on $\mathbb{R}^d$ it follows thus from Corollary 5.8 that the Lévy measure $\tilde{\nu}_\alpha := \Upsilon_{T_\alpha}^0(\nu)$ is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^d$. The density is given by

$$\tilde{r}_\alpha(z) = \int_{\mathbb{R}^d \setminus \{0\}} g_y^\alpha(z)\, \nu(\mathrm{d}y), \qquad (z \in \mathbb{R}^d),$$

where, for each y in $\mathbb{R}^d \setminus \{0\}$, g_y^α is a density of the d-dimensional random vector $T_\alpha y$ with respect to d-dimensional Lebesgue measure. To identify the distribution of $T_\alpha y$ we consider for any $u \in \mathbb{R}^d$ the matrix $\Theta = yu^*$ of rank one. Using (3.48) we then find that

$$\mathbb{E}\{\mathrm{e}^{\mathrm{i}\langle u, T_\alpha y\rangle}\} = \mathbb{E}\{\mathrm{e}^{\mathrm{i}\mathrm{tr}(T_\alpha y u^*)}\} = \det(\mathrm{I}_d - \mathrm{i}yu^*)^{-\alpha/2} = (1 - \mathrm{i}\,\langle y, u\rangle)^{-\alpha/2},$$

which determines the characteristic function for $T_\alpha u$.

5.3. *The Blumenthal-Getoor index.* Let μ be an infinitely divisible distribution on $\mathbb{R}^d$. The Blumenthal-Getoor index $\alpha = \alpha_\mu$ for μ (or the corresponding Lévy process $(X(t))_{t\geq 0}$) is then defined by:

$$\alpha = \inf\left\{\delta > 0 \,\middle|\, \textstyle\int_{B_1} \|x\|^\delta \,\nu(\mathrm{d}x) < \infty\right\},$$

where ν is the Lévy measure for μ and B_1 denotes the unit disk of $\mathbb{R}^d$ with respect to the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^d$ (cf. (3.1)-(3.2)). Clearly $\alpha \in [0,2]$. If $d = 1$ and $(X(t))_{t\geq 0}$ is a pure jump process, it is well-known (see e.g. Todorov and Tauchen (2011)) that

$$\alpha = \inf\left\{r > 0 \,\middle|\, \textstyle\sum_{0\leq s\leq 1} |\Delta X(s)|^r < \infty\right\}$$

where $\Delta X(s)$ is the jump at time s. Thus, the Blumenthal-Getoor index α measures the jump activity of $X(t)$ (the larger the index the larger and the more jumps).

For the non-Gaussian stable distributions without drift the Blumenthal-Getoor index equals the index of stability (cf. Sato (1999, p. 362)). We show next (refining Proposition 3.12) that the Upsilon transforms preserve the index of stability for the stable distributions.

Proposition 5.10. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and consider a measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$. Let further α be a number in $(0,2)$, and let μ be an α-stable distribution in $\mathsf{dom}_{ID}(\Upsilon_T)$. Then $\Upsilon_T(\mu)$ is again an α-stable distribution.*

Proof: Let (η, A, ν) denote the characteristic triplet for μ. According to Sato (1999, Theorem 14.3) the fact that μ is α-stable means exactly that $A = 0$ and

$$D_r\nu = r^\alpha\nu \quad \text{for all } r \text{ in } (0,\infty), \tag{5.11}$$

where, as previously, $D_r\nu$ is the transformation of ν by the mapping $x \mapsto rx$. Since obviously $\int_S TAT^* \,\mathrm{d}\gamma = 0$, it suffices thus to show that property (5.11) is transferred to the Lévy measure $\Upsilon_T^0(\nu)$ of $\Upsilon_T(\mu)$. Given a positive number r and a Borel set B in $\mathbb{R}^d$ we find that

$$\begin{aligned}
[D_r\Upsilon_T^0(\nu)](B) = [\Upsilon_T^0(\nu)](r^{-1}B) &= \int_S\int_{\mathbb{R}^d} 1_{r^{-1}B\setminus\{0\}}(Ty)\,\nu(\mathrm{d}y)\,\mathrm{d}\gamma \\
&= \int_S\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(ry))\,\nu(\mathrm{d}y)\,\mathrm{d}\gamma = \int_S\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(Tz)\,D_r\nu(\mathrm{d}z)\,\mathrm{d}\gamma \\
&= r^\alpha\int_S\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(Tz)\,\nu(\mathrm{d}z)\,\mathrm{d}\gamma = r^\alpha[\Upsilon_T^0(\nu)](B),
\end{aligned}$$

as desired. □

For general infinitely divisible laws, the everywhere defined Upsilon transforms (cf. Proposition 3.6) decrease the Blumenthal-Getoor index, as the following proposition shows.

Proposition 5.11. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and consider a measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$ such that $\int_S \|T\|^2 \,\mathrm{d}\gamma < \infty$, and $\gamma(\{T \neq 0\}) < \infty$. Let further μ be an infinitely divisible distribution on $\mathbb{R}^d$, and let α and $\tilde{\alpha}$ denote, respectively, the Blumenthal-Getoor index for μ and $\Upsilon_T(\mu)$. Then $\tilde{\alpha} \leq \alpha$.*

Proof: Let ν denote the Lévy measure for μ, and put $\tilde{\nu} = \Upsilon_T^0(\nu)$. It suffices then to show that $\int_{B_1} \|u\|^\delta \, \tilde{\nu}(\mathrm{d}u) < \infty$ for any δ in $(\alpha, 2]$. Given such a δ we find, as in the proof of Proposition 3.1, that

$$\begin{aligned}
\int_{B_1} \|u\|^\delta \, \tilde{\nu}(\mathrm{d}u) &\leq \int_{\mathbb{R}^d} \min\{1, \|u\|^\delta\} \, \tilde{\nu}(\mathrm{d}u) \\
&= \int_S \Big(\int_{\mathbb{R}^d} \min\{1, \|T(s)u\|^\delta\} \, \nu(\mathrm{d}u) \Big) \, \gamma(\mathrm{d}s) \\
&\leq \int_{\{T \neq 0\}} \Big(\int_{\mathbb{R}^d} \min\{1, \|T(s)\|^\delta \|u\|^\delta\} \, \nu(\mathrm{d}u) \Big) \, \gamma(\mathrm{d}s) \\
&\leq \int_{\{T \neq 0\}} \Big(\int_{\mathbb{R}^d} \min\{1, \|u\|^\delta\} \max\{1, \|T(s)\|^\delta\} \, \nu(\mathrm{d}u) \Big) \, \gamma(\mathrm{d}s) \\
&= \int_{\{T \neq 0\}} \max\{1, \|T(s)\|^\delta\} \, \gamma(\mathrm{d}s) \int_{\mathbb{R}^d} \min\{1, \|u\|^\delta\} \, \nu(\mathrm{d}u) \\
&\leq \int_{\{T \neq 0\}} \max\{1, \|T(s)\|^2\} \, \gamma(\mathrm{d}s) \int_{\mathbb{R}^d} \min\{1, \|u\|^\delta\} \, \nu(\mathrm{d}u),
\end{aligned}$$

and the assumptions imply that the resulting expression is finite. □

6. Random integral representation

In this section we derive a representation of the Upsilon mapping Υ_T as a random integral with respect to an $\mathbb{R}^d$-valued Lévy basis L on a general space S. These are Wiener type integrals of deterministic measurable functions $T : S \to \mathbb{M}_d(\mathbb{R})$ with respect to L.

Integration of non-random functions with respect to a Lévy basis (also known as infinitely divisible, independently scattered, random measures (i.d.i.s.r.m.)) goes back to Urbanik and Woyczyński (1967) and Rosiński (1984). It was systematically studied by Rajput and Rosiński (1989) when L is $\mathbb{R}$-valued and S is general, and by Sato (2004) when L is $\mathbb{R}^d$-valued, $S = [0, \infty)$. See also the related construction of improper integrals with respect to additive processes in Sato (2006a,b, 2007).

We present here a review and a self-contained treatment of the constructions of the relevant integrals which builds on the connection to the Upsilon mappings studied in the foregoing sections

We start with a brief overview of notation and basics for Lévy bases and the particular case of factorisable Lévy bases.

6.1. *Background on Lévy bases.* Let $(S, \mathcal{S})$ be a measurable space such that $\mathcal{S} = \sigma(\mathcal{S}^0)$, where $\mathcal{S}^0$ is a δ-ring of subsets of S such that there exists a sequence $\{F_n\} \subseteq \mathcal{S}^0$ with $F_n \subseteq F_{n+1}$ and $\cup_n F_n = S$. A recurrent example in this theory is when S is a Borel subset of $\mathbb{R}^m$, $\mathcal{S} = \mathcal{B}(S)$ and $\mathcal{S}^0 = \mathcal{B}_b(S)$, the class of bounded Borel subsets of S.

Definition 6.1. A family $L = \big\{L(F) : F \in \mathcal{S}^0\big\}$ of $\mathbb{R}^d$-valued random variables (defined on some probability space $(\Omega, \mathcal{F}, P)$) is called an $\mathbb{R}^d$-valued Lévy basis on S if the following three conditions are satisfied:

(a) the distribution of $L(F)$ is in $\mathcal{ID}(\mathbb{R}^d)$ for all $F \in \mathcal{S}^0$,
(b) for any $n \in \mathbb{N}$ and pairwise disjoint sets $E_1, \ldots, E_n \in \mathcal{S}^0$ the random elements $L(E_1), ..., L(E_n)$ are independent,
(c) for any pairwise disjoints sets $E_n \in \mathcal{S}^0, n \in \mathbb{N}$, satisfying $\cup_{n\in\mathbb{N}} E_n \in \mathcal{S}^0$ the series $\sum_{n=1}^{\infty} L(E_n)$ converges almost surely, and it holds that

$$L\left(\cup_{n\in\mathbb{N}} E_n\right) = \sum_{n=1}^{\infty} L(E_n) \text{ a.s.}$$

For each F in $\mathcal{S}^0$, we denote by $C\{u \ddagger L(F)\}$ the cumulant transform of $L(F)$, i.e. $C\{u \ddagger L(F)\} = \log \mathbb{E}\left(\exp\left(\mathrm{i}\,\langle L(F), u\rangle\right)\right)$ for any $u \in \mathbb{R}^d$. The Lévy-Khintchine representation of $C\{u \ddagger L(F)\}$ is for each F in $\mathcal{S}^0$ given by

$$\begin{aligned} C\{u \ddagger L(F)\} &= \mathrm{i}\,\langle \boldsymbol{\eta}(F), u\rangle - \frac{1}{2}\,\langle \mathbf{A}(F)u, u\rangle \\ &\quad + \int_{\mathbb{R}^d} \left(\mathrm{e}^{i\langle y,u\rangle} - 1 - \mathrm{i}\,\langle y, u\rangle\, 1_{B_1}(y)\right) \mathbf{n}(\mathrm{d}y, F), \quad (u \in \mathbb{R}^d), \end{aligned} \tag{6.1}$$

where
(i) $\boldsymbol{\eta}(\cdot) = (\boldsymbol{\eta}_1(\cdot), ..., \boldsymbol{\eta}_d(\cdot))$ is an $\mathbb{R}^d$-valued measure on $\mathcal{S}^0$,
(ii) $\mathbf{A}(\cdot) = (\mathbf{A}_{ij}(\cdot))$ is a $\overline{\mathbb{S}}_d^+$-valued measure on $\mathcal{S}^0$,
(iii) $\mathbf{n}$ is a bimeasure such that for fixed $F \in \mathcal{S}^0$, $\mathbf{n}(\mathrm{d}y, F)$ is a Lévy measure on $\mathbb{R}^d$, and for fixed $\mathrm{d}y$ a measure on $\mathcal{S}^0$.

For $F \in \mathcal{S}^0$ let

$$c(F) = |\boldsymbol{\eta}|\,(F) + \mathrm{tr}(\mathbf{A})(F) + \int_{\mathbb{R}^d} \min\{1, |x|^2\}\, \mathbf{n}(\mathrm{d}y, F), \tag{6.2}$$

with $|\boldsymbol{\eta}|$ denoting the variation measure of $\boldsymbol{\eta}$. (Recall that the variation $|b|$ of a vector valued measure b is defined as

$$|b|\,(F) = \sup \sum_n \|b(E_n)\| \qquad (F \in \mathcal{S}^0),$$

where the supremum is taken over all the partitions $F = \cup_n E_n$ of F into a finite number of disjoint sets E_n in $\mathcal{S}^0$). It can be shown that $L(E_n) \stackrel{\mathrm{Pr}}{\to} 0$ when $E_n \downarrow \emptyset$, $E_n \in \mathcal{S}^0$. Hence c is continuous at the empty set, and since $c(F_n) < \infty$, for $n \geq 1$, we can extend c to a σ-finite measure on $(S, \mathcal{S})$. This extension is called the control measure of L and it is also denoted by c.

The measures $\boldsymbol{\eta}, \mathbf{A}$ and $\mathbf{n}(\mathrm{d}y, \cdot)$ are absolutely continuous with respect to c. We define the functions $\eta(s) = (\eta_1(s), ..., \eta_d(s))$, $A(s) = (a_{ij}(s))$ and $\nu(\mathrm{d}y, s)$, $s \in S$, using Radon-Nikodym derivatives as follow:

$$\eta_i(s) = \frac{\mathrm{d}\boldsymbol{\eta}_i}{\mathrm{d}c}(s), \quad i = 1, \ldots, d, \tag{6.3}$$

$$a_{ij}(s) = \frac{\mathrm{d}\mathbf{A}_{ij}(\cdot)}{\mathrm{d}c}(s), \quad i, j = 1, \ldots, d, \tag{6.4}$$

$$\nu(\mathrm{d}y, s) = \frac{\mathbf{n}(\mathrm{d}y, \cdot)}{\mathrm{d}c}(s), \tag{6.5}$$

$$\mathbf{n}(\mathrm{d}y, \mathrm{d}s) = \nu(\mathrm{d}y, s)\, c(\mathrm{d}s) \tag{6.6}$$

There is then no loss of generality in assuming that $\nu(\mathrm{d}y, s)$ is a Lévy measure for each fixed $s \in S$ and that

$$\widetilde{\nu}(F) = \int_F \nu(\mathrm{d}y, s)\, c(\mathrm{d}s), \quad \big(F \in \mathcal{B}(\mathbb{R}^d)\big) \tag{6.7}$$

is a Lévy measure on $\mathbb{R}^d$. Moreover, the quadruplet $(\eta(\cdot), A(\cdot),\ \nu(\mathrm{d}y, \cdot), c((\mathrm{d}\cdot))$ determines uniquely the Lévy basis L.

When $\nu(\mathrm{d}x, s)$, $\eta(s)$ and $A(s)$ do not depend on s, the Lévy basis is said to be *factorisable*. In this case there exists a characteristic triplet (η, A, ν) such that $L(F)$ has an infinitely divisible distribution on $\mathbb{R}^d$ with characteristic triplet $c(F)(\eta, A, \nu) := (c(F)\eta, c(F)A, c(F)\nu)$ for each $F \in \mathcal{S}^0$. In this case we also say that (η, A, ν, c) is the generating quadruplet of the factorisable Lévy basis L. If, in addition, $S = \mathbb{R}^d$ and c is proportional to the Lebesgue measure, the Lévy basis is said to be homogeneous.

Remark 6.2. Suppose γ is a σ-finite measure on $S = \sigma(\mathcal{S}^0)$, such that $\gamma(F) < \infty$ for all F in $\mathcal{S}_0$. Then for any characteristic triplet (η, A, ν) on $\mathbb{R}^d$ we can construct a factorisable Lévy basis $L = \{L(F) \mid F \in \mathcal{S}^0\}$, such that $L(F)$ has generating triplet $\gamma(F)(\eta, A, \nu)$ for any F in $\mathcal{S}^0$. We note that L has generating quadruplet (η, A, ν, c), where the control measure c is equal to $k_L\gamma$, with the constant k_L given by

$$k_L = \|\eta\| + \mathrm{tr}(A) + \int_{\mathbb{R}^d} \min\{1, \|y\|^2\}\, \nu(\mathrm{d}y). \tag{6.8}$$

6.2. *Integral representation when γ is finite.* Throughout this subsection we consider a fixed *finite* measure space $(S, \mathcal{S}, \gamma)$. Let further η be a vector in $\mathbb{R}^d$, let A be a symmetric non-negative definite $d \times d$ matrix and let ν be a Lévy measure on $\mathbb{R}^d$. Then, there exists a factorisable Lévy basis $L = \{L(F) \mid F \in \mathcal{S}\}$ in $\mathbb{R}^d$ (defined on some probability space $(\Omega, \mathcal{F}, P)$) with generating quadruplet $(\eta, A, \nu, k_L\gamma)$ (cf. Remark 6.2).

A simple measurable mapping $T : S \to \mathbb{M}_d(\mathbb{R})$ may be written in the form:

$$T(x) = \sum_{j=1}^{n} \alpha_j 1_{F_j}(x) \tag{6.9}$$

where $n \in \mathbb{N}, \alpha_j \in \mathbb{M}_d(\mathbb{R})$ and $F_1, \dots, F_n$ are disjoint sets from $\mathcal{S}$. In this case we define the Wiener integral of T with respect to the Lévy basis L introduced above as the $\mathbb{R}^d$-valued random vector

$$I_L(T) = \int_S T(s)\, L(\mathrm{d}s) = \sum_{j=1}^{n} \alpha_j L(F_j). \tag{6.10}$$

It follows by standard arguments that $I_L(T)$ does not depend on the specific representation (6.9) and that

$$I_L(\alpha T + T') = \alpha I_L(T) + I_L(T') \tag{6.11}$$

for any simple measurable mappings $T, T' \colon S \to \mathbb{M}_d(\mathbb{R})$ and any constant matrix α in $\mathbb{M}_d(\mathbb{R})$.

We note next the connection to Upsilon transforms for simple measurable mappings $T \colon S \to \mathbb{M}_d(\mathbb{R})$.

Lemma 6.3. *Let μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν) and consider the Lévy basis $L = \{L(F) \mid F \in \mathcal{S}\}$ introduced above. For any simple measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$ we then have that*

$$\mathcal{L}\Big\{ \int_S T \,\mathrm{d}L \Big\} = \Upsilon_T(\mu) \in \mathcal{ID}(\mathbb{R}^d), \tag{6.12}$$

where $\mathcal{L}\{X\}$ denotes the law of a random vector X.

Proof: Consider first T in the form: $T = \alpha 1_F$, where $\alpha \in \mathbb{M}_d(\mathbb{R})$ and $F \in \mathcal{S}$. Then $\int_S T \,\mathrm{d}L = \alpha L(F)$, where $L(F)$ has characteristic triplet $\gamma(F)(\eta, A, \nu)$. It follows then from Sato (1999, Proposition 11.10) that $\alpha L(F)$ has characteristic triplet $(\tilde{\eta}, \tilde{A}, \tilde{\nu})$, where

$$\tilde{A} = \gamma(F)\alpha A \alpha^* = \int_S T(s) A T(s)^* \,\gamma(\mathrm{d}s),$$

$$\begin{aligned}\tilde{\eta} &= \gamma(F)\alpha\eta + \gamma(F)\int_{\mathbb{R}^d} \alpha x(1_{B_1}(\alpha x) - 1_{B_1}(x))\,\nu(\mathrm{d}x) \\ &= \int_S T(s)\eta\,\gamma(\mathrm{d}s) + \int_S \Big(\int_{\mathbb{R}^d} T(s)x(1_{B_1}(T(s)x) - 1_{B_1}(x))\,\nu(\mathrm{d}x)\Big)\,\gamma(\mathrm{d}s),\end{aligned}$$

and for any Borel set B in $\mathbb{R}^d$

$$\tilde{\nu}(B) = \gamma(F)\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(\alpha x)\,\nu(\mathrm{d}x) = \int_S \Big(\int_{\mathbb{R}^d} 1_{B\setminus\{0\}}(T(s)x)\,\nu(\mathrm{d}x)\Big)\,\gamma(\mathrm{d}s).$$

Hence it follows from Definition 3.4 that (6.12) holds in this case. Proposition 3.5 (and Theorem 3.3) further implies that

$$C\big\{z \ddagger \alpha L(F)\big\} = \int_S C_\mu(T(s)^* z)\,\gamma(\mathrm{d}s) = \gamma(F) C_\mu(\alpha^* z), \qquad (z \in \mathbb{R}^d). \tag{6.13}$$

Consider now a general simple measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$ written in the form (6.9) with disjoint $F_1, \dots, F_n$. Then $I_L(T) = \sum_{j=1}^n \alpha_j L(F_j)$, where the terms on the right hand side are independent random vectors. It follows thus for any z in $\mathbb{R}^d$ that

$$\begin{aligned} C\big\{z \ddagger I_L(T)\big\} &= \sum_{j=1}^n C\big\{z \ddagger \alpha_j L(F_j)\big\} = \sum_{j=1}^n \gamma(F_j) C_\mu(\alpha_j^* z) \\ &= \int_S C_\mu(T(s)^* z)\,\gamma(\mathrm{d}s) = C_{\Upsilon_T(\mu)}(z),\end{aligned}$$

where we have used (6.13) and Proposition 3.5 □

Proposition 6.4. *Assume that $T\colon S \to \mathbb{M}_d(\mathbb{R})$ is a measurable mapping satisfying that $\int_S \|T\|^2 \,\mathrm{d}\gamma < \infty$. Then*

(i) *There exists a sequence (T_n) of simple measurable mappings $T_n\colon S \to \mathbb{M}_d(\mathbb{R})$ such that conditions (a) and (b) of Proposition 4.3 are satisfied.*

(ii) *For any sequence (T_n) of simple measurable mappings $T_n\colon S \to \mathbb{M}_d(\mathbb{R})$ satisfying conditions (a) and (b) of Proposition 4.3, the sequence $(I_L(T_n))$ converges in probability, as $n \to \infty$, to a measurable mapping $Y\colon \Omega \to \mathbb{R}^d$.*

(iii) *The limit Y described in (ii) is, up to P-nullsets, the same for any sequence (T_n) of simple measurable mappings satisfying (a) and (b) of Proposition 4.3.*

Proof: (i) For any i, j in $\{1, 2, \dots, d\}$ we let $t_{ij}\colon S \to \mathbb{R}$ denote the entry at position (i, j) of T. Then by standard methods we can choose a sequence $(t_{ij}^{(n)})_{n\in\mathbb{N}}$ of simple measurable functions $t_{ij}^{(n)}\colon S \to \mathbb{R}$ such that

$$\sup_{n\in\mathbb{N}} |t_{ij}^{(n)}(s)| \le |t_{ij}(s)|, \quad \text{and} \quad \lim_{n\to\infty} t_{ij}^{(n)}(s) = t_{ij}(s) \qquad \text{for all } s \text{ in } S.$$

For each n we let $T_n\colon S \to \mathbb{M}_d(\mathbb{R})$ denote the simple measurable mapping with entries $t_{ij}^{(n)}$, $1 \le i, j \le d$. Since all norms on $\mathbb{M}_d(\mathbb{R})$ are equivalent, it follows then that

$$\lim_{n\to\infty} \|T_n(s) - T(s)\| = 0 \quad \text{for all } s \text{ in } S,$$

and that

$$\|T_n(s)\| \le K_d \max_{1\le i,j\le d} |t_{ij}^{(n)}(s)| \le K_d \max_{1\le i,j\le d} |t_{ij}(s)| \le K_d' \|T(s)\|$$

for all s in S and n in $\mathbb{N}$, and where K_d and K_d' are positive constants (depending only on d). Thus conditions (a) and (b) of Proposition 4.3 are satisfied if we put $g = K_d'^2\|T\|^2$.

(ii) Assume that (T_n) is an arbitrary sequence of simple measurable mappings $T_n\colon S \to \mathbb{M}_d(\mathbb{R})$ satisfying conditions (a) and (b) of Proposition 4.3. Then for any n, m in $\mathbb{N}$ the mapping $T_n - T_m$ is again simple and measurable, and it follows by (6.11) and Lemma 6.3 that

$$C\{z \ddagger I_L(T_n) - I_L(T_m)\} = C\{z \ddagger I_L(T_n - T_m)\} = C_{\Upsilon_{T_n - T_m}(\mu)}(z), \qquad (z \in \mathbb{R}^d).$$

Hence, Proposition 4.3(ii) implies that

$$\mathcal{L}\{I_L(T_n) - I_L(T_m)\} = \Upsilon_{T_n - T_m}(\mu) \xrightarrow{\text{w}} \delta_0, \quad \text{as } n, m \to \infty,$$

so that $(I_L(T_n))_{n\in\mathbb{N}}$ is a Cauchy sequence and hence convergent in probability (see e.g. Lemma 1.2.4 in Barndorff-Nielsen et al. (2006a)).

(iii) Assume that (T_n) and (T_n') are two sequences of simple measurable mappings both satisfying conditions (a) and (b) of Proposition 4.3. Then by (ii) there exist random vectors $Y, Y'\colon \Omega \to \mathbb{R}^d$ such that $I_L(T_n) \to Y$ and $I_L(T_n') \to Y'$ in probability as $n \to \infty$. Now the mixed sequence $T_1, T_1', T_2, T_2', \dots$ also satisfies (a) and (b) in Proposition 4.3, so there exists a random vector $Y''\colon \Omega \to \mathbb{R}^d$ such that $I_L(T_1), I_L(T_1'), I_L(T_2), I_L(T_2'), \dots$ converges to Y'' in probability. Thus, by subsequence considerations, $Y = Y'' = Y'$ P-almost everywhere, and this completes the proof. □

Definition 6.5. Assume that $T\colon S \to \mathbb{M}_d(\mathbb{R})$ is a measurable mapping satisfying that $\int_S \|T\|^2 \,\mathrm{d}\gamma < \infty$. Then we denote by

$$I_L(T) = \int_S T(s)\, L(\mathrm{d}s)$$

the random vector Y described in Proposition 6.4(ii)-(iii).

As an immediate consequence of Definition 6.5 and (6.11) we note that the integral just introduced is linear in the sense that

$$\int_S (\alpha T_1(s) + T_2(s))\, L(\mathrm{d}s) = \alpha \int_S T_1(s)\, L(\mathrm{d}s) + \int_S T_2(s)\, L(\mathrm{d}s), \tag{6.14}$$

whenever $\alpha \in \mathbb{M}_d(\mathbb{R})$ and $T_1, T_2\colon S \to \mathbb{M}_d(\mathbb{R})$ are measurable functions satisfying that $\int_S \|T_j\|^2 \,\mathrm{d}\gamma < \infty$, $j = 1, 2$.

Theorem 6.6. *Let $(S, \mathcal{S}, \gamma)$ be a finite measure space, and assume that $T\colon S \to \mathbb{M}_d(\mathbb{R})$ is a measurable mapping satisfying that $\int_S \|T\|^2 \,\mathrm{d}\gamma < \infty$. Let further μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν). Then*

$$\mathcal{L}\Big\{ \int_S T(s)\, L(\mathrm{d}s) \Big\} = \Upsilon_T(\mu)$$

where $L = \{L(F) \mid F \in \mathcal{S}\}$ is an $\mathbb{R}^d$-valued factorisable Lévy basis with generating quadruplet $(\eta, A, \nu, k_L\gamma)$ (cf. Remark 6.2).

Proof: By Proposition 6.4(i) we may choose a sequence (T_n) of simple measurable mappings $T_n\colon S \to \mathbb{M}_d(\mathbb{R})$ satisfying conditions (a) and (b) of Proposition 4.3. Then by definition $I_L(T)$ is the limit in probability of the sequence $I_L(T_n)$, so in particular (cf. Lemma 6.3) $\mathcal{L}\{I_L(T)\} \in \mathcal{ID}(\mathbb{R}^d)$, and $C\{z \ddagger I_L(T)\} = \lim_{n\to\infty} C\{z \ddagger I_L(T_n)\}$ for all z in $\mathbb{R}^d$ (cf. Lemma 7.7 in Sato (1999)). Combining this with Lemma 6.3 and Proposition 4.3(i) we find thus that

$$C\{z \ddagger I_L(T)\} = \lim_{n\to\infty} C\{z \ddagger I_L(T_n)\} = \lim_{n\to\infty} C_{\Upsilon_{T_n}(\mu)}(z) = C_{\Upsilon_T(\mu)}(z)$$

for all z in $\mathbb{R}^d$, and this completes the proof. □

6.3. *Integral representation when γ is σ-finite.* Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space and consider the δ-ring

$$\mathcal{S}^0 = \{F \in \mathcal{S} \mid \gamma(F) < \infty\}.$$

Then choose a sequence $(F_n)_{n\in\mathbb{N}}$ of sets from $\mathcal{S}^0$, such that

$$F_1 \subseteq F_2 \subseteq F_3 \subseteq \cdots, \quad \text{and} \quad \bigcup_{n\in\mathbb{N}} F_n = S. \tag{6.15}$$

Let further μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν). Then on some probability space $(\Omega, \mathcal{F}, P)$ there exists (see Subsection 6.1) a Lévy basis $L = \{L(F) \mid F \in \mathcal{S}^0\}$ such that for all F in $\mathcal{S}^0$

$$L(F) \text{ has characteristic triplet } \gamma(F)(\eta, A, \nu). \tag{6.16}$$

For any F in $\mathcal{S}^0$ we may then further consider the Lévy basis $L_F = \{L_F(G) \mid G \in \mathcal{S}\}$ given by

$$L_F(G) = \{L(F \cap G) \mid G \in \mathcal{S}\}. \tag{6.17}$$

We note that L_F has quadruplet $(\eta, A, \nu, k_{L_F}\gamma_F)$, where γ_F is the finite measure given by

$$\gamma_F(G) = \gamma(F \cap G), \qquad (G \in \mathcal{S}).$$

Consider now additionally a measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$, and then put

$$G_n = F_n \cap \{\|T\| \le n\}, \quad \text{and} \quad T_n = T1_{G_n}, \qquad (n \in \mathbb{N}). \tag{6.18}$$

We note that

$$\lim_{n\to\infty} \|T_n(s) - T(s)\| = 0, \qquad (s \in S),$$

and that

$$\int_S \|T_n\|^2 \,\mathrm{d}\gamma_{G_n} = \int_S \|T\|^2 1_{G_n} \,\mathrm{d}\gamma < \infty$$

for all n. Hence the integral $\int_S T_n \,\mathrm{d}L_{G_n}$ is well-defined (cf. Definition 6.5) and we have that

$$C\big\{z \ddagger \textstyle\int T_n \,\mathrm{d}L_{G_n}\big\} = \displaystyle\int_S C_\mu(T_n(s)^* z)\, \gamma_{G_n}(\mathrm{d}s) = \int_{G_n} C_\mu(T(s)^* z)\, \gamma(\mathrm{d}s) \tag{6.19}$$

(cf. Theorem 6.6).

Proposition 6.7. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping and consider sequences (F_n) and (G_n) of sets from $\mathcal{S}^0$ as in (6.15) and (6.18). Let further μ be a measure in $\mathsf{dom}_{ID}(\Upsilon_T)$, and consider for each n in $\mathbb{N}$ the Lévy basis L_{G_n} given by (6.17).*

Then with $T_n = T1_{G_n}$ the integral $\int_S T_n \,\mathrm{d}L_{G_n}$ converges in probability, as $n \to \infty$, to a random vector Y, and (up to a null-set) Y does not depend on the choice of the sequence (F_n) satisfying condition (6.15).

Proof: To prove the existence of the $\mathbb{R}^d$-valued random vector Y it suffices to show that $(\int_S T_n \,\mathrm{d}L_{G_n})_{n\ge1}$ is a Cauchy sequence in probability. Given n, m in $\mathbb{N}$ such that $n \le m$, note that $\int_S \|T_n\|^2 \,\mathrm{d}\gamma_{G_m} \le \int_S \|T_m\|^2 \,\mathrm{d}\gamma_{G_m} < \infty$ so that the integral $\int_S T_n \,\mathrm{d}L_{G_m}$ is well-defined, and moreover

$$\int_S T_n \,\mathrm{d}L_{G_m} = \int_S T_n \,\mathrm{d}L_{G_n}, \tag{6.20}$$

which follows easily by approximation of T_n with simple functions as in Proposition 6.4. Now by (6.14)

$$\int_S T_m \,\mathrm{d}L_{G_m} - \int_S T_n \,\mathrm{d}L_{G_n} = \int_S (T_m - T_n) \,\mathrm{d}L_{G_m}$$

so that (cf. Theorem 6.6)

$$C\big\{z \ddagger \textstyle\int_S T_m \,\mathrm{d}L_{G_m} - \int_S T_n \,\mathrm{d}L_{G_n}\big\} = \displaystyle\int_S C_\mu((T_m - T_n)^* z) \,\mathrm{d}\gamma_m = \int_{G_m \setminus G_n} C_\mu(T^* z) \,\mathrm{d}\gamma,$$

and hence

$$\big|C\big\{z \ddagger \textstyle\int_S T_m \,\mathrm{d}L_{G_m} - \int_S T_n \,\mathrm{d}L_{G_n}\big\}\big| \le \displaystyle\int_{G_n^c} |C_\mu(T^* z)| \,\mathrm{d}\gamma \tag{6.21}$$

for all z in $\mathbb{R}$. Since $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$ we know that $\int_S |C_\mu(T^* z)| \,\mathrm{d}\gamma < \infty$, and hence the right hand side of (6.21) tends to 0 as $n \to \infty$. This implies (see e.g. Lemma 1.2.4 in Barndorff-Nielsen et al. (2006a)) that $(\int_S T_n \,\mathrm{d}L_{G_n})_{n\in\mathbb{N}}$ is a Cauchy sequence in probability, as desired.

It remains to show that the limit $Y = \lim_{n\to\infty} \int_S T_n \,\mathrm{d}L_{G_n}$ does not depend on the sequence (F_n) satisfying (6.15). Assume thus that (F'_n) is another sequence from $\mathcal{S}^0$ satisfying (6.15) and then put

$$G'_n = F'_n \cap \{\|T\| \le n\}, \quad \text{and} \quad T'_n = T1_{G'_n}, \qquad (n \in \mathbb{N}).$$

Then since $\int_S \|T_n\|^2 \,\mathrm{d}\gamma_{G_n \cup G'_n}, \int_S \|T'_n\|^2 \,\mathrm{d}\gamma_{G_n \cup G'_n} < \infty$, it follows in analogy with (6.20) that

$$\int_S T_n \,\mathrm{d}L_{G_n} = \int_S T_n \,\mathrm{d}L_{G_n \cup G'_n}, \quad \text{and} \quad \int_S T'_n \,\mathrm{d}L_{G'_n} = \int_S T'_n \,\mathrm{d}L_{G_n \cup G'_n},$$

so that by (6.14)

$$\int_S T_n \,\mathrm{d}L_{G_n} - \int_S T'_n \,\mathrm{d}L_{G'_n} = \int_S (T_n - T'_n) \,\mathrm{d}L_{G_n \cup G'_n}.$$

Using then Theorem 6.6 it follows that

$$\begin{aligned}\left|C\left\{z \ddagger \textstyle\int_S T_n \, \mathrm{d}L_{G_n} - \int_S T_n' \, \mathrm{d}L_{G_n'}\right\}\right| &= \left| \int_S C_\mu((T_n - T_n')^* z)) \, \mathrm{d}\gamma_{G_n \cup G_n'} \right| \\ &\leq \int_{G_n \Delta G_n'} |C_\mu(T^* z)| \, \mathrm{d}\gamma,\end{aligned} \tag{6.22}$$

where $G_n \Delta G_n' = (G_n \setminus G_n') \cup (G_n' \setminus G_n)$. Since both (G_n) and (G_n') are increasing sequences with union S, it follows that $1_{G_n \Delta G_n'}(s) \to 0$ as $n \to \infty$ for any s in S, since s is in $G_n \cap G_n'$ for all sufficiently large n. Thus it follows from (6.22) and dominated convergence that $\int_S T_n \, \mathrm{d}L_{G_n} - \int_S T_n' \, \mathrm{d}L_{G_n'} \to 0$ in probability as $n \to \infty$, and this yields the desired conclusion. □

Definition 6.8. Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping and let μ be a measure in $\mathsf{dom}_{ID}(\Upsilon_T)$ with characteristic triplet (η, A, ν). Consider further the Lévy basis $L = \{L(F) \mid F \in \mathcal{S}^0\}$ given by (6.16). Then we denote by

$$I_L(T) = \int_S T(s) \, L(\mathrm{d}s)$$

the random vector Y described in Proposition 6.7.

Theorem 6.9. *Let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, let $T\colon S \to \mathbb{M}_d(\mathbb{R})$ be a measurable mapping and let μ be a measure in $\mathsf{dom}_{ID}(\Upsilon_T)$ with characteristic triplet (η, A, ν). Consider further the factorisable Lévy basis $L = \{L(F) \mid F \in \mathcal{S}^0\}$ given by (6.16) with generating quadruplet $(\eta, A, \nu, k_L\gamma)$ (cf. Remark 6.2). We then have*

$$\mathcal{L}\Big\{ \int_S T(s) \, L(\mathrm{d}s) \Big\} = \Upsilon_T(\mu).$$

Proof: Choose a sequence (F_n) of sets from $\mathcal{S}^0$ satisfying (6.15) and define (G_n) and T_n as in (6.18). It follows then from Definition 6.8 and (6.19) that

$$\begin{aligned} C\left\{z \ddagger \textstyle\int_S T \, \mathrm{d}L\right\} &= \lim_{n \to \infty} C\left\{z \ddagger \textstyle\int_S T_n \, \mathrm{d}L_{G_n}\right\} = \lim_{n \to \infty} \displaystyle\int_{G_n} C_\mu(T^* z) \, \mathrm{d}\gamma \\ &= \int_S C_\mu(T^* z) \, \mathrm{d}\gamma = C_{\Upsilon_T(\mu)}(z), \end{aligned}$$

where the third equality follows by dominated convergence, since $1_{G_n}(s) \to 1$ as $n \to \infty$ for any s in S (cf. Proposition 3.5). This completes the proof. □

Example 6.10 (Construction of multivariate supOU processes). Let us consider again Example 3.8, where $S = \mathbb{M}_d^- \times \mathbb{R}$, $\mathcal{S} = \mathcal{B}(\mathbb{M}_d^- \times \mathbb{R})$, $\gamma = \pi \otimes \lambda$ with π a probability measure on $\mathcal{B}(\mathbb{M}_d^-)$, and λ is the Lebesgue measure on $\mathbb{R}$. In addition $T(Q, r) = \mathrm{e}^{rQ}$ for any $Q \in \mathbb{M}_d^-$ and $r \in \mathbb{R}$. Assume that (3.29)-(3.30) are satisfied.

In addition consider the δ-ring $\mathcal{S}^0 = \mathcal{B}(\mathbb{M}_d^-) \times \mathcal{B}_b(\mathbb{R})$ where $\mathcal{B}_b(\mathbb{R})$ are the bounded Borel subsets of $\mathbb{R}$. Let further μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν) and L be a factorizable Lévy basis on $\mathcal{S}^0$ with generating quadruplet $(\eta, A, \nu, k_L(\pi \otimes \lambda))$. Assuming that ν satisfies (3.31) we have that μ is $\mathsf{dom}_{ID}(\Upsilon_T)$.

From Theorem 3.1 in Barndorff-Nielsen and Stelzer (2011), the $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in\mathbb{R}}$ given by

$$X_t = \int_{\mathbb{M}_d^-} \int_{-\infty}^{t} \mathrm{e}^{(t-r)Q}\, L(\mathrm{d}Q, \mathrm{d}r)$$

is such that for all $t \in \mathbb{R}$ the distribution of X_t is infinitely divisible with characteristic triplet $(\tilde{\eta}, \tilde{A}, \tilde{\nu})$ given by (3.28). Hence, X_t has distribution $\Upsilon_T(\mu)$.

6.4. *Integral representation when γ is Lebesgue measure on $\mathbb{R}^+$.* In this subsection we consider exclusively the case $(S, \mathcal{S}, \gamma) = ([0,\infty), \mathcal{B}([0,\infty)), \lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra on $[0,\infty)$ and λ denotes Lebesgue measure on $[0,\infty)$. Given any measure μ from $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν) we may then consider a Lévy basis $L = \{L(F) \mid F \in \mathcal{B}([0,\infty))^0\}$, where $\mathcal{B}([0,\infty))^0$ is the family of Borel subsets of $[0,\infty)$ with finite Lebesgue measure, and where

$$L(F) \text{ has characteristic triplet } \lambda(F)(\eta, A, \nu) \tag{6.23}$$

for any F in $\mathcal{B}([0,\infty))^0$ (cf. Subsection 6.1). In this case it follows easily (and is well-known) that the formula

$$Z_t = L((0,t]), \qquad (t \in [0,\infty)) \tag{6.24}$$

defines an $\mathbb{R}^d$-valued Lévy process (in law), and the random integral $\int_{[0,\infty)} T(s)L(\mathrm{d}s)$ studied in the previous subsections coincides furthermore with the well-known integral $\int_{[0,\infty)} T(s)\,\mathrm{d}Z_s$ with respect to (Z_s).

Consider now further a measure ρ on $(0,\infty)$ such that $\int_0^\infty \max\{1, t^2\}\,\rho(\mathrm{d}t) < \infty$. Then define $T\colon (0,\infty) \to [0,\infty)$ by

$$T(s) = \inf\{t \geq 0 \mid \rho([t,\infty)) \leq s\}, \qquad (s \in (0,\infty)),$$

and note that T is non-increasing and hence Borel-measurable. In the following we identify T with the matrix-valued mapping $T\mathbf{I}_n\colon (0,\infty) \to \mathbb{M}_d(\mathbb{R})$. Note that $T(s) = 0$ whenever $s \geq M := \rho((0,\infty))$. It is well-known that ρ may be recovered as the transformation of λ by T, i.e.,

$$\rho(B) = \lambda(T^{-1}(B)) \quad \text{for any Borel subset } B \text{ of } (0,\infty).$$

Indeed, this follows by noting e.g. that

$$(0, \rho([\alpha,\infty))) \subseteq T^{-1}([\alpha,\infty)) \subseteq (0, \rho([\alpha,\infty))]$$

for any number α in $(0,\infty)$. Note now that

$$\int_0^\infty T(s)^2\,\lambda(\mathrm{d}s) = \int_0^\infty t^2\,\lambda \circ T^{-1}(\mathrm{d}t) = \int_0^\infty t^2\,\rho(\mathrm{d}t) < \infty,$$

and since $T(s) = 0$ whenever $s \geq M$, we also have that $\lambda(\{T \neq 0\}) \leq M < \infty$. It follows thus from Corollary 3.6 that $\mathsf{dom}_{ID}(\Upsilon_T) = \mathcal{ID}(\mathbb{R}^d)$, and for any μ from $\mathcal{ID}(\mathbb{R}^d)$ that

$$C_{\Upsilon_T(\mu)}(z) = \int_0^M C_\mu(T(s)z)\,\lambda(\mathrm{d}s) = \int_0^M C_\mu(tz)\,\rho(\mathrm{d}t),$$

which shows that Υ_T coincides with the mapping Υ_ρ studied in Barndorff-Nielsen et al. (2008). Moreover, if we consider the characteristic triplet (A, ν, η) for μ

and the Lévy basis L given by (6.23), then the integral $\int_{(0,\infty)} T(s)\,L(\mathrm{d}s)$ is well-defined, and with (Z_t) the Lévy process given by (6.24) we have by application of Theorem 6.9 that

$$\mathcal{L}\Big\{\int_{(0,\infty)} T(s)\,\mathrm{d}Z_s\Big\} = \mathcal{L}\Big\{\int_{(0,\infty)} T(s)\,L(\mathrm{d}s)\Big\} = \Upsilon_T(\mu) = \Upsilon_\rho(\mu), \tag{6.25}$$

thus recovering the random integral representation of Υ_ρ established in Barndorff-Nielsen et al. (2008).

In the particular case where $\rho(\mathrm{d}t) = \mathrm{e}^{-t}\,\mathrm{d}t$, we find that

$$T(s) = \begin{cases} \ln(s^{-1}), & \text{if } s \in (0,1), \\ 0, & \text{if } s \in [1,\infty), \end{cases}$$

and hence that

$$\Upsilon_\rho(\mu) = \mathcal{L}\Big\{\int_0^1 \ln(s^{-1})\,\mathrm{d}Z_s\Big\}$$

for any measure μ in $\mathcal{ID}(\mathbb{R}^d)$. In the case $d = 1$ this provides a random integral representation of the measures in the so-called Goldie-Steutel-Bondesson class, which was obtained in Barndorff-Nielsen et al. (2006b).

6.5. *Random integral representation of Lévy Processes.* In this subsection we extend the random integral representation of $\Upsilon_T(\mu)$ established in Subsection 6.3 to a representation of the entire Lévy process associated to $\Upsilon_T(\mu)$.

Throughout the subsection we let $(S, \mathcal{S}, \gamma)$ be a σ-finite measure space, and we denote by λ the Lebesgue measure on $[0,\infty)$. We consider further a fixed measurable mapping $T\colon S \to \mathbb{M}_d(\mathbb{R})$. For any t, t' in $[0,\infty)$ such that $t < t'$ we then define the mapping $\tilde{T}_{t,t'}\colon S \times [0,\infty) \to \mathbb{M}_d(\mathbb{R})$ by

$$\tilde{T}_{t,t'}(s,u) = T(s)1_{(t,t']}(u), \qquad ((s,u) \in S \times [0,\infty)).$$

Lemma 6.11. *Let $f\colon \mathbb{M}_d(\mathbb{R}) \to \mathbb{C}$ be a Borel function such that $f(0) = 0$. Then for any t, t' in $[0,\infty)$, such that $t < t'$, we have that*

$$f \circ \tilde{T}_{t,t'} \in \mathcal{L}^1(\gamma \otimes \lambda) \iff f \circ T \in \mathcal{L}^1(\gamma).$$

If $f \geq 0$ or $f \circ T \in \mathcal{L}^1(\gamma)$, we have furthermore that

$$\int_{S\times[0,\infty)} f \circ \tilde{T}_{t,t'}(s,u)\,\gamma\otimes\lambda(\mathrm{d}s,\mathrm{d}u) = (t'-t)\int_S f\circ T(s)\,\gamma(\mathrm{d}s).$$

Proof: For any Borel set B in $\mathbb{M}_d(\mathbb{R})$ we note first that

$$(\gamma\otimes\lambda)\big(\tilde{T}_{t,t'}^{-1}(B\setminus\{0\})\big) = (t'-t)\gamma\big(T^{-1}(B\setminus\{0\})\big).$$

Hence, if $f \geq 0$, we find by transformation that

$$\begin{aligned}
&\int_{S\times[0,\infty)} f\circ\tilde{T}_{t,t'}(s,u)\,\gamma\otimes\lambda(\mathrm{d}s,\mathrm{d}u) = \int_{\mathbb{M}_d(\mathbb{R})} f(w)\,(\gamma\otimes\lambda)\circ\tilde{T}_{t,t'}^{-1}(\mathrm{d}w) \\
&= \int_{\mathbb{M}_d(\mathbb{R})\setminus\{0\}} f(w)\,(\gamma\otimes\lambda)\circ\tilde{T}_{t,t'}^{-1}(\mathrm{d}w) = (t'-t)\int_{\mathbb{M}_d(\mathbb{R})\setminus\{0\}} f(w)\,\gamma\circ T^{-1}(\mathrm{d}w) \\
&= (t'-t)\int_{\mathbb{M}_d(\mathbb{R})} f(w)\,\gamma\circ T^{-1}(\mathrm{d}w) = (t'-t)\int_S f\circ T(s)\,\gamma(\mathrm{d}s).
\end{aligned}$$

The remaining statements in the lemma now follow by splitting a real-valued f in its positive and negative parts and subsequently a complex valued f in its real- and imaginary parts. $\square$

Lemma 6.12. *Let t, t' be non-negative numbers such that $t < t'$, and consider as above the matrix-valued mappings T and $\tilde{T}_{t,t'}$ defined on the σ-finite measure spaces $(S, \mathcal{S}, \gamma)$ and respectively $(S \times [0,\infty), \mathcal{S} \otimes \mathcal{B}([0,\infty)), \gamma \otimes \lambda)$. We then have that*

$$\mathsf{dom}_L(\Upsilon_T^0) = \mathsf{dom}_L(\Upsilon^0_{\tilde{T}_{t,t'}}), \quad \textit{and} \quad \mathsf{dom}_{ID}(\Upsilon_T) = \mathsf{dom}_{ID}(\Upsilon_{\tilde{T}_{t,t'}}).$$

For μ in $\mathsf{dom}_{ID}(\Upsilon_T)$ it holds furthermore that

$$\Upsilon_{\tilde{T}_{t,t'}}(\mu) = \Upsilon_T(\mu)^{t'-t},$$

where $(\Upsilon_T(\mu)^u)_{u \geq 0}$ is the convolution semi-group associated to the infinitely divisible measure $\Upsilon_T(\mu)$.

Proof: For any z in $\mathbb{R}^d$ we consider the function $g_z \colon \mathbb{M}_d(\mathbb{R}) \to [0,\infty)$ given by

$$g_z(w) = \min\{1, \|wz\|^2\}, \qquad (w \in \mathbb{M}_d(\mathbb{R})),$$

and we note that $g_z(0) = 0$. For any Leẃy measure ν on $\mathbb{R}^d$ it follows then by application of Tonellis theorem and Lemma 6.11 that

$$\begin{aligned}
&\int_{S\times[0,\infty)} \Big(\int_{\mathbb{R}^d} g_z(\tilde{T}_{t,t'}(s,u))\, \nu(\mathrm{d}z) \Big)\, \gamma \otimes \lambda(\mathrm{d}s, \mathrm{d}u) \\
&\qquad = \int_{\mathbb{R}^d} \Big(\int_{S\times[0,\infty)} g_z(\tilde{T}_{t,t'}(s,u))\, \gamma \otimes \lambda(\mathrm{d}s, \mathrm{d}u) \Big)\, \nu(\mathrm{d}z) \\
&\qquad = (t'-t) \int_S \Big(\int_{\mathbb{R}^d} g_z(T(s))\, \nu(\mathrm{d}z) \Big)\, \gamma(\mathrm{d}s).
\end{aligned}$$

Hence it follows from (3.12) that

$$\nu \in \mathsf{dom}_L(\Upsilon^0_{\tilde{T}_{t,t'}}) \iff \nu \in \mathsf{dom}_L(\Upsilon^0_T). \tag{6.26}$$

Consider next a measure μ in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν). For any z in $\mathbb{R}^d$ we introduce then the function $f_z \colon \mathbb{M}_d(\mathbb{R}) \to \mathbb{C}$ given by

$$f_z(w) = C_\mu(w^* z), \qquad (w \in \mathbb{M}_d(\mathbb{R})),$$

and we note that $f_z(0) = 0$. It follows thus from Lemma 6.11 that $f_z \circ \tilde{T}_{t,t'} \in \mathcal{L}^1(\gamma \otimes \lambda)$, if and only if $f \circ T \in \mathcal{L}^1(\gamma)$. In combination with (6.26) and Proposition 3.5 this shows that

$$\mu \in \mathsf{dom}_{ID}(\Upsilon_T) \iff \mu \in \mathsf{dom}_{ID}(\Upsilon_{\tilde{T}_{t,t'}}).$$

In the affirmative case Lemma 6.11 yields further in combination with Proposition 3.5 that

$$\begin{aligned}
C_{\Upsilon_{\tilde{T}_{t,t'}}(\mu)}(z) &= \int_{S\times[0,\infty)} C_\mu(\tilde{T}_{t,t'}(s,u)^* z)\, \gamma \otimes \lambda(\mathrm{d}s, \mathrm{d}u) = (t'-t) \int_S C_\mu(T(s)^* z)\, \gamma(\mathrm{d}s) \\
&= (t'-t) C_{\Upsilon_T(\mu)}(z) = C_{\Upsilon_T(\mu)^{t'-t}}(z),
\end{aligned} \tag{6.27}$$

which completes the proof. $\square$

Theorem 6.13. *Let μ be a probability measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (η, A, ν), and assume that $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$. Consider further the σ-finite measure space $(S \times [0,\infty), \mathcal{S} \otimes \mathcal{B}([0,\infty)), \gamma \otimes \lambda)$ and a factorizable Lévy basis*[4]

$$L = \{L(F) \mid F \in (\mathcal{S} \otimes \mathcal{B}([0,\infty)))^0\}$$

with generating quadruplet $(\eta, A, \nu, k_L(\gamma \otimes \lambda))$ (cf. Remark 6.2). For each t in $[0,\infty)$ we may then define

$$Z_t = \int_{S\times[0,\infty)} T(s)1_{(0,t]}(u)\, L(\mathrm{d}s, \mathrm{d}u),$$

and it follows that $(Z_t)_{t\geq 0}$ is a Lévy process in law with marginals given by

$$L\{Z_t\} = \Upsilon_T(\mu)^t, \qquad (t \in [0,\infty)),$$

where $(\Upsilon_T(\mu)^t)_{t\geq 0}$ is the convolution semi-group associated to the infinitely divisible measure $\Upsilon_T(\mu)$.

Proof: Since $\mu \in \mathsf{dom}_{ID}(\Upsilon_T)$, it follows from Lemma 6.12 that $\mu \in \mathsf{dom}_{ID}(\Upsilon_{\tilde{T}_{t,t'}})$ for all positive numbers t, t' such that $t < t'$. Hence Proposition 6.7 ensures that the integral

$$\int_{S\times[0,\infty)} T(s)1_{(t,t']}(u)\, L(\mathrm{d}s, \mathrm{d}u) = \int_{S\times[0,\infty)} \tilde{T}_{t,t'}(s,u)\, L(\mathrm{d}s, \mathrm{d}u)$$

is well-defined, and in particular Z_t is well-defined for all t. Lemma 6.12 yields further in combination with Theorem 6.9 that

$$L\{Z_t\} = \Upsilon_T(\mu)^t \quad \text{for all } t \text{ in } [0,\infty), \tag{6.28}$$

and it remains to show that $(Z_t)_{t\geq 0}$ is a Lévy process in law. Clearly (6.28) implies that $Z_0 = 0$ almost surely, and that $Z_t \to 0$ in distribution as $t \searrow 0$. With t, t' as above we note further that

$$Z_{t'} - Z_t = \int_{S\times[0,\infty)} T(s)1_{(t,t']}(u)\, L(\mathrm{d}s, \mathrm{d}u) = \int_{S\times[0,\infty)} \tilde{T}_{t,t'}(s,u)\, L(\mathrm{d}s, \mathrm{d}u),$$

and hence Theorem 6.9 and Lemma 6.12 yield that

$$L\{Z_{t'} - Z_t\} = \Upsilon_{\tilde{T}_{t,t'}}(\mu) = \Upsilon_T(\mu)^{t'-t},$$

so that $(Z_t)_{t\geq 0}$ has stationary increments. For positive numbers $t_1, t_2, \ldots, t_n$ such that $0 < t_1 < t_2 < \cdots < t_n$, we consider finally the corresponding increments of (Z_t):

$$Z_{t_1} = \int_{S\times[0,\infty)} T(s)1_{(0,t_1]}(u)\, L(\mathrm{d}s, \mathrm{d}u),$$

$$Z_{t_2} - Z_{t_1} = \int_{S\times[0,\infty)} T(s)1_{(t_1,t_2]}(u)\, L(\mathrm{d}s, \mathrm{d}u), \ldots$$

$$\ldots, Z_{t_n} - Z_{t_{n-1}} = \int_{S\times[0,\infty)} T(s)1_{(t_{n-1},t_n]}(u)\, L(\mathrm{d}s, \mathrm{d}u).$$

In case T is a simple function (cf. (6.9)) and γ is finite, it follows immediately from (6.10) and the definition of a Lévy basis that these increments are independent. For general T and σ-finite γ the same conclusion follows subsequently from the fact

[4]Here $(\mathcal{S} \otimes \mathcal{B}([0,\infty)))^0$ denotes the class of sets F from $\mathcal{S} \otimes \mathcal{B}([0,\infty))$ such that $\gamma \otimes \lambda(F) < \infty$.

that independence is preserved under limits in probability (cf. Propositions 6.4 and 6.7). This completes the proof. □

Summary.

This paper investigates an alternative to probability mixing, termed Lévy Mixing, which ensures infinite divisibility of the resulting distribution. The Lévy Mixing is defined in terms of Upsilon transformations $\Upsilon_T^0 \colon \mathfrak{M}_L(\mathbb{R}^d) \to \mathfrak{M}_L(\mathbb{R}^d)$ and $\Upsilon_T \colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ associated to a measurable mapping $T \colon S \to \mathbb{M}_d(\mathbb{R})$ defined on a measure space $(S, \mathcal{S}, \gamma)$. Basic properties of the Upsilon transformations are established, including continuity of the mapping $(T, \mu) \mapsto \Upsilon_T(\mu)$ in both variables (separately). It is further established that the mapping Υ_T^0 –and hence the process of Lévy Mixing– has regularising effects, such as ensuring absolute continuity of the resulting Lévy measure. Finally the measure $\Upsilon_T(\mu)$ is realized as the distribution of the stochastic integral of T with respect to a Lévy basis (depending on μ). The existence of the stochastic integral in question is derived simultaneously. The results mentioned above are all proved under rather mild conditions on the mapping T and the measure γ. The type of Levy mixing considered here is a special case of a general concept of Levy mixing that we hope to discuss, in a separate note, in relation to probability mixing and a third kind of mixing termed random object mixing.

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