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On hitting probabilities of beta coalescents and absorption times of coalescents that come down from infinity

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Abstract. Let $X = (X_k)_{k=0,1,\ldots}$ denote the jump chain of the block counting process of the Λ-coalescent with $\Lambda = \beta(2-\alpha, \alpha)$ being the beta distribution with parameter $\alpha \in (0,2)$. A solution for the hitting probability $h(n,m)$ that the chain X ever visits the state m, conditional that it starts in the state $X_0 = n$, is obtained via an analytic method based on generating functions. For $\alpha \in (1,2)$ the results are applied to characterize the distribution of the almost sure limit τ of the absorption times τ_n of the coalescent restricted to a sample of size n. The latter result is generalized to arbitrary exchangeable coalescents (Ξ-coalescents) that come down from infinity. The results generalize those obtained for the particular case $\alpha = 1$ in Möhle, M. (2014) Asymptotic hitting probabilities for the Bolthausen–Sznitman coalescent, *J. Appl. Probab.* **51A**, to appear. This article furthermore supplements the work of Hénard, O. (2013), The fixation line, Preprint, arXiv:1307.0784.

1. Introduction

Let $\Pi = (\Pi_t)_{t \geq 0}$ be a continuous-time exchangeable coalescent. Note that Π is a Markov process with state space $\mathcal{P}$, the set of partitions on $\mathbb{N} := \{1, 2, \ldots\}$. During each transition blocks merge together to form single blocks. For more information on such coalescent processes with multiple collisions (Λ-coalescent) we refer the reader to Pitman (1999), Sagitov (1999) and Schweinsberg (2000b). For the general class of exchangeable coalescents allowing for simultaneous multiple collisions (Ξ-coalescent) we refer the reader to Möhle and Sagitov (2001) and Schweinsberg (2000a).

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For $t \in [0,\infty)$ let $N_t = |\Pi_t|$ denote the number of blocks of the random partition Π_t. The process $N := (N_t)_{t\ge 0}$ is called the block counting process of the coalescent Π. It is well known that N is Markovian. Let $X = (X_k)_{k\in\mathbb{N}_0:=\{0,1,\ldots\}}$ denote the jump chain of the block counting process N. Note that X is a chain with the only absorbing state 1 having only downward jumps of at least size 1. For $n,m \in \mathbb{N}$ we are interested in the hitting probability

$$h(n,m) \;:=\; \mathbb{P}(X_k = m \text{ for some } k \in \mathbb{N}_0 \,|\, X_0 = n) \tag{1.1}$$

that the jump chain X ever visits the state m conditional that the chain starts in the state $X_0 = n$. Since X is a jump chain with the only absorbing state 1 it follows for all $2 \le m \le n$ that $h(n,m) = \sum_{k=0}^\infty \mathbb{P}(X_k = m \,|\, X_0 = n) = \mathbb{E}(\sum_{k=0}^\infty 1_{\{X_k=m\}} \,|\, X_0 = n)$. Thus, for $2 \le m \le n$ the hitting probability $h(n,m)$ coincides with the entry $g(n,m)$ of the Green matrix $G = (g(n,m))_{n,m\in\mathbb{N}}$. Note that $g(n,m)$ is defined (see, for example, Norris (1997, p. 145)) as the expected number of visits of X of the state m, conditional that the chain starts in the state $X_0 = n$. The problem of solving the hitting probabilities $h(n,m)$ of the chain X is therefore essentially equivalent to the problem of solving the Green matrix G of X.

The hitting probabilities $h(n,m)$, $n,m \in \mathbb{N}$, are the unique minimal non-negative solution of a certain system of linear equations (see, for example, Norris (1997, Theorem 1.3.2)). However, closed solutions for $h(n,m)$ are usually not straightforward to obtain.

Clearly, for the Kingman coalescent, where $\Lambda = \delta_0$ is the Dirac measure at 0, we have $h(n,m) = 1$ for all $n,m \in \mathbb{N}$ with $m \le n$, since the block counting process of the Kingman n-coalescent visits every state $m \in \{1,\ldots,n\}$ almost surely. For the star shaped coalescent ($\Lambda = \delta_1$), we have $h(1,m) = \delta_{1m}$ and $h(n,m) = \delta_{nm} + \delta_{1m}$ for all $n,m \in \mathbb{N}$ with $n > 1$, where δ_{nm} denotes the Kronecker symbol.

For the particular case when Λ is the uniform distribution on $[0,1]$ (Bolthausen–Sznitman coalescent) a formula for $h(n,m)$ in terms of the Bernoulli numbers of the second kind is provided in Möhle (2014, Eq. (11)) leading to an integral representation for the so called asymptotic hitting probabilities $h(m) := \lim_{n\to\infty} h(n,m)$, $m \in \mathbb{N}$.

In the first part of the paper (Section 2) we mainly focus on the case when the measure Λ of the coalescent is the beta distribution $\Lambda = \beta(2-\alpha,\alpha)$ with parameter $\alpha \in (0,2)$ having density $x \mapsto (B(2-\alpha,\alpha))^{-1}x^{1-\alpha}(1-x)^{\alpha-1}$, $x \in (0,1)$, with respect to Lebesgue measure on $(0,1)$, where $B(.,.)$ denotes the beta function. For $\alpha = 1$ we obtain the Bolthausen–Sznitman coalescent. The boundary case $\alpha \to 0$ corresponds to the star-shaped coalescent whereas for $\alpha \to 2$ we obtain the Kingman coalescent.

A dual forwards in time model is exploited by Hénard (2013, Corollary 3.4) in order to verify that, for the $\beta(2-\alpha,\alpha)$-coalescent with parameter $\alpha \in (0,2)$, the hitting probabilities $h(n,m)$ converge as $n \to \infty$ and he provides integral representations for the asymptotic hitting probabilities $h(m) = \lim_{n\to\infty} h(n,m)$, $m \in \mathbb{N}$. However, solutions for $h(n,m)$ for finite n are not provided.

In this paper we extend the analytic method used in Möhle (2014) based on generating functions and derive a solution for the hitting probabilities $h(n,m)$, or, equivalently, for the Green matrix G, for the case when the measure Λ of the coalescent is the beta distribution $\Lambda = \beta(2-\alpha,\alpha)$ with parameter $\alpha \in (0,2)$, supplementing the work of Hénard (2013). The approach chosen in this article to obtain results on (asymptotic) hitting probabilities is mainly analytical (as in

Möhle (2014)), whereas recent works (see, for example, Hénard (2013)) are dealing with probabilistic concepts such as lookdown graphs.

The second part of this article (Section 3) deals with the absorption time $\tau := \inf\{t > 0 : |\Pi_t| = 1\}$ of exchangeable Ξ-coalescents $\Pi = (\Pi_t)_{t\geq 0}$ that come down from infinity. For $n \in \mathbb{N}$ let $\Pi^{(n)} = (\Pi_t^{(n)})_{t\geq 0}$ denote the coalescent restricted to a sample of size n and let $\tau_n := \inf\{t > 0 : |\Pi_t^{(n)}| = 1\}$ denote the absorption time of $\Pi^{(n)}$. The main result (Theorem 3.3) states that $\tau_n \to \tau$ almost surely and in L^p for any $p \in (0,\infty)$ and it is shown that the distribution of τ is uniquely determined via its moments. A new formula (see (3.4)) is provided which relates the moments of τ with the hitting probabilities (1.1) and the asymptotic hitting probabilities $h(m) := \mathbb{P}(|\Pi_t| = m$ for some $t \geq 0)$, $m \in \mathbb{N}$. The proof of Theorem 3.3 in particular shows that $h(n,m) \to h(m)$ as $n \to \infty$ for any $m \in \mathbb{N}$.

The results on the hitting probabilities of the $\beta(2-\alpha,\alpha)$-coalescent obtained in the first part of the article (Section 2) are applied to obtain concrete information (see Corollaries 3.1 and 3.2) on the almost sure limit $\tau = \lim_{n\to\infty} \tau_n$ of the $\beta(2-\alpha,\alpha)$-coalescent restricted to a sample of size n.

The paper is organized as follows. The results on the hitting probabilities for the $\beta(2-\alpha,\alpha)$-coalescent are provided in the following Section 2. In Section 3 the results on the absorption times are provided. Proofs are provided in Sections 4 and 5.

2. Hitting probabilities

Throughout this section $D := \{z \in \mathbb{C} : |z| < 1\}$ denotes the open unit disk. It will turn out to be convenient to introduce for $\alpha \in (0,2)$ the function $L_\alpha : D \to \mathbb{C}$ via

$$L_\alpha(z) \;:=\; \sum_{n=1}^{\infty} \frac{\Gamma(n+\alpha-1)}{\Gamma(n+1)\Gamma(\alpha)} z^n \;=\; \begin{cases} \dfrac{1-(1-z)^{1-\alpha}}{1-\alpha} & \text{if } \alpha \in (0,2) \setminus \{1\}, \\ -\log(1-z) & \text{if } \alpha = 1. \end{cases} \tag{2.1}$$

Note that $L_\alpha(1-) = 1/(1-\alpha) < \infty$ for $\alpha \in (0,1)$ whereas $L_\alpha(1-) = \infty$ for $\alpha \in [1,2)$.

The first result, Theorem 2.1 below, provides a formula for the hitting probability $h(n,m)$ that the jump chain X of the block counting process of the $\beta(2-\alpha,\alpha)$-coalescent, started at the state $n \in \mathbb{N}$, ever visits the state $m \in \{1,\ldots,n\}$. For coalescents different from the $\beta(2-\alpha,\alpha)$-coalescent with parameter $\alpha \in (0,2)$ (and of course different from the Kingman coalescent and the star shaped coalescent), the problem of finding a solution for the hitting probabilities $h(n,m)$ remains open.

For a power series $f(z) = \sum_{n=0}^\infty f_n z^n$ we denote in the following with $[z^n]f(z) := f_n$ the coefficient in front of z^n in the series expansion of f. The proof of the following Theorem 2.1 is provided in Section 4.

Theorem 2.1 (Hitting probabilities)**.** *For the $\beta(2-\alpha,\alpha)$-coalescent with parameter $\alpha \in (0,2)$, the hitting probabilities (1.1) are given by $h(n,1) = 1$ for $n \in \mathbb{N}$, $h(n,m) = 0$ for $n < m$, and*

$$h(n,m) \;=\; \frac{1}{B(m-1,\alpha)} \sum_{k=m-1}^{n-1} \frac{B(k,n-k-1+\alpha)}{B(k,n-k)} [z^k] \int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \tag{2.2}$$

for $2 \le m \le n$*, where* $B(.,.)$ *denotes the beta function,* L_α *is defined via (2.1) and*

$$[z^k]\int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;=\; \frac{1}{k}\sum_{j=1}^{k-m+1}\frac{(-1)^j}{(\Gamma(\alpha))^j}\sum_{\substack{n_1,\ldots,n_j\in\mathbb{N}\\ n_1+\cdots+n_j=k-m+1}}\frac{\Gamma(n_1+\alpha)\cdots\Gamma(n_j+\alpha)}{\Gamma(n_1+2)\cdots\Gamma(n_j+2)}. \tag{2.3}$$

The following result (Corollary 2.2) is known from the literature. It was first verified for the particular case $\alpha = 1$ (Bolthausen–Sznitman coalescent) by Möhle (2014) via an analytic method based on generating functions. Note that this analytic method is extended in the present paper in order to verify Theorem 2.1. Hénard provided shortly later (Hénard (2013)) a different proof of Corollary 2.2 for $\alpha \in (0,2)$ by exploiting a time-reversed (lookdown) model based on the Poisson construction of the coalescent. We provide an alternative proof of Corollary 2.2 based on Theorem 2.1.

Corollary 2.2 (Asymptotic hitting probabilities)**.** *For the* $\beta(2-\alpha,\alpha)$*-coalescent with parameter* $\alpha \in (0,2)$*, the limiting hitting probabilities* $h(m) := \lim_{n\to\infty} h(n,m)$*,* $m \in \mathbb{N}$*, exist. Moreover,* $h(1) = 1$ *and*

$$h(m) \;=\; \frac{1}{B(m-1,\alpha)}\int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t, \qquad m \in \mathbb{N}\setminus\{1\}, \tag{2.4}$$

where $B(.,.)$ *denotes the beta function and* L_α *is defined via (2.1).*

Remark 2.3. For $m \ge 2$ let $q(m)$ denote the probability that exactly m blocks merge together during the last jump of the coalescent process (before the process reaches its absorbing state). The probability $q(m)$ is naturally obtained (see also Hénard (2013)) from the asymptotic hitting probability $h(m)$ via $q(m) = h(m)p_{m1}$, $m \ge 2$, where p_{m1} denotes the transition probability that the jump chain of the block counting process moves from m to 1. Formulas for p_{m1} are well known, see for example Eq. (4.4) provided later in this article. The distribution q of the number of blocks involved in the last coalescence is of its own interest and has relations to cuttings of random trees. For the Bolthausen–Sznitman coalescent we refer the reader to Goldschmidt and Martin (2005), where an edge cutting procedure for the random recursive tree is exploited. For constructions of beta coalescents via pruning procedures for binary trees we refer the reader to Abraham and Delmas (2013b) and Abraham and Delmas (2013a).

The next corollary provides an alternative formula for $h(m)$ which is in particular useful to compute $h(m)$ for small values of m. In the following $\Psi := \Gamma'/\Gamma$ denotes the logarithmic derivative of the gamma function (digamma function).

Corollary 2.4 (Alternative formula for the asymptotic hitting probabilities)**.** *For the* $\beta(2-\alpha,\alpha)$*-coalescent,* $\alpha \in (0,2)$*, the asymptotic hitting probabilities satisfy*

$$h(m) \;=\; \frac{1}{B(m-1,\alpha)}\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1} l_\alpha(i), \qquad m \in \mathbb{N}\setminus\{1\}, \tag{2.5}$$

where the function $l_\alpha : [0,\infty) \to \mathbb{R}$ *is defined via* $l_\alpha(x) := \Psi((x+1)/(1-\alpha))$ *for* $\alpha \in (0,1)$*,* $l_\alpha(x) := \log(x+1)$ *for* $\alpha = 1$*, and* $l_\alpha(x) := \Psi((\alpha+x)/(\alpha-1))$ *for* $\alpha \in (1,2)$*.*

Remark 2.5. For instance, $h(2) = \alpha\Psi(2/(1-\alpha)) - \alpha\Psi(1/(1-\alpha))$ for $\alpha \in (0,1)$, $h(2) = \log 2$ for $\alpha = 1$, and $h(2) = \alpha\Psi((\alpha+1)/(\alpha-1)) - \alpha\Psi(\alpha/(\alpha-1))$ for $\alpha \in (1,2)$. Note that the function l_α is slowly varying at infinity. For the asymptotic behavior

$$h(m) \sim \begin{cases} \dfrac{1-\alpha}{\Gamma(\alpha)} m^{\alpha-1} & \text{for } \alpha \in (0,1), \\ \dfrac{1}{\log m} & \text{for } \alpha = 1, \\ \alpha - 1 & \text{for } \alpha \in (1,2) \end{cases} \tag{2.6}$$

as $m \to \infty$ we refer the reader to Corollary 1.3 of Möhle (2014) for $\alpha = 1$, to Theorem 1.8 of Berestycki et al. (2008) for $\alpha \in (1,2)$, and to Corollary 3.5 of Hénard (2013) for $\alpha \in (0,2) \setminus \{1\}$.

3. Absorption times

Theorem 2.1, Corollary 2.2 and Corollary 2.4 have direct applications for functionals of the $\beta(2-\alpha,\alpha)$-coalescent (restricted to a sample of size $n \in \mathbb{N}$) such as the number of collisions C_n, the absorption time τ_n and the total branch length L_n (the sum of the lengths of all branches of the coalescent tree). For example, Theorem 2.1 can be used to compute the mean of these functionals, since $\mathbb{E}(C_n) = \sum_{m=2}^n h(n,m)$, $\mathbb{E}(\tau_n) = \sum_{m=2}^n g_m^{-1} h(n,m)$ and $\mathbb{E}(L_n) = \sum_{m=2}^n m g_m^{-1} h(n,m)$ for all $n \in \mathbb{N}$, where $g_1, g_2, \ldots$ denote the total rates of the coalescent given via (4.3). For an overview of convergence results for the main functionals C_n, τ_n, and L_n of beta coalescents we refer the reader to Gnedin et al. (2014) and Kersting (2012) and the references therein. As an application, we verify the following result (Corollary 3.1) for the absorption time τ_n until the coalescent, started in a state with n blocks, reaches its absorbing state. For $\alpha \in (1,2)$, which implies that the coalescent comes down from infinity, the almost sure convergence of τ_n as $n \to \infty$ is well known from the literature (Pitman (1999), Schweinsberg (2000b)). The following corollary provides a bit more information on the distribution of the limiting random variable τ, which partly fills a gap ($0 < a < 1$ and $b = 2-a$) in Table 2 of Gnedin et al. (2014). In the following $\gamma \approx 0.577216$ denotes the Euler constant.

Corollary 3.1 (Convergence of the absorption times)**.** *Fix $\alpha \in (1,2)$. For the $\beta(2-\alpha,\alpha)$-coalescent, as $n \to \infty$, $\tau_n \to \tau$ almost surely and in L^p for any $p \in (0,\infty)$, where the distribution of the limiting random variable τ is uniquely determined via its finite moments*

$$\mathbb{E}(\tau^j) \;=\; j! \sum_{2 \le m_1 \le \cdots \le m_j} \frac{h(m_j) h(m_j, m_{j-1}) \cdots h(m_2, m_1)}{g_{m_1} \cdots g_{m_j}}, \qquad j \in \mathbb{N}, \tag{3.1}$$

with $h(n,m)$, $h(m)$, and g_m given by (2.2), (2.4), and (4.3) respectively. In particular,

$$\mathbb{E}(\tau) \;=\; \sum_{m=2}^{\infty} \frac{h(m)}{g_m} \;=\; \alpha(\alpha-1) \int_0^1 \frac{t}{(1-t)^{2-\alpha} - (1-t)} \,\mathrm{d}t \;=\; \alpha\left(\Psi\left(\frac{\alpha}{\alpha-1}\right) + \gamma\right),$$

in agreement with Hénard (2013, Corollary 3.6).

For $\alpha \in (1,2)$ it does not seem to be straightforward to derive simple expressions for $\mathbb{E}(\tau^j)$, $j \ge 2$. The next result at least expresses $\mathbb{E}(\tau^j)$ in terms of the moments of the τ_m, $m \ge 2$.

Corollary 3.2 (Alternative formula for the moments of τ)**.** *Fix $\alpha\in(1,2)$. For the $\beta(2-\alpha,\alpha)$-coalescent, the moments of the limiting random variable τ satisfy*

$$\mathbb{E}(\tau^j) \;=\; j\sum_{m=2}^\infty \frac{h(m)}{g_m}\mathbb{E}(\tau_m^{j-1}) \;=\; \alpha j\sum_{m=2}^\infty w_m\mathbb{E}(\tau_m^{j-1}), \qquad j\in\mathbb{N}, \tag{3.2}$$

with weights

$$w_m \;:=\; w_m(\alpha) \;:=\; \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;=\; \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\Psi\Big(\frac{\alpha+i}{\alpha-1}\Big), \qquad m\ge 2.$$

In particular, $\mathbb{E}(\tau^j)\le j!(\mathbb{E}(\tau))^j$ for all $j\in\mathbb{N}$.

The main assertions stated in Corollary 3.1 and Corollary 3.2 are not very specific to the $\beta(2-\alpha,\alpha)$-coalescent. They generalize to coalescents with multiple collisions (Λ-coalescents) and even to coalescents allowing for simultaneous multiple collisions (Ξ-coalescents), as long as the coalescent comes down from infinity. In the following Ξ denotes a measure on the infinite simplex

$$\Delta \;:=\; \{x=(x_1,x_2,\ldots): x_1\ge x_2\ge\cdots\ge 0, \textstyle\sum_{i=1}^\infty x_i\le 1\} \tag{3.3}$$

satisfying $\Xi(\Delta)\in(0,\infty)$. As in Schweinsberg (2000a) we call a Ξ-coalescent $\Pi=(\Pi_t)_{t\ge 0}$ standard if Π_0 is the partition of $\mathbb{N}$ into singletons and we say that a standard coalescent Π comes down from infinity if $\mathbb{P}(|\Pi_t|<\infty$ for all $t\in(0,\infty))=1$. If a standard Ξ-coalescent comes down from infinity, then its absorption time $\tau:=\inf\{t>0:|\Pi_t|=1\}$ satisfies $\mathbb{E}(\tau)<\infty$. The converse is not true in general. There exist coalescents that satisfy $\mathbb{E}(\tau)<\infty$ but do not come down from infinity. For example, the standard star-shaped coalescent (the Λ-coalescent with $\Lambda=\delta_1$, the Dirac measure at 1) stays a standard exponential time τ in the partition of $\mathbb{N}$ into singletons and at this time τ all blocks merge together to form the block $\mathbb{N}$. Thus, this coalescent satisfies $\mathbb{E}(\tau)=1<\infty$ but does not come down from infinity. Under the additional assumption that $\Xi(\Delta_f)=0$, where $\Delta_f:=\{x=(x_1,x_2,\ldots)\in\Delta: x_1+\cdots+x_k=1$ for some $k\in\mathbb{N}\}$, the Ξ-coalescent comes down from infinity if and only if $\mathbb{E}(\tau)<\infty$ (see Schweinsberg (2000a)).

As before we denote with g_m, $m\in\mathbb{N}$, the total rates of the coalescent. Formulas for the total rates are available (see, for example, Schweinsberg (2000a, Eq. (70))). Furthermore, for $n\in\mathbb{N}$ we denote with ϱ_n the natural restriction from $\mathcal{P}$ to $\mathcal{P}_n$, the set of partitions on $\{1,\ldots,n\}$, and with $\Pi^{(n)}:=(\Pi_t^{(n)})_{t\ge 0}:=(\varrho_n\circ\Pi_t)_{t\ge 0}$ the coalescent restricted to a sample of size n.

Theorem 3.3 (Convergence of the absorption times)**.** *Let $\Pi=(\Pi_t)_{t\ge 0}$ be a standard Ξ-coalescent that comes down from infinity. Then, the absorption time $\tau:=\inf\{t>0:|\Pi_t|=1\}$ satisfies $\mathbb{E}(\tau^j)\le j!(\mathbb{E}(\tau))^j<\infty$ for all $j\in\mathbb{N}$. Moreover, the absorption times $\tau_n:=\inf\{t>0:|\Pi_t^{(n)}|=1\}$ of the restricted coalescent $\Pi^{(n)}$ satisfy $\tau_n\to\tau$ as $n\to\infty$ almost surely and in L^p for any $p\in(0,\infty)$, and the distribution of τ is uniquely determined via its moments*

$$\mathbb{E}(\tau^j) \;=\; j!\sum_{2\le m_1\le\cdots\le m_j}\frac{h(m_j)h(m_j,m_{j-1})\cdots h(m_2,m_1)}{g_{m_1}\cdots g_{m_j}}, \qquad j\in\mathbb{N}, \tag{3.4}$$

with $h(n,m)$ given via (1.1), i.e. $h(n,m)=\mathbb{P}(|\Pi_t^{(n)}|=m$ for some $t\ge 0)$ and with $h(m):=\mathbb{P}(|\Pi_t|=m$ for some $t\ge 0)$, $n,m\in\mathbb{N}$. Moreover, $\lim_{n\to\infty}h(n,m)=$

$h(m)$ for all $m \in \mathbb{N}$. Alternatively, the moments of τ can be expressed in terms of the moments of the random variables τ_m, $m \geq 2$, via

$$\mathbb{E}(\tau^j) \;=\; j \sum_{m=2}^{\infty} \frac{h(m)}{g_m} \mathbb{E}(\tau_m^{j-1}), \qquad j \in \mathbb{N}. \tag{3.5}$$

Remark 3.4. 1. Theorem 3.3 for example holds for the Λ-coalescent with $\Lambda := \beta(a,b)$ being the beta distribution with parameters $a \in (0,1)$ and $b \in (0,\infty)$. Note that this fills a gap in Table 2 of Gnedin et al. (2014).

2. In comparison to Corollary 3.1 and Corollary 3.2, the disadvantage of Theorem 3.3 is that the hitting probabilities $h(n,m)$ and $h(m)$ are (so far) not known explicitly for most coalescents. However, Theorem 3.3 holds for all standard Ξ-coalescents that come down from infinity and explains how (the distribution of) the absorption time depends on the hitting probabilities, which underlines the importance of the hitting probabilities.

3. Note that (3.5) implies that the Fourier transform φ_τ of τ can be expressed in terms of the Fourier transforms φ_{τ_m} of the random variables τ_m, $m \geq 2$, via $\varphi_\tau(t) = 1 + it\sum_{m=2}^{\infty} h(m) g_m^{-1} \varphi_{\tau_m}(t)$, $t \in \mathbb{R}$.

4. Theorem 3.3 in particular states that $\mathbb{E}(\tau) < \infty$ automatically implies that $\mathbb{E}(\tau^j) < \infty$ for all $j \in \mathbb{N}$. Verifying this property is one of the crucial steps in the proof of Theorem 3.3. The proof of Theorem 3.3 would be easier under the additional assumption that the total rates g_m, $m \in \mathbb{N}$, satisfy $\sum_{m=2}^{\infty} 1/g_m < \infty$. Note however that there exist coalescents that come down from infinity but satisfy $\sum_{m=2}^{\infty} 1/g_m = \infty$. For example, suppose that the measure Λ has density $x \mapsto -\log x$ with respect to Lebesgue measure on $(0,1)$. Schweinsberg (2000b, Example 14) verified that the corresponding Λ-coalescent comes down from infinity by showing that $\mathbb{E}(\tau) < \infty$. It is readily checked that this coalescent has total rates

$$g_m \;=\; \int_0^1 \frac{1-(1-x)^m - mx(1-x)^{m-1}}{x^2} (-\log x)\,\mathrm{d}x \;=\; m(h_m - 1), \qquad m \in \mathbb{N},$$

where $h_m := \sum_{j=1}^{m} 1/j$ denotes the mth harmonic number, $m \in \mathbb{N}$. Thus, $g_m \sim m \log m$ as $m \to \infty$, which implies that $\sum_{m=2}^{\infty} 1/g_m = \infty$.

4. Proofs concerning the hitting probabilities

Before we turn to the proof of Theorem 2.1 we have to study the reciprocal function of the power series $z \mapsto L_\alpha(z)/z$. More specifically, for $\alpha \in (0,2)$ define the function $f_\alpha : D \to \mathbb{C}$ via

$$f_\alpha(z) \;:=\; \begin{cases} 1 & \text{for } z = 0 \\ z/L_\alpha(z) & \text{for } z \in D \setminus \{0\}, \end{cases} \tag{4.1}$$

where L_α is defined via (2.1). The following lemma shows that f_α has a Taylor expansion around 0 which converges absolutely at least for $|z| < 1/2$. The lemma furthermore provides a solution for the Taylor coefficients $a_n = a_n(\alpha)$, $n \in \mathbb{N}_0$.

Lemma 4.1. *Fix $\alpha \in (0,2)$. The function (4.1) has Taylor expansion $f_\alpha(z) = \sum_{n=0}^{\infty} a_n z^n$ being absolutely convergent at least for $|z| < 1/2$, and the Taylor coefficients are $a_0 := 1$ and*

$$a_n \;:=\; a_n(\alpha) \;:=\; \sum_{k=1}^{n} (-1)^k \sum_{\substack{n_1,\ldots,n_k \in \mathbb{N} \\ n_1+\cdots+n_k = n}} u_{n_1} \cdots u_{n_k}, \qquad n \in \mathbb{N}, \tag{4.2}$$

where

$$u_n \;:=\; u_n(\alpha) \;:=\; \frac{\Gamma(n+\alpha)}{\Gamma(n+2)\Gamma(\alpha)} \;=\; \binom{n+\alpha-1}{n}\frac{1}{n+1}, \qquad n\in\mathbb{N}.$$

Remark 4.2. Alternatively, the Taylor coefficients $a_n = a_n(\alpha)$, $n \in \mathbb{N}_0$, can be computed iteratively via the recursion (4.14) provided later in this article. For $\alpha = 1$ there is an alternative formula for a_n in terms of Bernoulli numbers of the second kind available, see Möhle (2014, Lemma 3.1).

Proof: From $0<\alpha<2$ it follows that $0<u_n=\alpha(\alpha+1)\cdots(\alpha+n-1)/((n+1)!)\le 1$ for all $n\in\mathbb{N}$ and, hence,

$$|a_n| \;\le\; \sum_{k=1}^n \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}} \;=\; \sum_{k=1}^n \binom{n-1}{k-1} \;=\; 2^{n-1}.$$

The series $\sum_{n=0}^\infty a_n z^n$ is therefore absolutely convergent at least for $|z|<1/2$.

By (2.1), the map h_α, defined via $h_\alpha(0):=0$ and $h_\alpha(z):=1-L_\alpha(z)/z$ for $z\in D\setminus\{0\}$, has Taylor expansion $h_\alpha(z)=\sum_{n=1}^\infty(-u_n)z^n$ with $u_1,u_2,\ldots$ as defined in Lemma 4.1. Thus, for $|z|<1/2$,

$$\begin{aligned}
f_\alpha(z) &= \frac{1}{1-h_\alpha(z)} \;=\; \sum_{k=0}^\infty (h_\alpha(z))^k \;=\; 1+\sum_{k=1}^\infty\Bigg(\sum_{n=1}^\infty(-u_n)z^n\Bigg)^k\\
&= 1+\sum_{k=1}^\infty \sum_{n_1,\ldots,n_k\in\mathbb{N}} (-u_{n_1})\cdots(-u_{n_k})z^{n_1+\cdots+n_k}\\
&= 1+\sum_{k=1}^\infty\sum_{n=k}^\infty z^n \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}} (-u_{n_1})\cdots(-u_{n_k})\\
&= 1+\sum_{n=1}^\infty z^n\sum_{k=1}^n(-1)^k \sum_{\substack{n_1,\ldots,n_k\in\mathbb{N}\\ n_1+\cdots+n_k=n}} u_{n_1}\cdots u_{n_k} \;=\; \sum_{n=0}^\infty a_n z^n.
\end{aligned}$$

Note that the interchange of the sums in the second last equation is allowed since the series $\sum_{n=0}^\infty a_n z^n$ is absolutely convergent. □

Let us now prepare the proof of Theorem 2.1. We first summarize briefly a few basic properties of the $\beta(2-\alpha,\alpha)$-coalescent. It is well known that the block counting process $N=(N_t)_{t\ge 0}$ of the $\beta(2-\alpha,\alpha)$-coalescent moves from the state $n\in\{2,3,\ldots\}$ to the state $k\in\{1,\ldots,n-1\}$ at the rate

$$\begin{aligned}
g_{nk} &= \binom{n}{k-1}\int_{[0,1]} x^{n-k-1}(1-x)^{k-1}\,\Lambda(\mathrm{d}x)\\
&= \binom{n}{k-1}\frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\int_0^1 x^{n-k-\alpha}(1-x)^{k+\alpha-2}\,\mathrm{d}x\\
&= \binom{n}{k-1}\frac{\Gamma(n-k-\alpha+1)\Gamma(k+\alpha-1)}{\Gamma(2-\alpha)\Gamma(\alpha)\Gamma(n)}\\
&= n\frac{\Gamma(n-k-\alpha+1)\Gamma(k+\alpha-1)}{\Gamma(2-\alpha)\Gamma(\alpha)\Gamma(k)\Gamma(n-k+2)}
\end{aligned}$$

and (see, for example, Huillet and Möhle (2013, Eq. (18)) with $a := 2-\alpha$ and $b := \alpha$) that the block counting process N has total rates

$$g_n \;:=\; \sum_{k=1}^{n-1} g_{nk} \;=\; (n-1)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)}, \qquad n\in\mathbb{N}. \tag{4.3}$$

Note that $g_n \sim n^\alpha/\Gamma(\alpha+1)$ as $n\to\infty$. The jump chain X of the block counting process has therefore transition probabilities

$$\begin{aligned} p_{nk} &:= \mathbb{P}(X_{j+1}=k\,|\,X_j=n) \;=\; \frac{g_{nk}}{g_n} \\ &= \frac{n}{g_n}\frac{\Gamma(k+\alpha-1)}{\Gamma(k)}\frac{\Gamma(n-k-\alpha+1)}{\Gamma(2-\alpha)\Gamma(\alpha)\Gamma(n-k+2)}, \qquad 1\le k<n. \end{aligned} \tag{4.4}$$

This particular factorizing structure of p_{nk} will turn out to be crucial for the following proof of Theorem 2.1.

Proof: (of Theorem 2.1) We generalize the proof of Theorem 1.1 of Möhle (2014). For $m\in\mathbb{N}$ define the generating function

$$\phi_m(z) \;:=\; \sum_{n=m}^\infty h(n,m)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)}z^n, \qquad z\in D. \tag{4.5}$$

For $\alpha=1$ Eq. (4.5) reduces to the generating function $\phi_m(z)=\sum_{n=m}^\infty h(n,m)z^n$ used in the proof of Theorem 1.1 of Möhle (2014). Note that $\phi_m^{(j)}(0)=0$ for all $m\in\mathbb{N}$ and $j\in\{0,\ldots,m-1\}$ and that $\phi_m^{(m)}(0)=m\Gamma(m+\alpha-1)/\Gamma(\alpha+1)$ for all $m\in\mathbb{N}$. We have (see Norris (1997, Theorem 1.3.2)) $h(m,m)=1$ and $h(n,m)=\sum_{k=1}^n p_{nk}h(k,m)=\sum_{k=m}^{n-1}p_{nk}h(k,m)$ for $n>m$. Hence,

$$\begin{aligned} \sum_{n=m}^\infty h(n,m)\frac{g_n}{n}z^n &= \frac{g_m}{m}z^m + \sum_{n=m+1}^\infty h(n,m)\frac{g_n}{n}z^n \\ &= \frac{g_m}{m}z^m + \sum_{n=m+1}^\infty \sum_{k=m}^{n-1} p_{nk}h(k,m)\frac{g_n}{n}z^n \\ &= \frac{g_m}{m}z^m + \sum_{k=m}^\infty h(k,m)z^k \sum_{n=k+1}^\infty p_{nk}\frac{g_n}{n}z^{n-k}. \end{aligned}$$

Plugging in (4.4) yields

$$\begin{aligned} &\sum_{n=m}^\infty h(n,m)\frac{g_n}{n}z^n \\ &= \frac{g_m}{m}z^m + \frac{1}{\Gamma(2-\alpha)\Gamma(\alpha)}\sum_{k=m}^\infty h(k,m)\frac{\Gamma(k+\alpha-1)}{\Gamma(k)}z^k \sum_{n=k+1}^\infty \frac{\Gamma(n-k-\alpha+1)}{\Gamma(n-k+2)}z^{n-k} \\ &= \frac{g_m}{m}z^m + \frac{\alpha}{\Gamma(2-\alpha)}\sum_{k=m}^\infty h(k,m)\frac{\Gamma(k+\alpha-1)}{\Gamma(\alpha+1)\Gamma(k)}z^k \sum_{j=1}^\infty \frac{\Gamma(j-\alpha+1)}{\Gamma(j+2)}z^j \\ &= \frac{g_m}{m}z^m + \phi_m(z)a(z), \end{aligned} \tag{4.6}$$

where the auxiliary function $a : D \to \mathbb{C}$ is defined via

$$\begin{aligned} a(z) &:= \frac{\alpha}{\Gamma(2-\alpha)}\sum_{j=1}^\infty \frac{\Gamma(j-\alpha+1)}{\Gamma(j+2)} z^j \\ &= \begin{cases} 1+\dfrac{(1-z)\log(1-z)}{z} & \text{if } \alpha=1 \\ \dfrac{(1-z)^\alpha-1+\alpha z}{(\alpha-1)z} & \text{if } \alpha\in(0,2)\setminus\{1\} \end{cases} \\ &= 1-\frac{(1-z)^\alpha L_\alpha(z)}{z}. \end{aligned}$$

On the other hand, by (4.3),

$$\begin{aligned} \sum_{n=m}^\infty h(n,m)\frac{g_n}{n} z^n &= \sum_{n=m}^\infty h(n,m)\Big(1-\frac{1}{n}\Big)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)} z^n \\ &= \sum_{n=m}^\infty h(n,m)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)} z^n - \sum_{n=m}^\infty h(n,m)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)}\frac{z^n}{n} \\ &= \phi_m(z) - \int_0^z \frac{\phi_m(t)}{t}\,\mathrm{d}t. \end{aligned} \tag{4.7}$$

Since (4.6) and (4.7) are equal we conclude that

$$\phi_m(z) - \int_0^z \frac{\phi_m(t)}{t}\,\mathrm{d}t \;=\; \frac{g_m}{m} z^m + \phi_m(z)a(z)$$

or, equivalently, $\int_0^z \phi_m(t)/t\,\mathrm{d}t = (1-a(z))\phi_m(z) - (g_m/m)z^m$. Taking the derivative with respect to z yields $\phi_m(z)/z = -a'(z)\phi_m(z) + (1-a(z))\phi_m'(z) - g_m z^{m-1}$ or, equivalently,

$$(1-a(z))\phi_m'(z) \;=\; \Big(\frac{1}{z}+a'(z)\Big)\phi_m(z) + g_m z^{m-1}.$$

Distinguishing the two cases $\alpha=1$ and $\alpha\in(0,2)\setminus\{1\}$ it is readily checked that $(1/z+a'(z))/(1-a(z)) = 1/z+\alpha/(1-z)$. Thus, ϕ_m satisfies the differential equation

$$\phi_m'(z) \;=\; \Big(\frac{1}{z}+\frac{\alpha}{1-z}\Big)\phi_m(z) + r_m(z), \tag{4.8}$$

where

$$r_m(z) \;:=\; \frac{g_m z^{m-1}}{1-a(z)} \;=\; \frac{g_m z^m}{(1-z)^\alpha L_\alpha(z)}.$$

Note that $g_1=0$ and, hence, $r_1\equiv 0$. For $m=1$ the solution of the (homogeneous) differential equation (4.8) with initial conditions $\phi_1(0)=0$ and $\phi_1'(0)=1/\alpha$ is $\phi_1(z)=z/(\alpha(1-z)^\alpha)$, in agreement with $h(n,1)=1$ for all $n\in\mathbb{N}$. Assume now that $m\ge 2$. Then, the solution of the (inhomogeneous) differential equation (4.8) with initial conditions $\phi_m(0)=\phi_m'(0)=\cdots=\phi_m^{(m-1)}(0)=0$ and $\phi_m^{(m)}(0)=m\Gamma(m+\alpha-1)/\Gamma(\alpha+1)$ is $\phi_m(z)=c_m(z)\phi_1(z)$, where

$$\begin{aligned} c_m(z) &:= \int_0^z \frac{r_m(t)}{\phi_1(t)}\,\mathrm{d}t = \int_0^z \frac{\alpha(1-t)^\alpha}{t}\frac{g_m t^m}{(1-t)^\alpha L_\alpha(t)}\,\mathrm{d}t \\ &= \alpha g_m \int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t = \frac{1}{B(m-1,\alpha)}\int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t, \qquad m\ge 2. \end{aligned} \tag{4.9}$$

Recall the definition of $[z^n]f(z)$. Using this notation we obtain

$$\begin{aligned} h(n,m)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)} &= [z^n]\phi_m(z) \;=\; [z^n](\phi_1(z)c_m(z)) \\ &= \sum_{k=m-1}^{n-1}\left([z^{n-k}]\frac{z}{\alpha(1-z)^\alpha}\right)([z^k]c_m(z)) \\ &= \frac{1}{\alpha}\sum_{k=m-1}^{n-1}([z^{n-k-1}](1-z)^{-\alpha})([z^k]c_m(z)). \end{aligned}$$

Since

$$(1-z)^{-\alpha} \;=\; \sum_{j=0}^{\infty}\binom{-\alpha}{j}(-z)^j \;=\; \sum_{j=0}^{\infty}\frac{\Gamma(j+\alpha)}{\Gamma(\alpha)\Gamma(j+1)}z^j$$

it follows that

$$h(n,m)\frac{\Gamma(n+\alpha-1)}{\Gamma(\alpha+1)\Gamma(n)} \;=\; \frac{1}{\alpha}\sum_{k=m-1}^{n-1}\frac{\Gamma(n-k-1+\alpha)}{\Gamma(\alpha)\Gamma(n-k)}[z^k]c_m(z),$$

or, equivalently,

$$\begin{aligned} h(n,m) &= \sum_{k=m-1}^{n-1}\frac{\Gamma(n)\Gamma(n-k-1+\alpha)}{\Gamma(n-k)\Gamma(n+\alpha-1)}[z^k]c_m(z) \\ &= \sum_{k=m-1}^{n-1}\frac{B(k,n-k-1+\alpha)}{B(k,n-k)}[z^k]c_m(z). \end{aligned}$$

Plugging in (4.9) yields (2.2). It remains to verify (2.3). By Lemma 4.1 the map f_α defined via (4.1) has Taylor expansion $z/L_\alpha(z) = \sum_{j=0}^\infty a_j z^j$ with coefficients a_j given via (4.2). Thus, for $m \geq 2$,

$$\int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;=\; \int_0^z \sum_{j=0}^{\infty} a_j t^{m+j-2}\,\mathrm{d}t \;=\; \sum_{j=0}^{\infty} a_j\frac{z^{m+j-1}}{m+j-1} \;=\; \sum_{k=m-1}^{\infty}\frac{a_{k-m+1}}{k}z^k. \tag{4.10}$$

and, hence,

$$[z^k]\int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;=\; \frac{a_{k-m+1}}{k}, \qquad k \geq m-1 \geq 1. \tag{4.11}$$

Plugging in the expression (4.2) for the coefficient a_{k-m+1} yields (2.3). □

Remark 4.3. Note that (4.10) implies that

$$\sum_{k=m-1}^{\infty}\frac{a_{k-m+1}}{k} \;=\; \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;\in\; (0,\infty), \qquad m\in\mathbb{N}\setminus\{1\}, \alpha\in(0,2). \tag{4.12}$$

Eq. (4.12) will be useful in the proofs later.

The following lemma shows that the coefficients $a_1, a_2, \ldots$ in the Taylor expansion of f_α defined in (4.1) are all strictly negative. For more general information on the sign of the coefficients of reciprocal power series we refer the reader to Lamperti (1958). Similar arguments as used in the following proof can be traced back at least to Kaluza (1928, p. 162–163).

Lemma 4.4. *Fix $\alpha \in (0,2)$. Then, except for a_0 (which is equal to 1), the Taylor coefficients $a_1, a_2, \ldots$ of the function f_α defined in (4.1) are all strictly negative.*

Proof: We prove this by induction on n. Clearly, for $0 \neq |z| < 1/2$,

$$\bigg(\sum_{n=0}^\infty a_n z^n\bigg)\bigg(\sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{\Gamma(n+2)\Gamma(\alpha)} z^n\bigg) \;=\; \frac{z}{L_\alpha(z)}\,\frac{L_\alpha(z)}{z} \;=\; 1, \tag{4.13}$$

so $a_0 = 1$, $a_1 = -\alpha/2$ and so on. In particular, a_1 is negative. Eq. (4.13) implies that $\sum_{j=0}^n a_j \Gamma(\alpha+n-j)/(\Gamma(n+2-j)\Gamma(\alpha)) = 0$ for all $n \in \mathbb{N}$. Replacing n by $n+1$ we conclude that the coefficients a_n, $n \in \mathbb{N}_0$, satisfy the recursion

$$a_{n+1} \;=\; -\sum_{j=0}^n a_j \frac{\Gamma(n+1-j+\alpha)}{\Gamma(n-j+3)\Gamma(\alpha)}, \qquad n \in \mathbb{N}_0. \tag{4.14}$$

Suppose that $a_1, \ldots, a_n$ are negative. We have

$$\begin{aligned}
0 \;&=\; \bigg(\sum_{j=0}^n a_j \frac{\Gamma(\alpha+n-j)}{\Gamma(n+2-j)\Gamma(\alpha)}\bigg)\frac{n+\alpha}{n+2}\\
&=\; \bigg(\frac{\Gamma(\alpha+n)}{\Gamma(n+2)\Gamma(\alpha)} - \sum_{j=1}^n |a_j| \frac{\Gamma(\alpha+n-j)}{\Gamma(n+2-j)\Gamma(\alpha)}\bigg)\frac{n+\alpha}{n+2}\\
&=\; \frac{\Gamma(\alpha+n+1)}{\Gamma(n+3)\Gamma(\alpha)} - \sum_{j=1}^n |a_j| \frac{\Gamma(n+1-j+\alpha)}{\Gamma(n-j+3)\Gamma(\alpha)} \underbrace{\frac{n+2-j}{n+\alpha-j}\,\frac{n+\alpha}{n+2}}_{>1}\\
&<\; \frac{\Gamma(\alpha+n+1)}{\Gamma(n+3)\Gamma(\alpha)} - \sum_{j=1}^n |a_j| \frac{\Gamma(n+1-j+\alpha)}{\Gamma(n-j+3)\Gamma(\alpha)} \;=\; -a_{n+1},
\end{aligned}$$

which completes the induction. □

We are now able to verify Corollary 2.2.

Proof: (of Corollary 2.2) By Theorem 2.1 and (4.11), for $2 \le m \le n$,

$$\begin{aligned}
h(n,m) \;&=\; \frac{1}{B(m-1,\alpha)} \sum_{k=m-1}^{n-1} b_n(k)\,[z^k] \int_0^z \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t\\
&=\; \frac{1}{B(m-1,\alpha)} \sum_{k=m-1}^{n-1} b_n(k)\,\frac{a_{k-m+1}}{k},
\end{aligned}$$

where

$$b_n(k) \;:=\; \frac{B(k,n-k-1+\alpha)}{B(k,n-k)} \;=\; \frac{\Gamma(n)\Gamma(n-k+\alpha-1)}{\Gamma(n-k)\Gamma(n+\alpha-1)} \;=\; \prod_{j=1}^k \frac{n-j}{n-j+\alpha-1}$$

and $a_n = a_n(\alpha)$, $n \in \mathbb{N}_0$, are the coefficients of the Taylor expansion of the map f_α defined in (4.1). It remains to verify that

$$\sum_{k=m-1}^{n-1} b_n(k) \frac{a_{k-m+1}}{k} \;\to\; \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t, \qquad n \to \infty. \tag{4.15}$$

Note that, by Lemma 4.4, $a_0 = 1$ and $a_j < 0$ for all $j \in \mathbb{N}$. It turns out to be convenient to treat the three cases $\alpha = 1$, $\alpha \in (1,2)$, and $\alpha \in (0,1)$ separately.

Case 1: Assume that $\alpha = 1$. Then $b_n(k) = 1$ and, since the coefficients $a_1, a_2, \ldots$ are all strictly negative, the sum on the left hand side in (4.15) is strictly decreasing in n. In particular, this sum converges to $\sum_{k=m-1}^\infty a_{k-m+1}/k = \int_0^1 t^{m-1}/L_1(t)\,\mathrm{d}t$ by (4.12). Note that this argument was already given in Möhle (2014).

Case 2: Assume that $\alpha \in (1,2)$. It is readily checked that $b_n(k) \to 1$ as $n \to \infty$ and that $0 \le b_n(k) \le 1$ for all $n \in \mathbb{N}$ and all $k \in \{1, \ldots, n-1\}$. Thus, by dominated convergence it follows that the sum on the left hand side in (4.15) converges to $\sum_{k=m-1}^\infty a_{k-m+1}/k = \int_0^1 t^{m-1}/L_\alpha(t)\,\mathrm{d}t$ as required.

Case 3: Assume that $\alpha \in (0,1)$. Taking the limit $t \nearrow 1$ in the series expansion (see the remark after Lemma 4.4) $t/L_\alpha(t) = \sum_{j=0}^\infty a_j t^j$ it follows that $1 - \alpha = 1/L_\alpha(1-) = \sum_{j=0}^\infty a_j = 1 - \sum_{j=1}^\infty |a_j|$. Thus, $\sum_{j=0}^\infty |a_j| = \alpha + 1$.

It is readily checked that there exists a constant $C = C(\alpha) > 0$ (which does not depend on n and k) such that $|b_n(k) - 1| \le Ck/n$ for all $n \in \mathbb{N}$ and all $k \in \{1, \ldots, n-1\}$. Thus,

$$\bigg|\sum_{k=m-1}^{n-1} b_n(k)\frac{a_{k-m+1}}{k} - \sum_{k=m-1}^{n-1} \frac{a_{k-m+1}}{k}\bigg| \;\le\; \sum_{k=m-1}^{n-1} |b_n(k)-1|\frac{|a_{k-m+1}|}{k}$$

$$\le\; \frac{C}{n}\sum_{k=m-1}^{n-1} |a_{k-m+1}| \;\le\; \frac{C}{n}\sum_{j=0}^{\infty}|a_j| \;=\; \frac{C}{n}(\alpha+1).$$

Since $a_1, a_2, \ldots$ are all strictly negative it follows that $\sum_{k=m-1}^{n-1} a_{k-m+1}/k$ is strictly decreasing in n and converges to $\sum_{k=m-1}^\infty a_{k-m+1}/k = \int_0^1 t^{m-1}/L_\alpha(t)\,\mathrm{d}t$. Thus, (4.15) holds as well for $\alpha \in (0,1)$ which completes the proof. $\square$

Proof: (of Corollary 2.4) For $\alpha = 1$ Eq. (2.5) is known from Möhle (2014, Corollary 1.3). For $\alpha \in (0,2) \setminus \{1\}$ we have

$$w_m \;:=\; \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t \;=\; (1-\alpha)\int_0^1 \frac{t^{m-1}}{1-(1-t)^{1-\alpha}}\,\mathrm{d}t, \qquad m \ge 2.$$

If $\alpha \in (0,1)$ then the substitution $t = 1 - x^{\frac{1}{1-\alpha}}$ yields

$$w_m \;=\; \int_0^1 \frac{(1-x^{\frac{1}{1-\alpha}})^{m-1}x^{\frac{\alpha}{1-\alpha}}}{1-x}\,\mathrm{d}x \;=\; \int_0^1 \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^i\frac{x^{\frac{i+\alpha}{1-\alpha}}}{1-x}\,\mathrm{d}x.$$

Since $\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^i = 0$ for $m \ge 2$, we can rewrite this as

$$w_m \;=\; \int_0^1 \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\frac{1-x^{\frac{i+\alpha}{1-\alpha}}}{1-x}\,\mathrm{d}x \;=\; \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\Psi\Big(\frac{i+1}{1-\alpha}\Big),$$

where the last equality follows from the well known integral representation

$$\Psi(z) + \gamma = \int_0^1 \frac{1-x^{z-1}}{1-x}\,\mathrm{d}x, \qquad \mathrm{Re}(z) > 0, \tag{4.16}$$

of the digamma function (see, for example, Abramowitz and Stegun (1964, 6.3.22)). Thus, (2.5) holds for $\alpha \in (0,1)$. Assume now that $\alpha \in (1,2)$. Then the substitution

$t = 1 - x^{\frac{1}{\alpha-1}}$ yields

$$w_m = \int_0^1 \frac{(1-x^{\frac{1}{\alpha-1}})^{m-1}x^{\frac{1}{\alpha-1}}}{1-x}\,dx = \int_0^1 \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^i\frac{x^{\frac{i+1}{\alpha-1}}}{1-x}\,dx.$$

Again, since $\sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^i = 0$ for $m \geq 2$, we can rewrite this as

$$w_m = \int_0^1 \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\frac{1-x^{\frac{i+1}{\alpha-1}}}{1-x}\,dx = \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\Psi\Big(\frac{\alpha+i}{\alpha-1}\Big),$$

where the last equality follows from (4.16). Thus, (2.5) holds as well for $\alpha \in (1,2)$. □

5. Proofs concerning the absorption times

The results on the absorption times stated in Section 3 rely on the following basic lemma.

Lemma 5.1. *Let Ξ be a measure on the infinite simplex Δ satisfying $\Xi(\Delta) \in (0,\infty)$ and let $\Pi = (\Pi_t)_{t\geq 0}$ be a standard Ξ-coalescent. Then,*

$$\mathbb{E}(\tau_n^j) = j\sum_{m=2}^{n}\frac{h(n,m)}{g_m}\mathbb{E}(\tau_m^{j-1}), \qquad n \geq 2, j \in \mathbb{N}, \tag{5.1}$$

where $g_1, g_2, \ldots$ are the total rates of the coalescent and $h(n,m)$ denotes the hitting probability defined via (1.1). Moreover,

$$\mathbb{E}(\tau_n^j) = j!\sum_{2\leq m_1\leq\cdots\leq m_j\leq n}\frac{h(n,m_j)h(m_j,m_{j-1})\cdots h(m_2,m_1)}{g_{m_1}\cdots g_{m_j}}, \qquad n \geq 2, j \in \mathbb{N}. \tag{5.2}$$

Proof: It is well known that the total rate g_n is non-decreasing in n. Thus, $g_n \geq g_2 = \Xi(\Delta) > 0$ for all $n \geq 2$. As in Drmota et al. (2007, p. 1407) (see also p. 395 of Freund and Möhle (2009)) it follows that the moments of τ_n satisfy the relation

$$\mathbb{E}(\tau_n^j) = \frac{j}{g_n}\mathbb{E}(\tau_n^{j-1}) + \sum_{m=2}^{n}p_{nm}\mathbb{E}(\tau_m^j), \qquad n \geq 2, j \in \mathbb{N}, \tag{5.3}$$

where the p_{nm}, $n, m \in \mathbb{N}$ with $m \leq n$, are the transition probabilities of the jump chain $X = (X_k)_{k\in\mathbb{N}_0}$ of the block counting process of the coalescent Π. Note that $p_{nn} = 0$ for all $n \geq 2$. For convenience, we rewrite (5.3) for arbitrary but fixed $j \in \mathbb{N}$ in the form $a_n = r_n + \sum_{m=2}^n p_{nm}a_m$, $n \geq 2$, with $a_n := \mathbb{E}(\tau_n^j)$ and $r_n := jg_n^{-1}\mathbb{E}(\tau_n^{j-1})$. By induction on N it follows that

$$a_n = \sum_{m=2}^{n}\sum_{k=0}^{N-1}p_{nm}^{(k)}r_m + \sum_{m=2}^{n}p_{nm}^{(N)}a_m, \qquad n \geq 2, N \in \mathbb{N}, \tag{5.4}$$

where the $p_{nm}^{(k)}$ denote the k-step transition probabilities of the jump chain X. For $N = 1$ Eq. (5.4) coincides with (5.3). The induction step from N to $N+1$ is performed by making use of the Chapman Kolmogoroff equations. Now note that $p_{nm}^{(N)} \to \delta_{m1}$ as $N \to \infty$, since the state 1 is absorbing and all other states

$m \in \{2, \ldots, n\}$ are transient. Thus, taking the limit $N \to \infty$ on both sides in (5.4) yields

$$a_n \;=\; \sum_{m=2}^{n} \sum_{k=0}^{\infty} p_{nm}^{(k)} r_m \;=\; \sum_{m=2}^{n} h(n,m) r_m, \qquad n \geq 2, \tag{5.5}$$

since

$$\begin{aligned} h(n,m) &= \mathbb{P}(\bigcup_{k=0}^{\infty} \{X_k = m\} \,|\, X_0 = n) \\ &= \sum_{k=0}^{\infty} \mathbb{P}(X_k = m \,|\, X_0 = n) \;=\; \sum_{k=0}^{\infty} p_{nm}^{(k)}, \quad 2 \leq m \leq n. \end{aligned}$$

Thus, $\mathbb{E}(\tau_n^j) = a_n = \sum_{m=2}^n h(n,m) r_m = j \sum_{m=2}^n h(n,m) g_m^{-1} \mathbb{E}(\tau_m^{j-1})$ for all $j \in \mathbb{N}$ and all $n \geq 2$, which is (5.1). Now, (5.2) follows easily from (5.1) by induction on $j \in \mathbb{N}$. □

We now first prove Theorem 3.3 and will verify Corollary 3.1 and Corollary 3.2 afterwards.

Proof: (of Theorem 3.3) Clearly, $0 = \tau_1 \leq \tau_2 \leq \cdots$ and, hence, $\tau_n \to \tau_\infty := \lim_{n\to\infty} \tau_n$. It is straightforward to check that τ_∞ almost surely coincides with $\tau := \inf\{t > 0 : N_t = 1\}$. Thus, $\tau_n \to \tau$ almost surely. By (5.1), for all $j \in \mathbb{N}$ and all $n \geq 2$,

$$\begin{aligned} \mathbb{E}(\tau_n^j) &= j \sum_{m=2}^{n} \frac{h(n,m)}{g_m} \mathbb{E}(\tau_m^{j-1}) \;\leq\; j\mathbb{E}(\tau^{j-1}) \sum_{m=2}^{n} \frac{h(n,m)}{g_m} \\ &= j\mathbb{E}(\tau^{j-1})\mathbb{E}(\tau_n) \;\leq\; j\mathbb{E}(\tau^{j-1})\mathbb{E}(\tau), \end{aligned}$$

where the last equality holds by (5.2). Taking the limit $n \to \infty$ it follows by monotone convergence that $\mathbb{E}(\tau^j) \leq j\mathbb{E}(\tau^{j-1})\mathbb{E}(\tau)$ for all $j \in \mathbb{N}$. By induction on j we conclude that $\mathbb{E}(\tau^j) \leq j!(\mathbb{E}(\tau))^j$ for all $j \in \mathbb{N}$. Together with the assumption that the coalescent comes down from infinity (which implies that $\mathbb{E}(\tau) < \infty$) this yields $\mathbb{E}(\tau^j) < \infty$ for all $j \in \mathbb{N}$. Thus $\mathbb{E}(\tau^p) < \infty$ for all $p \in (0, \infty)$. From $\tau_n^p \leq \tau^p$ and the integrability of τ^p it follows that the sequence $(\tau_n^p)_{n\in\mathbb{N}}$ is uniformly integrable. This uniform integrability together with the almost sure convergence $\tau_n \to \tau$ implies (see, for example, Chung (2001, Theorem 4.5.4) or Kallenberg (2002, Proposition 4.12)) the convergence $\tau_n \to \tau$ in L^p. In particular, we have convergence $\mathbb{E}(\tau_n^p) \to \mathbb{E}(\tau^p)$ as $n \to \infty$ of all moments. Note that the moment generating function $z \mapsto \sum_{j=0}^\infty \mathbb{E}(\tau^j) z^j / j!$ of τ has at least radius of convergence $1/\mathbb{E}(\tau) > 0$. The distribution of τ is hence uniquely determined by its moments.

In order to verify (3.4) fix $j \in \mathbb{N}$ and let $\mathcal{R}$ denote the range of the process $(N_t)_{t>0}$. Note that $t = 0$ is excluded. By assumption the coalescent comes down from infinity. Thus, almost surely, the range $\mathcal{R}$ does not contain the element ∞ and is hence almost surely a random subset of $\mathbb{N}$. Furthermore, let T_m denote the sojourn time of the block counting process $(N_t)_{t\geq 0}$ in the state $m \geq 2$. Since the coalescent comes down from infinity, τ has the same distribution as $\sum_{m=2}^\infty T_m 1_{\{m\in\mathcal{R}\}}$. Note that for the Kingman coalescent this distributional representation of τ boils down to $\sum_{m=2}^\infty T_m$. Formulas of this form have been used in the literature, see for example the last displayed expression on p. 171 in the proof of Bertoin (2006, Theorem 4.1). We conclude that

$$\begin{aligned}\mathbb{E}(\tau^j) &= \mathbb{E}\Bigg(\bigg(\sum_{m=2}^\infty T_m 1_{\{m\in\mathcal{R}\}}\bigg)^j\Bigg)\\ &= \sum_{m_1,\ldots,m_j=2}^\infty \mathbb{E}(T_{m_1}\cdots T_{m_j})\,\mathbb{P}(\{m_1,\ldots,m_j\}\subseteq\mathcal{R})\\ &= \sum_{m_1,\ldots,m_j=2}^\infty \bigg(\prod_{m=2}^\infty \mathbb{E}(T_m^{a_m})\bigg)\mathbb{P}(\{m_1,\ldots,m_j\}\subseteq\mathcal{R})\\ &= \sum_{2\le m_1\le\cdots\le m_j} \frac{j!}{\prod_{m=2}^\infty a_m!}\bigg(\prod_{m=2}^\infty \mathbb{E}(T_m^{a_m})\bigg)\mathbb{P}(\{m_1,\ldots,m_j\}\subseteq\mathcal{R}),\end{aligned}$$

where, for $m\ge 2$, a_m denotes the number of indices $m_1,\ldots,m_j$ being equal to m. Note that $\sum_{m=2}^\infty a_m = j$, so at most j of the a_m's are not equal to 0. Since $\mathbb{E}(T_m^{a_m}) = a_m!/g_m^{a_m}$ for all $m\ge 2$, the above expression simplifies to

$$\mathbb{E}(\tau^j) \;=\; j!\sum_{2\le m_1\le\cdots\le m_j}\frac{\mathbb{P}(\{m_1,\ldots,m_j\}\subseteq\mathcal{R})}{g_{m_1}\cdots g_{m_j}}.$$

Thus, (3.4) holds, since $\mathbb{P}(\{m_1,\ldots,m_j\}\subseteq\mathcal{R}) = h(m_j)h(m_j,m_{j-1})\cdots h(m_2,m_1)$ with $h(.,.)$ defined via (1.1) and $h(m_j) := \mathbb{P}(N_t = m_j \text{ for some } t\ge 0)$. Obviously, (3.4) can be rewritten as

$$\begin{aligned}\mathbb{E}(\tau^j) &= j\sum_{m_j=2}^\infty \frac{h(m_j)}{g_{m_j}}(j-1)!\sum_{2\le m_1\le\cdots\le m_{j-1}\le m_j}\frac{h(m_j,m_{j-1})\cdots h(m_2,m_1)}{g_{m_1}\cdots g_{m_{j-1}}}\\ &= j\sum_{m_j=2}^\infty \frac{h(m_j)}{g_{m_j}}\mathbb{E}(\tau_{m_j}^{j-1}) \;=\; j\sum_{m=2}^\infty\frac{h(m)}{g_m}\mathbb{E}(\tau_m^{j-1}),\end{aligned}$$

where the second last equality follows from (5.2). Thus, (3.5) holds. It remains to show that the hitting probabilities satisfy

$$\lim_{n\to\infty} h(n,m) \;=\; h(m), \qquad m\in\mathbb{N}. \tag{5.6}$$

Let $E := \mathbb{N}\cup\{\infty\}$ and for arbitrary but fixed $m\in\mathbb{N}$ define

$$A \;:=\; \{x\in D_E([0,\infty)) : x(t) = m \text{ for some } t\in[0,\infty)\}.$$

Clearly, the closure of A is all of $D_E([0,\infty))$, so A is not closed. In order to verify that the complement A^c is closed choose $x_n\to x$ in $D_E([0,\infty))$ such that $x_n\in A^c$ for each n. Now suppose $x\notin A^c$. Then $x\in A$, so there exists a continuity point t of x such that $x(t)=m$. By Proposition 5.2 in Chapter 3 of Ethier and Kurtz (1986) it follows from $x_n\to x$ in $D_E([0,\infty))$ that $\lim_{n\to\infty} x_n(t) = x(t) = m$. But then $x_n\in A$ for n sufficiently large, using the fact that $\mathbb{N}$ has discrete metric. This contradiction shows that $x\in A^c$. Thus A^c is closed, A is open and has boundary $\partial A = A^c$.

We now turn to the block counting processes. Theorem 1 of Donnelly (1991) (applied with $B^N(t) := |\Pi_t^{(N)}|$, $D(t) := |\Pi_t|$ and $n_N := N$, and afterwards formally N replaced by n) shows that, as $n\to\infty$, the block counting process $(N_t^{(n)})_{t\ge 0}$ weakly converges in $D_E([0,\infty))$ to $(N_t)_{t\ge0}$. Note that the assumptions of Donnelly (1991, Theorem 1) are satisfied. Assumption (A1) of Donnelly (1991, Theorem

1) holds, since $N_t^{(l)} = |\varrho_l \circ \Pi_t| \le |\varrho_n \circ \Pi_t| = N_t^{(n)}$ for all $t \in [0,\infty)$ and all $l \le n$. Assumption (A2) of Donnelly (1991, Theorem 1) holds since, for $n \le m$, the process $(N_t^{(m)})_{t\ge 0}$, conditional on $N_0^{(m)} = n$, has the same distribution as the process $(N_t^{(n)})_{t\ge 0}$, so this distribution does not depend on m $(\ge n)$. Assumption (A3) of Donnelly (1991, Theorem 1) is checked as follows. For all $t, M \in (0,\infty)$ we have $\lim_{n\to\infty} \mathbb{P}(N_t^{(n)} \le M) = \mathbb{P}(\bigcap_{n\in\mathbb{N}}\{N_t^{(n)} \le M\}) = \mathbb{P}(N_t \le M)$ and, therefore,

$$\begin{aligned}\lim_{M\to\infty}\lim_{n\to\infty} \mathbb{P}(N_t^{(n)} \le M) &= \lim_{M\to\infty} \mathbb{P}(N_t \le M)\\ &= \mathbb{P}\Big(\bigcup_{M\in\mathbb{N}}\{N_t \le M\}\Big) \;=\; \mathbb{P}(N_t < \infty) \;=\; 1,\end{aligned}$$

since the coalescent comes down from infinity.

Since A is open, the Portmanteau Theorem (see, for example, Billingsley (1999, Theorem 2.1)) gives for any subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ with $\lim_{k\to\infty} n_k = \infty$ as $k \to \infty$ the inequality

$$\begin{aligned}\liminf_{k\to\infty} h(n_k, m) &= \liminf_{k\to\infty} \mathbb{P}(N_t^{(n_k)} = m \text{ for some } t \in [0,\infty))\\ &\ge \mathbb{P}(N_t = m \text{ for some } t \in [0,\infty)) \;=:\; h(m), \qquad m \in \mathbb{N}. \end{aligned} \tag{5.7}$$

Now let $(n_k)_{k\in\mathbb{N}}$ be a subsequence with $n_k \to \infty$ as $k \to \infty$ such that the limit $\tilde{h}(m) := \lim_{k\to\infty} h(n_k, m)$ exists for each $m \in \mathbb{N}$. Note that such a subsequence exists since $0 \le h(n,m) \le 1$ for all $n, m \in \mathbb{N}$. Clearly, $\tilde{h}(1) = 1 = h(1)$, since the state 1 is absorbing. In order to verify that $\tilde{h}(m) = h(m)$ for all $m \ge 2$ define $f_k(m) := h(n_k, m)/g_m$ for $2 \le m \le n_k$ and $f_k(m) := 0$ otherwise. Let μ denote the counting measure on $\{2, 3, \ldots\}$. By Fatou's lemma we have

$$\begin{aligned}\sum_{m=2}^{\infty} \frac{\tilde{h}(m)}{g_m} &= \int \frac{\tilde{h}(m)}{g_m}\,\mu(\mathrm{d}m) \;=\; \int \lim_{k\to\infty} f_k(m)\,\mu(\mathrm{d}m)\\ &\le \liminf_{k\to\infty} \int f_k(m)\,\mu(\mathrm{d}m) \;=\; \liminf_{k\to\infty} \sum_{m=2}^{n_k} \frac{h(n_k, m)}{g_m}\\ &= \liminf_{k\to\infty} \mathbb{E}(\tau_{n_k}) \;=\; \mathrm{E}(\tau) \;=\; \sum_{m=2}^{\infty} \frac{h(m)}{g_m} \;<\; \infty.\end{aligned}$$

On the other hand, by (5.7), $\tilde{h}(m) \ge h(m)$ for all $m \ge 2$. Therefore, $\tilde{h}(m) = h(m)$ for all $m \ge 2$. In particular, the limits $\tilde{h}(m)$, $m \in \mathbb{N}$, do not depend on the specific subsequence $(n_k)_{k\in\mathbb{N}}$. Thus, for all $m \in \mathbb{N}$, the limit $\lim_{n\to\infty} h(n,m)$ exists and is equal to $h(m)$, and (5.6) is established. The proof is complete. □

Proof: (of Corollary 3.1) It is well known that the $\beta(2-\alpha,\alpha)$-coalescent with $\alpha \in (1,2)$ comes down from infinity which is equivalent to $\mathbb{E}(\tau) < \infty$. Thus, Theorem 3.3 (which is already verified) is applicable. Theorem 3.3 yields essentially all the results stated in Corollary 3.1. In particular, (3.1) holds, where – thanks to Theorem 2.1 and Corollary 2.2 – $h(n,m)$ and $h(m)$ are now explicitly given by (2.2) and (2.4) respectively. Choosing $j = 1$ in (3.1) we conclude from Corollary

2.2 and from $g_m B(m-1,\alpha) = 1/\alpha$ that

$$\begin{aligned}\mathbb{E}(\tau) &= \sum_{m=2}^\infty \frac{h(m)}{g_m} = \sum_{m=2}^\infty \frac{1}{g_m B(m-1,\alpha)} \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t = \alpha \sum_{m=2}^\infty \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t\\ &= \alpha \int_0^1 \frac{t}{(1-t)L_\alpha(t)}\,\mathrm{d}t = \alpha(\alpha-1)\int_0^1 \frac{t}{(1-t)^{2-\alpha}-(1-t)}\,\mathrm{d}t,\end{aligned}$$

in agreement with Corollary 3.6 of Hénard (2013). The substitution $t = 1 - x^{\frac{1}{\alpha-1}}$ yields

$$\mathbb{E}(\tau) = \alpha \int_0^1 \frac{1-x^{\frac{1}{\alpha-1}}}{1-x}\,\mathrm{d}x = \alpha\biggl(\Psi\Bigl(\frac{\alpha}{\alpha-1}\Bigr)+\gamma\biggr),$$

where the last equality follows from (4.16). □

Proof: (of Corollary 3.2) Again, Theorem 3.3 is applicable. In particular, $\mathbb{E}(\tau^j) = j\sum_{m=2}^\infty h(m) g_m^{-1}\mathbb{E}(\tau_m^{j-1})$. Plugging in the formula for $h(m)$ from Corollary 3.1 and noting that $g_m B(m-1,\alpha) = 1/\alpha$ we conclude that $\mathbb{E}(\tau^j) = \alpha j \sum_{m=2}^\infty w_m \mathbb{E}(\tau_m^{j-1})$ with

$$w_m := \int_0^1 \frac{t^{m-1}}{L_\alpha(t)}\,\mathrm{d}t = \sum_{i=0}^{m-1}\binom{m-1}{i}(-1)^{i+1}\Psi\Bigl(\frac{\alpha+i}{\alpha-1}\Bigr),$$

where the last equality was already shown in the proof of Corollary 2.4. □

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