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Percolation in a multiscale Boolean model

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Abstract. We consider percolation in a multiscale Boolean model. This model is defined as the union of scaled independent copies of a given Boolean model. The scale factor of the n^{th} copy is ρ^{-n}. We prove, under optimal integrability assumptions, that no percolation occurs in the multiscale Boolean model for large enough ρ if the rate of the Boolean model is below some critical value.

1. Introduction and statement of the main result

1.1. *The Boolean model.* Let $d \geq 2$ and μ be a finite measure on $]0, +\infty[$ having μ positive mass. Let ξ be a Poisson point process on $\mathbb{R}^d \times]0, +\infty[$ whose intensity is the product of the Lebesgue measure on $\mathbb{R}^d$ by μ. With ξ we associate a random set $\Sigma(\mu)$ defined as follows:

$$\Sigma(\mu) = \bigcup_{(c,r)\in\xi} B(c, r)$$

where $B(c, r)$ is the open Euclidean ball of radius r centered at c. The random set $\Sigma(\mu)$ is the Boolean model with parameter μ. We shall sometimes write Σ to simplify the notations.

The following description may be more intuitive. Let χ denote the projection of ξ on $\mathbb{R}^d$. With probability one this projection is one-to-one. We can therefore write:

$$\xi = \{(c, r(c)), c \in \chi\}.$$

Write $\mu = m\nu$ where ν is a probability measure. Then, χ is a Poisson point process on $\mathbb{R}^d$ with density m. Moreover, given χ, the sequence $(r(c))_{c\in\chi}$ is a sequence of independent random variable with common distribution ν.

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1.2. *Percolation in the Boolean model.* Let C denote the connected component of Σ that contains the origin. We say that Σ percolates if C is unbounded with positive probability. We refer to the book by Meester and Roy (1996) for background on continuum percolation. Set:

$$\lambda_c(\mu) = \inf\{\lambda > 0 : \Sigma(\lambda\mu) \text{ percolates}\}.$$

One easily check that $\lambda_c(\mu)$ is finite as soon as μ has a positive mass. In Gouéré (2008) we proved that $\lambda_c(\mu)$ is positive if and only if:

$$\int r^d \mu(dr) < \infty.$$

The only if part had been proved earlier by Hall (1985). For all $A, B \subset \mathbb{R}^d$, we write $A \leftrightarrow_\Sigma B$ if there exists a path in Σ from A to B. We denote by $S(c,r)$ the Euclidean sphere of radius r centered at c :

$$S(c,r) = \{x \in \mathbb{R}^d : \|x - c\|_2 = r\}.$$

We write $S(r)$ when $c = 0$.

The critical parameter $\lambda_c(\mu)$ can also be defined as follows:

$$\lambda_c(\mu) = \sup\left\{\lambda > 0 : P\left(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)\right) \to 0 \text{ as } r \to \infty\right\},$$

We shall need two other critical parameters:

$$\begin{aligned} \widehat{\lambda}_c(\mu) &= \sup\left\{\lambda > 0 : P\left(S(r/2) \leftrightarrow_{\Sigma(\lambda\mu)} S(r)\right) \to 0 \text{ as } r \to \infty\right\}, \\ \widetilde{\lambda}_c(\mu) &= \sup\left\{\lambda > 0 : r^d P\left(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)\right) \to 0 \text{ as } r \to \infty\right\}. \end{aligned}$$

We have (see Lemma A.1) :

$$\widetilde{\lambda}_c(\mu) \le \widehat{\lambda}_c(\mu) \le \lambda_c(\mu). \tag{1.1}$$

When the support of μ is bounded,

$$P\left(\{0\} \leftrightarrow_{\Sigma(\lambda\mu)} S(r)\right)$$

decays exponentially fast to 0 as soon as $\lambda < \lambda_c(\mu)$ (see for example Meester and Roy (1996), Section 12.10 in Grimmett (1999) in the case of constant radii or the papers Meester et al. (1994); Menshikov and Sidorenko (1987); Zuev and Sidorenko (1985a,b)). Therefore:

$$\widetilde{\lambda}_c(\mu) = \widehat{\lambda}_c(\mu) = \lambda_c(\mu) \text{ as soon as the support of } \mu \text{ is bounded.} \tag{1.2}$$

The unbounded case is much less understood and, even for example when μ has exponential tail, it is not known whether the different critical parameters coincide or not.

Remarks.

- The threshold parameter $\widehat{\lambda}_c(\mu)$ is positive if and only if $\int r^d \mu(dr)$ is finite (i.e., if and only if $\lambda_c(\mu)$ is positive). See Lemma A.2.
- Using ideas of Gouéré (2008), we can check that $\widetilde{\lambda}_c(\mu)$ is positive if and only if

  $$x^d \int_x^\infty r^d \mu(dr) \to 0 \text{ as } x \to \infty.$$

  If we only use results stated in Gouéré (2008), we can easily get the following weaker statements. Let $D(\lambda\mu)$ denote the Euclidean diameter of the connected component of $\Sigma(\lambda\mu)$ that contains the origin. Note that $\widetilde{\lambda}_c(\mu)$ is positive if and only if there exists λ such that:

$$r^d P(D(\lambda\mu) \geq r) \to 0, \text{ as } r \to \infty. \tag{1.3}$$

If $E(D(\lambda\mu)^d)$ is finite then (1.3) holds. If (1.3) holds then $E(D(\lambda\mu)^{d-\varepsilon})$ is finite for any small enough $\varepsilon > 0$. By Theorem 2.2 of Gouéré (2008) we thus get the following implications:

$$\int_0^{+\infty} r^{2d}\mu(dr) < \infty \text{ implies } \widetilde{\lambda}_c(\mu) > 0 \text{ implies } \forall \varepsilon > 0 : \int_0^{+\infty} r^{2d-\varepsilon}\mu(dr) < \infty.$$

1.3. *A multiscale Boolean model.* Let $\rho > 1$ be a scale factor. Let $(\Sigma_n)_{n\geq 0}$ be a sequence of independent copies of $\Sigma(\mu)$. In this paper, we are interested in percolation properties of the following multiscale Boolean model:

$$\Sigma^\rho(\mu) = \bigcup_{n\geq 0} \rho^{-n}\Sigma_n. \tag{1.4}$$

We shall sometimes write Σ^ρ to simplify the notations. As before, we say that Σ^ρ percolates if the connected component of Σ^ρ that contains the origin is unbounded with positive probability.

This model seems to have been first introduced as a model of failure in geophysical medias in the $80'$. We refer to the paper by Molchanov et al. (1990) for an account of those studies. For more recent results we refer to Broman and Camia (2010); Meester and Roy (1996); Meester et al. (1994); Menshikov et al. (2001, 2003); Popov and Vachkovskaia (2002).

This model is related to a discrete model introduced by Mandelbrot (1974). We refer to the survey by Chayes (1995) and, for more recent results, to Broman and Camia (2008); Orzechowski (1996); White (2001).

In Menshikov et al. (2001), Menshikov, Popov and Vachkovskaia considered the case where the radii of the unscaled process Σ_0 equal 1. They proved the following result.

Theorem 1.1 (Menshikov et al. (2001))**.** *If $\lambda < \lambda_c(\delta_1)$ then, for all large enough ρ, $\Sigma^\rho(\lambda\mu)$ does not percolate.*

In Menshikov et al. (2003) the same authors considered the case where the radii are random and can be unbounded. They considered the following sub-autosimilarity assumption on the measure μ:

$$\lim_{a\to\infty} \sup_{r\geq 1/2} \frac{a^d \mu([ar, +\infty[)}{\mu([r, +\infty[)} = 0 \tag{1.5}$$

with the convention $0/0 = 0$. They proved the following result.

Theorem 1.2 (Menshikov et al. (2003))**.** *Assume that the measure μ satisfies (1.5) and that $\widetilde{\lambda}_c(\mu)$ is positive. If $\lambda < \widetilde{\lambda}_c(\mu)$ then, for all large enough ρ, $\Sigma^\rho(\lambda\mu)$ does not percolate.*

Note that (1.5) is fulfilled for any measure with bounded support. Because of (1.2), Theorem 1.2 is then a generalization of Theorem 1.1.

In Gouéré (2009) we proved the following related result in which ρ is fixed.

Theorem 1.3 (Gouéré (2009)). *Let $\rho > 1$. There exists $\lambda > 0$ such that $\Sigma^\rho(\lambda\mu)$ does not percolate if and only if:*

$$\int_{[1,+\infty[} r^d \ln(r)\mu(r) < \infty. \tag{1.6}$$

The main result of this paper is the first item of the following theorem in which, in particular, we remove the technical assumption (1.5). The second item is easy and already contained in Theorem 1.3. Recall that, by Lemma A.2, $\widehat{\lambda}_c(\mu)$ is positive as soon as $\int r^d\mu(dr)$ is finite and therefore as soon as (1.6) holds.

Theorem 1.4.

(1) *Assume (1.6). Then, for all $\lambda < \widehat{\lambda}_c(\mu)$, there exists $\rho(\lambda) > 1$ such that, for all $\rho \geq \rho(\lambda)$:*

$$P\left(S(r/2) \leftrightarrow_{\Sigma^\rho(\lambda\mu)} S(r)\right) \to 0 \text{ as } r \to \infty \tag{1.7}$$

and therefore $\Sigma^\rho(\lambda\mu)$ does not percolate.

(2) *Assume that (1.6) does not hold. Then, for all $\lambda > 0$ and for all $\rho > 1$, $\Sigma^\rho(\lambda\mu)$ percolates.*

The proof is given in Section 2. The ideas of its proof and the ideas of the proofs of Theorems 1.1 and 1.2 are given in Subsection 2.2.

The first item of Theorem 1.4 is a generalization of Theorem 1.2 and thus of Theorem 1.1. Indeed, by (1.1), one has $\lambda < \widehat{\lambda}_c$ as soon as $\lambda < \widetilde{\lambda}_c$. Moreover, by the second item of Theorem 1.4, (1.6) has to be a consequence of the assumptions of Theorem 1.2. For example, one can check that (1.6) is a consequence of (1.5) [1]. Alternatively, one can check that (1.6) is a consequence of $\widetilde{\lambda}_c(\mu) > 0$ (see the remarks at the end of Section 1.2).

Let us denote by $\lambda_c(m^\rho_\infty)$ and $\widehat{\lambda}_c(m^\rho_\infty)$ the λ_c and $\widehat{\lambda}_c$ critical thresholds for the multiscale model with scale parameter ρ. Theorems 1.3 and 1.4 yield the following result:

(1) If (1.6) holds then $\lambda_c(m^\rho_\infty) > 0$ (and actually the proof of Theorem 1.3 yields $\widehat{\lambda}_c(m^\rho_\infty) > 0$) for all $\rho > 1$ and $\widehat{\lambda}_c(m^\rho_\infty) \to \widehat{\lambda}_c(\mu) > 0$ as $\rho \to \infty$.
(2) Otherwise, $\widehat{\lambda}_c(m^\rho_\infty) = \lambda_c(m^\rho_\infty) = 0$ for all $\rho > 1$.

Let us denote by $D^\rho(\lambda\mu)$ the diameter of the connected component of $\Sigma^\rho(\lambda\mu)$ that contains the origin. The following result is an easy consequence of Theorem 1.4 above and Theorems 2.9 and 1.2 in Gouéré (2009).

Theorem 1.5. *Let $s > 0$, $\lambda > 0$ and $\rho > 1$.*

(1) *If $\int_{[1,+\infty[} r^{d+s}\mu(dr) < \infty$ and (1.7) holds, then $E\big((D^\rho(\lambda\mu))^s\big) < \infty$.*
(2) *If $\int_{[1,+\infty[} r^{d+s}\mu(dr) = \infty$ then $E\big((D^\rho(\lambda\mu))^s\big) = \infty$.*

The proof is given is Section 3.

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[1] From (1.5) one gets the existence of $a > 1$ such that, for all $r \geq a$, one has $\mu([r,+\infty[) \leq 2^{-1}a^{-d}\mu([r/a,+\infty[)$. By induction and standard computations this yields, for all $r \geq a$, $\mu([r,+\infty[) \leq Ar^{-\ln(2)/\ln(a)-d}$. Therefore, for a small enough $\eta > 0$, one has $\int r^{d+\eta}\mu(dr) < \infty$.

1.4. *Superposition of Boolean models with different laws.* Using the same arguments as in the proof of Theorem 1.4, we could prove similar results for infinite superpositions

$$\bigcup_{n\geq 0} \rho^{-n}\Sigma_n$$

where the Boolean models Σ_n are independent but not identically distributed. We will not give such a result here. However, we wish to give a weaker result for the superposition of two independent Boolean models at different scales. As we consider only two scales the proof is easier than the proof of Theorem 1.4. The proof uses Lemmas 2.1, 2.2 and 2.3 and is given in Section 4. The result gives some insight on the critical threshold in the case of balls of random radii.

This result, in the case where the supports of ν_1 and ν_2 are bounded, is already implicit in Meester et al. (1994) in their proof of non universality of critical covered volume (see (1.8) below). See also Molchanov et al. (1990).

For all $\rho > 1$, we denote by $H^\rho\mu$ the measure defined by $H^\rho\mu(A) = \rho^d\mu(\rho A)$. With this definition, $\rho^{-1}\Sigma(\mu)$ is a Boolean model driven by the measure $H^\rho\mu$.

Proposition 1.6. *Let ν_1 and ν_2 be two finite measures on $]0,+\infty[$. We assume that the masses of ν_1 and ν_2 are positive. Let $0<\alpha<1$. Then, for all $\rho>1$,*

$$\widehat{\lambda}_c(\alpha\nu_1+(1-\alpha)H^\rho\nu_2) \leq \min\big(\widehat{\lambda}_c(\alpha\nu_1), \widehat{\lambda}_c((1-\alpha)H^\rho\nu_2)\big) = \min\left(\frac{\widehat{\lambda}_c(\nu_1)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)}{1-\alpha}\right).$$

Moreover,

$$\widehat{\lambda}_c(\alpha\nu_1+(1-\alpha)H^\rho\nu_2) \to \min\left(\frac{\widehat{\lambda}_c(\nu_1)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)}{1-\alpha}\right) \text{ as } \rho\to\infty.$$

The above convergence is uniform in α.

We now make some remarks about this result and about some related numerical results. For a finite measure μ on $]0,+\infty[$, we denote by $\phi_c(\mu)$ the critical covered volume:

$$\phi_c(\mu) = P\big(0\in\Sigma(\lambda_c(\mu)\mu)\big) = 1-\exp\left(-\lambda_c(\mu)\int v_d r^d\mu(dr)\right) \tag{1.8}$$

where v_d is the volume of the unit Euclidean ball in $\mathbb{R}^d$. This is the mean volume occupied by the critical Boolean model and this is a scale invariant quantity. Let us assume that $\nu_1=\nu_2=\delta_1$. By (1.2), by Proposition 1.6 and with the above notation we have:

$$\phi_c(\alpha\delta_1+(1-\alpha)H^\rho\delta_1) \to 1-\exp\left(-v_d\lambda_c(\delta_1)\min\left(\frac{1}{\alpha},\frac{1}{1-\alpha}\right)\right). \tag{1.9}$$

There are several numerical studies of the above critical covered volume when $d=2$ and $d=3$. To the best of our knowledge, the most acccurate values when $d=2$ are given in Quintanilla and Ziff (2007). Let us assume henceforth that $d=2$. In Quintanilla and Ziff (2007), the authors give:

$$\phi_c(\delta_1) = 1-\exp(-v_2\lambda_c(\delta_1)) \approx 0.6763475(6). \tag{1.10}$$

In Figure 1.1 we reproduce the graph of the critical covered volume $\phi(\alpha,\rho) := \phi_c(\alpha\delta_1+(1-\alpha)H^\rho\delta_1)$ as a function of α when $\rho=2$, $\rho=5$ and $\rho=10$ (see Quintanilla and Ziff (2007) for more results). We also represent the graph of the right-hand side of (1.9), that we denote by $\phi(\alpha, \infty)$, as a function of α. We use (1.10) to get an approximate value of $v_2\lambda_c(\delta_1)$.

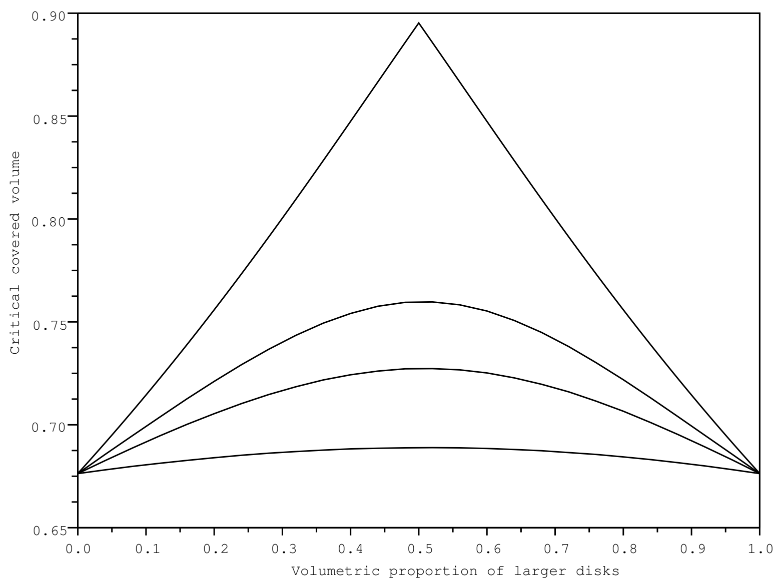

FIGURE 1.1. Critical covered volume as a function of α for different values of ρ. From bottom to top: $\rho = 2, \rho = 5, \rho = 10$ and the limit as $\rho \to \infty$.

Remarks.

- When $\rho \to \infty$, the critical covered volume $\phi(\cdot, \rho)$ converges to $\phi(\cdot, \infty)$ which is symmetric: $\phi(\alpha, \infty) = \phi(1-\alpha, \infty)$. When ρ is finite, the critical covered volume may also look symmetric but Quintanilla and Ziff (2007) showed, based on their numerical simulations and statistical analysis, that this is not the case.
- When ρ is finite, the critical covered fraction looks concave as a function of α. However $\phi(\cdot, \infty)$ is not concave as soon as $\phi_c(\delta_1) < 1 - \exp(-2)$. Based on (1.10), $\phi(\cdot, \infty)$ is therefore not concave. As a consequence, at least for large enough ρ, $\phi(\cdot, \rho)$ is not concave.
- The numerical results suggests that the minimum of the critical covered fraction is reached when all the disks have the same radius. (Equivalently, for all ρ and all α, $\phi(\alpha, \rho) \geq \phi(0, \rho) = \phi(1, \rho) = \phi_c(\delta_1)$.) There is neither a proof nor a disproof of such a result. However, is it known that this property does not hold in sufficiently high dimension, see Gouéré and Marchand (2011).
- The numerical results also suggest some monotonicity in ρ. This has not been proven nor disproven.

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2. Proof of Theorem 1.4

2.1. *Some notations.* In the whole of Section 2, we make the following assumptions:

- μ satisfies (1.6).
- $1 < \widehat{\lambda_c}(\mu)$.

For all $\eta > 0$, we denote by $T_\eta \mu$ the measure defined by $T_\eta \mu(A) = \mu(A - \eta)$. In other words, we can build $\Sigma(T_\eta \mu)$ from $\Sigma(\mu)$ by adding η to each radius.

For all $\rho > 1$, we denote by $H^\rho \mu$ the measure defined by $H^\rho \mu(A) = \rho^d \mu(\rho A)$. With this definition, $\rho^{-1}\Sigma(\mu)$ is a Boolean model driven by the measure $H^\rho \mu$. For all $n \geq 0$, we let:

$$m_n^\rho = \sum_{k=0}^{n} H^{\rho^k} \mu.$$

With this definition and the same notation as in (1.4),

$$\bigcup_{k=0}^{n} \rho^{-k} \Sigma_k$$

is a Boolean model driven by m_n^ρ. We also let:

$$m_\infty^\rho = \sum_{k \geq 0} H^{\rho^k} \mu.$$

So, $\Sigma^\rho(\mu)$ is a Boolean model driven by the locally finite measure m_∞^ρ.

Let $p(a, \mu)$ denote the probability of existence of a path from $S(a/2)$ to $S(a)$ in $\Sigma(\mu)$:

$$p(a, \mu) = P(S(a/2) \leftrightarrow_{\Sigma(\mu)} S(a)).$$

We aim at proving that, for large enough ρ, $p(a, m_\infty^\rho) \to 0$ as a tends to infinity and $\Sigma^\rho(\mu)$ does not percolate. The first item of Theorem 1.4 follows by applying this result to the measure $\lambda\mu$. Recall that the second item of Theorem 1.4 is contained in Theorem 1.3.

2.2. *Ideas.* In this subsection we first sketch the proof of the existence of ρ and a such that $p(a, m_\infty^\rho)$ is small. This gives the main ingredients of the proof of the first item of Theorem 1.4. A full proof is given in Subsection 2.3.

We then give the ideas of the proof of Theorems 1.1 and 1.2 by Menshikov, Popov and Vachkovskaia. Our aim is to emphasize the similarities and differencies between the proofs.

The proof of Theorems 1.1 and 1.2 and our proof share several ideas but are otherwise quite different. In particular, we do not use their stochastic comparison argument and do not require the technical assumption (1.5).

Sketch of the proof of the first item of Theorem 1.4. Consider a small $\varepsilon_1 > 0$. Fix a small $\eta > 0$ and a large a such that (see Lemma 2.1):

$$p(a, T_\eta \mu) \leq \varepsilon_1 / 2. \tag{2.1}$$

For all $n \geq 1$, write:

$$m_n^\rho = H^\rho m_{n-1}^\rho + \mu.$$

If the event $\{S(a/2) \leftrightarrow_{\Sigma(m_n^\rho)} S(a)\}$ occurs, then either the event $\{S(a/2) \leftrightarrow_{T_\eta \mu} S(a)\}$ occurs (with a natural coupling between the Boolean models) either in

$\Sigma(H^\rho m^\rho_{n-1}) \cap B(a)$ one can find a component of diameter at least η. We use this observation through its following crude consequence (see Lemma 2.2):

$$p(a, m^\rho_n) \leq p(a, T_\eta \mu) + Ca^d \eta^{-d} p(\eta/2, H^\rho m^\rho_{n-1}).$$

By scaling and by (2.1), this yields:

$$p(a, m^\rho_n) \leq \varepsilon_1/2 + Ca^d \eta^{-d} p(\rho\eta/2, m^\rho_{n-1}). \tag{2.2}$$

But for any ε_2, any small enough ε_1 and any large enough a we can find τ such that (see Lemmas 2.3 and 2.4):

$$p(\tau a, m^\rho_{n-1}) \leq \varepsilon_2 \text{ as soon as } p(a, m^\rho_{n-1}) \leq \varepsilon_1. \tag{2.3}$$

An important fact is that τ does not depends on n nor on ρ, provided $\rho \geq \rho_0$ where ρ_0 is an arbitrary constant strictly larger than 1. Here we use assumption (1.6) to bound error terms due to the existence of large balls.

We choose ε_2 such that:

$$Ca^d \eta^{-d} \varepsilon_2 = \varepsilon_1/2.$$

We set $\rho = 2\tau a/\eta$. Then, (2.2) and (2.3) can be rewritten as follows:

$$\begin{aligned} p(a, m^\rho_n) &\leq \varepsilon_1/2 + Ca^d \eta^{-d} p(\tau a, m^\rho_{n-1}) &(2.4)\\ Ca^d \eta^{-d} p(\tau a, m^\rho_{n-1}) &\leq \varepsilon_1/2 \text{ as soon as } p(a, m^\rho_{n-1}) \leq \varepsilon_1. &(2.5)\end{aligned}$$

As moreover (2.1) implies $p(a, m^\rho_0) \leq \varepsilon_1$ we get, by induction and then sending n to infinity (see Lemma 2.5):

$$p(a, m^\rho_\infty) \leq \varepsilon_1.$$

The convergence of $p(a, m^\rho_\infty)$ to 0 is then extracted from the above result for a small enough ε_2 and from arguments behind (2.3) applied to m^ρ_∞ and other ε.

Sketch of the proofs of Theorems 1.1 and 1.2 by Menshikov, Popov and Vachkovskaia. Let us quickly describe the ideas of the proofs of Menshikov, Popov and Vachkovskaia. Those ideas are used in their papers Menshikov et al. (2001) and Menshikov et al. (2003) through a discretization of space (definition of different notions of good boxes at different scales); we describe them in a slightly more geometric way. For simplicity we only consider two scales: $\rho^{-1}\Sigma_1$ and Σ_0. For simplicity, we also assume that the radius is one in the unscaled model ($\mu = \lambda\delta_1$). We assume that the scale factor ρ is large enough. Assume that C is a connected component of $\rho^{-1}\Sigma_1 \cup \Sigma_0$ whose diameter is a least α (it can be much larger) for a small enough constant $\alpha > 0$. Then, C is included in the union of the following kind of sets:

(1) connected components of $\rho^{-1}\Sigma_1$ whose diameter is at least α;
(2) balls of Σ_0 enlarged by α (same centers but the radii are $1+\alpha$ instead of 1).

Then, they show that the union of all those sets is stochastically dominated by a Boolean model similar to Σ_0 but with radii enlarged by a factor α and with density of centers $1+\alpha'$ times the corresponding density for Σ_0 for a suitable $\alpha' > 0$. This part uses $\lambda < \widehat{\lambda}_c$. In some sense, one can therefore control percolation in the union of two models by percolation in one model. Iterating the argument with care in the constants α and α', one sees that – for large enough ρ – one can control percolation in the multiscale model by percolation in a subcritical model. This yields the result when the radius is constant.

When the radius is random and non bounded, the proof is more involved and require in particular the technical assumption (1.5).

2.3. *Proof of Theorem 1.4.* As $1 < \widehat{\lambda}_c(\mu)$, we know that $p(a,\mu)$ tends to 0 as a tends to infinity. We need the following slightly stronger consequence.

Lemma 2.1. *There exists $\eta > 0$ such that $p(a, T_\eta \mu)$ tends to 0 as a tends to ∞.*

Proof. Let $\varepsilon > 0$ and $x > 0$. We have:

$$\begin{aligned} H^{1+\varepsilon} T_{\varepsilon^2}\mu([x,+\infty[) &= (1+\varepsilon)^d T_{\varepsilon^2}\mu([x(1+\varepsilon),+\infty[) \\ &= (1+\varepsilon)^d \mu([x(1+\varepsilon)-\varepsilon^2,+\infty[) \\ &\leq \kappa(\varepsilon)(1+\varepsilon)^d \mu([x,+\infty[) \end{aligned} \quad (2.6)$$

where

$$\kappa(\varepsilon) = \frac{\mu(]0,+\infty[)}{\mu([\varepsilon,+\infty[)}.$$

The inequality is proven as follows. If $x \geq \varepsilon$, then $[x(1+\varepsilon)-\varepsilon^2,+\infty[\subset [x,+\infty[$ and the result follows from $\kappa(\varepsilon) \geq 1$. If, on the contrary, $x < \varepsilon$, then the left-hand side is bounded above by $(1+\varepsilon)^d\mu(]0,+\infty[)$ which is itself bounded above by the right-hand side.

Note that $\kappa(\varepsilon)(1+\varepsilon)^d$ tends to 1 as ε tends to 0. Let us say that a measure ν is subcritical if $\widehat{\lambda}_c(\nu) > 1$. As μ is subcritical, we get that $\kappa(\varepsilon)(1+\varepsilon)^d\mu$ is subcritical for small enough ε. We fix such an ε. By (2.6) we can couple a Boolean model driven by $H^{1+\varepsilon}T_{\varepsilon^2}\mu$ and a Boolean model driven by $\kappa(\varepsilon)(1+\varepsilon)^d\mu$ in such a way that the first one is contained in the second one. Therefore the first one is subcritical. By scaling, a Boolean model driven by $T_{\varepsilon^2}\mu$ is then subcritical. We take $\eta = \varepsilon^2$. □

Lemma 2.2. *Let ν_1 and ν_2 be two finite measures on $]0,+\infty[$. One has, for all $\eta > 0$ and $a \geq 4\eta$:*

$$p(a,\nu_1+\nu_2) \leq p(a,T_\eta\nu_1) + C_1 a^d \eta^{-d} p(\eta/2,\nu_2)$$

where $C_1 = C_1(d) > 0$ depends only on the dimension d.

Proof. Let $(x_i)_{i\in I}$ be a family of points such that :

- The balls $B(x_i,\eta/4)$, $i \in I$, cover $B(a) = B(0,a)$.
- There are at most $C_1 a^d \eta^{-d}$ points in the family where $C_1 = C_1(d)$ depends only on the dimension d.

We couple the different Boolean models as follows. Let $\Sigma(\nu_1)$ be a Boolean model driven by ν_1. Let $\Sigma(\nu_2)$ be a Boolean model driven by ν_2. Assume that $\Sigma(\nu_1)$ and $\Sigma(\nu_2)$ are independent. Then $\Sigma(\nu_1)\cup\Sigma(\nu_2)$ is a Boolean model driven by $\nu_1+\nu_2$. We set $\Sigma(\nu_1+\nu_2) = \Sigma(\nu_1)\cup\Sigma(\nu_2)$. We also consider $\Sigma(T_\eta\nu_1)$, the Boolean model obtained by adding η to the radius of each ball of $\Sigma(\nu_1)$. Thus $\Sigma(T_\eta\nu_1)$ is driven by $T_\eta\nu_1$.

Let us prove the following property:

$$\begin{aligned} &\{S(a/2) \leftrightarrow_{\Sigma(\nu_1+\nu_2)} S(a)\} \\ &\qquad \subset \{S(a/2) \leftrightarrow_{\Sigma(T_\eta\nu_1)} S(a)\} \cup \bigcup_{i\in I}\{S(x_i,\eta/4) \leftrightarrow_{\Sigma(\nu_2)} S(x_i,\eta/2)\}. \end{aligned} \quad (2.7)$$

Assume that $\Sigma(\nu_1+\nu_2) = \Sigma(\nu_1) \cup \Sigma(\nu_2)$ connects $S(a/2)$ with $S(a)$. Recall $a \geq 4\eta$. If the diameter of all connected components of $\Sigma(\nu_2) \cap B(a)$ are less or equal to η, then $\Sigma(T_\eta \nu_1)$ connects $S(a/2)$ with $S(a)$. Otherwise, let C be a connected component of $\Sigma(\nu_2) \cap B(a)$ with diameter strictly larger than η. Let x, y be two points of C such that $\|x-y\| > \eta$. The point x belongs to a ball $B(x_i, \eta/4)$. As y does not belong to $B(x_i, \eta/2)$, the component C connects $S(x_i, \eta/4)$ to $S(x_i, \eta/2)$. Therefore, $\Sigma(\nu_2)$ connects $S(x_i, \eta/4)$ to $S(x_i, \eta/2)$. We have proven (2.7). The lemma follows (using the union bound, translation invariance and the upper bound on the cardinality of I). $\square$

The following lemma is essentially the first item of Proposition 3.1 in Gouéré (2008). For the sake of completeness we nevertheless provide a proof.

Lemma 2.3. *Let ν be a finite measure on $]0, +\infty[$. There exists a constant $C_2 = C_2(d) > 0$ such that, for all $a > 0$:*

$$p(10a, \nu) \leq C_2 p(a, \nu)^2 + C_2 \int_{[a,+\infty[} r^d \nu(dr).$$

Proof. Let K be a finite subset of $S(5)$ such that $K + B(1/2)$ covers $S(5)$. Let L be a finite subset of $S(10)$ such that $L + B(1/2)$ covers $S(10)$. Let A be the following event: there exists a random ball $B(c, r)$ of $\Sigma(\nu)$ such that $r \geq a$ and $B(c, r) \cap B(10a)$ is non empty. We have:

$$\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \subset \{S(5a) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(10a)\}$$

where, in the last event, we ask for the existence of a path contained in balls of $\Sigma(\nu)$ of radius at most a. Let us prove the following:

$$\begin{aligned}&\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\} \setminus A \\ &\quad \subset \bigcup_{k \in K, l \in L} \{S(ak, a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(ak, a)\} \cap \{S(al, a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(al, a)\}. \qquad (2.8)\end{aligned}$$

Assume that the event on the left-hand side occurs. Then, by the previous remark, there exists a path from a point $x \in S(5a)$ to a point $y \in S(10a)$ that is contained in balls of $\Sigma(\nu)$ of radius at most a. As $Ka + B(a/2)$ covers $S(5a)$, there exists $k \in K$ such that x belongs to $B(ka, a/2)$. Using the previous path, one gets that the event

$$\{S(ak, a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(ak, a)\}$$

occurs. By a similar arguments involving y we get (2.8).

Observe that, for all $k \in K$ and $l \in L$, the events

$$\{S(ak, a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(ak, a)\} \text{ and } \{S(al, a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(al, a)\}$$

are independent. Indeed, the first one depends only on balls with centers in $B(ak, 2a)$, the second one depends only on balls with centers in $B(al, 2a)$, and $\|ak - al\| \geq 5a$. Using this independence, stationarity and (2.8), we then get:

$$P(\{S(5a) \leftrightarrow_{\Sigma(\nu)} S(10a)\}) \leq CP(S(a/2) \leftrightarrow^{\leq a}_{\Sigma(\nu)} S(a)\})^2 + P(A)$$

where C is the product of the cardinality of K by the cardinality of L. We thus have:

$$p(10a, \nu) \leq Cp(a, \nu)^2 + P(A).$$

The probability $P(A)$ is bounded above by standard Poisson point process computations. $\square$

From the previous lemma, we deduce the following result.

Lemma 2.4. *Let $\varepsilon > 0$. There exists $C_3 = C_3(d) > 0$, $a_0 = a_0(d,\mu)$ and $k_0 = k_0(d,\mu,\varepsilon)$ such that, for all N, all $\rho \geq 2$ and all $a \geq a_0$: if $p(a, m_N^\rho) \leq C_3$ then for all $k \geq k_0$, $p(a10^k, m_N^\rho) \leq \varepsilon$.*

Proof. For all $\rho \geq 2$ and all $a \geq 1$ we have:

$$\begin{aligned}
\int_{[a,+\infty[} r^d m_\infty^\rho(dr) &= \sum_{k\geq 0} \rho^{kd} \int_{]0,+\infty[} 1_{[a,+\infty[}(r\rho^{-k})(r\rho^{-k})^d \mu(dr) \\
&= \int_{]0,+\infty[} \sum_{k\geq 0} 1_{[a,+\infty[}(r\rho^{-k}) r^d \mu(dr) \\
&= \int_{[a,+\infty[} \left(\lfloor \ln(r/a)\ln(\rho)^{-1} \rfloor + 1\right) r^d \mu(dr) \\
&\leq \int_{[a,+\infty[} (\ln(r)\ln(2)^{-1} + 1) r^d \mu(dr).
\end{aligned}$$

Let C_2 be the constant given by Lemma 2.3. By (1.6) we can chose $a_0 = a_0(d,\mu) \geq 1$ such that

$$C_2^2 \int_{[a_0,+\infty[} (\ln(r)\ln(2)^{-1} + 1) r^d \mu(dr) \leq \frac{1}{4}. \tag{2.9}$$

Let $C_3 = (2C_2)^{-1}$. Let N, ρ and a be as in the statement of the lemma. From Lemma 2.3 we get:

$$\begin{aligned}
C_2 p(10a, m_N^\rho) &\leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a,+\infty[} r^d m_N^\rho(dr) &(2.10)\\
&\leq (C_2 p(a, m_N^\rho))^2 + C_2^2 \int_{[a,+\infty[} (\ln(r)\ln(2)^{-1} + 1) r^d \mu(dr) &(2.11)
\end{aligned}$$

Let (u_k) be a sequence defined by $u_0 = 1/2$ and, for all $k \geq 0$:

$$u_{k+1} = u_k^2 + C_2^2 \int_{[a_0 10^k,+\infty[} (\ln(r)\ln(2)^{-1} + 1) r^d \mu(dr). \tag{2.12}$$

Note that the sequence (u_k) only depends on d and μ.

Assume that $p(a, m_N^\rho) \leq C_3$. We then have $C_2 p(a, m_N^\rho) \leq u_0$. Using $a \geq a_0$ and (2.11), we then get $C_2 p(a10^k, m_N^\rho) \leq u_k$ for all k. Therefore, it sufficies to show that the sequence (u_k) tends to 0.

Using (2.12), (2.9) and $u_0 = 1/2$ we get $0 \leq u_k \leq 1/2$ for all k. Therefore, $0 \leq \limsup u_k \leq 1/2$. By (2.12) and by the convergence of the integral we also get $\limsup u_k \leq (\limsup u_k)^2$. As a consequence, $\limsup u_k = 0$ and the lemma is proven. $\square$

Lemma 2.5. *For all $a > 0$ and $\rho > 1$ the following convergence holds:*

$$p(a, m_\infty^\rho) = \lim_{N\to\infty} p(a, m_N^\rho).$$

Proof. The sequence of events

$$A_N = \{S(a/2) \leftrightarrow_{\Sigma(m_N^\rho)} S(a)\}$$

is increasing (we use the natural coupling between our Boolean models). Therefore, it suffices to show that the union of the previous events is

$$A = \{S(a/2) \leftrightarrow_{\Sigma(m_\infty^\rho)} S(a)\}.$$

If A occurs, then there is a path from $S(a/2)$ to $S(a)$ that is contained in $\Sigma(m_\infty^\rho)$. By a compactness argument, this path is included in a finite union of ball of $\Sigma(m_\infty^\rho)$. Therefore, there exists N such that the path is included in $\Sigma(m_N^\rho)$ and A_N occurs. This proves $A \subset \cup A_N$. The other inclusion is straightforward. □

Proof of the second item of Theorem 1.4. By Lemma 2.1, we can fix $\eta_1 > 0$ such that $p(a, T_{10\eta_1}\mu)$ tends to 0 as a tends to ∞. Let C_1 be given by Lemma 2.2. Let a_0 and C_3 be as given by Lemma 2.4. Fix $a_1 \geq \max(40\eta_1, a_0, 1)$ such that $p(a, T_{10\eta_1}\mu) \leq C_3/2$ for all $a \geq a_1$. Let k_0 be given by Lemma 2.4 with the choice:

$$\varepsilon = C_1^{-1}(10a_1)^{-d}\eta_1^d C_3/2.$$

Therefore, for all $\rho \geq 2$, all N, all $a \in [a_1, 10a_1]$ and all $\eta \in [\eta_1, 10\eta_1]$:

$$C_1 a^d \eta^{-d} p(a10^k, m_N^\rho) \leq \frac{C_3}{2} \text{ for all } k \geq k_0 \text{ as soon as } p(a, m_N^\rho) \leq C_3.$$

Fix $k \geq k_0$, $a \in [a_1, 10a_1]$ and $\eta \in [\eta_1, 10\eta_1]$. Set:

$$\rho = 2a10^k\eta^{-1}.$$

Note $\rho \geq 8 \geq 2$ as $a \geq a_1 \geq 40\eta_1 \geq 4\eta$. By Lemma 2.2 we have, for all N:

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq p(a, T_\eta\mu) + C_1 a^d \eta^{-d} p(\eta/2, m_{N+1}^\rho - \mu) \\ &= p(a, T_\eta\mu) + C_1 a^d \eta^{-d} p(\eta/2, H^\rho m_N^\rho). \end{aligned}$$

By definition of a_1, by $a_1 \leq a$, by $\eta \leq 10\eta_1$, by scaling and by definition of ρ we get, for all N:

$$\begin{aligned} p(a, m_{N+1}^\rho) &\leq \frac{C_3}{2} + C_1 a^d \eta^{-d} p(\rho\eta/2, m_N^\rho) \\ &= \frac{C_3}{2} + C_1 a^d \eta^{-d} p(a10^k, m_N^\rho). \end{aligned}$$

Combining this inequality with the property defining k_0, we get that $p(a, m_N^\rho) \leq C_3$ implies $p(a, m_{N+1}^\rho) \leq C_3$. As $p(a, m_0^\rho) = p(a, \mu) \leq p(a, T_{10\eta_1}\mu) \leq C_3/2$ we get $p(a, m_N^\rho) \leq C_3$ for all integer N.

Let $\varepsilon' > 0$. Using again Lemma 2.4 we get the existence of an integer k_0' such that $p(a10^{k'}, m_N^\rho) \leq \varepsilon'$ for all $k' \geq k_0'$ as soon as $p(a, m_N^\rho) \leq C_3$. But we have proven the latter property. Therefore $p(a10^{k'}, m_N^\rho) \leq \varepsilon'$ for all N and all $k' \geq k_0'$. By Lemma 2.5, we get $p(a10^{k'}, m_\infty^\rho) \leq \varepsilon'$ for all $k' \geq k_0'$. Using the freedom on the choice of $k \geq k_0$ and $\eta \in [\eta_1, 10\eta_1]$, we get that the previous result holds for all $\rho \geq 2a10^{k_0-1}\eta_1^{-1}$ and then for all $\rho \geq 2a_1 10^{k_0}\eta_1^{-1}$. Moreover, using the freedom on the choice of $a \in [a_1, 10a_1]$ and $k' \geq k_0'$, we get:

$$p(r, m_\infty^\rho) \leq \varepsilon' \text{ for all } r \geq a_1 10^{k_0'} \text{ and all } \rho \geq 2a_1 10^{k_0}\eta_1^{-1}.$$

Therefore, $p(r, m_\infty^\rho)$ tends to 0 as r tends to infinity. As a consequence, $\Sigma^\rho(\mu)$ does not percolate for any $\rho \geq 2a_1 10^{k_0}\eta_1^{-1}$. □

3. Proof of Theorem 1.5

Lemma 3.1. *Let $s > 0$ and $\rho > 1$. The following assumptions are equivalent:*

(1) $\int_{]0,+\infty[} r^{d+s}\mu(dr) < \infty$.
(2) $\int_{[1,+\infty[} r^{d+s}m^\rho_\infty(dr) < \infty$.

Proof. We have:

$$\begin{aligned}\int_{[1,+\infty[} r^{d+s}m^\rho_\infty(dr) &= \sum_{k\geq 0}\rho^{kd}\int_{]0,+\infty[} 1_{[1,+\infty[}(r\rho^{-k})(r\rho^{-k})^{d+s}\mu(dr)\\ &= \int_{[1,+\infty[}\sum_{k\geq 0} 1_{[1,+\infty[}(r\rho^{-k})\rho^{-ks}r^{d+s}\mu(dr).\end{aligned}$$

Therefore:

$$\int_{[1,+\infty[} r^{d+s}\mu(dr) \leq \int_{[1,+\infty[} r^{d+s}m^\rho_\infty(dr) \leq \frac{1}{1-\rho^{-s}}\int_{[1,+\infty[} r^{d+s}\mu(dr).$$

This yields the result. □

Proof of the first item of Theorem 1.5. By the discussion at the beginning of Section 1.5 in Gouéré (2009), $\Sigma^\rho(\lambda\mu)$ is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure λm^ρ_∞. Let us check the three assumptions of Theorem 2.9 in Gouéré (2009) with $\rho = 10$ (ρ is not used in the same way in Gouéré (2009)). We refer to Section 2.1 of Gouéré (2009) for definitions.

(1) The first assumption is fulfilled thanks to (1.7).
(2) For all $\beta > 0$ and all $x \in \mathbb{R}^d$, the event $G(x, 0, \beta)$ only depends on balls $B(c, r)$ of $\Sigma^\rho(\lambda\mu)$ such that c belongs to $B(x, 3\beta)$. By the independance property of Poisson point processes, we then get that $G(0, 0, \beta)$ and $G(x, 0, \beta)$ are independent whenever $\|x\| \geq 10\beta$. Therefore $I(10, 0, \beta) = 0$ and the second assumption of Theorem 2.9 is fulfilled.
(3) The third assumption (note that μ in Gouéré (2009) is m^ρ_∞ in this paper) is fulfilled thanks to Lemma 3.1.

Theorem 2.9 in Gouéré (2009) yields the result. □

Proof of the second item of Theorem 1.5. If $\int r^d\mu(dr)$ is infinite then, $\Sigma(\lambda\mu)$ percolates for all $\lambda > 0$ (see the dicussion of Section 1.2). Therefore $\Sigma^\rho(\lambda\mu)$ percolates for all $\rho > 1$ and $\lambda > 0$. Therefore $D^\rho(\lambda\mu) = \infty$ with positive probability for all $\rho > 1$ and $\lambda > 0$.

Now, assume that $\int r^d\mu(dr)$ is finite. Then, by the discussion at the beginning of Section 1.5 in Gouéré (2009), $\Sigma^\rho(\lambda\mu)$ is driven by a Poisson point process whose intensity is the product of the Lebesgue measure by the locally finite measure λm^ρ_∞. We can therefore apply Theorem 1.2 in Gouéré (2009). By Lemma 3.1, assumption (A3) of Theorem 1.2 in Gouéré (2009) is not fulfilled (note that μ in Gouéré (2009) is m^ρ_∞ in this paper). Theorem 1.2 in Gouéré (2009) then yields the result. □

4. Proof of Proposition 1.6

We first need a lemma, which is a consequence of Lemmas 2.2 and 2.3.

Lemma 4.1. *Let ν_1 and ν_2 be two finite measures on $]0,+\infty[$. Let $\eta > 0$, $a_0 \geq 4\eta$ and $\rho > 1$. There exists $C_4 = C_4(d) > 0$ such that $\widehat{\lambda}_c(\nu_1 + H^\rho(\nu_2)) \geq 1$ as soon as the following conditions hold:*

(1) $p(a, T_\eta\nu_1) \leq C_4$ *for all* $a \in [a_0, 10a_0]$.
(2) $a_0^d \eta^{-d} p(\rho\eta/2, \nu_2) \leq C_4$.
(3) $\int_{[a_0,+\infty[} r^d \nu_1(dr) \leq C_4$ *and* $\int_{[a_0,+\infty[} r^d \nu_2(dr) \leq C_4$.

Proof. Let $C_4 = C_4(d) > 0$ be such that $C_4 C_2(1+10^d C_1) \leq 1/2$ and $2C_2^2 C_4 \leq 1/4$, where C_1 appears in Lemma 2.2 and C_2 appears in Lemma 2.3. Set $\nu = \nu_1 + H^\rho(\nu_2)$.

For all $a \in [a_0, 10a_0]$ we have, by Lemma 2.2 applied to ν_1 and $H^\rho(\nu_2)$, by scaling and by the assumptions of the lemma:

$$\begin{aligned} p(a,\nu) &\leq p(a, T_\eta\nu_1) + C_1 a^d \eta^{-d} p(\eta/2, H^\rho\nu_2) \\ &= p(a, T_\eta\nu_1) + C_1 a^d \eta^{-d} p(\rho\eta/2, \nu_2) \\ &\leq C_4(1+10^d C_1) \\ &\leq 1/(2C_2). \end{aligned} \tag{4.1}$$

But for all $a \geq a_0$ we have, by Lemma 2.3 and by the assumptions of the lemma:

$$\begin{aligned} C_2 p(10a,\nu) &\leq (C_2 p(a,\nu))^2 + C_2^2 \int_{[a,+\infty[} r^d \nu(dr) \\ &= (C_2 p(a,\nu))^2 + C_2^2 \int_{[a,+\infty[} r^d \nu_1(dr) + C_2^2 \int_{[a\rho,+\infty[} r^d \nu_2(dr) \quad (4.2) \\ &\leq (C_2 p(a,\nu))^2 + 2C_2^2 C_4 \\ &\leq (C_2 p(a,\nu))^2 + 1/4. \end{aligned} \tag{4.3}$$

By (4.1) and (4.3) we get $C_2 p(a,\nu) \leq 1/2$ for all $a \geq a_0$ and therefore $0 \leq \limsup C_2 p(a,\nu) \leq 1/2$. By (4.2) and the third assumption of the lemma, we get $\limsup C_2 p(a,\nu) \leq (\limsup C_2 p(a,\nu))^2$. Therefore, we must have $\limsup C_2 p(a,\nu) = 0$ and the lemma is proven. □

Proof of Proposition 1.6. The inequality is straightforward. To prove the equality, we note that, by scaling, $\widehat{\lambda}_c(H^\rho\nu_2) = \widehat{\lambda}_c(\nu_2)$. Let us prove the convergence. We can assume $\widehat{\lambda}_c(\nu_1) > 0$ and $\widehat{\lambda}_c(\nu_2) > 0$, otherwise the convergence is obvious. Therefore, by Lemma A.2, the integrals $\int r^d \nu_1(dr)$ and $\int r^d \nu_2(dr)$ are finite.

Let C_4 be the constant given by Lemma 4.1. Let $0 < \varepsilon < 1$. Note that:

$$\widehat{\lambda}_c\big((1-\varepsilon)\widehat{\lambda}_c(\nu_1)\nu_1\big) = (1-\varepsilon)^{-1}\widehat{\lambda}_c(\nu_1)^{-1}\widehat{\lambda}_c(\nu_1) > 1.$$

Therefore, by Lemma 2.1 (in which (1.6) is not used), we can fix $\eta > 0$ such that

$$p(a, T_\eta(1-\varepsilon)\widehat{\lambda}_c(\nu_1)\nu_1) \to 0.$$

We can then fix $a_0 \geq 4\eta$ such that:

$$p(a, T_\eta(1-\varepsilon)\widehat{\lambda}_c(\nu_1)\nu_1) \leq C_4 \text{ for all } a \geq a_0 \tag{4.4}$$

and such that

$$\int_{[a_0,+\infty[} r^d \widehat{\lambda}_c(\nu_1)\nu_1(dr) \leq C_4 \tag{4.5}$$

and

$$\int_{[a_0,+\infty[} r^d \widehat{\lambda}_c(\nu_2)\nu_2(dr) \leq C_4. \tag{4.6}$$

Now we fix $\rho_0 > 1$ such that :

$$a_0^d \eta^{-d} p(\rho\eta/2, (1-\varepsilon)\widehat{\lambda}_c(\nu_2)\nu_2) \le C_4 \text{ for all } \rho \ge \rho_0. \tag{4.7}$$

Now, let $0 < \alpha < 1$ and set

$$\lambda = \min\left(\frac{\widehat{\lambda}_c(\nu_1)(1-\varepsilon)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)(1-\varepsilon)}{1-\alpha}\right).$$

By (4.4), (4.7), (4.5) and (4.6) we get that Assumptions 1 , 2 and 3 of Lemma 4.1 are fulfilled for the measures $\alpha\lambda\nu_1$ and $(1-\alpha)\lambda\nu_2$ and for $\rho \ge \rho_0$. Therefore, we get

$$\widehat{\lambda}_c(\alpha\lambda\nu_1 + (1-\alpha)\lambda H^\rho \nu_2) \ge 1$$

and thus:

$$\widehat{\lambda}_c(\alpha\nu_1 + (1-\alpha)H^\rho\nu_2) \ge \lambda = (1-\varepsilon)\min\left(\frac{\widehat{\lambda}_c(\nu_1)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)}{1-\alpha}\right).$$

Therefore, as soon as $\rho \ge \rho_0$, we have:

$$\begin{aligned} 0 &\le \min\left(\frac{\widehat{\lambda}_c(\nu_1)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)}{1-\alpha}\right) - \widehat{\lambda}_c(\alpha\nu_1 + (1-\alpha)H^\rho\nu_2) \\ &\le \varepsilon \min\left(\frac{\widehat{\lambda}_c(\nu_1)}{\alpha}, \frac{\widehat{\lambda}_c(\nu_2)}{1-\alpha}\right) \\ &\le \varepsilon \max(2\widehat{\lambda}_c(\nu_1), 2\widehat{\lambda}_c(\nu_2)). \end{aligned}$$

This yields the proposition. □

Appendix A. Critical parameters

Lemma A.1. *The critical parameters satisfy:*

$$\widetilde{\lambda}_c(\mu) \le \widehat{\lambda}_c(\mu) \le \lambda_c(\mu).$$

Proof. The second inequality is a consequence of the following inclusion:

$$\{\{0\} \leftrightarrow_\Sigma S(r)\} \subset \{S(r/2) \leftrightarrow_\Sigma S(r)\}.$$

The first inequality can be proven as follows. Let $r \ge 1$. By the FKG inequality, we get:

$$\begin{aligned} P(\{0\} \leftrightarrow_\Sigma S(r)) &\ge P(B(0,1) \subset \Sigma \text{ and } S(1) \leftrightarrow_\Sigma S(r)) \\ &\ge CP(S(1) \leftrightarrow_\Sigma S(r)) \end{aligned}$$

where $C = P(B(0,1) \subset \Sigma) > 0$ does not depend on r. For all large enough r, we can cover $S(2r)$ by at most $C'r^d$ balls $B(x_i,1)$ where C' only depends on the dimension d. If there is a path in Σ from $S(2r)$ to $S(4r)$, then there exists i and a path in Σ from $S(x_i,1)$ to $S(x_i,r)$. (Consider the ball $B(x_i,1)$ that contains the initial point of the path.) By stationarity and by the previous inequality we thus get:

$$\begin{aligned} P(S(2r) \leftrightarrow_\Sigma S(4r)) &\le C'r^d P(S(1) \leftrightarrow_\Sigma S(r)) \\ &\le C'C^{-1}r^d P(\{0\} \leftrightarrow_\Sigma S(r)). \end{aligned}$$

The first inequality stated in the lemma follows. □

Lemma A.2. *The threshold parameter $\widehat{\lambda_c}(\mu)$ is positive if and only if $\int r^d \mu(dr)$ is finite.*

Proof. If $\widehat{\lambda_c}(\mu)$ is positive, then there exists $\lambda > 0$ such that $\Sigma(\lambda\mu)$ does not percolate. By Theorem 2.1 of Gouéré (2008) this implies that $\int r^d \mu(dr)$ is finite.

Let us assume now that $\int r^d \mu(dr)$ is finite. We need to prove the existence of $\lambda > 0$ such that $p(a, \lambda\mu)$ tends to 0. This is proven, as an intermediate result, in the proof of Theorem 1.1 in Gouéré (2009). As the result is an easy consequence of Lemma 2.3, we find it more convenient to provide a proof here. Let C_2 be the constant given by Lemma 2.3. For all $a > 0$ and $\lambda > 0$ we have:

$$C_2 p(10a, \lambda\mu) \leq (C_2 p(a, \lambda\mu))^2 + \lambda C_2^2 \int_{[a,+\infty[} r^d \mu(dr). \tag{A.1}$$

For all $0 < a \leq 1$ we have, by standard computations:

$$C_2 p(a, \lambda\mu) \leq C_2 P(\text{a ball of } \Sigma(\lambda\mu) \text{ touches } B(a)) \leq C_2 v_d \lambda \int_{]0,+\infty[} (1+r)^d \mu(dr)$$

where v_d is the volume of the unit Euclidean ball. As $\int r^d \mu(dr)$ is finite, we can therefore fix $\lambda > 0$ such that:

$$\lambda C_2^2 \int_{[a,+\infty[} r^d \mu(dr) \leq 1/4 \text{ for all } a > 0 \text{ and } C_2 p(a, \lambda\mu) \leq 1/2 \text{ for all } 0 < a \leq 1. \tag{A.2}$$

By (A.1), (A.2) and by induction we get $C_2 p(a, \lambda\mu) \leq 1/2$ for all $a > 0$. Therefore, we have $0 \leq \limsup C_2 p(a, \lambda\mu) \leq 1/2$. But (A.1) also yields the inequality $\limsup C_2 p(a, \lambda\mu) \leq (\limsup C_2 p(a, \lambda\mu))^2$. As a consequence we must have $\limsup C_2 p(a, \lambda\mu) = 0$ and then $p(a, \lambda\mu) \to 0$. □

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