ALEA, Lat. Am. J. Probab. Math. Stat. 11 (1), 331–340 (2014)

Limit theory for bivariate extreme generalized order statistics and dual generalized order statistics

H.M. Barakat, E.M. Nigm and M.A. Abd Elgawad

Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt
E-mail address: hbarakat2@hotmail.com

Department of Mathematics, Faculty of Science, Zagazig University, Zagazig, Egypt

Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

Abstract. In Kamps (1995) generalized order statistics (gos) have been introduced as a unifying theme for several models of ascendingly ordered random variables (rv's). Following Kamps (1995), Burkschat et al. (2003) have introduced the concept of dual generalized order statistics (dgos) to unify several models that produce ordered rv's. In this paper we study the asymptotic bivariate df of the (lower-lower), (upper-upper) and (lower-upper) extreme $m-$gos and $m-$dgos (i.e, $m_1 = m_2 = ... = m_{n-1} = m \neq -1$).

1. Introduction

Generalized order statistics (gos), as well as the dual generalized order statistics (dgos), have been introduced as a unified distribution theoretical set-up which contains a variety of models of order random variables (rv's). Since Kamps (1995) had introduced the concept of gos as a unification of several models of ascendingly ordered rv's, the use of such concept has been steadily growing along the years. Actually, in the past decade, properties of gos have attracted considerable attention in the literature. This is due to the fact that such concept includes important well-known concepts that have been separately treated in statistical literature. Theoretically, many of the models of ordered rv's are contained in the gos model, such as ordinary order statistics (oos), order statistics with non-integral sample size, sequential order statistics (sos), record values, Pfeifer's record model and progressive type II censored order statistics (pos). These models can be applied in reliability theory. For instance, the sos model is an extension of the oos model and serves as a model describing certain dependencies or interactions among the system components caused by failures of components and the pos model is an important method of obtaining data in lifetime tests. Live units removed early on can

Received by the editors February 6, 2013; accepted June 10, 2014.

Key words and phrases. Generalized order statistics, Dual generalized order statistics, Extreme generalized order statistics.

be readily used in other tests, thereby saving cost to the experimenter. Random variables that are decreasingly ordered cannot be integrated into the framework of gos. Therefore, Burkschat et al. (2003) have introduced the concept of dgos to enable a common approach to desendingly ordered rv's. Each of the concepts of gos and dgos enables a common approach to structural similarities and analogies. Known results in submodels can be subsumed, generalized, and integrated within a general framework. Kamps (1995) defined gos by first defining what he called uniform gos and then using the quantile transformation to obtain the general gos $X(r,n,\tilde{m},k), r=1,2,...,n$, based on a df F, which are defined by their probability density function (pdf)

$$f^{(\tilde{m},k)}_{1,2,\ldots,n:n}(x_1,x_2,...,x_n)=\left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}(1-F(x_j))^{\gamma_j-\gamma_{j+1}-1}f(x_j)\right)\times(1-F(x_n))^{\gamma_n-1}f(x_n),$$

where $F^{-1}(0)\le x_1\le ...\le x_n\le F^{-1}(1)$, $\gamma_n=k>0$, $\gamma_r=k+n-r+\sum_{j=r}^{n-1}m_j$, $r=1,2,...,n-1$, and $\tilde{m}=(m_1,m_2,\cdots,m_{n-1})\in\Re^{n-1}$. Particular choices of the parameters $\gamma_1,\gamma_2,...,\gamma_n$ lead to different models, e.g., $m-$gos ($\gamma_r=k+(n-r)(m+1), r=1,2,...,n-1$), oos ($k=1,\gamma_r=n-r+1, r=1,2,...,n-1$) and sos ($k=\alpha_n,\gamma_r=(n-r+1)\alpha_r, r=1,2,...,n-1$) (see Kamps (1995)).

By a similar way, the dgos $X_d(r,n,\tilde{m},k), r=1,2,...,n$, based on a df F, are defined by their pdf

$$f^{d(\tilde{m},k)}_{1,2,\ldots,n:n}(x_1,x_2,...,x_n)=\left(\prod_{j=1}^{n}\gamma_j\right)\left(\prod_{j=1}^{n-1}F^{\gamma_j-\gamma_{j+1}-1}(x_j)f(x_j)\right)F^{\gamma_n-1}(x_n)f(x_n),$$

where $F^{-1}(1)\ge x_1\ge ...\ge x_n\ge F^{-1}(0)$.

Nasri-Roudsari (1996) (see also Barakat (2007)) has derived the marginal df of the rth $m-$gos, $m\neq-1$, in the form $\Phi^{(m,k)}_{r:n}(x)=I_{G_m(x)}(r,N-r+1)$, where $G_m(x)=1-(1-F(x))^{m+1}=1-\bar{F}^{m+1}(x)$, $I_x(a,b)=\frac{1}{\beta(a,b)}\int_o^x t^{a-1}(1-t)^{b-1}dt$ denotes the incomplete beta ratio function and $N=\frac{k}{m+1}+n-1$. By using the well-known relation $I_x(a,b)=1-I_{\overline{x}}(b,a)$, where $\overline{x}=1-x$, the marginal df of the $(n-r+1)$th $m-$gos, $m\neq-1$, is given by $\Phi^{(m,k)}_{n-r+1:n}(x)=I_{G_m(x)}(N-R_r+1,R_r)$, where $R_r=\frac{k}{m+1}+r-1$. Similarly, by putting $T_m(x)=F^{m+1}(x)$, the marginal df's of the rth and $(n-r+1)$th $m-$dgos, $m\neq-1$, can be written, respectively by

$$\Phi^{d(m,k)}_{r:n}(x)=I_{T_m(x)}(N-r+1,r) \tag{1.1}$$

and

$$\Phi^{d(m,k)}_{n-r+1:n}(x)=I_{T_m(x)}(R_r,N-R_r+1). \tag{1.2}$$

Moreover, by using the results of Kamps (1995) and Burkschat et al. (2003), we can write explicitly the joint pdf's of the rth and sth $m-$gos and $m-$dgos, $m\neq-1$, $1\le r<s\le n$, respectively as:

$$f^{(m,k)}_{r,s:n}(x_r,x_s)=\frac{C_{s-1,n}}{\Gamma(r)\Gamma(s-r)}\overline{F}^{m}(x_r)g_m^{r-1}(F(x_r))(g_m(F(x_s))-g_m(F(x_r)))^{s-r-1}\overline{F}^{\gamma_s-1}(x_s)f(x_r)f(x_s),\ -\infty<x_r<x_s<\infty, \tag{1.3}$$

and

$$f_{r,s:n}^{d(m,k)}(x_r,x_s)=\frac{C_{s-1,n}}{\Gamma(r)\Gamma(s-r)}F^m(x_r)(g_m(\overline{F}(x_s))-g_m(\overline{F}(x_r)))^{s-r-1}g_m^{r-1}(\overline{F}(x_r))$$
$$F^{\gamma_s-1}(x_s)f(x_r)f(x_s), -\infty<x_s<x_r<\infty, \tag{1.4}$$

where $C_{r-1,n}=\prod_{i=1}^{r}\gamma_i, r=1,2,...,n$, and $g_m(x)=\frac{1}{m+1}[1-\overline{x}^{m+1}]$.

Asymptotic theory of extreme $m-$gos and $m-$dgos, $m\neq-1$. The following two theorems extend the well-known results concerning the asymptotic theory of extreme (lower-upper) ordinary order statistics to the case of (upper-lower) extreme $m-$gos and $m-$dgos. These theorems can be easily proved by applying the following asymptotic relations, due to Smirnov (1952) (see also Barakat (1997)):

$$\Gamma_r(nA_n)-\delta_{1n}\leq I_{A_n}(r,n-r+1)\leq\Gamma_r(nA_n)-\delta_{2n},$$

if $nA_n\sim A<\infty$, as $n\to\infty$ (we mean by $a_n\sim b_n$, as $n\to\infty$, $\frac{a_n}{b_n}\to 1$, as $n\to\infty$) and

$$1-\Gamma_r(n\overline{A}_n)-\delta_{2n}\leq I_{A_n}(n-r+1,r)\leq 1-\Gamma_r(n\overline{A}_n)-\delta_{1n},$$

if $n\overline{A}_n\sim\overline{A}<\infty$, as $n\to\infty$, where $\Gamma_r(x)=\frac{1}{\Gamma(r)}\int_0^x t^{r-1}e^{-t}dt$ is the incomplete gamma function, $\delta_{in}>0,\delta_{in}\to 0$, as $n\to\infty$, $i=1,2$, and $0<A_n<1$. However, the results concerning gos are originally derived by Nasri-Roudsari (1996) and Nasri-Roudsari and Cramer (1999) (see also Barakat (2007)), while those concerning the dgos can be easily derived by using (1.1), (1.2) and the relations between gos and dgos, see Burkschat et al. (2003).

Theorem 1.1. *Let $m>-1$ and $r\in\{1,2,...,n\}$. Then, there exist normalizing constants $c_n,\tilde{c}_n>0$ and $d_n,\tilde{d}_n$, for which*

$$\Phi_{r:n}^{(m,k)}(c_nx+d_n)=I_{G_m(c_nx+d_n)}(r,N-r+1)\ \xrightarrow[n]{w}\ \Phi_r^{(m,k)}(x) \tag{1.5}$$

and

$$\Phi_{n-r+1:n}^{d(m,k)}(\tilde{c}_nx+\tilde{d}_n)=I_{T_m(\tilde{c}_nx+\tilde{d}_n)}(R_r,N-R_r+1)\ \xrightarrow[n]{w}\ \hat{\Phi}_r^{d(m,k)}(x), \tag{1.6}$$

where $\Phi_r^{(m,k)}(x)$ and $\hat{\Phi}_r^{d(m,k)}(x)$ are nondegenerate df's and $\xrightarrow[n]{w}$ denotes the weak convergence, as $n\to\infty$, if, and only if, there exist normalizing constants $\alpha_n>0$ and β_n, for which

$$\Phi_{r:n}^{(0,1)}(\alpha_nx+\beta_n)=\Phi_{n-r+1:n}^{d(0,1)}(\alpha_nx+\beta_n)\ \xrightarrow[n]{w}\ \Gamma_r(\mathcal{V}_{j,\beta}(x)),\beta>0.$$

In this case $\Phi_r^{(m,k)}(x)=\Gamma_r(\mathcal{V}_{j,\beta}(x))$ and $\hat{\Phi}_r^{d(m,k)}(x)=\Gamma_{R_r}(\mathcal{V}_{j,\beta}^{m+1}(x)), j\in\{1,2,3\}$, where

Type I) : $\mathcal{V}_1(x)=\mathcal{V}_{1;\beta}(x)=e^x,\ \forall\ x;$

Types II) : $\mathcal{V}_{2;\beta}(x)=\begin{cases}(-x)^{-\beta}, & x\leq 0,\\ \infty, & x>0;\end{cases}$

Types III) : $\mathcal{V}_{3;\beta}(x)=\begin{cases}0, & x\leq 0,\\ x^{\beta}, & x>0.\end{cases}$

Moreover, $c_n,d_n,\tilde{c}_n$ and $\tilde{d}_n$ may be chosen such that $c_n=\alpha_{\psi(n)},d_n=\beta_{\psi(n)},\tilde{c}_n=\alpha_{\phi(n)}$ and $\tilde{d}_n=\beta_{\phi(n)}$, where $\phi(n)=n^{1/(m+1)}$ and $\psi(n)=n(m+1)$. Finally, (1.5) and (1.6) hold if, and only if, $NG_m(c_nx+d_n)\to\mathcal{V}_{j,\beta}(x)$ and $NT_m(\tilde{c}_nx+\tilde{d}_n)\to\mathcal{V}_{j,\beta}^{m+1}(x)$, as $n\to\infty$ (note that $N\sim n$, as $n\to\infty$), respectively.

Theorem 1.2. *Let $m > -1$ and $r \in \{1, 2, ..., n\}$. Then, there exist normalizing constants $a_n, \tilde{a}_n > 0$ and $b_n, \tilde{b}_n$, for which*

$$\Phi^{(m,k)}_{n-r+1:n}(a_n x + b_n) = I_{G_m(a_n x+b_n)}(N - R_r + 1, R_r) \xrightarrow[n]{w} \hat{\Phi}^{(m,k)}_r(x) \tag{1.7}$$

and

$$\Phi^{d(m,k)}_{r:n}(\tilde{a}_n x + \tilde{b}_n) = I_{T_m(\tilde{a}_n x+\tilde{b}_n)}(N - r + 1, r) \xrightarrow[n]{w} \Phi^{d(m,k)}_r(x), \tag{1.8}$$

where $\hat{\Phi}^{(m,k)}_r(x)$ and $\Phi^{d(m,k)}_r(x)$ are nondegenerate df's if, and only if, there exist normalizing constants $\hat{\alpha}_n > 0$ and $\hat{\beta}_n$, for which

$$\Phi^{(0,1)}_{n-r+1:n}(\hat{\alpha}_n x + \hat{\beta}_n) = \Phi^{d(0,1)}_{r:n}(\hat{\alpha}_n x + \hat{\beta}_n) \xrightarrow[n]{w} 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x)), \alpha > 0.$$

In this case $\hat{\Phi}^{(m,k)}_r(x) = 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$ and $\Phi^{d(m,k)}_r(x) = 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x)), i \in \{1, 2, 3\}$, where

Type I$)$: $\mathcal{U}_1(x) = \mathcal{U}_{1;\alpha}(x) = e^{-x}, \ \forall \ x;$

Types II$)$: $\mathcal{U}_{2;\alpha}(x) = \begin{cases} \infty, & x \le 0, \\ x^{-\alpha}, & x > 0; \end{cases}$

Types III$)$: $\mathcal{U}_{3;\alpha}(x) = \begin{cases} (-x)^{\alpha}, & x \le 0, \\ 0, & x > 0. \end{cases}$

Moreover, $a_n, b_n, \tilde{a}_n$ and $\tilde{b}_n$ may be chosen such that $a_n = \hat{\alpha}_{\phi(n)}, b_n = \hat{\beta}_{\phi(n)}, \tilde{a}_n = \hat{\alpha}_{\psi(n)}$ and $\tilde{b}_n = \hat{\beta}_{\psi(n)}$. Finally, (1.7) and (1.8) hold if, and only if, $N\bar{G}_m(a_n x + b_n) \to \mathcal{U}^{m+1}_{i,\alpha}(x)$ and $N\overline{T}_m(\tilde{a}_n x + \tilde{b}_n) \to \mathcal{U}_{i,\alpha}(x)$, as $n \to \infty$, respectively.

2. The joint df's of two lower and two upper extreme $m-$gos and $m-$dgos, $m \ne -1$

Throughout this section we assume that $1 \le r < s \le n$ in Theorems 2.1 and 2.2, while $1 \le s < r \le n$ in Theorems 2.3 and 2.4.

Theorem 2.1. *Let $c_n > 0$ and d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x))$ and $\Phi^{(m,k)}_{s:n}(y_n) \xrightarrow[n]{w} \Gamma_s(\mathcal{V}_{j,\beta}(y)), j \in \{1, 2, 3\}$, hold, where $x_n = c_n x + d_n$ and $y_n = c_n y + d_n$. Then the normalized joint df $\Phi^{(m,k)}_{r,s:n}(x_n, y_n)$ of the rth and sth $m-$gos, $m \ne -1$, satisfies the relation*

$$\Phi^{(m,k)}_{r,s:n}(x_n, y_n) \xrightarrow[n]{w} \begin{cases} \Gamma_s(\mathcal{V}_{j,\beta}(y)), & x \ge y, \\ \frac{1}{(r-1)!} \int_0^{\mathcal{V}_{j,\beta}(x)} \Gamma_{s-r}(\mathcal{V}_{j,\beta}(y) - u) u^{r-1} e^{-u} du, & x \le y. \end{cases} \tag{2.1}$$

Proof. By using (1.3), the joint df of rth and sth $m-$gos is given by

$$\Phi^{(m,k)}_{r,s:n}(x_n, y_n) = \begin{cases} \Phi^{(m,k)}_{s:n}(y_n), & y \le x, \\ \frac{C_{s-1,n}}{\Gamma(r)\Gamma(s-r)} \int_{-\infty}^{x_n} \int_u^{y_n} \overline{F}^m(u) \overline{F}^{\gamma_s - 1}(v) g^{r-1}_m(F(u)) & \\ (g_m(F(v)) - g_m(F(u)))^{s-r-1} f(u) f(v) dv du, & x \le y. \end{cases}$$

Therefore, for the case $y \le x$, the theorem follows by using Theorem 1.1 (relation (1.5)). For the case $x \le y$, consider the transformation $\xi = F(u)$, $\eta = F(v)$, we get

$$\Phi^{(m,k)}_{r,s:n}(x_n, y_n) = C^\star_n \int_0^{F(x_n)} \int_\xi^{F(y_n)} \overline{\xi}^m \overline{\eta}^{\gamma_s - 1} (1 - \overline{\xi}^{m+1})^{r-1} (\overline{\xi}^{m+1} - \overline{\eta}^{m+1})^{s-r-1} d\eta d\xi, \tag{2.2}$$

where $\bar{\eta} = 1-\eta$, $\bar{\xi} = 1-\xi$ and $C_n^\star = \frac{C_{s-1,n}}{(m+1)^{s-2}(r-1)!(s-r-1)!}$. Again, by using the transformation $1-\overline{\xi}^{m+1} = z$, $1-\overline{\eta}^{m+1} = w$, we get

$$\Phi_{r,s:n}^{(m,k)}(x_n,y_n) = C_n^{\star\star}\int_0^{G_m(x_n)}\int_z^{G_m(y_n)}(1-w)^{\frac{\gamma_s-m-1}{m+1}}z^{r-1}(w-z)^{s-r-1}dwdz,$$

where $C_n^{\star\star} = \frac{C_n^\star}{(m+1)^2}$. On the other hand, we have $\frac{\gamma_s-m-1}{m+1} = N-s$ and

$$\begin{aligned}\frac{(r-1)!(s-r-1)!C_n^{\star\star}}{(N-s)^s} &= \frac{\prod_{j=1}^s\gamma_j}{(N-s)^s(m+1)^s} = \frac{\prod_{j=1}^s(N-j+1)}{(N-s)^s}\\ &= \frac{\prod_{j=1}^s(1-\frac{j-1}{N})}{(1-\frac{s}{N})^s}\\ &= (1+\frac{s^2}{N}(1+o(1)))(1-\sum_{j=2}^s\frac{j-1}{N}(1+o(1)))\\ &= 1+[\frac{s^2}{N}-\frac{1}{N}(\frac{s^2-s}{2})](1+o(1)) = 1+\rho_N,\end{aligned} \tag{2.3}$$

where $0<\rho_N = \frac{1}{2N}(s^2+s)(1+o(1))\to 0$, as $N\to\infty$. Therefore, by using the transformation $w = \frac{\theta}{N-s}$, $z = \frac{\phi}{N-s}$ and the relations $(N-s)G_m(x_n)\sim NG_m(x_n)\to \mathcal{V}_{j,\beta}(x)$, $(N-s)G_m(y_n)\sim NG_m(y_n)\to\mathcal{V}_{j,\beta}(y)$ and $(1-\frac{\theta}{N-s})^{N-s}\to e^{-\theta}$, as $n\to\infty$, we get

$$\begin{aligned}\Phi_{r,s:n}^{(m,k)}(x_n,y_n) &= \frac{(1+\rho_N)}{(r-1)!(s-r-1)!}\\ &\times\int_0^{(N-s)G_m(x_n)}\int_\phi^{(N-s)G_m(y_n)}(1-\frac{\theta}{N-s})^{N-s}\phi^{r-1}(\theta-\phi)^{s-r-1}d\theta d\phi\\ &\sim\frac{1}{(r-1)!(s-r-1)!}\int_0^{NG_m(x_n)}\int_\phi^{NG_m(y_n)}e^{-\theta}\phi^{r-1}(\theta-\phi)^{s-r-1}d\theta\, d\phi\\ &= \frac{1}{(r-1)!}\int_0^{NG_m(x_n)}\Gamma_{s-r}(NG_m(y_n)-u)u^{r-1}e^{-u}du.\\ &\to\frac{1}{(r-1)!}\int_0^{\mathcal{V}_{j,\beta}(x)}\Gamma_{s-r}(\mathcal{V}_{j,\beta}(y)-u)u^{r-1}e^{-u}du.\end{aligned}$$

The theorem is proved.

Theorem 2.2. *Let $\tilde{a}_n>0$ and $\tilde{b}_n$ be suitable normalizing constants, for which the limit relations $\Phi_{r:n}^{d(m,k)}(\tilde{x}_n)\xrightarrow[n]{w}1-\Gamma_r(\mathcal{U}_{i,\alpha}(x))$ and $\Phi_{s:n}^{d(m,k)}(\tilde{y}_n)\xrightarrow[n]{w}1-\Gamma_s(\mathcal{U}_{i,\alpha}(y))$, $i\in\{1,2,3\}$, hold, where $\tilde{x}_n = \tilde{a}_nx+\tilde{b}_n$ and $\tilde{y}_n = \tilde{a}_ny+\tilde{b}_n$. Then, the normalized joint df of the rth and sth m−dgos, $m\neq -1$, satisfies the relation*

$$\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n)\xrightarrow[n]{w}\begin{cases}1-\Gamma_r(\mathcal{U}_{i,\alpha}(x)), & x\le y,\\ 1-\Gamma_s(\mathcal{U}_{i,\alpha}(y))-\frac{1}{(s-1)!} & \\ \int_{\mathcal{U}_{i,\alpha}(y)}^\infty I_{\frac{\mathcal{U}_{i,\alpha}(x)}{t}}(r,s-r)t^{s-1}e^{-t}dt, & x\ge y.\end{cases} \tag{2.4}$$

Proof. By using (1.4) the joint df of rth and sth $m-$dgos is given by

$$\Phi_{r,s:n}^{d(m,k)}(\tilde{x}_n,\tilde{y}_n) = \begin{cases}\Phi_{r:n}^{d(m,k)}(\tilde{x}_n), & x\le y,\\ \frac{C_{s-1,n}}{\Gamma(r)\Gamma(s-r)}\int_{-\infty}^{\tilde{y}_n}\int_v^{\tilde{x}_n}F^m(u)F^{\gamma_s-1}(v)g_m^{r-1}(\bar{F}(u)) & \\ (g_m(\bar{F}(v)-g_m(\bar{F}(u))^{s-r-1}f(u)f(v)dudv, & y\le x.\end{cases}$$

Therefore, for the case $x \leq y$, the theorem follows by using Theorem 1.2 (relation (1.8)). For the case $y \leq x$, consider the transformation $\xi = F(u)$, $\eta = F(v)$, we get

$$\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n) = C^\star_n \int_0^{F(\tilde{y}_n)} \int_\eta^{F(\tilde{x}_n)} \xi^m \eta^{\gamma_s - 1}(1-\xi^{m+1})^{r-1}(\xi^{m+1} - \eta^{m+1})^{s-r-1} d\xi d\eta. \tag{2.5}$$

Again, by using the transformation $1 - \xi^{m+1} = z$, $1 - \eta^{m+1} = w$, we get

$$\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n) = C^{\star\star}_n \int_{\bar{T}_m(\tilde{y}_n)}^1 \int_{\bar{T}_m(\tilde{x}_n)}^w (1-w)^{N-s} z^{r-1}(w-z)^{s-r-1} dz dw.$$

Therefore, by using the transformation $w = \frac{\theta}{N-s}$, $z = \frac{\phi}{N-s}$, the relations (2.3), $(N-s)\bar{T}_m(x_n) \sim N\bar{T}_m(x_n) \to \mathcal{U}_{i,\alpha}(x)$, $(N-s)\bar{T}_m(y_n) \sim N\bar{T}_m(y_n) \to \mathcal{U}_{i,\alpha}(y)$ and $(1 - \frac{\theta}{N-s})^{N-s} \to e^{-\theta}$, as $n \to \infty$, we get

$$\Phi^{d(m,k)}_{r,s:n}(\tilde{x}_n, \tilde{y}_n) = \frac{(1+\rho_N)}{(r-1)!(s-r-1)!}$$

$$\times \int_{(N-s)\bar{T}_m(\tilde{y}_n)}^{N-s} \int_{(N-s)\bar{T}_m(\tilde{x}_n)}^{\theta} (1 - \frac{\theta}{N-s})^{N-s} \phi^{r-1}(\theta - \phi)^{s-r-1} d\phi d\theta$$

$$\sim \frac{1}{(r-1)!(s-r-1)!} \int_{N\bar{T}_m(\tilde{y}_n)}^{N} \int_{N\bar{T}_m(\tilde{x}_n)}^{\theta} e^{-\theta} \phi^{r-1}(\theta-\phi)^{s-r-1} d\phi d\theta$$

$$= \frac{1}{(r-1)!(s-r-1)!} \int_{N\bar{T}_m(\tilde{y}_n)}^{N} e^{-\theta}\theta^{s-1} d\theta \int_{\frac{N\bar{T}_m(\tilde{x}_n)}{\theta}}^{1} z^{r-1}(1-z)^{s-r-1} dz$$

$$= \Gamma_s(N) - \Gamma_s(N\bar{T}_m(\tilde{y}_n)) - \frac{1}{(s-1)!} \int_{N\bar{T}_m(\tilde{y}_n)}^{N} \theta^{s-1} e^{-\theta} I_{\frac{N\bar{T}_m(\tilde{x}_n)}{\theta}}(r, s-r) d\theta.$$

Therefore, an application of Theorem 1.2 (the relation (1.8)), thus yields the limit relation (2.4), in the case $y \leq x$. The theorem is proved.

Theorem 2.3. *Let $a_n > 0$ and b_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(x_n) \xrightarrow[n]{w} 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))$ and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} 1 - \Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y))$, $i \in \{1,2,3\}$, hold, where $x_n = a_n x + b_n$, $y_n = a_n y + b_n$ and $\grave{r} = n - r + 1 < n - s + 1 = \grave{s}$. Then the joint df of the $\grave{r}$th and $\grave{s}$th $m-$gos, $m \neq -1$, satisfies the relation*

$$\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n, y_n) \xrightarrow[n]{w} \begin{cases} 1 - \Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y)), & x \geq y, \\ 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x)) - \frac{1}{\Gamma(R_r)} \int_{\mathcal{U}^{m+1}_{i,\alpha}(x)}^{\infty} & \\ I_{\frac{\mathcal{U}^{m+1}_{i,\alpha}(y)}{t}}(R_s, R_r - R_s) t^{R_r - 1} e^{-t} dt, & x \leq y. \end{cases} \tag{2.6}$$

Proof. In view of Theorem 1.2 (relation (1.7)), the relation (1.3) and the condition of Theorem 2.3, we have $\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n, y_n) = \Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} 1 - \Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y)), y \leq x$. Thus, the theorem is proved in the case of $y \leq x$. For the case $x \leq y$, we begin with the relation (2.2), after replacing r and s by $\grave{r}$ and $\grave{s}$, respectively. By using the transformation $\overline{\xi}^{m+1} = z, \overline{\eta}^{m+1} = w$ and noting that $n - r = N - R_r, n - s = N - R_s$,

$\gamma_{n-s+1} = (m+1)R_s$ and $C_{\grave{s}-1,n} = C_{N-R_s,n} = (m+1)^{N-R_s+1}\prod_{j=1}^{N-R_s+1}(N-j+1) = (m+1)^{N-R_s+1}\frac{\Gamma(N+1)}{\Gamma(R_s)}$, we get

$$\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) = \grave{C}_n\int_{\bar{G}_m(x_n)}^{1}\int_{\bar{G}_m(y_n)}^{z} w^{R_s-1}(1-z)^{N-R_r}(z-w)^{R_r-R_s-1}dwdz,$$

where $\grave{C}_n = \frac{\Gamma(N+1)}{\Gamma(N-R_r+1)\Gamma(R_r-R_s)\Gamma(R_s)}$. Again by using the transformation $w = \frac{\theta}{N-R_r}$, $z = \frac{\phi}{N-R_r}$ and the limit relation $(1-\frac{\phi}{N-R_r})^{N-R_r} \to e^{-\phi}$, as $n\to\infty$, we get

$$\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) \sim \frac{\grave{C}_n}{(N-R_r)^{R_r}} \times\int_{(N-R_r)\bar{G}_m(x_n)}^{(N-R_r)}\int_{(N-R_r)\bar{G}_m(y_n)}^{\phi} e^{-\phi}\theta^{R_s-1}(\phi-\theta)^{R_r-R_s-1}d\theta d\phi.$$

Now, by using Striling's formula (c.f. Lebedev (1965)), we have $\frac{\Gamma(R_r-R_s)\Gamma(R_s)\grave{C}_n}{(N-R_r)^{R_r}} \sim e^{-R_r}(1-\frac{R_r}{N})^{-N+\frac{1}{2}} \sim 1$, as $N\to\infty$ (i.e., as $n\to\infty$), and noting that $(N-R_r)\bar{G}_m(x_n) \sim N\bar{G}_m(x_n) \to \mathcal{U}^{m+1}_{i,\alpha}(x)$, $(N-R_r)\bar{G}_m(y_n) \sim N\bar{G}_m(y_n) \to \mathcal{U}^{m+1}_{i,\alpha}(y)$, as $n\to\infty$, we get

$$\begin{aligned}\Phi^{(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) &\sim \frac{1}{\Gamma(R_r-R_s)\Gamma(R_s)} \times\int_{N\bar{G}_m(x_n)}^{N}\int_{N\bar{G}_m(y_n)}^{\phi} e^{-\phi}\theta^{R_s-1}(\phi-\theta)^{R_r-R_s-1}d\theta d\phi.\\ &= \frac{1}{\Gamma(R_r)}\int_{N\bar{G}_m(x_n)}^{N}\phi^{R_r-1}e^{-\phi}\left(1-I_{\frac{N\bar{G}_m(y_n)}{\phi}}(R_s,R_r-R_s)\right)d\phi\\ &= 1-\Gamma_{R_r}(N\bar{G}_m(x_n))\\ &\quad -\frac{1}{\Gamma(R_r)}\int_{N\bar{G}_m(x_n)}^{N} I_{\frac{N\bar{G}_m(y_n)}{t}}(R_s,R_r-R_s)t^{R_r-1}e^{-t}dt.\end{aligned}$$

An application of Theorem 1.2 (relation (1.7)), thus yields the limit relation (2.6), in the case $x\le y$.

Theorem 2.4. *Let $\tilde{c}_n>0$ and $\tilde{d}_n$ be normalizing constants, for which the limit relations $\Phi^{d(m,k)}_{\grave{r}:n}(\tilde{x}_n) \xrightarrow[n]{w} \Gamma_{R_r}(\mathcal{V}^{m+1}_{j,\beta}(x))$ and $\Phi^{d(m,k)}_{\grave{s}:n}(\tilde{y}_n) \xrightarrow[n]{w} \Gamma_{R_s}(\mathcal{V}^{m+1}_{j,\beta}(y))$, $j\in\{1,2,3\}$, hold, where $\tilde{x}_n = \tilde{c}_n x+\tilde{d}_n$, $\tilde{y}_n = \tilde{c}_n y+\tilde{d}_n$ and $\grave{r} = n-r+1 < n-s+1 = \grave{s}$. Then the joint df of the $\grave{r}$th and $\grave{s}$th $m-$dgos, $m\ne -1$, satisfies the relation*

$$\Phi^{d(m,k)}_{\grave{r},\grave{s}:n}(\tilde{x}_n,\tilde{y}_n) \xrightarrow[n]{w} \begin{cases}\Gamma_{R_r}(\mathcal{V}^{m+1}_{j,\beta}(x)), & x\le y,\\ \frac{1}{\Gamma(R_s)}\int_0^{\mathcal{V}^{m+1}_{j,\beta}(y)}\Gamma_{R_r-R_s}(\mathcal{V}^{m+1}_{j,\beta}(x)-u)u^{R_s-1}e^{-u}du, & x\ge y.\end{cases} \tag{2.7}$$

Proof. In view of Theorem 1.1 (relation (1.6)), the relation (1.4) and the condition of Theorem 2.4, we have $\Phi^{d(m,k)}_{\grave{r},\grave{s}:n}(\tilde{x}_n,\tilde{y}_n) = \Phi^{d(m,k)}_{\grave{r}:n}(\tilde{x}_n) \xrightarrow[n]{w} \Gamma_{R_r}(\mathcal{V}^{m+1}_{i,\beta}(x))$, $x\le y$. Thus, the theorem is proved in the case of $x\le y$. For the case $y\le x$, we begin with the relation (2.5), after replacing r and s by $\grave{r}$ and $\grave{s}$, respectively. By using the transformation $\xi^{m+1}=z, \eta^{m+1}=w$ and noting that $n-r=N-R_r, n-s=N-R_s$,

$\gamma_{n-s+1} = (m+1)R_s$ and $C_{\grave{s}-1,n} = (m+1)^{N-R_s+1}\frac{\Gamma(N+1)}{\Gamma(R_s)}$, we get

$$\Phi^{d(m,k)}_{\grave{r},\grave{s}:n}(\tilde{x}_n,\tilde{y}_n) = \grave{C}_n\int_0^{T_m(\tilde{y}_n)}\int_w^{T_m(\tilde{x}_n)} w^{R_s-1}(1-z)^{N-R_r}(z-w)^{R_r-R_s-1}dzdw.$$

Thus, by using the transformation $w = \frac{\theta}{N-R_r}$, $z = \frac{\phi}{N-R_r}$ and the relations $\frac{\Gamma(R_r-R_s)\Gamma(R_s)\grave{C}_n}{(N-R_r)^{R_r}} \sim 1$, $(N-R_r)T_m(\tilde{x}_n) \sim NT_m(\tilde{x}_n) \to \mathcal{V}^{m+1}_{j,\beta}(x)$, $(N-R_r)T_m(\tilde{y}_n) \sim NT_m(\tilde{y}_n) \to \mathcal{V}^{m+1}_{j,\beta}(y)$ and $(1-\frac{\phi}{N-R_r})^{(N-R_r)} \to e^{-\phi}$, as $n\to\infty$, we get

$$\Phi^{d(m,k)}_{\grave{r},\grave{s}:n}(x_n,y_n) = \frac{\grave{C}_n}{(N-R_r)^{R_r}}\int_0^{(N-R_r)T_m(\tilde{y}_n)}\int_\theta^{(N-R_r)T_m(\tilde{x}_n)}\theta^{R_s-1}$$

$$(1-\frac{\phi}{N-R_r})^{(N-R_r)}(\phi-\theta)^{R_r-R_s-1}d\phi d\theta$$

$$\sim \frac{1}{\Gamma(R_r-R_s)\Gamma(R_s)}\int_0^{NT_m(\tilde{y}_n)}\int_\theta^{NT_m(\tilde{x}_n)}\theta^{R_s-1}e^{-\phi}(\phi-\theta)^{R_r-R_s-1}d\phi d\theta$$

$$= \frac{1}{\Gamma(R_s)}\int_0^{NT_m(\tilde{y}_n)}\Gamma_{R_r-R_s}(NT_m(\tilde{x}_n)-u)u^{R_s-1}e^{-u}du.$$

Therefore, by applying Theorem 1.1 (relation (1.6)), we get the limit relation (2.7), in the case $y \le x$.

Example 2.5. For the oos model, where $k=1$ and $m=0$, it is easy to show that both of Theorems 2.1 and 2.4 give the limit df of the rth and sth order statistics by the relation (2.1), with normalizing constants $\alpha_n > 0$ and β_n. Moreover, both of Theorems 2.2 and 2.3 give the limit df of the $\grave{r}$th and $\grave{s}$th order statistics, where $\grave{r} = n-r+1 < n-s+1 = \grave{s}$, by the relation (2.4), with normalizing constants $\hat{\alpha}_n > 0$ and $\hat{\beta}_n$.

3. The joint df of lower-upper extreme $m-$gos and $m-$dgos, $m \neq -1$

Throughout this section we assume that $1 \le r, s \le n$, $\grave{s} = n-s+1$.

Theorem 3.1. *Let $a_n, c_n > 0$ and b_n, d_n be suitable normalizing constants, for which the limit relations $\Phi^{(m,k)}_{r:n}(x_n) \xrightarrow[n]{w} \Phi^{(m,k)}_r(x) = \Gamma_r(\mathcal{V}_{j,\beta}(x)), j \in \{1,2,3\}$, and $\Phi^{(m,k)}_{\grave{s}:n}(y_n) \xrightarrow[n]{w} \hat{\Phi}^{(m,k)}_s(y) = 1-\Gamma_{R_s}(\mathcal{U}^{m+1}_{i,\alpha}(y)), i \in \{1,2,3\}$, hold, where $x_n = c_nx+d_n$ and $y_n = a_ny+b_n$. Then, $\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) \xrightarrow[n]{w} \Phi^{(m,k)}_r(x)\hat{\Phi}^{(m,k)}_s(y)$. This means that the lower and upper $m-$gos are asymptotically independent.*

Proof. In view of (1.3), we have

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n,y_n) = D_n\int_0^{F(x_n)}\int_\xi^{F(y_n)}\overline{\xi}^m\overline{\eta}^{\gamma_{n-s+1}-1}(1-\overline{\xi}^{m+1})^{r-1} \times (\overline{\xi}^{m+1}-\overline{\eta}^{m+1})^{n-s-r}d\eta\, d\xi, \tag{3.1}$$

$\forall\, x_n \le y_n$, where $D_n = \frac{C_{n-s,n}}{(m+1)^{n-s-1}(r-1)!(n-s-r)!}$. Now, in view of the conditions of the theorem, it is easy to show that $\forall (x,y)$, for which $\mathcal{V}_{j,\beta}(x), \mathcal{U}_{i,\alpha}(y) < \infty$, we have $y_n \to \omega(F) = \sup\{x : F(x) < 1\} > \inf\{x : F(x) > 0\} = \alpha(F) \leftarrow x_n$, as $n \to \infty$. Therefore, for all large n, the relation (3.1) holds, $\forall x, y$, for which $\mathcal{V}_{j,\beta}(x), \mathcal{U}_{i,\alpha}(y) < \infty$. Now, by using the transformation $1 - \overline{\xi}^{m+1} = v$, $\overline{\eta}^{m+1} = u$ and noting that $\frac{\gamma_{n-s+1} - m - 1}{m+1} = R_s - 1$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n, y_n) = \frac{D_n}{(m+1)^2} \int_0^{G_m(x_n)} \int_{\bar{G}_m(y_n)}^{1-v} u^{R_s - 1} v^{r-1} (1 - u - v)^{n-s-r} du dv.$$

Therefore, by using the transformation $u = \frac{w}{N - R_s - r}$, $v = \frac{z}{N - R_s - r}$, we get

$$\Phi^{(m,k)}_{r,\grave{s}:n}(x_n, y_n) \sim \tilde{C}_n \int_0^{(N-R_s-r)G_m(x_n)} \int_{(N-R_s-r)\bar{G}_m(y_n)}^{N-R_s-r} w^{R_s-1} z^{r-1} e^{-(w+z)} dw dz,$$

where $\tilde{C}_n = \frac{D_n}{(m+1)^2 (N - R_s - r)^{R_s + r}}$. On the other hand, by using Striling's formula, we get

$$\begin{aligned}\Gamma(r)\tilde{C}_n &= \frac{C_{N-R_s,n}}{(m+1)^{N-R_s+1}(N - R_s - r)^{R_s+r}\Gamma(N - R_s - r + 1)} \\ &= \frac{\Gamma(N+1)}{\Gamma(N - R_s - r + 1)(N - R_s - r)^{R_s+r}\Gamma(R_s)} \sim \frac{1}{\Gamma(R_s)}.\end{aligned}$$

Therefore, the proof of the theorem follows upon noting that $(N - R_s - r)G_m(x_n) \sim NG_m(x_n) \to \mathcal{V}_{j,\beta}(x)$ and $(N - R_s - r)\bar{G}_m(y_n) \sim N\bar{G}_m(y_n) \to \mathcal{U}^{m+1}_{i,\alpha}(y)$, as $n \to \infty$. The theorem is proved.

Theorem 3.2. *Let $\tilde{a}_n, \tilde{c}_n > 0$ and $\tilde{b}_n, \tilde{d}_n$ be suitable normalizing constants, for which the limit relations $\Phi^{d(m,k)}_{r:n}(\tilde{x}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}_{r}(x) = 1 - \Gamma_r(\mathcal{U}_{i,\alpha}(x)), i \in \{1,2,3\}$, and $\Phi^{d(m,k)}_{\grave{s}:n}(\tilde{y}_n) \xrightarrow[n]{w} \hat{\Phi}^{d(m,k)}_{s}(y) = \Gamma_{R_s}(\mathcal{V}^{m+1}_{j,\beta}(y)), j \in \{1,2,3\}$, hold, where $\tilde{x}_n = \tilde{a}_n x + \tilde{b}_n$ and $\tilde{y}_n = \tilde{c}_n y + \tilde{d}_n$. Then, $\Phi^{d(m,k)}_{r,\grave{s}:n}(\tilde{x}_n, \tilde{y}_n) \xrightarrow[n]{w} \Phi^{d(m,k)}_{r}(x)\hat{\Phi}^{d(m,k)}_{s}(y)$. This means that the lower and upper $m-$dgos are asymptotically independent.*

Proof. The proof of this theorem is similar as the proof of Theorem 3.1, except only the obvious changes. Therefore, we do not repeat the details.

Example 3.3 (the limit df's of the generalized range and midrange). Under the conditions of Theorems 3.1 and 3.2, the lower and the upper extreme m-gos, as well as m-dgos, are asymptotically independent. Therefore, if there exist normalizing constants $a_n, c_n > 0$ and b_n, d_n, for which $a_n/c_n \to c > 0$, as $n \to \infty$, and the limit relations $\Phi^{(m,k)}_{\grave{r}:n}(a_n x + b_n) \xrightarrow[n]{w} 1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x)), i \in \{1,2,3\}$, and $\Phi^{(m,k)}_{r:n}(c_n x + d_n) \xrightarrow[n]{w} \Gamma_r(\mathcal{V}_{j,\beta}(x)), j \in \{1,2,3\}$, hold, then in view of lemma 2.9.1 in Galambos (1987), the generalized quasi-ranges $R(r,n,m,k) = X(\grave{r},n,m,k) - X(r,n,m,k)$ and the generalized quasi-midranges $M(r,n,m,k) = \frac{X(\grave{r},n,m,k) + X(r,n,m,k)}{2}, r = 1,2,..,$ satisfy the relations

$$P(R(r,n,m,k) \le a_n x + b_n - d_n) \xrightarrow[n]{w} [1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))] \star [1 - \Gamma_r(\mathcal{V}_{j,\beta}(-cx))]$$

and

$$P(2M(r,n,m,k) \le a_n x + b_n + d_n) \xrightarrow[n]{w} [1 - \Gamma_{R_r}(\mathcal{U}^{m+1}_{i,\alpha}(x))] \star [\Gamma_r(\mathcal{V}_{j,\beta}(cx))],$$

respectively, where the symbol $\star$ denotes the convolution operation. The same result can be easily obtained for the dual generalized quasi-ranges $R_d(r,n,m,k) = X_d(r,n,m,k) - X_d(\grave{r},n,m,k)$ and the dual generalized quasi-midranges $M_d(r,n,m,k) = \frac{1}{2}(X_d(r,n,m,k) + X_d(\grave{r},n,m,k)), r = 1,2,...$

Acknowledgements. The authors would like to thank the anonymous referee and the editor-in-chief for their efforts and the constructive suggestions that improved the representation substantially.

References

H.M. Barakat. Asymptotic properties of bivariate random extremes. *J. Statist. Plann. Inference* **61** (2), 203–217 (1997). MR1457718.

H.M. Barakat. Limit theory of generalized order statistics. *J. Statist. Plann. Inference* **137** (1), 1–11 (2007). MR2292835.

M. Burkschat, E. Cramer and U. Kamps. Dual generalized order statistics. *Metron* **61** (1), 13–26 (2003). MR1994107.

J. Galambos. *The asymptotic theory of extreme order statistics*. Robert E. Krieger Publishing Co., Inc., Melbourne, FL, second edition (1987). ISBN 0-89874-957-3. MR936631.

U. Kamps. *A concept of generalized order statistics*. Teubner Skripten zur Mathematischen Stochastik. [Teubner Texts on Mathematical Stochastics]. B. G. Teubner, Stuttgart (1995). ISBN 3-519-02736-4. MR1329319.

N.N. Lebedev. *Special functions and their applications*. Revised English edition. Translated and edited by Richard A. Silverman. Prentice-Hall, Inc., Englewood Cliffs, N.J. (1965). MR0174795.

D. Nasri-Roudsari. Extreme value theory of generalized order statistics. *J. Statist. Plann. Inference* **55** (3), 281–297 (1996). MR1422134.

D. Nasri-Roudsari and E. Cramer. On the convergence rates of extreme generalized order statistics. *Extremes* **2** (4), 421–447 (2000) (1999). MR1776857.

N.V. Smirnov. Limit distributions for the terms of a variational series. *Amer. Math. Soc. Translation* **1952** (67), 64 (1952). MR0047277.