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Hausdorff dimension of the level sets of some stochastic PDEs from fluid dynamics

Lorenzo Baglioni and Marco Romito

Dipartimento di Matematica, Università di Pisa
Largo Bruno Pontecorvo 5,
56127 Pisa, Italia.
E-mail address: baglioni@dm.unipi.it,romito@dm.unipi.it
URL: http://www.dm.unipi.it/pages/romito

Abstract. We determine with positive probability the Hausdorff dimension of the level sets of a class of Navier–Stokes α–models at finite viscosity, forced by mildly rough Gaussian white noise.

1. Introduction

This paper addresses the problem of determining the Hausdorff dimension of the level sets of the solutions of some stochastic PDEs from fluid dynamics in two space dimensions. Consider the following stochastic PDE,

$$\dot\theta + \nu(-\Delta)^\alpha\theta + u\cdot\nabla\theta = \dot\eta, \qquad t \geq 0, \tag{1.1}$$

on $[-\pi,\pi]^2$, with periodic boundary conditions and zero spatial mean, where ν, α and M are suitable parameters, $\dot\eta$ is Gaussian noise and the transport velocity is given by $u = \nabla^\perp(-\Delta)^{-M}\theta$. We prove almost sure upper bounds, as well as lower bounds with positive probability, on the Hausdorff and packing dimension of the level sets of the random field $\theta(t)$ at any positive time $t > 0$.

Our equation (1.1) belongs to the class of *Navier–Stokes α–like models* (see for instance Olson and Titi (2007) and the reference therein) and, when $\alpha = 1$, the 2D Navier–Stokes equations in the vorticity formulation correspond to $M = 1$ (see for instance Majda and Bertozzi (2002)), the surface quasi–geostrophic equation corresponds to $M = \frac{1}{2}$, while $M = -1$ describes the large scale flows of a rotating shallow fluid (see Fal′kovich (2007)). In this paper we will assume $\alpha > 1$ and $M \geq 1$. Unfortunately, our results do not cover the above mentioned cases, although they are the motivating examples.

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Our interest in level sets for equations from fluid dynamics is inspired by the theory developed for the 0–level set at a physical and numerical level in Fal′kovich (2007). Our technique is not powerful enough to be able to say something about the "physical case", for a series of reasons: we work at finite viscosity and not in the vanishing viscosity regime, we assume hyper–dissipation, due to the limits of the techniques we employ, we are not able to capture all of the interesting cases of the transport velocity, we are dealing with rough noise. On the other hand, this paper is a preliminary work and extensions are currently being developed.

The results presented here focus on any level set. On the other hand the zero spatial mean condition gives a privileged status to the zero level set. It would be expected to obtain stronger results on the zero level set than on any other level set. We conjecture (see Remark 2.4) that indeed the zero level set should have a "deterministic" dimension, that is, determined almost surely.

We give a few details on the techniques used to achieve our results. As in Dalang et al. (2007, 2009), the non–linear problem is reduced to the linear problem (namely, the same equation without the non–linearity or, in other words, the linearization at 0) by means of an absolute continuity result between the laws of the two processes. In the above references the equivalence is provided by the Girsanov theorem. In our problem Girsanov's transformation cannot be applied (see Remark 4.1), and we apply a weaker result from Da Prato and Debussche (2004), using the polynomial moments from Es-Sarhir and Stannat (2010). On the one hand this gives equivalence of the laws at the level of the single time rather than of the full path, but on the other hand this is enough for our purpose. The equivalence result is already known from Mattingly and Suidan (2005); Watkins (2010) and ours is an alternative proof.

Once the problem is reduced to a linear equation, it becomes more amenable and one can use the theory developed for Gaussian processes in Kahane (1985) (see also Xiao (1995, 1997); Wu and Xiao (2006)) to show almost sure upper bounds on the Hausdorff and packing dimension, as well as lower bounds with positive probability.

1.1. *Notations.* Let $\mathbb{T}_2 = [-\pi,\pi]^2$ be the 2–dimensional torus. For every $\gamma \in \mathbf{R}$ denote by $H^\gamma_\#(\mathbb{T}_2)$ the Sobolev space of periodic functions on $\mathbb{T}_2$ with mean zero on $\mathbb{T}_2$, defined in terms of the complex Fourier coefficients with respect to the Fourier basis $\{\mathrm{e}^{\mathrm{i}k\cdot x} : k \in \mathbf{Z}^2\}$, as

$$H^\gamma_\#(\mathbb{T}_2) = \big\{(\xi_k)_{k\in\mathbf{Z}^2} \subset \mathbf{C} : \xi_0 = 0, \quad \|\xi\|^2_\gamma := \sum_{k\in\mathbf{Z}^2} |k|^{2\gamma}|\xi_k|^2 < \infty\big\},$$

with norm $\|\cdot\|_\gamma$. In particular, when $\gamma = 0$, we use the standard notation $L^2_\#(\mathbb{T}_2)$. Define moreover the spaces of divergence–free vector fields V_γ as

$$V_\gamma = \big\{(u_k)_{k\in\mathbf{Z}^2_\star} \subset \mathbf{C}^2 : k\cdot u_k = 0 \text{ for all } k,\ u^i = (u^i_k)_{k\in\mathbf{Z}^2_\star} \in H^\gamma_\# \text{ for i=1,2}\big\},$$

where $\mathbf{Z}^2_\star = \mathbf{Z}^2 \setminus \{(0,0)\}$, with norm $\|u\|^2_\gamma := \|u^1\|^2_\gamma + \|u^2\|^2_\gamma$, and $u = (u^1,u^2)$. In particular, set $H = V_0$.

Denote by A the realization of the Laplace operator $-\Delta$ on $L^2_\#(\mathbb{T}_2)$ with periodic boundary conditions. A real orthonormal basis of $L^2_\#(\mathbb{T}_2)$ (and hence of each $H^\gamma_\#$) of eigenvectors of A is given as follows. Set

$$\mathbf{Z}^2_+ = \{k \in \mathbf{Z}^2 : k_2 > 0\} \cup \{k \in \mathbf{Z}^2 : k_1 > 0, k_2 = 0\},$$

$\mathbf{Z}^2_- = -\mathbf{Z}^2_+$ and $\mathbf{Z}^2_\star = \mathbf{Z}^2_+ \cup \mathbf{Z}^2_-$, and $e_k = c_1 \sin k \cdot x$ for $k \in \mathbf{Z}^2_+$ and $e_k = c_1 \cos k \cdot x$ for $k \in \mathbf{Z}^2_-$, where $c_1 = \sqrt{2}(2\pi)^{-1}$, then $(e_k)_{k\in\mathbf{Z}^2_\star}$ is an orthonormal basis of $L^2_\#(\mathbb{T}_2)$ (see for instance Majda and Bertozzi (2002)). With these positions, given $\gamma \in \mathbf{R}$, $A^\gamma x = \sum_{k\in\mathbf{Z}^2_\star} |k|^{2\gamma} x_k e_k$ for $x = \sum_k x_k e_k$. Set $E_k = \frac{k^\perp}{|k|} e_k$ for every $k \in \mathbf{Z}^2_\star$, then $(E_k)_{k\in\mathbf{Z}^2_\star}$ is an orthonormal basis of H. With a slight abuse of notations, we will also denote by A the realization of the Laplace operator on H.

Given $y \in \mathbf{R}$ and a field $v : \mathbb{T}_2 \to \mathbf{R}$, denote by $\mathcal{L}_y(v)$ the y–level set of v, namely $\mathcal{L}_y(v) = \{x \in \mathbb{T}_2 : v(x) = y\}$.

We denote by $\dim_H$ and by $\dim_P$ the Hausdorff and the packing dimension, respectively. We refer to Falconer (2003) for their definition and properties.

2. Formulation of the problem and main results

2.1. *Formulation of the problem.* Fix $\nu > 0$, $\alpha \geq 1$ and $M \in \mathbf{R}$. Consider on $\mathbb{T}_2$ the following stochastic PDE,

$$\dot\theta + u \cdot \nabla\theta = -\nu A^\alpha \theta + \dot\eta, \qquad t \geq 0, x \in \mathbb{T}_2, \tag{2.1}$$

with periodic boundary conditions and with $\iint_{\mathbb{T}_2} \theta\, dx = 0$, where u is given as $u = \nabla^\perp A^{-M}\theta$. By its definition it turns out that $\operatorname{div}(u) = 0$ and $\iint u(x)\, dx = 0$.

The non–linearity. Set for a vector function v, with $\operatorname{div}(v) = 0$, and a scalar function f,

$$B(v, f) = v \cdot \nabla f = \operatorname{div}(vf).$$

We use the same notation $B(v, v')$ when v is a vector and the operator B is understood component–wise, namely $[B(v, v')]_i = B(v, v'_i)$. Set moreover

$$B_M(x) = B(\nabla^\perp A^{-M} x, x)$$

We stress two important properties of the non–linear term. Their proofs are standard using integration by parts arguments.

Lemma 2.1. *For every smooth x and v, with* $\operatorname{div} v = 0$,

$$\langle x, B(v, x)\rangle_{L^2} = 0, \qquad\qquad \langle A^{-M}x, B_M(x)\rangle_{L^2} = 0. \tag{2.2}$$

Proof: The first is classical in the theory of Navier–Stokes equations (see for instance Majda and Bertozzi (2002)). For the second,

$$\langle A^{-M}x, B_M(x)\rangle = \int (A^{-M}x)\operatorname{div}(x\nabla^\perp A^{-M}x) = -\int x(\nabla A^{-M}x)\cdot(\nabla^\perp A^{-M}x) = 0,$$

by integrating by parts. □

The random forcing term. The random forcing term $\dot\eta$ is modeled as a coloured in space and white in time Gaussian noise, namely η is a Wiener process with covariance $\mathcal{C} \in \mathscr{L}(L^2_\#(\mathbb{T}_2))$. For our purposes, we will assume that $\mathcal{C}$ has a smoothing effect.

Assumption 2.2 (on the covariance). The operator $\mathcal{C}$ is positive linear bounded on $L^2_\#(\mathbb{T}_2)$. The driving noise is *homogeneous* in space, hence $\mathcal{C}$ has the same eigenvectors of the operator A. Under these assumptions, there are numbers $(\sigma_k)_{k\in\mathbf{Z}^2_\star}$

such that for $x = \sum_k x_k e_k \in L^2_\#$,

$$\mathcal{C}x = \sum_{k \in \mathbf{Z}^2_\star} \sigma_k^2 x_k e_k.$$

Assume additionally that there exists $\delta \in (1-\alpha, 2-\alpha)$ such that

$$\frac{c_2}{|k|^\delta} \leq |\sigma_k| \leq \frac{c_3}{|k|^\delta}. \tag{2.3}$$

The abstract formulation. In conclusion (2.1) can be recast in its abstract form as

$$d\theta + \big(\nu A^\alpha \theta + B_M(\theta)\big)\, dt = \mathcal{C}^{\frac{1}{2}}\, dW. \tag{2.4}$$

2.2. *The linear problem.* Consider the linear version of the problem under examination, namely,

$$\begin{cases} dz + \nu A^\alpha z = \mathcal{C}^{\frac{1}{2}}\, dW, \\ z(0) = 0. \end{cases} \tag{2.5}$$

Under our Assumption 2.2 existence and uniqueness of a solution of problem (2.5) above is classical (see for instance Da Prato and Zabczyk (1992)). Continuity in space, although classical, can be derived directly from our Lemma 3.1 and the Kolmogorov continuity theorem.

With continuity at hand, it is meaningful to consider spatial level sets for the solution of the problem above. Our first result gives upper and lower bounds for the dimension of the level sets of the solution of (2.5).

Theorem 2.3. *Let $\alpha \geq 1$ and let Assumption 2.2 be true. For every $t > 0$ and $y \in \mathbf{R}$,*

- $\mathbb{P}[\dim_P(\mathcal{L}_y(z_t)) \leq 3 - \alpha - \delta] = 1$,
- $\mathbb{P}[\dim_P(\mathcal{L}_y(z_t)) = \dim_H(\mathcal{L}_y(z_t)) = 3 - \alpha - \delta] > 0$.

The proof of the theorem will be given through Propositions 3.5 and 3.7, and is based on well–known techniques (Kahane (1985); Xiao (1995, 1997)) for the dimension of level sets.

Remark 2.4. Theorem 2.3 above says that $\mathcal{L}_y(z_t)$ has dimension equal to $3-\alpha-\delta$ with positive probability. It should not be expected that in general this may hold with probability one. Indeed, if $y \neq 0$, the set $\mathcal{L}_y(z_t)$ is empty with positive probability, since z_t is continuous and defined on a compact set. On the other hand if $y = 0$ then $\mathcal{L}_0(z_t) \neq \emptyset$ with probability one, since z_t is non–zero and with zero average. We conjecture that $\dim_P(\mathcal{L}_0(z_t)) = \dim_H(\mathcal{L}_0(z_t)) = 3 - \alpha - \delta$ with probability one.

An effective way to prove that a random set has an almost sure dimension is to show that the set contains a *limsup random fractal* as proposed in Khoshnevisan et al. (2000). It is easy to construct a limsup random fractal contained into $\mathcal{L}_0(z_t)$ by using the sets $\{|z_t(x_k^n)| \leq \epsilon\}$ as building blocks, where $(x_k^n)_k$ is a dyadic grid. Unfortunately, the correlation between distant blocks is too strong to apply Khoshnevisan et al. (2000, Corollary 3.3).

A different approach to prove the conjecture could be based on existence and regularity of the occupation density of z_t at 0. First, the random field z_t has an occupation density ℓ due to Geman and Horowitz (1980, Theorem 22.1) (see also Pitt (1978, Theorem 3)). Moreover, it is not difficult to see that z_t satisfies the property of *local* $2(\alpha+\delta-1)$–*nondeterminism* (see Monrad and Pitt (1987) for the definition, and Xiao (2008, 2009) for a recent account) and so, by Pitt (1978, Theorem 4) (or Geman and Horowitz (1980, Theorem 26.1)), the occupation density ℓ is Hölder continuous. By Monrad and Pitt (1987, Theorem 1), in order to prove the "probability one" statement, it is sufficient to show that $\ell(0) > 0$ with probability one. For instance this is true if there is a random constant $c_4 = c_4(\omega) > 0$ such that

$$\mathrm{Leb}_{\mathbb{T}_2}(\{x \in \mathbb{T}_2 : \ |z_t(x)| \leq \epsilon\}) \geq c_4 \epsilon.$$

We have not been able to prove that $\mathbb{P}[\ell(0) > 0] = 1$.

2.3. *The non–linear problem.* We extend the results on the dimension of level sets by means of an absolute continuity result. The random perturbation we consider is not "strong" enough (in terms of regularization) to apply Girsanov's theorem (which is a standard method when dealing with non–linear terms of order zero, see for instance Dalang et al. (2007, 2009)). We use an absolute continuity theorem of Da Prato and Debussche (2004) to translate the dimension results on the linear problem to the non–linear problem. We remark that another option to prove the absolute continuity could be given by the idea in Mattingly and Suidan (2005) (see also Mattingly and Suidan (2008); Watkins (2010)).

Existence and strong uniqueness of a solution of problem (2.4) is proved in Lemma 4.3, while continuity with respect to the space variable is proved in Lemma 4.7. In particular, continuity ensures that the level sets of θ are properly defined.

Theorem 2.5. *Let $\nu > 0$, $\alpha > 1$, and $M \geq 1$, and assume the covariance $\mathcal{C}$ satisfies Assumption 2.2. Let θ be the solution of problem (2.1) with $\theta(0) \in L^2_{\#}$, then for every $y \in \mathbf{R}$ and $t > 0$,*

- $\mathbb{P}[\dim_P(\mathcal{L}_y(\theta_t)) \leq 3 - \alpha - \delta] = 1$,
- $\mathbb{P}[\dim_P(\mathcal{L}_y(\theta_t)) = \dim_H(\mathcal{L}_y(\theta_t)) = 3 - \alpha - \delta] > 0$.

The theorem will be proved in Section 4. As we shall see in the course of the proof of the above result, the same holds when θ is the stationary solution.

3. Linear results

In this section we prove Theorem 2.3. Existence and uniqueness for the solution z of (2.5), as well as of its invariant measure and strong mixing are a standard matter, see Da Prato and Zabczyk (1992). In the next lemma we summarize a few results concerning point–wise properties of z that we will need in the rest of the section.

Lemma 3.1. *Under the assumptions of Theorem 2.3, for every $t > 0$ and $x \in \mathbb{T}_2$, $z(t,x)$ is a centred Gaussian random variable such that $c_2\sigma_t^2 \leq \mathbb{E}|z(t,x)|^2 \leq c_3\sigma_t^2$, with $\sigma_t > 0$. Moreover, there is $g_t : \mathbf{R} \to \mathbf{R}$ such that*

$$\mathbb{E}[|z(t,x) - z(t,y)|^2] = g_t(x-y),$$

and $c_5|x|^{2(\alpha+\delta-1)} \leq g_t(x) \leq c_6|x|^{2(\alpha+\delta-1)}$.

Proof: We can write $z(t,x)$ as

$$z(t,x) = \sum_{k\in\mathbf{Z}^2_\star} \Big(\sigma_k \int_0^t \mathrm{e}^{-\nu|k|^{2\alpha}(t-s)}\; d\beta_k(s)\Big) e_k(x),$$

where $(\beta_k)_{k\in\mathbf{Z}^2_\star}$ are independent standard Brownian motions. Given $x \in \mathbb{T}_2$, $t > 0$, the real valued random variable $z(t,x)$ is Gaussian and,

$$\operatorname{Var}(z(t,x)) = \sum_{k\in\mathbf{Z}^2_\star} \frac{\sigma_k^2 e_k(x)^2}{2\nu|k|^{2\alpha}}\big(1-\mathrm{e}^{-2\nu|k|^{2\alpha}t}\big) \approx \sigma_t^2 = \sum_{k\in\mathbf{Z}^2_\star} \frac{1-\mathrm{e}^{-2\nu|k|^{2\alpha}t}}{4\nu|k|^{2(\alpha+\delta)}}.$$

The expectation of the increments yields

$$\mathbb{E}[|z(t,x)-z(t,y)|^2] = \sum_{k\in\mathbf{Z}^2_\star} \frac{\sigma_k^2}{2\nu|k|^{2\alpha}}\big(1-\mathrm{e}^{-2\nu|k|^{2\alpha}t}\big)\sin^2\frac{k}{2}(x-y) = g_t(x-y).$$

Using (2.3) and the fact that $(1-\mathrm{e}^{-2\nu|k|^{2\alpha}t})$ is bounded from above and below by constants independent of k (but not t), we see that

$$g_t(x) \approx \sum_{k\in\mathbf{Z}^2_\star} \frac{1}{|k|^{2\alpha+2\delta}}\sin^2\frac{k}{2}(x-y),$$

hence $g_t(x) \approx |x|^{2(\alpha+\delta-1)}$ by Lemma 3.2 below. □

Lemma 3.2. *Let $\gamma > 0$. Then for all $x \in \mathbf{R}^d$ with $|x| < 1$,*

$$\sum_{k\in\mathbf{Z}^d_\star} \frac{1}{|k|^{d+\gamma}}\sin^2(k\cdot x) \sim h_\gamma(x),$$

where $h_2(x) = -|x|^2\log|x|$ and $h_\gamma(x) = |x|^{\gamma\wedge 2}$ otherwise.

Proof: For the upper bound,

$$\sum_{k\in\mathbf{Z}^d_\star} \frac{1}{|k|^{d+\gamma}}\sin^2\big(k\cdot x\big) \le \sum_{|k|\cdot|x|\le 1} \frac{|x|^2}{|k|^{d+\gamma-2}} + \sum_{|k|\cdot|x|\ge 1} \frac{1}{|k|^{d+\gamma}} \le \boxed{\mathrm{s}} + \boxed{\mathrm{l}}.$$

The term $\boxed{\mathrm{l}}$ can be easily estimated by comparison with an integral, yielding $\boxed{\mathrm{l}} \le c_7|x|^\gamma$. For $\boxed{\mathrm{s}}$ we use the elementary result,

$$\sum_{|k|\le A} \frac{1}{|k|^\beta} \sim \begin{cases} A^{(d-\beta)_+} & \beta \ne d, \\ \log A & \beta = d, \end{cases} \qquad A \text{ large},$$

to obtain immediately that $\boxed{\mathrm{s}} \sim h_\gamma(x)$.

For the lower bound, an elementary computation shows that there is $c_8 = c_8(\gamma) > 0$ such that,

$$\frac{1}{|k|^{d+\gamma}} = c_8 \int_0^\infty \mathrm{e}^{-t|k|^2}\, t^{\frac{d+\gamma-2}{2}}\, dt,$$

hence

$$\begin{aligned}\sum_{k\in\mathbf{Z}^d_\star}\frac{1}{|k|^{d+\gamma}}\sin^2\big(\tfrac12 k\cdot x\big) &= c_8\int_0^\infty\Big(\sum_{k\in\mathbf{Z}^d_\star}\mathrm{e}^{-t|k|^2}\sin^2\big(\tfrac12 k\cdot x\big)\Big)t^{\frac{d+\gamma-2}{2}}\,dt\\ &=\frac12 c_8\int_0^\infty\Big(\sum_{k\in\mathbf{Z}^d_\star}\mathrm{e}^{-t|k|^2}(1-\cos(k\cdot x)\Big)t^{\frac{d+\gamma-2}{2}}\,dt\\ &\geq \frac12 c_8\int_0^1\big(\phi(t,0)-\phi(t,x)\big)t^{\frac{d+\gamma-2}{2}}\,dt,\end{aligned}$$

where

$$\phi(t,x)=\sum_{k\in\mathbf{Z}^d_\star}\mathrm{e}^{-t|k|^2}\cos(k\cdot x).$$

The function ϕ is the fundamental solution of the heat equation with periodic boundary conditions and mean zero. In particular, $\phi(t,0)-\phi(t,x)\geq c_9 t^{-d/2}\big(1\wedge |x|^2/t\big)$, hence

$$\sum_{k\in\mathbf{Z}^d_\star}\frac{1}{|k|^{d+\gamma}}\sin^2\big(\tfrac12 k\cdot x\big)\geq\int_0^1\frac{c_{10}}{t^{\frac{d}{2}}}\Big(1\wedge\frac{|x|^2}{t}\Big)t^{\frac{d+\gamma-2}{2}}\,dt\geq c_{11}h_\gamma(x),$$

by a direct computation. □

Remark 3.3. If we replace the usual Euclidean distance with the "torus distance", namely $|x-y|_{\mathbb{T}_2}=\inf_{k\in\mathbf{Z}^2}|x-y+2\pi k|$ in the statement of Lemma 3.1 and Lemma 3.2 , the conclusions of both lemmata still hold true.

Remark 3.4. If $d=1$, the above lemma admits a probabilistic proof, using a Fourier series expansion of the fractional Brownian motion. Indeed, by Iglói (2005) it follows, by simple computations that exploit the explicit form of the covariance function of the process, that

$$\sum_{k=1}^\infty\frac{1}{k^{1+2H}}\sin^2\big(\tfrac12 kt\big)\sim t^{2H}.$$

We have not been able to find a similar proof in the multi–dimensional case.

3.1. *The upper bound.* The following proposition contains the first part of Theorem 2.3.

Proposition 3.5. *Under the assumptions of Theorem 2.3,*

$$\dim_H(\mathcal{L}_y(z_t))\leq\dim_P(\mathcal{L}_y(z_t))\leq 3-\alpha-\gamma,$$

almost surely.

Proof: We know by Lemma 3.1 and Gaussianity that z_t is γ–Hölder continuous for every $\gamma<\alpha+\delta-1$ and that the Hölder coefficient L_γ has finite polynomial moments by Kunita (1990, Theorem 1.4.1). Let $\gamma<\alpha+\delta-1$ and consider a ball $B_\epsilon(x)$ in $\mathbb{T}_2$, then

$$\mathbb{P}[B_\epsilon(x)\cap\mathcal{L}_y(z_t)]\leq\mathbb{P}[y\in z_t(B_\epsilon(x)),\ |z_t(x)-y|\geq\epsilon^\gamma]+\mathbb{P}[|z_t(x)-y|\leq\epsilon^\gamma].$$

On the first event there is $x_y \in B_\epsilon(x)$ such that $z_t(x_y) = y$, hence for a γ' such that $\gamma < \gamma' < \alpha + \delta - 1$, we have that $\epsilon^\gamma \le |z_t(x) - z_t(x_y)| \le L_{\gamma'}\epsilon^{\gamma'}$. Therefore for n large enough,

$$\mathbb{P}[B_\epsilon(x) \cap \mathcal{L}_y(z_t)] \le \mathbb{P}[L_{\gamma'} \ge \epsilon^{\gamma-\gamma'}] + c_{12}\epsilon^\gamma \le \epsilon^{n(\gamma'-\gamma)}\mathbb{E}[L_{\gamma'}^n] + c_{13}\epsilon^\gamma \le c_{14}\epsilon^\gamma.$$

Consider now a covering of $\mathbb{T}_2$ of 2^{2k} balls B_k of radius 2^{-k}, and let N_k be the smallest number of balls of radius 2^{-k} covering $\mathcal{L}_y(z_t)$. Clearly, $N_k \le \sum \mathbb{1}_{\{B_k \cap \mathcal{L}_y \neq \emptyset\}}$, hence $\mathbb{E}[N_k] \le c_{15}2^{2k-\gamma k}$. By the first Borel–Cantelli lemma, $N_k \le c_{15}2^{2k-\gamma'' k}$ for k large enough, a. s., where $\gamma'' < \gamma$. Therefore, $\dim_P \mathcal{L}_y(z_t) \le 2 - \gamma''$, and the upper bound follows by taking $\gamma'' \uparrow \alpha + \delta - 1$. □

3.2. *The lower bound.* We prove the second part of Theorem 2.3. To this end we first give an estimate of the two–points covariance.

Lemma 3.6. *Under the same assumptions of Theorem 2.3, let $t > 0$. Then there is $c_{16} > 0$ such that for every $x, x' \in \mathbb{T}_2$, with $x \neq x'$, $\det(q_{xx'}) \ge c_{16}|x - x'|^{2(\alpha+\delta-1)}$, where $q_{xx'}$ is the covariance matrix of $(z_t(x), z_t(x'))$.*

Proof: Given $t > 0$, define the numbers $a_k(t)$ by

$$\sigma_t(x)^2 = \sum_{k \in \mathbf{Z}^2_\star} \frac{\sigma_k^2}{2\nu|k|^{2\alpha}}\big(1 - \mathrm{e}^{-2\nu|k|^{2\alpha}t}\big)e_k(x)^2 = \sum_{k \in \mathbf{Z}^2_\star} a_k(t)^2 e_k(x)^2.$$

If $x, x' \in \mathbb{T}_2$, define $\sigma_t(x, x')$ by

$$\sigma_t(x, x') = \mathbb{E}[z_t(x)z_t(x')] = \sum_{k \in \mathbf{Z}^2_\star} a_k(t)^2 e_k(x)e_k(x').$$

Then a few elementary computations (using the fact that $a_{-k} = a_k$ and the symmetries of $\mathbf{Z}^2_\star$) show that

$$\det(q_{xx'}) = \frac{1}{2}\sum_{m,n \in \mathbf{Z}^2_\star} a_m(t)^2 a_n(t)^2 \sin^2\big((m+n)\tfrac{x-x'}{2}\big)$$

By re–arranging the sum, we finally obtain

$$\det(q_{xx'}) = \sum_{k \in \mathbf{Z}^2_\star} A_k \sin^2(k \cdot \tfrac{x-x'}{2}), \quad \text{where} \quad A_k = \frac{1}{2}\sum_{m+n=k; m,n \in \mathbf{Z}^2_\star} a_n(t)^2 a_m(t)^2.$$

Since $a_k(t) \sim |k|^{-2(\alpha+\delta)}$ and $\alpha + \delta > 1$, it is easy to see that $A_k \sim |k|^{-2(\alpha+\delta)}$ and

$$\det(q_{xx'}) \ge \sum_{k \in \mathbf{Z}^2_\star} \frac{c_{17}}{|k|^{2+2(\alpha+\delta-1)}}\sin^2(k \cdot \tfrac{x-x'}{2}) \ge c_{18}|x - x'|^{2(\alpha+\delta-1)}$$

where the last inequality follows from Lemma 3.2. □

Proposition 3.7. *Under the assumptions of Theorem 2.3,*

$$\dim_P(\mathcal{L}_y(z_t)) \ge \dim_H(\mathcal{L}_y(z_t)) \ge 3 - \alpha - \delta,$$

with positive probability.

Proof: We use Frostman's ideas, see Kahane (1985). To this end, given a non–negative measure μ on $\mathbb{T}_2$ and a number $\gamma > 0$, define the γ–energy of μ as

$$\|\mu\|_\gamma = \int_{\mathbb{T}_2}\int_{\mathbb{T}_2} \frac{\mu(dx)\mu(dx')}{|x - x'|^\gamma},$$

where $|\cdot|$ denotes the distance on $\mathbb{T}_2$ (see Remark 3.3).

We proceed as in Wu and Xiao (2006) (see also Xiao (1995, 1997)) and define the measures $\mu_n = \sqrt{2\pi n}\exp(-\frac{1}{2}n|z_t(x)-y|^2)\,dx$. For our purposes, it is sufficient to show that there are c_{19}, c_{20} (independent of n), such that $\mathbb{E}[\mu_n(\mathbb{T}_2)] \geq c_{19}$, $\mathbb{E}[\mu_n(\mathbb{T}_2)^2] \leq c_{20}$ and $\mathbb{E}[\|\mu_n\|_\gamma] < \infty$ for every $\gamma < 3-\alpha-\delta$. Indeed, by these facts it follows that there is a sub–sequence converging to a measure μ. Moreover, μ is non–zero with probability at least $c_{19}^2 c_{20}^{-1}$ (see Kahane (1985)). By continuity of z_t (Lemma 3.1), it follows that μ has support in $\mathcal{L}_y(z_t)$ and hence Frostman's lemma (see Kahane (1985, Theorem 10.3.2)) yields $\mathbb{P}[\dim_H \mathcal{L}_y(z_t) \geq \gamma] \geq c_{19}^2 c_{20}^{-1}$. We notice preliminarily that

$$\sqrt{2\pi n}\,\mathrm{e}^{-\frac{1}{2}n|z_t(x)-y|^2} = \int_{\mathbf{R}} \mathrm{e}^{-\frac{1}{2n}u^2 + \mathrm{i}u(z_t(x)-y)}\,du.$$

As in Wu and Xiao (2006), simple computations yield,

$$\begin{aligned}\mu_n(\mathbb{T}_2) &= \int_{\mathbb{T}_2}\int_{\mathbf{R}} \mathrm{e}^{-\frac{1}{2n}u^2 - \mathrm{i}uy}\,\mathbb{E}[\mathrm{e}^{\mathrm{i}uz_t(x)}]\,du\,dx = \\ &= \int_{\mathbb{T}_2}\int_{\mathbf{R}} \mathrm{e}^{-\mathrm{i}uy}\,\mathrm{e}^{-\frac{1}{2}(\frac{1}{n}+\sigma_t(x)^2)u^2}\,du\,dx = \int_{\mathbb{T}_2}\sqrt{\frac{2\pi}{\frac{1}{n}+\sigma_t(x)^2}}\,\mathrm{e}^{-\frac{y^2}{2(\frac{1}{n}+\sigma_t(x)^2)}}\,dx \geq \\ &\geq (1+c_6^2\sigma_t^2)^{-\frac{1}{2}}\,\mathrm{e}^{-\frac{y^2}{2c_5^2\sigma_t^2}} = c_{19}.\end{aligned}$$

With similar computations, involving this time two dimensional Gaussian random variables, we see that

$$\begin{aligned}\mu_n(dx)\mu_n(dx') &= \Big(\iint_{\mathbf{R}^2} \mathrm{e}^{-\frac{1}{2}u\cdot g_{xx'}\cdot u^T}\,\mathrm{e}^{-\mathrm{i}y(u_1+u_2)}\,du\Big)\,dx\,dx' = \\ &= \frac{2\pi}{\sqrt{\det g_{xx'}}}\,\mathrm{e}^{-\frac{1}{2}(y,y)\cdot g_{xx'}^{-1}\cdot(y,y)^T}\,dx\,dx' \leq \frac{2\pi}{\sqrt{\det g_{xx'}}}\,dx\,dx',\end{aligned}$$

where $g_{xx'} = \frac{1}{n}I + q_{xx'}$ and $q_{xx'}$ is the covariance matrix of $(z_t(x), z_t(x'))$. By Lemma 3.6, $\det(g_{xx'}) \geq \det(q_{xx'}) \geq c_{16}|x-x'|^{2(\alpha+\delta-1)}$, hence $\mu_n(dx)\mu_n(dx') \leq c_{21}|x-x'|^{-(\alpha+\delta-1)}$ and it is immediate to deduce that $\mathbb{E}[\mu_n(\mathbb{T}_2)^2] \leq c_{20}$. Likewise, we deduce that $\mathbb{E}[\|\mu_n\|_\gamma]$ is bounded uniformly in n if $\gamma < 3-\alpha-\delta$. □

4. Non–linear results

We turn to the proof of Theorem 2.5. Our strategy is based on the idea that if θ is the solution of (2.4) and z of (2.5), and if the laws of $\theta(t)$ and $z(t)$ are equivalent measures, then Theorem 2.3 immediately implies Theorem 2.5.

Remark 4.1. A standard way to prove absolute continuity of laws of solutions of stochastic PDEs is the Girsanov transformation. In our case, to apply Girsanov's transformation, the quantity

$$\int_0^t \|\mathcal{C}^{-1/2}B_M(\theta)\|_{L^2}^2\,ds < \infty, \qquad \mathbb{P}-a.\,s.$$

should be finite, at the very least. This happens when $\alpha > 2$ (see Mattingly and Suidan (2005, Theorem 3)).

The following theorem could be proved by means of the same method in Mattingly and Suidan (2005), which is indeed the case $\alpha > 1$, $M = 1$ (see also Watkins (2010)). Here we present an alternative proof based on the method introduced in Da Prato and Debussche (2004) and on the polynomial moments proved in Lemma 4.4.

Theorem 4.2. *Let $\alpha > 1$, $M \geq 1$ and assume the covariance is as in Assumption 2.2. Let $x \in L^2_\#(\mathbb{T}_2)$, $\theta(\cdot; x)$ be the solution of (2.1) with initial condition x and z be the solution of (2.5) with initial condition $z(0) = 0$. Then for every $t > 0$ the law of $\theta(t; x)$ is equivalent to the law of $z(t)$.*

We first state some preliminary results that will be necessary for the absolute continuity theorem given above. The first result ensures existence and uniqueness for the solutions of (2.4). Its proof is quite standard and follows the lines of the proof of Flandoli (2008, Theorem 2.9).

Lemma 4.3. *Let $\alpha > 1$ and $M \geq 1$, and let Assumption 2.2 be in force. Given a probability μ on $H^{-\alpha}_\#(\mathbb{T}_2)$ with all polynomial moments finite in $H^{-\alpha}_\#(\mathbb{T}_2)$, there exists a unique (path–wise) solution θ of (2.4) with initial distribution μ such that $\theta \in C([0,\infty); H^{-\alpha}_\#(\mathbb{T}_2)) \cap L^2_{loc}([0,\infty); L^2_\#(\mathbb{T}_2))$. Moreover for every $m \geq 1$ and $T > 0$,*

$$\mathbb{E}\Big[\sup_{[0,T]} \|\theta(t)\|^{2m}_{-M}\Big] + \nu\mathbb{E}\Big[\int_0^T \|\theta(t)\|^{2m-2}_{-M}\|\theta(t)\|^2_{\alpha-M}\,dt\Big] < \infty, \tag{4.1}$$

$$\mathbb{E}\Big[\log\Big(1 + \sup_{[0,T]} \|\theta(t)\|^2_{-\alpha} + \nu\int_0^T \|\theta(t)\|^2_{L^2}\,dt\Big)\Big] < \infty. \tag{4.2}$$

Denote by $\theta(\cdot; x)$ the solution with initial distribution concentrated at x. Then the process $(\theta(\cdot; x))_{x\in H^{-\alpha}_\#(\mathbb{T}_2)}$ is a Markov process and the associated transition semigroup is Feller in $H^{-\alpha}_\#(\mathbb{T}_2)$.

If additionally μ has second moment finite in $L^2_\#(\mathbb{T}_2)$, then for every $\gamma < \alpha+\delta-1$,

$$\mathbb{E}\Big[\log\Big(1 + \sup_{[0,T]} \|\theta(t)\|^2_{L^2} + \nu\int_0^T \|\theta(t)\|^2_{\gamma}\,dt\Big)\Big] < \infty. \tag{4.3}$$

Moreover, the process $(\theta(\cdot; x))_{x\in L^2_\#(\mathbb{T}_2)}$ is Feller in $L^2_\#(\mathbb{T}_2)$.

Proof: The proof is standard and it is based on finite dimensional approximations. For every $N \geq 1$ let $H_N = \operatorname{span}[e_k : |k|_\infty \leq N]$, denote by π_N the projection onto H_N and consider the following finite dimensional approximation of (2.4),

$$d\theta_N + (\nu A^\alpha\theta_N + \pi_N B_M(\theta_N))\,dt = \pi_N\mathcal{C}^{\frac{1}{2}}dW. \tag{4.4}$$

Using (2.2), it is easy to see (as in the first chapter of Flandoli (2008)) that the approximated problem admits a unique solution for any initial condition in H_N. Given a probability measure μ on $H^{-\alpha}_\#$, let θ_N be the solution of (4.4) with initial distribution $\pi_N\mu$, and denote by $\mathbb{P}_N$ its law.

Step 1: inequality (4.1) for u_N. The estimate (4.1) for u_N follows by applying Itô's formula to $\|\theta_N(t)\|^{2m}_{-M}$, using the second of (2.2) and standard stopping time arguments (as in Flandoli (2008)). Notice that the estimate will pass to limit as $N \uparrow \infty$ yielding (4.1) for the infinite dimensional problem.

Step 2: inequality (4.2) *for* u_N. To prove (4.2), we notice that if $M = 1$, (4.1) provides already a stronger inequality. There is no loss of generality then if we assume $M > 1$ here. To prove (4.2) for u_N, set $\eta_N = \theta_N - \pi_N z$, where z is the solution of (2.5), then η_N is solution of $\dot\eta_N + \nu A^\alpha \eta_N + \pi_N B_M(\theta_N) = 0$. Set

$$\varphi_N(t) = \|\eta_N(t)\|_{-\alpha}^2 + \nu \int_0^t \|\eta_N(s)\|_{L^2}^2\, ds,$$

then

$$\frac{d}{dt}\log(1+\varphi_N) \le c_{22}(\|\theta_N\|_{\alpha-M}^2 + \|z\|_{L^2}^2).$$

Here the non–linear term is the only non standard part, and we estimate it as follows. By the Hölder inequality and the embedding $H_\#^{1+\epsilon} \hookrightarrow L^\infty$, with $\epsilon \le M-1$ so that $2 - 2M + \epsilon \le \alpha - M$,

$$\begin{aligned} 2\langle A^{-\alpha}\eta_N, B_M(\theta_N)\rangle &\le c_{25}\|\eta_N\|_{1-2\alpha}\|\theta_N\|_{2-2M+\epsilon}\|\theta_N\|_{L^2} \le \\ &\le \nu\|\eta_N\|_{L^2}^2 + c_{26}\varphi(\|\theta_N\|_{\alpha-M}^2 + \|z\|_{L^2}^2). \end{aligned}$$

By integrating in time and taking expectation of the inequality for $\log(1+\varphi_N)$, and by using the inequality $\log(1+x+y) \le \log(1+x) + \log(1+y)$ (for positive x, y), as well as that $z \in C([0,T]; H_\#^\gamma(\mathbb{T}_2)$ for every $\gamma < \alpha + \delta - 1$, we obtain (4.2) for θ_N. Notice again that this inequality is stable in the limit $N \uparrow \infty$ by semi–continuity and will yield (4.2) for θ.

Step 3: inequality (4.3) *for* u_N. The computations are essentially the same of the previous step, but with

$$\psi_N(t) = \|\eta_N(t)\|_{L^2}^2 + \nu \int_0^t \|\eta_N(s)\|_\alpha^2\, ds,$$

and we use the first equality in (2.2), the Hölder inequality, interpolation inequalities, and Sobolev's embeddings (with $\epsilon < \alpha + \delta - 1$ and $\epsilon < \frac12$) to estimate the non–linear part as

$$\begin{aligned} \langle \eta_N, B_M(\theta_N)\rangle_{L^2} &= \langle B(\nabla^\perp A^{-M}\theta_N, \pi_N z\rangle_{L^2} \le \\ &\le c_{30}\|\eta_N\|_1\|\theta_N\|_\epsilon\|z\|_{L^2} \le c_{31}(\|\eta_N\|_1^{1+\epsilon}\|\eta_N\|_{L^2}^{1-\epsilon}\|z\|_{L^2} + \|\eta_N\|_1\|z\|_\epsilon^2) \le \\ &\le c_{32}(1 + \|z\|_\epsilon^4)\psi_N. \end{aligned}$$

Step 4: time regularity. Here we show that there are $\beta \in (0, \frac12)$ and $p > 1$, such that θ_N has a uniform (in N) finite moment in $W^{\beta,p}(0,T; H_\#^{-2\alpha})$. Indeed, write $\theta_N(t) = \theta_N(0) + \pi_N \mathcal{C}^{1/2}W_t + J_N(t)$, where J_N is the time integral of the drift of (4.4). Clearly, $\pi_N\mathcal{C}^{1/2}W_t$ has all polynomial moments uniformly bounded in $W^{\beta,p}(0,T; H_\#^{-2\alpha})$, for every $p > 1$ and $\beta \in (0, \frac12)$. By duality, $\|B_M(x)\|_{-2\alpha} \le \|x\|_{L^2}\|x\|_{-M}$, since if $\|y\|_{L^2} \le 1$, by the Hölder inequality and Sobolev's embeddings,

$$\langle A^{-\alpha}y, B_M(x)\rangle \le \|A^{-\alpha+\frac12}y\|_\infty \|\nabla^\perp A^{-M}x\|_{L^2}\|x\|_{L^2} \le c_{33}\|x\|_{-M}\|x\|_{L^2}.$$

Therefore $B_M(\theta_N)$ has uniformly (in N) bounded logarithmic moments in the space $L^2(0,T; H_\#^{-1-\alpha})$ by the above computations and the bounds in the first and second steps. Since the same holds trivially for $A^\alpha\theta_N$, we deduce that $J(\theta_N)$ has uniformly bounded logarithmic moments in $W^{\beta,p}(0,T; H_\#^{-2\alpha})$ with p as before.

Step 5: the limit. By the previous steps the sequence of laws $(\mathbb{P}_N)_{N\ge0}$ is tight in the space $\mathcal{S}_T = L^2(0,T; L_\#^2) \cap W^{\beta,p}(0,T; H_\#^{-2\alpha})$, for suitable β, p. Hence there are a sub–sequence (that we keep denoting by $(\mathbb{P}_N)_{N\geq 0}$ for simplicity) and a probability measure $\mathbb{P}_\infty$ such that $\mathbb{P}_N \rightharpoonup \mathbb{P}_\infty$. By Skorokhod's theorem there are, on a new probability space, a cylindrical Wiener process (that we keep denoting by W), a sequence of random variables (denoted by θ_N for simplicity) and a random variable θ_∞, with law $\mathbb{P}_\infty$, such that each θ_N has distribution $\mathbb{P}_N$. Moreover $(\theta_N)_{N\geq 1}$ converges a. s. to θ_∞ weakly in $\mathcal{S}_T$. Since the space $\mathcal{S}_T$ is compactly embedded in $C([0,T]; H_\#^{-2\alpha})$ and in $L^2(0,T; H_\#^{1-2M})$, (see Flandoli (2008)) it turns out that $\theta_N \to \theta$ converges a. s. in these topologies, and in particular implies that $u_N \to u_\infty$ converges a. s. strongly in $L^2(0,T; L^2_\#)$, where $u_\infty = \nabla^\perp A^{-M}\theta_\infty$. Such convergence properties readily allow to show that θ_∞ is a (distributional) solution of (2.4).

Step 6: uniqueness. Let θ_1, θ_2 two solutions with the same initial condition. Set $u_i = \nabla^\perp A^{-M}\theta_i$, $i = 1, 2$, $\xi = \theta_2 - \theta_1$ and $v = u_2 - u_1$, then

$$\dot{\xi} + \nu A^\alpha \xi + B(v, \theta_2) + B(u_1, \xi) = 0.$$

With the same methods of the fourth step, it is easy to see that $\dot{\xi} \in L^2(0,T; H_\#^{-2\alpha})$. Since $\xi \in L^2(0,T; L^2_\#)$, it turns out that $t \mapsto \|\xi(t)\|^2_{-\alpha}$ is differentiable with derivative $2\langle A^{-\alpha}\xi, \dot{\xi}\rangle$ (see Temam (2001)), hence

$$\frac{d}{dt}\|\xi\|^2_{-\alpha} + 2\nu\|\xi\|^2_{L^2} + 2\langle A^{-\alpha}\xi, B(v,\theta_2) + B(u_1,\xi)\rangle = 0,$$

and using the Hölder inequality and Sobolev's embeddings,

$$\begin{aligned}2|\langle A^{-\alpha}\xi, B(v,\theta_2) + B(u_1,\xi)\rangle| &\leq c_{34}\big(\|\theta_1\|_{L^2} + \|\theta_2\|_{L^2}\big)\|\xi\|_{L^2}\|\xi\|_{-\alpha}\\ &\leq \nu\|\xi\|_{L^2} + c_{35}\big(\|\theta_1\|_{L^2} + \|\theta_2\|_{L^2}\big)^2\|\xi\|^2_{-\alpha},\end{aligned}$$

in conclusion by Gronwall's lemma we have that a. s., $\xi \equiv 0$. □

The next preliminary ingredient is to prove that there exists an invariant measure for problem (2.4) which has all polynomial moments finite in L^2 (and in smaller spaces of higher regularity). This is done following (almost) Es-Sarhir and Stannat (2010).

Lemma 4.4. *Let $\alpha > 1$, $M \geq 1$ and let Assumption 2.2 be true. Then there exists an invariant measure μ for the transition semigroup associated to problem (2.4). Moreover, for every $\gamma < \delta + \alpha - 1$ and $m \geq 1$ there is a number $c_{36} > 0$ such that*

$$\int \|x\|^{2m-2}_{L^2}\|x\|^2_\gamma\,\mu(dx) \leq c_{36}. \tag{4.5}$$

Proof: Consider the Galerkin approximations (4.4) of (2.4). It is fairly standard (see Flandoli (2008)) to prove, using the bounds and methods as in Lemma 4.3 that for every N the system (4.4) admits an invariant measure μ_N. If we are able to prove (4.5) for each μ_N with a constant c_{36} independent of N, then the lemma is proved. Indeed, the uniform bound ensures tightness of $(\mu_N)_{N\geq 1}$ in $L^2_\#(\mathbb{T}_2)$ and, consequently, of the laws of each of the stationary solution of (4.4) with initial condition μ_N in $C([0,\infty]; H_\#^{-\alpha}) \cap L^2_{\text{loc}}([0,\infty); L^2_\#)$. The same methods of the previous lemma ensure that, up to a sub–sequence, there is a solution of (2.4) which is limit of stationary laws, hence stationary itself. Its marginal μ at fixed time turns out to be an invariant measure for (2.4) and a limit point of $(\mu_N)_{N\geq 1}$. By semi–continuity μ verifies (4.5).

It remains to prove (4.5) for the Galerkin system. Given $N \geq 1$, let θ_N be the stationary solution of (4.4) with marginal law μ_N.

Step 1: estimates for the linear part. For every $\lambda > 0$ consider the solution $z_{\lambda,N}$ of the following problem,

$$dz_{\lambda,N} + (\nu A^\alpha z_{\lambda,N} + \lambda z_{\lambda,N})\,dt = \pi_N \mathcal{C}^{\frac{1}{2}}\,dW,$$

with initial condition $z_{\lambda,N}(0) = 0$, and recall that $W(t) = \sum_{k\in\mathbf{Z}^2_\star} \beta_k e_k$, with $(\beta_k)_{k\in\mathbf{Z}^2_\star}$ independent standard Brownian motions. Set for every $T > 0$, $a \in (0, \frac{1}{2})$, $\epsilon \in (0, \alpha+\delta-1)$ and $\beta \in (0,1)$ such that $\alpha(1-2a(1-\beta)) < (\alpha+\delta-1-\epsilon)$,

$$M_{a,\epsilon,\beta}(T)^2 = \sum_{k\in\mathbf{Z}^2_\star} \frac{1}{|k|^{4a\alpha(1-\beta)+2\delta-2\epsilon}} \Big(\sup_{0\leq s<t\leq T} \tfrac{|\beta_k(t)-\beta_k(s)|}{|t-s|^a}\Big)^2.$$

From Proposition 2.1 and Corollary 2.2 of Es-Sarhir and Stannat (2010), $\mathbb{E}[M_{a,\epsilon,\beta}(T)^{2m}] \leq c_{37} T^{m(1-2a)}$ and, $\mathbb{P}-a.\,s.$, $\|z_{\lambda,N}(t)\|^2_\epsilon \leq c_{38}\lambda^{-2a\beta} M_{a,\epsilon,\beta}(T)^2$. For the rest of the proof fix values of a and β as required above.

Step 2: estimates for the non–linear part. Set $\eta_{\lambda,N} = \theta_N - z_{\lambda,N}$, then $\eta_{\lambda,N}$ solves

$$\dot\eta_{\lambda,N} + \nu A^\alpha \eta_{\lambda,N} + \pi_N B_M(\theta_N) = \lambda z_{\lambda,N},$$

with initial condition $\eta_{\lambda,N}(0) = \theta_N(0)$. For every $m \geq 1$,

$$\begin{aligned}\frac{d}{dt}(1+\|\eta_{\lambda,N}\|^2_{L^2})^m = 2m(1+\|\eta_{\lambda,N}\|^2_{L^2})^{m-1}\big(-\nu\|\eta_{\lambda,N}\|^2_\alpha +\\ -\langle\eta_{\lambda,N}, B_M(\theta_N)\rangle + \lambda\langle z_{\lambda,N},\eta_{\lambda,N}\rangle\big),\end{aligned}$$

hence using (2.2), Hölder's inequality and Sobolev's embeddings,

$$\langle\eta_{\lambda,N}, B_M(\theta_N)\rangle \leq \frac{\nu}{4}\|\eta_{\lambda,N}\|^2_\alpha + c_{39}\|z_{\lambda,N}\|^4_\epsilon + c_{39}\|z_{\lambda,N}\|^2_\epsilon\|\eta_{\lambda,N}\|^2_{L^2},$$

where $\epsilon \in (0,\alpha+\delta-1)$ can be chosen arbitrarily small (and $c_{39} = c_{39}(\epsilon)$, although is independent from N). Young's inequality and the inequalities of the previous step yield,

$$\begin{aligned}\frac{d}{dt}(1+\|\eta_{\lambda,N}\|^2_{L^2})^m &+ \frac{3}{2}\nu m(1+\|\eta_{\lambda,N}\|^2_{L^2})^{m-1}\|\eta_{\lambda,N}\|^2_\alpha \leq\\ &\leq c_{40} m(1+\|\eta_{\lambda,N}\|^2_{L^2})^{m-1}(\lambda^4 + \|z_{\lambda,N}\|^4_\epsilon + \|z_{\lambda,N}\|^2_\epsilon\|\eta_{\lambda,N}\|^2_{L^2})\\ &\leq c_{40} m(1+\|\eta_{\lambda,N}\|^2_{L^2})^{m-1}(\lambda^4 + \|z_{\lambda,N}\|^4_\epsilon + c_{38}\lambda^{-2a\beta}M_{a,\epsilon,\beta}(T)^2\|\eta_{\lambda,N}\|^2_{L^2}).\end{aligned}$$

Consider $\omega \in \{M_{a,\epsilon,\beta}(T) \leq R\}$ and choose $\lambda = \lambda_R$ so that

$$c_{40}c_{38}R^2\lambda_R^{-2a\beta} = \frac{\nu}{4},$$

then by using the Poincaré inequality and again Young's inequality,

$$\frac{d}{dt}(1+\|\eta_{\lambda,N}\|^2_{L^2})^m + \nu m(1+\|\eta_{\lambda,N}\|^2_{L^2})^{m-1}\|\eta_{\lambda,N}\|^2_\alpha \leq c_{41}(1+\lambda_R^{4m} + \|z_{\lambda,N}\|^{4m}_\epsilon).$$

Finally, by integrating in $[0,T]$, on the event $\{M_{a,\epsilon,\beta}(T) \leq R\}$,

$$\nu m\int_0^T \|\eta_{\lambda_R,N}\|^{2m-2}_{L^2}\|\eta_{\lambda_R,N}\|^2_\alpha\,dt \leq (1+\|\theta(0)\|^2_{L^2})^m + c_{41}\int_0^T (1+\lambda_R^{4m}+\|z_{\lambda_R,N}\|^{4m}_{L^2})\,dt. \tag{4.6}$$

Step 3: estimates in λ_R. Define for every integer $R \geq 1$ the events $A_R = \{R-1 < M_{a,\epsilon,\beta}(T) \leq R\}$. By Fernique's theorem (see for instance Da Prato and Zabczyk (1992)), for every $q \geq 2$ and every $\epsilon < \alpha + \delta - 1$,

$$\mathbb{E}[\|z_{\lambda_R,N}(t)\|_\epsilon^q] \leq c_{42}\big(\mathbb{E}[\|z_{\lambda_R,N}\|_\epsilon^2]\big)^{\frac{q}{2}} \leq c_{43}\Big(\sum_{k\in\mathbf{Z}^2_\star} \frac{|k|^{2\epsilon-2\delta}}{\lambda_R + \nu|k|^{2\alpha}}\Big)^{\frac{q}{2}} \leq c_{44}\lambda_R^{-\frac{q(\alpha+\delta-1-\epsilon)}{2\alpha}},$$

since $\lambda_R + \nu|k|^2 \geq \lambda_R \vee (\nu|k|^2)$. By our choice of λ_R, a, and β, if $q \geq 2$,

$$\mathbb{E}\Big[\sum_{R=1}^\infty \mathbb{1}_{A_R}\int_0^T \|z_{\lambda_R,N}(t)\|_\epsilon^q\,dt\Big] \leq \mathbb{E}\Big[\sum_{R=1}^\infty \int_0^T \|z_{\lambda_R,N}(t)\|_\epsilon^q\,dt\Big] \leq c_{45}T. \tag{4.7}$$

Likewise,

$$\mathbb{E}\Big[\sum_{R=1}^\infty \mathbb{1}_{A_R}\lambda_R^q\Big] \leq c_{46}\mathbb{E}\Big[(1 + M_{a,\beta}(T))^{\frac{q}{a\beta}}\Big] \leq c_{47}(1 + T^{q\frac{1-2a}{2a\beta}}). \tag{4.8}$$

Finally, recall (4.6) and use (4.7) and (4.8) to obtain

$$\mathbb{E}\Big[\sum_{R=1}^\infty \mathbb{1}_{A_R}\int_0^T \|\eta_{\lambda_R,N}\|_{L^2}^{2m-2}\|\eta_{\lambda_R,N}\|_\alpha^2\,dt\Big] \leq c_{48}(1 + \mathbb{E}_{\mu_N}[\|x\|_{L^2}^{2m}] + T + T^{2m\frac{1-2a}{a\beta}}). \tag{4.9}$$

Step 4: conclusion. By assumption $\mathbb{E}_{\mu_N}[\|x\|_{L^2}^{2m}]$ is uniformly bounded in N for every $m \geq 1$, due to the estimate in $H^{-M}_\#(\mathbb{T}_2)$. Fix $\gamma \in (0, \alpha+\delta-1)$, then by (4.7) and (4.9),

$$\begin{aligned}
&T\mathbb{E}_{\mu_N}[\|x\|_{L^2}^{2m-2}\|x\|_\gamma^2] = \\
&\quad = \mathbb{E}\Big[\sum_{R=1}^\infty \mathbb{1}_{A_R}\int_0^T \|\theta_N(t)\|_{L^2}^{2m-2}\|\theta_N\|_\gamma^2\,dt\Big] \\
&\quad \leq c_{49}\mathbb{E}\Big[\sum_{R=1}^\infty \mathbb{1}_{A_R}\Big(\int_0^T \|z_{\lambda_R,N}(t)\|_\gamma^{2m}\,dt + \int_0^T \|\eta_{\lambda_R,N}(t)\|_{L^2}^{2m-2}\|\eta_{\lambda_R,N}(t)\|_\alpha^2\,dt\Big)\Big] \\
&\quad \leq c_{50}\big(1 + T + T^{2m\frac{1-2a}{a\beta}} + \mathbb{E}_{\mu_N}[\|x\|_{L^2}^{2m}]\big).
\end{aligned}$$

The above inequality holds for all $T > 0$, hence if we take $T = 2c_{50}$ and use the Poincaré inequality, we obtain (4.5) for μ_N (with a constant uniform in N). □

Corollary 4.5. *Let $\alpha > 1$, $M \geq 1$ and let Assumption 2.2 be true. Given $\gamma < \delta + \alpha - 1$ and an integer $m \geq 1$, there are c_{51}, $c_{52} > 0$ such that if $x \in L^2_\#(\mathbb{T}_2)$, and if θ is the solution of (2.4) with initial condition x, then*

$$\mathbb{E}\Big[\int_0^T \|\theta(t)\|_{L^2}^{2m-2}\|\theta(t)\|_\gamma^2\,dt\Big] \leq c_{51}(1 + T^{c_{52}m} + \|x\|_{L^2}^{2m}), \tag{4.10}$$

for every $T > 0$.

Proof: The proof proceeds as in the previous Lemma. Indeed, steps 1–3 hold regardless of the stationarity of the solution. For the last step, we follow the same lines and, without stationarity, inequality (4.10) follows. □

Next, we show that problem (2.4) has a unique invariant measure which is strongly mixing. Moreover, the strong Feller property ensures that the law of $\theta(t)$ is equivalent to the law of the invariant measure, for every $t > 0$. This allows us to reduce absolute continuity of laws at each time to absolute continuity of the invariant measures.

Lemma 4.6. *Let $\alpha > 1$, $M \geq 1$ and let Assumption 2.2 be true. Then the transition semigroup is strong Feller in $H_{\#}^{-\alpha}(\mathbb{T}_2)$ and problem (2.4) admits a unique invariant measure which is strongly mixing.*

Proof: Uniqueness and strong mixing follow immediately from Doob's theorem (see Da Prato and Zabczyk (1996)) if we prove that the transition semigroup is strong Feller and irreducible in $H_{\#}^{-\alpha}(\mathbb{T}_2)$. The proof is very similar to Flandoli and Maslowski (1995), we give only a sketch and the key estimates.

Preliminarily, we prove the strong Feller property in $L^2_{\#}$. Let $\chi : [0,\infty) \to \mathbf{R}$ be a smooth non–increasing function such that $\chi \equiv 1$ on $[0,1]$ and $\chi \equiv 0$ on $[2,\infty)$ and set $\chi_R(r) = \chi(r/R^2)$ and $B_M^R(x) = \chi_R(\|x\|_{L^2}^2)B_M(x)$. The cut–off problem is

$$d\theta_R + (\nu A^\alpha \theta_R + B_M^R(\theta_R))\,dt = \mathcal{C}^{\frac{1}{2}}\,dW.$$

It is easy to show, as in Lemma 4.3, that the above equation has a unique solution in $C([0,\infty); L^2_{\#})$ for each initial condition in $L^2_{\#}(\mathbb{T}_2)$. Strong Feller in $L^2_{\#}$ follows from the Bismut–Elworthy–Li formula (see Elworthy and Li (1994)), which yields for every bounded measurable φ and every $x, h \in H^{\epsilon}_{\#}(\mathbb{T}_2)$,

$$\begin{aligned}|P_t^R\varphi(x+h) - P_t^R\varphi(x)| &\leq \frac{c_{53}}{\sqrt{t}}\|\varphi\|_\infty \mathbb{E}\Big[\Big(\int_0^t \|\mathcal{C}^{-\frac{1}{2}} D_h\theta_R(s;y)\|_{L^2}^2\,ds\Big)^{\frac{1}{2}}\Big]\\ &\leq \frac{c_{54}}{\sqrt{t}}\|\varphi\|_\infty \mathbb{E}\Big[\int_0^t \|D_h\theta_R(s;y)\|_\delta^2\,ds\Big]^{\frac{1}{2}},\end{aligned}$$

where $(P_t^R)_{t\geq 0}$ is the transition semigroup corresponding to θ_R and $D_h\theta_R(\cdot;y)$ is the Gateaux derivative (with respect to the initial condition) of θ_R in the direction h. Let $\xi(s;y) = D_h\theta_R(s;y)$, then it is sufficient to compute the "energy estimate" of ξ in $L^2(\mathbb{T}_2)$, and the "dissipative" term will provide the estimate we need. Clearly, the most troublesome term is the non–linearity, which is estimated as follows, using the Hölder inequality with exponents p, q, with $2 - \frac{1}{p} \leq \alpha$,

$$\begin{aligned}&\langle DB_M^R(\theta_R)\xi, \xi\rangle_{L^2} \leq\\ &\quad\leq c_{58}\chi_R'(\|\theta_R\|_{L^2}^2)\|\theta_R\|_{L^2}^3\|\xi\|_{L^2}\|\xi\|_\alpha + c_{59}\chi_R(\|\theta_R\|_{L^2}^2)\|\theta_R\|_{L^2}\|\xi\|_{L^2}\|\xi\|_\alpha\\ &\quad\leq \nu\|\xi\|_\alpha^2 + c_{60}R^6\|\xi\|_{L^2}^2.\end{aligned}$$

Next, we prove strong Feller in $H_{\#}^{-\alpha}$, following an idea in Romito (2011). We know that if $\varphi : H_{\#}^{-\alpha}(\mathbb{T}_2) \to \mathbf{R}$ is bounded measurable, then $P_t\varphi \in C_b(L^2_{\#})$, and we want to prove that $P_t\varphi \in C_b(H_{\#}^{-\alpha})$. To this end, let $x_n \to x$ in $H_{\#}^{-\alpha}$, then it is sufficient to show that $\theta(t;x_n) \to \theta(t;x)$ a.s. in $L^2_{\#}$, since by the Markov property and the Lebesgue theorem, $P_t\varphi(x_n) = \mathbb{E}[P_{t/2}\varphi(\theta(t/2;x_n))]$ converges to $\mathbb{E}[P_{t/2}\varphi(\theta(t/2;x))] = P_t\varphi(x)$.

Set $w_n(t) = \theta(t;x_n) - \theta(t;x) = \theta_n - \theta$. The computations that follow are only formal and can be made rigorous through suitable approximations (for instance the Galerkin approximations in Lemma 4.3). The final inequality (4.11) we get is stable under the approximation by semi–continuity and by the fact that the right–hand side of (4.11) is finite at the level of the original problem.

The energy inequality in $L^2_\#$ and the first property in (2.2) yield

$$\begin{aligned}\frac{d}{dt}\|w_n\|_{L^2}^2 + 2\nu\|w_n\|_\alpha^2 &= -2\langle w_n, B_M(\theta_n) - B_M(\theta)\rangle\\ &\leq \nu\|w_n\|_\alpha^2 + c_{61}\|\theta\|_{L^2}^2\|w_n\|_{L^2}^2,\end{aligned}$$

hence by Gronwall's lemma, for every $s \leq t$,

$$\|w_n(t)\|_{L^2}^2 \leq \|w_n(s)\|_{L^2}^2 \,\mathrm{e}^{c_{61}\int_s^t \|\theta\|_{L^2}^2} \leq \|w_n(s)\|_{L^2}^2 \,\mathrm{e}^{c_{61}\int_0^t \|\theta\|_{L^2}^2}.$$

Notice that the exponential term is $\mathbb{P}$–a.s. finite by (4.2). Integrate the above inequality for $s \in [0,t]$ to get

$$\|w_n(t)\|_{L^2}^2 \leq \frac{1}{t}\Big(\int_0^t \|w_n(s)\|_{L^2}^2\,ds\Big)\,\mathrm{e}^{c_{61}\int_0^t \|\theta\|_{L^2}^2\,ds}. \tag{4.11}$$

So it is sufficient to prove that $\int_0^t \|w_n(s)\|_{L^2}^2\,ds \to 0$. Use the Hölder inequality (with exponent p such that $2 - \frac{1}{p} \leq \alpha$ as above) and Sobolev's embeddings on the energy inequality in $H^{-\alpha}_\#$ for w_n, to get

$$\begin{aligned}\frac{d}{dt}\|w_n\|_{-\alpha}^2 + 2\nu\|w_n\|_{L^2}^2 &= -\langle w_n, B(\nabla^\perp A^{-M}\theta_n, w_n) + B(\nabla^\perp A^{-M} w_n, \theta)\rangle_{-\alpha}\\ &\leq \nu\|w_n\|_{L^2}^2 + c_{62}(\|\theta\|_{L^2}^2 + \|\theta_n\|_{L^2}^2)\|w_n\|_{-\alpha}^2.\end{aligned}$$

The Gronwall inequality finally yields

$$\|w_n(t)\|_{-\alpha}^2 + \nu\int_0^t \|w_n(s)\|_{L^2}^2\,ds \leq \|x_n - x\|_{-\alpha}\,\mathrm{e}^{c_{62}\int_0^t (\|\theta\|_{L^2}^2 + \|\theta_n\|_{L^2}^2)\,ds},$$

and the right hand side converges to 0, $\mathbb{P}$–a.s., since $x_n \to x$ in $H^{-\alpha}_\#$.

Similar computations also yield irreducibility as in Flandoli and Maslowski (1995) (see also Ferrario (1997, 1999)). □

Lemma 4.7. *Let $\alpha > 1$, $M \geq 1$ and let Assumption 2.2 be in force. Given $\epsilon \in (0,1)$ with $\epsilon < \alpha - 1$, $x \in L^2_\#(\mathbb{T}_2)$, and $t > 0$, there is $c_{63} = c_{63}(\epsilon, t, \|x\|_{L^2})$ such that if θ is the solution of (2.4) with initial condition x, and if $\eta = \theta - z$, where z is the solution of (2.5), then*

$$\mathbb{E}[\|\eta(t)\|_{1+\epsilon}] \leq \frac{c_{63}}{t^{\frac{1+\epsilon}{2\alpha}}}.$$

In particular, $\theta(t)$ is continuous on $\mathbb{T}_2$ for every $t > 0$.

Proof: Fix T and ϵ as in the statement, and consider Galerkin approximations of the equation for η, as in Lemma 4.4. At the level of approximations, we know that $\mathbb{E}[\sup_{[0,T]}\|\eta_N(t)\|_{1+\epsilon}] < \infty$ (for instance by Lemma 4.3), although with a constant that may possibly depend on N. We claim that there is $c_{63} = c_{63}(\epsilon, T, \|x\|_{L^2}) > 0$ (in particular not depending on N) such that

$$\mathbb{E}\Big[\sup_{[0,T]} t^{\frac{1+\epsilon}{2\alpha}}\|\eta_N(t)\|_{1+\epsilon}\Big] \leq c_{63}.$$

Before proving the claim, we show how this proves the lemma. Indeed if $t \in (0,T]$, then by semi–continuity,

$$\begin{aligned}\mathbb{E}[\|\eta(t)\|_{1+\epsilon}] &\leq \liminf_N \mathbb{E}[\|\eta_N(t)\|_{1+\epsilon}] \leq\\ &\leq t^{-\frac{1+\epsilon}{2\alpha}}\liminf_N \mathbb{E}\Big[\sup_{[0,T]} t^{\frac{1+\epsilon}{2\alpha}}\|\eta_N(t)\|_{1+\epsilon}\Big] \leq c_{63} t^{-\frac{1+\epsilon}{2\alpha}}.\end{aligned}$$

We conclude with the proof of the claim above. For simplicity for the rest of the proof we drop the subscript ${}_N$. Set $\mathcal{M} = \sup_{[0,T]} t^{\frac{1+\epsilon}{2\alpha}} \|\eta(t)\|_{1+\epsilon}$. We have that

$$\eta(t) = \mathrm{e}^{-\nu A^\alpha t} x - \int_0^t \mathrm{e}^{-\nu A^\alpha (t-s)} B_M(\theta)\, ds = \boxed{1} + \boxed{2}.$$

For the first term we use standard analytic semigroup bounds, $t^{\frac{1+\epsilon}{2\alpha}} \|\boxed{1}\|_{1+\epsilon} \leq c_{64} \|x\|_{L^2}$. For the second term we use also the inequality (we use here the Sobolev embedding $H^{1+\epsilon} \subset L^\infty$)

$$\|(\nabla^\perp A^{-M}\theta)\cdot\theta\|_{L^2} \leq \|\theta\|_{L^2} \|\nabla^\perp A^{-M}\theta\|_{L^\infty} \leq c_{65} \|\theta\|_{L^2} \|\theta\|_{2-2M+\epsilon} \leq c_{66} \|\theta\|_{L^2} \|\theta\|_\epsilon,$$

to get

$$\|\boxed{2}\|_{1+\epsilon} \leq \int_0^t \|A^{\frac{1}{2}(1+\epsilon)} \mathrm{e}^{-\nu A^\alpha (t-s)} B_M(\theta)\|_{1+\epsilon}\, ds \leq \int_0^t \frac{c_{67}}{(t-s)^{\frac{2+\epsilon}{2\alpha}}} \|\theta\|_{L^2} \|\theta\|_\epsilon.$$

We use interpolation to estimate

$$\|\theta\|_{L^2}\|\theta\|_\epsilon \leq \|\theta\|_{L^2}(\|z\|_\epsilon + \|\eta\|_\epsilon) \leq \|\theta\|_{L^2}\|z\|_\epsilon + \|\theta\|_{L^2} \|\eta\|_{L^2}^{\frac{1}{1+\epsilon}} \|\eta\|_{1+\epsilon}^{\frac{\epsilon}{1+\epsilon}}.$$

Now, choose $p \geq 2$ such that $\frac{2+\epsilon}{2\alpha} + \frac{1}{p} < 1$, and let q be the Hölder conjugate exponent of p, then

$$\begin{aligned}
\int_0^t \frac{\|\theta\|_{L^2}}{(t-s)^{\frac{2+\epsilon}{2\alpha}}} \|\eta\|_{L^2}^{\frac{1}{1+\epsilon}} \|\eta\|_{1+\epsilon}^{\frac{\epsilon}{1+\epsilon}}\, ds &\leq \mathcal{M}^{\frac{\epsilon}{1+\epsilon}} \Big(\int_0^t \frac{ds}{(t-s)^{\frac{2+\epsilon}{2\alpha}q} s^{\frac{\epsilon q}{2\alpha}}}\Big)^{\frac{1}{q}} \Big(\int_0^t \|\theta\|_{L^2}^p \|\eta\|_{L^2}^{\frac{p}{1+\epsilon}}\Big)^{\frac{1}{p}} \\
&\leq c_{68} t^{\frac{1}{q}-\frac{1+\epsilon}{\alpha}} \mathcal{M}^{\frac{\epsilon}{1+\epsilon}} \Big(\int_0^T \|\theta\|_{L^2}^p \|\eta\|_{L^2}^{\frac{p}{1+\epsilon}}\, ds\Big)^{\frac{1}{p}} \\
&= c_{68} t^{\frac{1}{q}-\frac{1+\epsilon}{\alpha}} \mathcal{M}^{\frac{\epsilon}{1+\epsilon}} \mathcal{A}_T^{\frac{1}{p}}.
\end{aligned}$$

with obvious definition of the random variable $\mathcal{A}_T$. Likewise,

$$\int_0^t \frac{ds}{(t-s)^{\frac{2+\epsilon}{2\alpha}}} \|\theta\|_{L^2}\|z\|_\epsilon \leq c_{69} t^{\frac{1}{q}-\frac{2+\epsilon}{2\alpha}} \Big(\int_0^T \|\theta\|_{L^2}^p \|z\|_\epsilon^p\, ds\Big)^{\frac{1}{p}} = c_{69} t^{\frac{1}{q}-\frac{2+\epsilon}{2\alpha}} \mathcal{B}_T^{\frac{1}{p}}.$$

In conclusion the above estimates yield

$$\mathcal{M} \leq c_{70}\big(\|x\|_H + T^{\frac{1}{q}-\frac{1}{2\alpha}} \mathcal{A}_T^{\frac{1}{p}} + T^{\frac{1}{q}-\frac{1+\epsilon}{2\alpha}} \mathcal{B}_T^{\frac{1}{p}} \mathcal{M}^{\frac{\epsilon}{1+\epsilon}}\big).$$

Take expectation of both sides of the above inequality, use the Hölder inequality and Lemma 4.4 to obtain

$$\mathbb{E}[\mathcal{M}] \leq K(1 + \mathbb{E}[\mathcal{M}]^{\frac{2\epsilon}{1+\epsilon}}),$$

where $K = K(q, \epsilon, T, \|x\|_{L^2}) \geq 1$. Recall that at the level of approximations we know that $\mathbb{E}[\mathcal{M}] < \infty$. Let us prove that $\mathbb{E}[\mathcal{M}] \leq (2K)^{\frac{1+\epsilon}{1-\epsilon}}$. Indeed if on the contrary $\mathbb{E}[\mathcal{M}] \geq (2K)^{\frac{1+\epsilon}{1-\epsilon}}$, then $\mathbb{E}[\mathcal{M}] \geq 2K\mathbb{E}[\mathcal{M}]^{\frac{2\epsilon}{1+\epsilon}}$ and so

$$\mathbb{E}[\mathcal{M}] \leq K + K\mathbb{E}[\mathcal{M}]^{\frac{2\epsilon}{1+\epsilon}} \leq K + \frac{1}{2}\mathbb{E}[\mathcal{M}],$$

that is $\mathbb{E}[\mathcal{M}] \leq 2K < (2K)^{\frac{1+\epsilon}{1-\epsilon}}$, a contradiction. □

We have now all elements to prove Theorem 4.2. Since both problems (2.5) and (2.4) satisfy the strong Feller property and have irreducible transition probabilities, the law of each process at some time $t > 0$ and the corresponding invariant measure are equivalent measures. It is then sufficient to show equivalence of the invariant measures. To this end choose $\epsilon \in (0,1)$ such that $\epsilon > 2-\alpha-\delta$ and $\epsilon < \alpha-\delta$ (which is possible since $\alpha > 1$), and let $\theta_\epsilon = A^{-\epsilon/2}\theta$ and $z_\epsilon = A^{-\epsilon/2}z$. The new process θ_ϵ solves

$$d\theta_\epsilon + \nu A^\alpha \theta_\epsilon + B_{M,\epsilon}(\theta_\epsilon) = \mathcal{C}_\epsilon^{\frac{1}{2}}\, dW, \tag{4.12}$$

where $B_{M,\epsilon}(x) = A^{-\epsilon/2}B_M(A^{\epsilon/2}x)$ and $\mathcal{C}_\epsilon = A^{-\epsilon}\mathcal{C}$, and similarly for z_ϵ. We will deduce equivalence of the invariant measures of θ_ϵ and z_ϵ by Da Prato and Debussche (2004, Theorem 3.4) (and an additional argument) applied to θ_ϵ and z_ϵ.

Proof of Theorem 4.2: We briefly summarize the several assumptions of Theorem 3.4 of Da Prato and Debussche (2004). As it concerns the linear problem, it is required that (see Da Prato and Debussche (2004, Hypothesis 2.1))

- $\mathcal{C}_{\epsilon,\infty}$ is trace class, and $\mathrm{e}^{-tA^\alpha}\, L^2_\# \subset \mathcal{C}_{\epsilon,t}^{1/2} L^2_\#$,
- $\|\Lambda_t\|$ has Laplace transform in $(-1,\infty)$,

where $\Lambda_t = \mathcal{C}_{\epsilon,t}^{-1/2}\,\mathrm{e}^{-tA^\alpha}$ and

$$\mathcal{C}_{\epsilon,t} = \int_0^t \mathrm{e}^{-sA^\alpha}\,\mathcal{C}_\epsilon\,\mathrm{e}^{-s(A^\alpha)^*}\, ds, \qquad 0 < t \leq \infty.$$

For the nonlinear problem it is required that

- there is a unique solution with initial condition in $L^2_\#$,
- the transition semigroup is Feller and admits a strongly mixing invariant measure μ_ϵ,
- there are a sequence $(B_{M,\epsilon,N})_{N\geq 1}$ of Lipschitz continuous maps on $L^2_\#$ and $g \in L^2(L^2_\#, \mu_\epsilon)$ such that $\|B_{M,\epsilon,N}(x)\|_{L^2} \leq g(x)$, μ_ϵ –a. s., and $B_{M,\epsilon,N}(x) \to B_{M,\epsilon}(x)$ μ_ϵ–a. s.,
- if $\theta_{\epsilon,N}$ is the solution of (4.12) with $B_{M,\epsilon}$ replaced by $B_{M,\epsilon,N}$, then for μ_ϵ–a. e. x and for all $t > 0$, $\theta_{\epsilon,N}(t;x) \to \theta_\epsilon(t;x)$ $\mathbb{P}$–a. s. in $L^2_\#$,
- for every $\lambda > 0$ and μ_ϵ–a. e. x,

$$\int_0^\infty \mathrm{e}^{-\lambda t}\,\|B_{M,\epsilon,N}(\theta_{\epsilon,N}(t;x)) - B_{M,\epsilon,N}(\theta_\epsilon(t;x))\|_{L^2}\, ds \longrightarrow 0, \qquad n \to \infty.$$

If the above assumptions hold, Theorem 3.4 of Da Prato and Debussche (2004) ensures that the invariant measure of θ_ϵ is absolutely continuous with respect to the invariant measure of z_ϵ (and thus the same statement hold for the invariant measures of θ and z, as well as for the laws at each time by the strong Feller property).

Before checking that the assumptions of the aforementioned theorem are satisfied, we show how to deduce the equivalence of measures (notice that this statement is not given in Da Prato and Debussche (2004)). Indeed, the theorem in Da Prato and Debussche (2004) shows the following formula

$$(\lambda - L_\epsilon)^{-1} = (\lambda - N_\epsilon)^{-1} - (\lambda - N_\epsilon)^{-1}\langle B_{M,\epsilon}, D(\lambda - L_\epsilon)^{-1}\rangle, \qquad \lambda > 0,$$

where L_ϵ and N_ϵ are the generators of the Markov semigroups associated to z_ϵ and θ_ϵ, respectively. To prove equivalence it is sufficient to show that for every bounded measurable function f on $L^2_\#$, $(\lambda - L_\epsilon)^{-1}f = 0$ if and only if $(\lambda - N_\epsilon)^{-1}f = 0$. If $(\lambda - L_\epsilon)^{-1}f = 0$, the conclusion is immediate (it is already in Da Prato and Debussche (2004)). Assume that $(\lambda - N_\epsilon)^{-1}f = 0$ and set $\phi_\lambda = (\lambda - L_\epsilon)^{-1}f = 0$. By the formula above,

$$\phi_\lambda + (\lambda - N_\epsilon)^{-1}\langle B_{M,\epsilon}, D\phi_\lambda\rangle = 0.$$

Clearly, $\phi_\lambda \in D(L_\epsilon)$ and, by Proposition 2.4 of Da Prato and Debussche (2004), $\phi_\lambda \in C_b^1$. Therefore $\phi_\lambda \in D(N_\epsilon)$ and $(\lambda - N_\epsilon)\phi_\lambda = (\lambda - L_\epsilon)\phi_\lambda - \langle B_{M,\epsilon}, D\phi_\lambda\rangle$. By the formula above it follows that $(\lambda - L_\epsilon)\phi_\lambda = 0$, that is $\phi_\lambda = 0$. This proves the equivalence.

We turn to the verification of the assumptions of Theorem 3.4 of Da Prato and Debussche (2004). The assumptions for the linear problem are standard and can be verified as in Section 4 of the mentioned paper. We only give a few details on the last hypothesis, namely that $\|\Lambda_t\|$ admits Laplace transform defined on $(-1,\infty)$. By our assumptions each e_k (the sine–cosine orthonormal basis) is an eigenvector of Λ_t. Denote by $\lambda_{t,k}$ the corresponding eigenvalue. Elementary computations yield

$$\lambda_{t,k} = \Big(|\sigma_k|^2 \frac{1-\mathrm{e}^{-2t|k|^{2\alpha}}}{2|k|^{2(\alpha+\epsilon)}}\Big)^{-\frac{1}{2}} \mathrm{e}^{-t|k|^{2\alpha}} \sim \frac{c_{71}}{t^{\frac{\alpha+\delta+\epsilon}{2\alpha}}}$$

for t small. Existence of the Laplace transform follows, since $\|\Lambda_t\| = \sup_k \lambda_{t,k}$ and $\delta + \epsilon < 2 - \alpha \leq \alpha$.

As it regards the assumptions for the non–linear problem, Lemma 4.3 and Lemma 4.6 ensure that problem (4.12) has a unique solution and generates a Feller semigroup in $L^2_\#$. Moreover the semi–group has a unique invariant measure μ_ϵ which is strongly mixing. Let

$$B_{M,\epsilon,N}(x) = \frac{N}{N + \|x\|_{L^2}} \pi_N B_{M,\epsilon}(\pi_N x),$$

where π_N is the projection onto span$[e_k : |k| \leq N]$, then $B_{M,\epsilon,N}$ is Lipschitz–continuous in $L^2_\#$. Moreover, $\theta_{\epsilon,N}(t) \to \theta_\epsilon(t)$ for all $t > 0$ and all $x \in L^2_\#$, whenever $\theta_{\epsilon,N}(0) = \theta_\epsilon(0)$. Let us prove the main assumption of Da Prato and Debussche (2004, Theorem 3.4), namely that $B_{M,\epsilon,N}(x) \to B_{M,\epsilon}(x)$, μ–a.s. in $L^2_\#$, and that there is a $g \in L^2(L^2_\#, \mu)$ such that $\|B_{M,\epsilon,N}(x)\|_{L^2} \leq g(x)$, μ–a.s.. Let $\epsilon_M = 0$ if $M > 1$, and an arbitrary value in $\big(0, \epsilon - (2 - \delta - \alpha)\big)$ if $M = 1$, and let $g_\epsilon = c_{72}\|\cdot\|_{\epsilon+\epsilon_M}\|\cdot\|_1$. Lemma 4.4 and interpolation immediately imply that $g_\epsilon \in L^2(L^2_\#, \mu_\epsilon)$ since,

$$\mathbb{E}^{\mu_\epsilon}[g_\epsilon^2(x)] \leq c_{72}\mathbb{E}^{\mu}[\|x\|^2_{L^2}\|x\|^2_{1-\epsilon+\epsilon_M}].$$

Moreover, by choosing c_{72} large enough, we have that $\|B_{M,\epsilon,N}(x)\|_{L^2} \leq g_\epsilon(x)$. Indeed, by using the embedding of L^∞ into $H^{1+\gamma}$, for $\gamma \leq \epsilon_M \vee 2(M-1)$, we have that

$$\|B_M(x)\|_{L^2} \leq c_{73}\|x\|_{\epsilon_M}\|x\|_1, \qquad \|B_M(x)\|_{-1} \leq c_{74}\|x\|_{\epsilon_M}\|x\|_{L^2},$$

and hence $\|B_M(x)\|_{-\epsilon} \leq c_{75}\|x\|_{\epsilon_M}\|x\|_{1-\epsilon}$.

Finally, the last assumption of Da Prato and Debussche (2004, Theorem 3.4) follows from the convergence of $\theta_{\epsilon,N}$ to θ_ϵ in $L^2_\#$ (which in turns gives convergence in stronger norms using the same methods of Lemma 4.7), and the bounds in Corollary 4.5, that give uniform integrability. □

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