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Monotonicity in first-passage percolation

Jean-Baptiste Gouéré

Université d'Orléans, MAPMO, B.P. 6759, 45067 Orléans Cedex 2, France
E-mail address: jbgouere@univ-orleans.fr
URL: http://www.univ-orleans.fr/mapmo/membres/gouere/

Abstract. We consider standard first-passage percolation on $\mathbb{Z}^d$. Let e_1 be the first coordinate vector. Let $a(n)$ be the expected passage time from the origin to ne_1. In this short paper, we note that $a(n)$ is increasing under some strong condition on the support of the distribution of the passage times on the edges.

1. Introduction and results

First passage percolation. We consider the graph $\mathbb{Z}^d$, $d \geq 2$, obtained by taking $\mathbb{Z}^d$ as vertex set and by puting an edge between two vertices if the Euclidean distance between them is 1. We consider a family of non-negative i.i.d. random variables $\tau = (\tau(e))_{e\in\mathcal{E}}$ indexed by the set of edges $\mathcal{E}$ of the graph. We interpret $\tau(e)$ as the time needed to travel along the edge e. (The graph is unoriented.)

If a and b are two vertices of $\mathbb{Z}^d$, we call path from a to b any finite sequence of vertices $r = (a = x_0, ..., x_k = b)$ such that, for all $i \in \{0, ..., k-1\}$, the vertices x_i et x_{i+1} are linked by an edge. We denote by $\mathcal{C}(a, b)$ the set of such paths. The time needed to travel along a path $r = (x_0, ..., x_k)$ is defined by:

$$\tau(r) = \sum_{i=0}^{k-1} \tau(x_i, x_{i+1}).$$

Then, the time needed to go from a to b is defined by:

$$T(a, b) = \inf\{\tau(r) : r \in \mathcal{C}(a, b)\}.$$

Let $e_1, \ldots, e_d$ denote the canonical basis vectors of $\mathbb{R}^d$. We are interested in the sequence $(a(n))_{n\geq 0}$ defined by:

$$a(n) = E(T(0, ne_1)).$$

We write $T'(0, ne_1)$ and $a'(n)$ for the passage times and expected passage times obtained when the paths are restricted to $\{(x_1, \ldots, x_d) : 0 \leq x_1 \leq n\}$.

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Main result and related results. We denote by S_- the infimum of the support of the distribution of the $\tau(e)$. We denote by S_+ the supremum of the support.

Theorem 1.1. *Assume $0 < S_-$ and $S_+ \leq 2S_-$. Then the sequence (a_n) is non-decreasing. More precisely, we have:*

$$a(n) \geq a(n-1) + S_- \left[1 - \frac{(S_+ - S_-)^2}{{S_-}^2}\right].$$

We can prove monotonicity under some slightly different sets of assumptions. See further remarks below.

As soon as the distribution of the $\tau(e)$ is not a Dirac distribution there exists, with probability one, infinitely many random N such that [1] $T(0,(N-1)e_1) > T(0,Ne_1)$. However, monotonicity of expected passage times seems quite natural and was already conjectured by Hammersley and Welsh in Hammersley and Welsh (1965). In Alm and Wierman (1999), Alm and Wierman proved the monotonicity for $\mathbb{Z} \times \mathbb{N}$ and other 2 dimensional models. In Ahlberg (2014+), Ahlberg made a detailed study of first passage percolation on essentially one-dimensional graphs, of which $\mathbb{Z} \times \{-K,\dots,K\}^{d-1}$ is an example. In particular, he proved the existence of a constant n_0, depending on the graph, such that $n \geq n_0$ implies $a(n) \geq a(n-1)$. When the time constant is positive, both of the previous arguments in Alm and Wierman (1999) and Ahlberg (2014+) prove strict monotonicity of the sequence $(a(n))$. In Howard (2001), Howard proved the monotocity for an Euclidean first-passage percolation model. We are not aware of any other positive results.

On the other hand, van den Berg proved in van den Berg (1983) that, when $d = 2$, one has $a'(2) < a'(1)$ when $\tau(e) = 1$ with small probability and $\tau(e) = 0$ otherwise. Note that we still have $a'(2) < a'(1)$ if, instead of setting $\tau(e) = 0$ we set $\tau(e) = \varepsilon$ for a small enough [2] ε. A related result was given by Joshi in Joshi (1977).

We refer to the review by Howard (2004) for a more detailed account.

Further remarks.

- The same result holds for the $a'(n)$.
- The proof gives that $T(0,ne_1)$ stochastically dominates the mean of n dependent copies of $T(0,(n-1)e_1)$ (see (2.7) and (2.1)).
- With the same strategy one can prove for example the following result:

$$a(n) \geq a(n-1) \text{ as soon as } S_- > 0 \text{ and } \left(\frac{a(n)n^{-1}}{S_-} - 1\right)\left(\frac{E(\tau(e))}{S_-} - 1\right) \leq \frac{1}{2} \tag{1.1}$$

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[1] Let us sketch a proof.

Fix a and b such that $S_- < a < b < S_+$. For each n, consider a box $\{n-C,\dots,n\} \times \{-D,\dots,D\}^{d-1}$. Let A_n be the following event: $\tau(e) \leq a$ for edges along the boundary of the box and $\tau(e) \geq b$ for edges inside the box. For suitably chosen large C and D and for $n > C$, we have $T(0,(n-1)e_1) > T(0,ne_1)$ as soon as A_n occurs. As the A_n are local event of fixed positive probability, the result follows.

[2] Indeed, $a'(1)$ can only increase while $a'(2)$ increases by at most $\varepsilon E(N)$ where N is the length of a geodesic for the initial passage times. Using $T'(0,2e_1) \leq 2$ one can check that any geodesic must remain in a random box of subgeometrical height. Therefore $E(N)$ is finite and the result follows.

where e is a fixed edge. We show how to adapt the proof of Theorem 1.1 to prove this result below the proof of Theorem 1.1. In particular, using the inequality $a(n) \leq nE(\tau(e))$, we get that a is non-decreasing as soon as :

$$S_- > 0 \text{ and } E(\tau(e)) \leq (1 + 2^{-1/2})S_-.$$

This gives a sufficient condition with no assumption on S_+ which can be infinite. However, this sufficient condition is still strong and we do not see how to give any significantly weaker condition.

- Fix the distribution of $\tau(e)$. Assume $S_- > 0$ and $E\tau(e) < \infty$. Then the conditions in (1.1) are true for large enough n and d. This is due to the fact that $a(n)n^{-1}$ can be made arbitrarily close to S_-.

2. Proofs

2.1. *Proof of Theorem 1.1.* For all i we consider the following sets of edges:

- H^i: the set of edges $(x, x + e_1)$ where $x = (x_1, \dots, x_d)$ is such that $x_1 = i$.
- V^i: the set of edges $(x, x + e_k)$ where $x_1 = i$ and k belongs to $\{2, \dots, d\}$.

We define new passage times $\tau^i(e)$ as follows:

- If e belongs to H^i then $\tau^i(e) = 0$.
- If e belongs to V^i then $\tau^i(e) = +\infty$.
- Otherwise, $\tau^i(e) = \tau(e)$.

We denote by $\tau^i(r)$ the time needed to travel along a path r with the passage times $\tau^i(e)$. We denote by $T^i(a, b)$ the time needed to travel from a to b with the passage times $\tau^i(e)$. Note, for all $n \geq 1$ and all $i \in \{0, n-1\}$, the following:

$$T^i(0, ne_1) \text{ and } T(0, (n-1)e_1) \text{ have the same distribution.} \tag{2.1}$$

We now compare $T^i(0, ne_1)$ and $T(0, ne_1)$. Let π be a path from 0 to ne_1 such that $\tau(\pi) = T(0, ne_1)$. (The existence of such a path is an easy consequence of the fact that the passage times on edges are finite and bounded from below.) We modify this path as follows. Each time the path goes, in this order, through an edge $(x, y) \in V^i$, we replace this part of the path by $(x, x+e_1, y+e_1, y)$. We denote by π^i the modified path. We have

$$\tau^i(\pi^i) \leq \tau(\pi) - S_-\mathrm{card}(\pi \cap H^i) + (S_+ - S_-)\mathrm{card}(\pi \cap V^i)$$

where, for example, $\mathrm{card}(\pi \cap H^i)$ denotes the number of edges of H^i used by π. The term involving H^i is due to the time saved by the modification of the passage times. The term involving V^i is partly due to the time left by the modification of the path. We thus get

$$T^i(0, ne_1) \leq T(0, ne_1) - S_-\mathrm{card}(\pi \cap H^i) + (S_+ - S_-)\mathrm{card}(\pi \cap V^i) \tag{2.2}$$

and then

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap H^i) + (S_+ - S_-) \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap V^i). \tag{2.3}$$

Note

$$\sum_{i=0}^{n-1} \mathrm{card}(\pi \cap H^i) \geq n, \tag{2.4}$$

as π is a path from 0 to ne_1. But

$$\begin{aligned} T(0,ne_1) &= \tau(\pi) \\ &\geq S_- \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap V^i) + S_- \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap H^i) \\ &\geq S_- \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap V^i) + S_- n \qquad (2.5) \end{aligned}$$

and, moreover,

$$\begin{aligned} T(0,ne_1) &\leq \tau(0,e_1,2e_2,\dots,ne_1) \\ &\leq nS_+. \end{aligned}$$

Therefore:

$$\begin{aligned} \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap V^i) &\leq \frac{T(0,ne_1) - nS_-}{S_-} \qquad (2.6) \\ &\leq \frac{nS_+ - nS_-}{S_-}. \end{aligned}$$

From (2.3) and (2.6) we get:

$$\sum_{i=0}^{n-1} T^i(0,ne_1) \leq nT(0,ne_1) - nS_- + \frac{n(S_+ - S_-)^2}{S_-}. \qquad (2.7)$$

Taking expectations and using (2.1) we get:

$$na(n-1) \leq na(n) - nS_- \left[1 - \frac{(S_+ - S_-)^2}{{S_-}^2}\right].$$

The proof follows. □

2.2. *Proof of* (1.1). The proof is essentially the same. The main difference lies in the definition of the new passage times $\tau^i(e)$. We let $\widetilde{\tau}$ be an independent copy of τ. We then set:

- If e belongs to H^i then $\tau^i(e) = 0$.
- If e belongs to V^i then $\tau^i(e) = +\infty$.
- If e belongs to V^{i+1} then $\tau^i(e) = \widetilde{\tau}^i(e)$.
- Otherwise, $\tau^i(e) = \tau(e)$.

Instead of (2.2) we can write, after taking conditional expectation w.r.t. τ :

$$\begin{aligned} T^i(0,ne_1) \leq & T(0,ne_1) - S_- \mathrm{card}(\pi \cap H^i) + (E(\tau(e)) - S_-)\mathrm{card}(\pi \cap V^i) \\ & + (E(\tau(e)) - S_-)\mathrm{card}(\pi \cap V^{i+1}). \end{aligned} \qquad (2.8)$$

For example, the term $(E(\tau(e)) - S_-)\mathrm{card}(\pi \cap V^{i+1})$ is due to the fact that for each edge $e \in \pi \cap V^{i+1}$:

- We save $\tau(e)$ and then at least S_-.
- We lose $\tau^i(e)$ which is independent of π.

Instead of (2.3) we can get :

$$\sum_{i=0}^{n-1} T^i(0, ne_1) \leq nT(0, ne_1) - S_- \sum_{i=0}^{n-1} \mathrm{card}(\pi \cap H^i) + 2(E(\tau(e)) - S_-) \sum_{i=1}^{n} \mathrm{card}(\pi \cap V^i). \tag{2.9}$$

Using (2.4), an equality similar to (2.5) and taking expectation, we get:

$$na(n-1) \leq na(n) - S_- n + 2(E(\tau(e)) - S_-) \frac{a(n) - nS_-}{S_-}$$

and thus:

$$a(n-1) \leq a(n) - S_- \left(1 - 2\left(\frac{E(\tau(e))}{S_-} - 1\right)\left(\frac{a(n)n^{-1}}{S_-} - 1\right)\right).$$

The proof follows. □

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