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Uniform spanning trees on Sierpiński graphs

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Abstract. We study spanning trees on Sierpiński graphs (i.e., finite approximations to the Sierpiński gasket) that are chosen uniformly at random. We construct a joint probability space for uniform spanning trees on every finite Sierpiński graph and show that this construction gives rise to a multi-type Galton-Watson tree. We derive a number of structural results, for instance on the degree distribution. The connection between uniform spanning trees and loop-erased random walk is then exploited to prove convergence of the latter to a continuous stochastic process. Some geometric properties of this limit process, such as the Hausdorff dimension, are investigated as well. The method is also applicable to other self-similar graphs with a sufficient degree of symmetry.

1. Introduction

The Sierpiński gasket is certainly one of the most famous fractals, and the Sierpiński graphs, which can be seen as finite approximations of the Sierpiński

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gasket, are among the most thoroughly studied self-similar graphs. The number of spanning trees in the n-th Sierpiński graph G_n (starting with a single triangle G_0, see Figure 1.1) turns out to be given by the remarkable explicit formula

$$\tau(G_n) = \left(\frac{3}{20}\right)^{1/4} \cdot \left(\frac{3}{5}\right)^{n/2} \cdot 540^{3^n/4},$$

which was obtained by different methods in several recent works: by setting up and solving a system of recursions (Teufl and Wagner, 2006, 2011a, Chang et al., 2007), or by electrical network theory (Teufl and Wagner, 2011b). In the book of Lawler and Limic (2010), a proof using probabilistic results is sketched. Moreover, the Laplacian spectrum of G_n can be described rather explicitly by means of a technique known as "spectral decimation" (Shima, 1991, Fukushima and Shima, 1992), from which another proof can be derived, see Anema (2012).

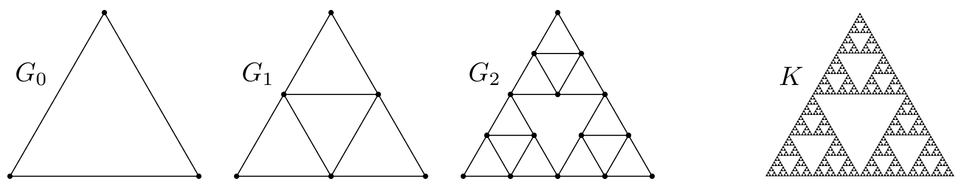


Figure 1.1. Sierpiński graphs G_0, G_1, G_2, and the Sierpiński gasket K.

Once the counting problem is solved, it is natural to consider *uniformly random spanning trees* of G_n and to study their structure. Uniform spanning trees are known to have strong connections to other probabilistic models, such as loop-erased random walk (Wilson's celebrated algorithm to construct uniform spanning trees being a particular application, see Wilson, 1996, Lawler and Limic, 2010), and they are also of interest in mathematical physics. For this reason, the structure of uniformly random spanning trees in other important families of graphs such as square grids has been studied thoroughly, see Burton and Pemantle (1993).

The recursive nature of Sierpiński graphs and the strong symmetry enables us to derive a number of results on uniform spanning trees, as will be shown in this paper. After some preliminaries, we construct a joint probability space for uniform spanning trees on every finite Sierpiński graph. An important tool in this context is the theory of (a rather general kind of) Galton-Watson processes. Making use of this tool, we also prove some structural results on uniform spanning trees of G_n, for instance a strong law of large numbers for the degree distribution of a uniform spanning tree. This extends the work of Chang and Chen (2010), who prove convergence of expected values (for which they also give explicit formulae). Similar results for the two-dimensional square lattice were obtained by Manna et al. (1992).

Loop-erased random walk on the Sierpiński gasket was studied in the paper of Hattori and Mizuno (2014); our results on uniform spanning trees provide an alternative approach to this topic and were obtained independently of Hattori and Mizuno and approximately at the same time (see for instance Teufl, 2011). The expected length of such a walk from one corner to another was studied earlier in the physics literature by Dhar and Dhar (1997); it grows asymptotically like $\left(\frac{4}{3} + \frac{1}{15}\sqrt{205}\right)^n$. As it was also shown by Hattori and Mizuno, we find that, upon renormalization, loop-erased random walk converges to a limit process. The analogue of this process for the square lattice is the celebrated Schramm-Loewner evolution (see Schramm, 2000, Lawler et al., 2004), whose analysis is notoriously complicated. However, the different geometry of the Sierpiński graphs makes it possible to prove rather strong theorems on the shape of this limiting process comparatively easily, including parameters such as the *Hausdorff dimension*. Similar results on the limit process of the self-avoiding walk were obtained by Hattori et al. (1990, 1993), Hattori and Hattori (1991), Hattori and Kusuoka (1992) and by Hambly et al. (2002) for the self-repelling walk.

In Section 8, we study the metric induced by a random spanning tree on the Sierpiński graph G_n. We prove almost sure convergence to a limit metric, and show that the resulting metric space is a so-called $\mathbb{R}$-tree. We also study the *interface*, which is, loosely speaking, the set where different branches of a spanning tree embedded in the plane "touch", and estimate its Hausdorff dimension.

In the following list, the main results of this paper are summarised. For the sake of simplicity, all results and their derivation are only given for the Sierpiński gasket, but there are other fractals to which the same approach applies, see Section 9.

- We construct a joint probability space for uniform spanning trees on every finite Sierpiński graph using a projective limit. As part of the construction, we also have to consider spanning forests with the property that each of the components contains one of the three corner vertices. We show that the distribution of the component sizes in random spanning forests of this type converges (upon renormalization) to a limiting distribution—see Section 5.1.
- We prove almost sure convergence of the *degree distribution* (see Section 5.2): the proportion of vertices of degree i ($i \in \{1, 2, 3, 4\}$ fixed) in a random spanning tree of G_n converges almost surely to a limit constant $w(i)$.
- Section 6 is concerned with loop-erased random walk on Sierpiński graphs G_n: using the connection between spanning trees and loop-erased random walk, we recover the aforementioned result that the length of such a walk from one corner to another grows asymptotically like $\left(\frac{4}{3} + \frac{1}{15}\sqrt{205}\,\right)^n$, and that the renormalized length has a limit distribution (cf. Hattori and Mizuno, 2014, Theorem 5). We also provide tail estimates for this limit distribution, see Lemma 6.5.
- In Section 7, we study the limit process and prove some geometric properties: specifically, we show that the limit curve is almost surely self-avoiding (Theorem 7.6), and has Hausdorff dimension $\log_2\left(\frac{4}{3} + \frac{1}{15}\sqrt{205}\,\right) \approx 1.193995$ (Theorem 7.10, (5)). These results were also obtained in the aforementioned paper of Hattori and Mizuno (2014, Theorems 9 and 10). Moreover, we prove Hölder continuity with an explicit exponent (Theorem 7.10, (4)).
- The limit of the tree metric is the main topic of Section 8. It is shown (Theorem 8.1) that we almost surely obtain a random metric on the "rational points" (i.e., all points which are vertices in some finite approximation) of the Sierpinski gasket whose Cauchy completion is an $\mathbb{R}$-tree, i.e., a metric space in which there is a unique arc between any two points and this arc is geodesic (that is an isometric embedding of a real interval).

2. Notation and Preliminaries

A *graph* G is a pair (VG, EG), where VG is the vertex set and

$$EG \subseteq \big\{\{x,y\} : x,y \in VG, x \neq y\big\}$$

is the edge set. Two vertices x, y are *adjacent* if $\{x,y\} \in EG$. The *degree* $\deg x$ of a vertex x is the number of adjacent vertices. A *walk* in G is a finite or infinite sequence $(x_0, x_1, \dots)$ of vertices in G, such that consecutive entries are adjacent. A walk is called *self-avoiding* if its entries are mutually distinct. The edge set $E(x)$ of a walk $x = (x_0, x_1, \dots)$ is the set

$$E(x) = \big\{\{x_0,x_1\},\{x_1,x_2\},\dots\big\}.$$

Equipped with the edge set $E(x)$ a walk x gives rise to a subgraph of G. The *length* of the walk $(x_0, \dots, x_n)$ is equal to n, the number of edges. The *distance* $d_G(v,w)$ of two vertices v, w is the least integer n such that there is a walk of length n in G connecting v and w.

A *tree* is a connected graph without cycles. A *spanning tree* of a graph G is a subgraph of G which is a tree and contains all vertices of G. Similarly, a *forest* is a graph without cycles and a *spanning forest* of a graph G is a subgraph of G which is a forest and contains all vertices of G. Let F be a forest and v, w be two vertices in F. If v, w are in the same component of F, then we write vFw to denote the unique self-avoiding walk in F connecting v and w.

Next we need some ingredients from probability theory. We use multi-index notation: If $r \in \mathbb{N}$, $\boldsymbol{z} = (z_1, \dots, z_r)$, and $\boldsymbol{k} = (k_1, \dots, k_r)$, then $\boldsymbol{z}^{\boldsymbol{k}} = z_1^{k_1} \cdots z_r^{k_r}$. If $\boldsymbol{X} = (X_1, \dots, X_r)$ is a random vector in $\mathbb{N}_0^r$, then

$$\mathsf{PGF}(\boldsymbol{X}, \boldsymbol{z}) = \mathbb{E}\big(\boldsymbol{z}^{\boldsymbol{X}}\big) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^r} \mathbb{P}(\boldsymbol{X} = \boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}$$

is the (multivariate) *probability generating function* of $\boldsymbol{X}$.

An *r-type Galton-Watson process* $(\boldsymbol{X}_n)_{n\geq 0} = (X_{1,n}, \dots, X_{r,n})_{n \geq 0}$ is a stochastic process that starts with one or more individuals, each of which has a type associated to it. Each individual gives birth to zero or more children according to the offspring probabilities

$$\boldsymbol{p}(\boldsymbol{k}) = \big(p_1(\boldsymbol{k}), \dots, p_r(\boldsymbol{k})\big)$$

for $\boldsymbol{k} = (k_1, \dots, k_r) \in \mathbb{N}_0^r$. Here $p_i(\boldsymbol{k})$ is the probability that an individual of type i has k_j children of type j for $j \in \{1, \dots, r\}$. The vector $\boldsymbol{X}_n$ represents the number of individuals in the n-th generation by their type (i.e., $X_{i,n}$ is the number of individuals of type i in the n-th generation). It is convenient to describe the offspring distributions by their multidimensional multivariate probability generating function $\boldsymbol{f} = (f_1, \dots, f_r)$, which is called *offspring generating function* and given by

$$\boldsymbol{f}(\boldsymbol{z}) = \sum_{\boldsymbol{k} \in \mathbb{N}_0^r} \boldsymbol{p}(\boldsymbol{k}) \boldsymbol{z}^{\boldsymbol{k}}$$

for $\boldsymbol{z} = (z_1, \dots, z_r)$. Then

$$\mathsf{PGF}(\boldsymbol{X}_n, \boldsymbol{z}) = \mathsf{PGF}(\boldsymbol{X}_{n-1}, \boldsymbol{f}(\boldsymbol{z})) = \cdots = \mathsf{PGF}(\boldsymbol{X}_0, \boldsymbol{f}^n(\boldsymbol{z})),$$

where $\boldsymbol{f}^n$ is the n-fold iteration of $\boldsymbol{f}$. The *mean matrix* $\boldsymbol{M} = (m_{ij})_{1 \leq i,j \leq r}$ is given by $m_{ij} = (\partial f_i / \partial z_j)(\boldsymbol{1})$. The process is called

- *positively regular* if $\boldsymbol{M}$ is primitive (i.e., all entries of $\boldsymbol{M}^k$ are positive for some k),
- *singular* if $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{M}\boldsymbol{z}$,
- *subcritical*, *critical* or *supercritical* depending on whether the largest eigenvalue of the mean matrix is less than, equal to, or greater than 1.

see for instance Mode (1971) for these notions and the theory of multi-type Galton-Watson processes.

Let $\mathbb{W}$ be a subset of $\mathbb{N}$ (in the following $\mathbb{W}$ will always be $\{1,2,3\}$); elements of $\mathbb{W}^n$ are written as words over the alphabet $\mathbb{W}$, e.g. 12133 means $(1,2,1,3,3)$. Let $\mathbb{W}^0 = \{\varnothing\}$ consist of the empty word $\varnothing$ only and set

$$\mathbb{W}^* = \biguplus_{n\geq 0} \mathbb{W}^n.$$

Concatenation of two words $v, w \in \mathbb{W}^*$ is written by juxtaposition vw. The set $\mathbb{W}^*$ carries a graph structure in a canonical way: two words $v, w \in \mathbb{W}^*$ are adjacent if $w = v\iota$ or $v = w\iota$ for some $\iota \in \mathbb{W}$. This turns $\mathbb{W}^*$ into a tree with root $\varnothing$. If $w = v\iota$, then w is called *child* or *offspring* of v and ι is the *suffix* of w.

Let R be a finite set and fix some $\mathbb{W} \subseteq \mathbb{N}$. Consider the set of all subtrees with an element of R associated to each vertex, i.e.,

$$\mathcal{W}_R = \big\{(W,f) \,:\, W \subseteq \mathbb{W}^* \text{ induces a subtree}, \varnothing \in W, f = (f_w)_{w\in W} \in R^W\big\}.$$

If $(W,f) \in \mathcal{W}_R$ and $w \in W$, then we say that the word w is the *label* and f_w is the *type* of the pair (w, f_w). A *labelled multi-type Galton-Watson tree* with labels in $\mathbb{W}^*$ and types in R is a random element of the set $\mathcal{W}_R$, whose distribution is uniquely determined by the following:

- The type of the root (or *ancestor*) $\varnothing$ is given by a fixed distribution on R.
- The random offspring generation of a vertex (or *individual*) w only depends on the type of w. It is given by a probability distribution (depending on the type of w) on the set

  $$\big\{(S,g) \,:\, S \subseteq \mathbb{W}, g = (g_\iota)_{\iota\in S} \in R^S\big\}.$$

  The interpretation is that once a pair (S,g) is chosen, the individual w gives birth to $|S|$ children with labels $w\iota$ for $\iota \in S$, and type g_ι is assigned to child $w\iota$.

A labelled multi-type Galton-Watson tree is denoted by $(F_w)_{w\in W} \in \mathcal{W}_R$. Notice that in this notation $W \subseteq \mathbb{W}^*$ is the random set of individuals and F_w is the random type of an individual $w \in W$.

To every labelled multi-type Galton-Watson tree $(F_w)_{w\in W}$ with labels in $\mathbb{W}^*$ and types in R, the (random) number of individuals of a certain type in the n-th generation yields a multi-type Galton-Watson process with $r = |R|$ types. To this end, let $a_1, \ldots, a_r$ be the elements of R and set

$$X_{i,n} = |\{w \in W \cap \mathbb{W}^n \,:\, F_w = a_i\}|, \qquad \boldsymbol{X}_n = (X_{1,n}, \ldots, X_{r,n})$$

for $n \geq 0$ and $i \in \{1, \ldots, r\}$. Then $(\boldsymbol{X}_n)_{n\geq 0}$ is an r-type Galton-Watson process.

3. Construction of Sierpiński graphs

The Sierpiński gasket K (see Sierpiński (1915) for its origin in mathematical literature) can be defined formally by means of the following three similitudes:

$$\psi_i(x) = \tfrac{1}{2}(x - u_i) + u_i$$

for $i \in \{1, 2, 3\}$, where $u_1 = (0, 0)$, $u_2 = (1, 0)$, and $u_3 = \frac{1}{2}(1, \sqrt{3})$. Then K is the unique non-empty compact set such that

$$K = \psi_1(K) \cup \psi_2(K) \cup \psi_3(K).$$

Its Hausdorff dimension is given by

$$\dim_H K = \frac{\log 3}{\log 2} = 1.5849625\ldots$$

The Sierpiński graphs $G_0, G_1, \ldots$ are discrete approximations to K and are constructed inductively: The vertex set VG_0 and edge set EG_0 of G_0 are given by

$$VG_0 = \{u_1, u_2, u_3\} \qquad \text{and} \qquad EG_0 = \{\{u_1, u_2\}, \{u_2, u_3\}, \{u_3, u_1\}\},$$

respectively. Then, for any $n \geq 0$, the sets VG_{n+1} and EG_{n+1} are defined as follows:

$$VG_{n+1} = \psi_1(VG_n) \cup \psi_2(VG_n) \cup \psi_3(VG_n),$$
$$EG_{n+1} = \psi_1(EG_n) \cup \psi_2(EG_n) \cup \psi_3(EG_n).$$

Notice that G_{n+1} is an amalgam of three scaled images of G_n, which we denote by $\psi_1(G_n)$, $\psi_2(G_n)$, and $\psi_3(G_n)$, i.e.,

$$G_{n+1} = \psi_1(G_n) \cup \psi_2(G_n) \cup \psi_3(G_n).$$

The vertices in $VG_0 \subset VG_n$ are often called *corner vertices* or *boundary vertices* of the graph G_n. The vertex sets are nested, i.e., $VG_0 \subset VG_1 \subset VG_2 \subset \cdots$, and the Sierpiński gasket K is the closure of the union $VG_0 \cup VG_1 \cup VG_2 \cdots$. Figure 1.1 shows the graphs G_0, G_1, G_2 and the Sierpiński gasket K. The self-similar nature and the fact that the three scaled images only intersect in the three points

$$\tfrac{1}{2}(u_2 + u_3), \qquad \tfrac{1}{2}(u_3 + u_1), \qquad \tfrac{1}{2}(u_1 + u_2),$$

allows to solve many problems concerning the Sierpiński gasket and Sierpiński graphs exactly.

As explained above, we may view G_n as an amalgam of three copies of G_{n-1}. More generally, we may consider G_n as an amalgam of 3^{n-k} copies of G_k ($0 \leq k \leq n$). For any word $w = w_1 \cdots w_n \in \mathbb{W}^n$ ($n \geq 1$), set $\psi_w = \psi_{w_1} \circ \cdots \circ \psi_{w_n}$, and let $\psi_\varnothing$ be the identity map. Then, for $0 \leq k \leq n$,

$$G_n = \bigcup_{w \in \mathbb{W}^k} \psi_w(G_{n-k}).$$

If $w \in \mathbb{W}^k$, we call $\psi_w(G_{n-k})$ (respectively $\psi_w(K)$) a *k-part* of G_n (respectively K). Note that the k-parts are in one-to-one correspondence with the words in $\mathbb{W}^k$. For any word $w \in \mathbb{W}^k$ and any subgraph $H \subseteq G_n$ the *restriction* $\pi_w(H)$ is the subgraph of G_{n-k} given by

$$\pi_w(H) = \psi_w^{-1}(H \cap \psi_w(G_{n-k})).$$

4. Spanning trees on Sierpiński graphs

For the sake of completeness we reproduce the computation of the number of spanning trees of the Sierpiński graphs following the approach given by Teufl and Wagner (2006, 2011a), Chang et al. (2007). Using a decomposition of certain spanning forests a recursion for the number of spanning trees and two other quantities is derived. In the physics literature this approach is often called the renormalization group. We write

- $\mathcal{T}_n$ to denote the set of spanning trees of G_n,
- $\mathcal{S}_n^i$ ($i \in \{1,2,3\}$) to denote the set of spanning forests in G_n with two connected components, so that one component contains u_i and the other contains $VG_0 \setminus \{u_i\}$, and
- $\mathcal{R}_n$ to denote the set of spanning forests of G_n with three connected components, each of which contains exactly one vertex from the set VG_0.

By symmetry, the sets $\mathcal{S}_n^1$, $\mathcal{S}_n^2$, and $\mathcal{S}_n^3$ all have the same cardinality. For notational convenience, we set

$$\mathcal{Q}_n = \mathcal{T}_n \uplus \mathcal{S}_n^1 \uplus \mathcal{S}_n^2 \uplus \mathcal{S}_n^3 \uplus \mathcal{R}_n.$$

The crucial observation is that the restriction of a spanning forest in $\mathcal{Q}_{n+1}$ to one of $\psi_1(G_n)$, $\psi_2(G_n)$, or $\psi_3(G_n)$ can be identified with a spanning forest in $\mathcal{Q}_n$. If $f \in \mathcal{Q}_{n+1}$, then $\pi_1(f), \pi_2(f), \pi_3(f) \in \mathcal{Q}_n$ and

$$f = \psi_1(\pi_1(f)) \cup \psi_2(\pi_2(f)) \cup \psi_3(\pi_3(f)). \tag{4.1}$$

Here and in the following we use lowercase letters for elements of $\mathcal{Q}_n$ and capital letters for random elements of $\mathcal{Q}_n$. Since $\mathcal{T}_0$ consists of the three elements , , , whereas $|\mathcal{S}_0^i| = |\mathcal{R}_0| = 1$ for $i \in \{1,2,3\}$, the subdivision of $\mathcal{T}_n$ into three families of equal size turns out to be advantageous. In the following we describe one subdivision which is convenient and induced by symmetry. Set

$\mathcal{T}_0^1 = \{$$\}$, $\qquad \mathcal{T}_0^2 = \{$$\}$, $\qquad \mathcal{T}_0^3 = \{$$\}$,

and in general, for $n \geq 1$ and $i \in \{1,2,3\}$,

$$\mathcal{T}_n^i = \big\{ t \in \mathcal{T}_n \,:\, u_i t u_j \subseteq \psi_i(G_{n-1}) \cup \psi_j(G_{n-1}) \text{ for all } j \in \{1,2,3\} \setminus \{i\} \big\}.$$

Here we consider the self-avoiding walk $u_i t u_j$ as the subgraph consisting of the vertices and the edges connecting consecutive vertices. In words, $\mathcal{T}_n^i$ is the set of spanning trees with the property that the unique paths from u_i to the other corner vertices u_j, $j \neq i$, only pass through $\psi_i(G_{n-1})$ and $\psi_j(G_{n-1})$ and do not "make a detour". Then

$$\mathcal{T}_n = \mathcal{T}_n^1 \uplus \mathcal{T}_n^2 \uplus \mathcal{T}_n^3$$

and $|\mathcal{T}_n| = 3\,|\mathcal{T}_n^i|$ for $i \in \{1,2,3\}$. Define

$$\tau_n = |\mathcal{T}_n^1| = |\mathcal{T}_n^2| = |\mathcal{T}_n^3|, \qquad \sigma_n = |\mathcal{S}_n^1| = |\mathcal{S}_n^2| = |\mathcal{S}_n^3|, \qquad \rho_n = |\mathcal{R}_n|.$$

Lemma 4.1. *If $n \geq 0$, then*

$$\begin{aligned} \tau_{n+1} &= 18\tau_n^2\sigma_n, \\ \sigma_{n+1} &= 21\tau_n\sigma_n^2 + 9\tau_n^2\rho_n, \\ \rho_{n+1} &= 14\sigma_n^3 + 36\tau_n\sigma_n\rho_n, \end{aligned}$$

and

$$\tau_n = \left(\tfrac{5}{3}\right)^{-n/2} 540^{(3^n-1)/4},$$
$$\sigma_n = \left(\tfrac{5}{3}\right)^{n/2} 540^{(3^n-1)/4},$$
$$\rho_n = \left(\tfrac{5}{3}\right)^{3n/2} 540^{(3^n-1)/4}.$$

Proof: The recursion satisfied by τ_n, σ_n, ρ_n follows from the decomposition (4.1). For a graphical explanation of the specific terms, see Figure 4.2. The initial values are $(\tau_0, \sigma_0, \rho_0) = (1, 1, 1)$, and using induction, it is easy to verify that the formulae stated in the lemma are indeed the explicit solution to the recursion. □

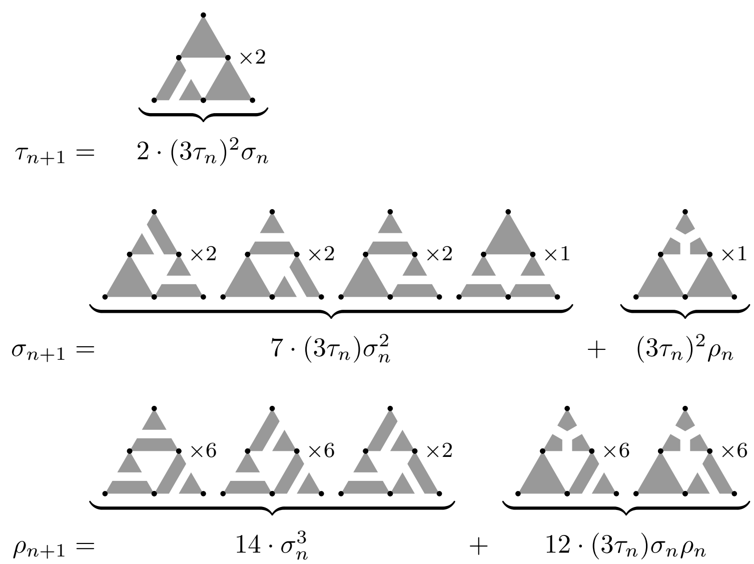
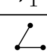
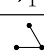
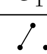
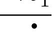


FIGURE 4.2. All arrangements (up to symmetry) for the construction of spanning trees and spanning forests used in the recursion for τ_n, σ_n, and ρ_n. Shaded area indicates connected parts.

We define the *trace* $\mathsf{Tr}\, f \in \mathcal{Q}_n$ of a spanning forest $f \in \mathcal{Q}_{n+1}$ as follows: For $f \in \mathcal{Q}_1$, the trace is given in Table 4.1.

TABLE 4.1. Traces of spanning forests in $\mathcal{Q}_1$.

M	$\mathcal{T}_1^1$	$\mathcal{T}_1^2$	$\mathcal{T}_1^3$	$\mathcal{S}_1^1$	$\mathcal{S}_1^2$	$\mathcal{S}_1^3$	$\mathcal{R}_1$
$\mathsf{Tr}\, f$ for $f \in M$							

If $n > 0$ and $f \in \mathcal{Q}_{n+1}$, then consider the 3^n n-parts of G_{n+1}, which are isomorphic to G_1. On each of these parts f induces (up to scaling and translation) a forest in $\mathcal{Q}_1$. In order to obtain the trace $\mathsf{Tr}\, f$, replace each of these small forests by their respective trace:

$$\mathsf{Tr}\, f = \bigcup_{w \in \mathbb{W}^n} \psi_w(\mathsf{Tr}\, \pi_w(f)).$$

Note that Tr maps $\mathcal{T}_{n+1}^i$ onto $\mathcal{T}_n^i$, $\mathcal{S}_{n+1}^i$ onto $\mathcal{S}_n^i$ ($i \in \{1,2,3\}$), and $\mathcal{R}_{n+1}$ onto $\mathcal{R}_n$. In order to emphasize the dependence on n, we write $\mathsf{Tr}_n^{n+1} t$ instead of $\mathsf{Tr}\, f$ if $f \in \mathcal{Q}_{n+1}$. For $m \geq n$ define $\mathsf{Tr}_n^m = \mathsf{Tr}_n^{n+1} \circ \cdots \circ \mathsf{Tr}_{m-1}^m$. Then

$$\mathcal{T}_n^1 = \{t \in \mathcal{T}_n : \mathsf{Tr}_0^n t = \},$$
$$\mathcal{T}_n^2 = \{t \in \mathcal{T}_n : \mathsf{Tr}_0^n t = \},$$
$$\mathcal{T}_n^3 = \{t \in \mathcal{T}_n : \mathsf{Tr}_0^n t = \}.$$

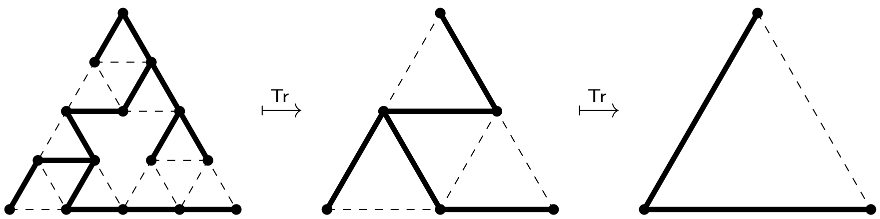


FIGURE 4.3. A spanning tree t on G_2 and the traces $\mathsf{Tr}\, t = \mathsf{Tr}_1^2 t$ and $\mathsf{Tr}_0^2 t$.

Figure 4.3 shows a spanning tree on G_2 and its traces on G_1 and G_0. The importance of the trace stems from the fact that $(\mathcal{Q}_n, \mathsf{Tr}_n^m)$ is a projective system. Hence we can define $\mathcal{Q}_\infty = \varprojlim \mathcal{Q}_n$ and write Tr_n^∞ to denote the canonical projection from $\mathcal{Q}_\infty$ to $\mathcal{Q}_n$. Similarly, set

$$\mathcal{T}_\infty^i = \varprojlim \mathcal{T}_n^i, \qquad \mathcal{S}_\infty^i = \varprojlim \mathcal{S}_n^i, \qquad \mathcal{R}_\infty = \varprojlim \mathcal{R}_n$$

for $i \in \{1,2,3\}$. Then

$$\mathcal{T}_\infty = \varprojlim \mathcal{T}_n = \mathcal{T}_\infty^1 \uplus \mathcal{T}_\infty^2 \uplus \mathcal{T}_\infty^3$$

and

$$\mathcal{Q}_\infty = \mathcal{T}_\infty \uplus \mathcal{S}_\infty^1 \uplus \mathcal{S}_\infty^2 \uplus \mathcal{S}_\infty^3 \uplus \mathcal{R}_\infty.$$

Let $w \in \mathbb{W}^*$ be a word of length $n \geq 0$ and let $f \in \mathcal{Q}_\infty$. Then

$$\pi_w(f) = (\pi_w(\mathsf{Tr}_n^\infty f), \pi_w(\mathsf{Tr}_{n+1}^\infty f), \dots) \in \mathcal{Q}_\infty$$

extends the definition of the restriction operator π_w to $\pi_w \colon \mathcal{Q}_\infty \to \mathcal{Q}_\infty$.

Next we define the *type* of an element of $\mathcal{Q}_\infty$ (or a part of it). Set

$$\mathcal{C} = \{ , , , , , , \}.$$

For $f \in \mathcal{Q}_\infty$ let $\chi_\varnothing(f) \in \mathcal{C}$ be given by Table 4.2. The symbol $\chi_\varnothing(f)$ gives a crude indication of the shape of f.

TABLE 4.2. The definition of $\chi_\varnothing(f)$ for $f \in \mathcal{Q}_\infty$.

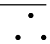
M	$\mathcal{T}_\infty^1$	$\mathcal{T}_\infty^2$	$\mathcal{T}_\infty^3$	$\mathcal{S}_\infty^1$	$\mathcal{S}_\infty^2$	$\mathcal{S}_\infty^3$	$\mathcal{R}_\infty$
$\chi_\varnothing(f)$ for $f \in M$							

For any non-empty word $w \in \mathbb{W}^*$ define $\chi_w(f)$ by $\chi_w(f) = \chi_\varnothing(\pi_w(f))$. This yields a map $\boldsymbol{\chi} \colon \mathcal{Q}_\infty \to \mathcal{C}^{\mathbb{W}^*}$ given by $\boldsymbol{\chi}(f) = (\chi_w(f))_{w \in \mathbb{W}^*}$: $\boldsymbol{\chi}(f)$ encodes the shape of f at every level. In order to reconstruct $\mathsf{Tr}_n^\infty f$ from $\boldsymbol{\chi}(f)$, let η be the map from $\mathcal{C}$ to the set of subgraphs of G_0 defined in the obvious way, see Table 4.3. Then

$$\mathsf{Tr}_n^\infty f = \bigcup_{w \in \mathbb{W}^n} \psi_w(\eta(\chi_w(f))).$$

Hence $\boldsymbol{\chi}$ is one-to-one.

Table 4.3. The mappings η and ν.

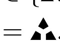
x							
$\eta(x)$							
$\nu(x)$	1	2	3	4	5	6	7

Let ν be the bijection from $\mathcal{C}$ to $\{1,\dots,7\}$ given in Table 4.3. Define the functions $\chi^{\#}_{i,n}(f) = |\{w \in \mathbb{W}^n : \nu(\chi_w(f)) = i\}|$, which count the number of n-parts of type $i \in \{1,2,\dots,7\}$, and set

$$\boldsymbol{\chi}^{\#}_n(f) = \big(\chi^{\#}_{1,n}(f), \dots, \chi^{\#}_{7,n}(f)\big)$$

for $n \geq 0$. Of course all these definitions also make sense for finite forests $f \in \mathcal{Q}_m$ (where m is some nonnegative integer). For $w \in \mathbb{W}^n$, $n \leq m$, define $\chi_w(f)$ and $\boldsymbol{\chi}^{\#}_n(f)$ in analogy to the definitions above.

Finally, we define the number of connected components $c(x)$, where x is a symbol in $\mathcal{C}$, in the canonical way as follows:

$c(x) = 1$ if $x \in \{$  $\}$,
$c(x) = 2$ if $x \in \{$  $\}$,
$c(x) = 3$ if $x =$ .

For $f \in \mathcal{Q}_\infty$, we set $c(f) = c(\chi_\varnothing(f))$, which is also the number of components of $\mathsf{Tr}_n^\infty f$ for any $n \geq 0$. The following simple lemma relates all our counting functions (cf. Teufl and Wagner, 2011a, Lemma 5.1). We write $\boldsymbol{v}^t$ to denote the transpose of a vector $\boldsymbol{v}$.

Lemma 4.2. *For any $f \in \mathcal{Q}_\infty$ and any $n \geq 0$,*

$$\boldsymbol{\chi}^{\#}_n(f) \cdot (1,1,1,1,1,1,1)^t = 3^n,$$
$$\boldsymbol{\chi}^{\#}_n(f) \cdot (2,2,2,1,1,1,0)^t = \tfrac{3}{2}(3^n+1) - c(f),$$
$$\boldsymbol{\chi}^{\#}_n(f) \cdot (1,1,1,-1,-1,-1,-3)^t = 3 - 2c(f).$$

Proof: The first equation is immediate. In order to prove the second, notice that $\boldsymbol{\chi}^{\#}_n(f) \cdot (2,2,2,1,1,1,0)^t$ is the number of edges in the spanning forest $\mathsf{Tr}_n^\infty f$ of G_n. As this spanning forest has $c(f)$ components, its number of edges is given by $|VG_n| - c(f) = \frac{3}{2}(3^n+1) - c(f)$. The third equation follows from the first and the second by elimination of 3^n. □

5. Uniform spanning trees

We now come to the core part of this paper: the discussion of the structure of uniform spanning trees of G_n. Let us write $\mathsf{Unif}\,\mathcal{X}$ to denote the uniform distribution on a finite, non-empty set $\mathcal{X}$. For $i \in \{1,2,3\}$ let T_n^i be a uniformly random element in $\mathcal{T}_n^i$. If $B \sim \mathsf{Unif}\{1,2,3\}$ is independent of T_n^i, then T_n^B is clearly a uniform spanning tree of G_n, i.e., $T_n^B \sim \mathsf{Unif}\,\mathcal{T}_n$. In the following lemma, we prove the important fact that the trace preserves probabilities:

Lemma 5.1. *Let $i \in \{1,2,3\}$. If T_n^i is uniformly random on $\mathcal{T}_n^i$, S_n^i is uniformly random on $\mathcal{S}_n^i$, and R_n is uniformly random on $\mathcal{R}_n$, then*

$$\mathbb{P}(\mathsf{Tr}\,T_{n+1}^i \in A) = \mathbb{P}(T_n^i \in A),$$
$$\mathbb{P}(\mathsf{Tr}\,S_{n+1}^i \in B) = \mathbb{P}(S_n^i \in B),$$
$$\mathbb{P}(\mathsf{Tr}\,R_{n+1} \in C) = \mathbb{P}(R_n \in C)$$

for any $A \subseteq \mathcal{T}_n^i$, $B \subseteq \mathcal{S}_n^i$, $C \subseteq \mathcal{R}_n$.

Proof: In order to prove the first identity, we have to show that

$$\mathbb{P}(\mathsf{Tr}\,T_{n+1}^i = t) = \mathbb{P}(T_n^i = t)$$

for any $t \in \mathcal{T}_n^i$. This is equivalent to

$$|\mathsf{Tr}^{-1}\,t| = \frac{\tau_{n+1}}{\tau_n}$$

for any $t \in \mathcal{T}_n^i$. Since

$$|\mathcal{T}_1^k| = \tau_1 = 18, \qquad |\mathcal{S}_1^k| = \sigma_1 = 30, \qquad |\mathcal{R}_1| = \rho_1 = 50$$

for $k \in \{1,2,3\}$, Lemma 4.2 implies that

$$|\mathsf{Tr}^{-1}\,t| = 18^{\chi^{\#}_{n,1}(t)+\chi^{\#}_{n,2}(t)+\chi^{\#}_{n,3}(t)} \cdot 30^{\chi^{\#}_{n,4}(t)+\chi^{\#}_{n,5}(t)+\chi^{\#}_{n,6}(t)} \cdot 50^{\chi^{\#}_{n,7}(t)} = 18 \cdot 540^{(3^n-1)/2}.$$

Using Lemma 4.1, it is easy to see that

$$\frac{\tau_{n+1}}{\tau_n} = 18 \cdot 540^{(3^n-1)/2}.$$

The same argument applies to the second and third identity, too. $\square$

In light of Lemma 5.1 and Kolmogorov's Extension Theorem there is a probability measure P_{T^i} on $\mathcal{T}_\infty^i$ such that $P_{T^i}(\{t \in \mathcal{T}_\infty^i : \mathsf{Tr}_n^\infty\, t \in \cdot\}) = \mathbb{P}(T_n^i \in \cdot)$. Let P_{S^i} and P_R be the analogous measures on $\mathcal{S}_\infty^i$ and $\mathcal{R}_\infty$, respectively. Set

$$\Omega = \{1,2,3\} \times \mathcal{T}_\infty^1 \times \mathcal{T}_\infty^2 \times \mathcal{T}_\infty^3 \times \mathcal{S}_\infty^1 \times \mathcal{S}_\infty^2 \times \mathcal{S}_\infty^3 \times \mathcal{R}_\infty,$$
$$P = \mathsf{Unif}\{1,2,3\} \times P_{T^1} \times P_{T^2} \times P_{T^3} \times P_{S^1} \times P_{S^2} \times P_{S^3} \times P_R.$$

Let $B, T_\infty^i, S_\infty^i, R_\infty$ be the canonical projections from Ω to $\{1,2,3\}$, $\mathcal{T}_\infty^i$, $\mathcal{S}_\infty^i$, $\mathcal{R}_\infty$, respectively. Set $T_\infty = T_\infty^B$ and, for $n \geq 0$, $T_n = \mathsf{Tr}_n^\infty\, T_\infty$. Then T_n is a uniform spanning tree on G_n and $T_n = \mathsf{Tr}_n^m\, T_m = \mathsf{Tr}_n^\infty\, T_\infty$ for $m \geq n \geq 0$. Analogous statements hold for $S_n^i = \mathsf{Tr}_n^\infty\, S_\infty^i$ and $R_n = \mathsf{Tr}_n^\infty\, R_\infty$.

In the following we write $\mathbb{P}$ instead of P and always use Ω equipped with $\mathbb{P}$ as probability space, whenever the random elements T_n, T_n^i, S_n^i, etc. are considered.

Let $[\blacktriangle, \blacktriangle, \blacktriangle]$ (suppressing the dependence on n) be a shorthand for the set

$$\{\psi_1(f_1) \cup \psi_2(f_2) \cup \psi_3(f_3) \,:\, f_1 \in \mathcal{T}_{n-1}^1, f_2 \in \mathcal{T}_{n-1}^1, f_3 \in \mathcal{S}_{n-1}^2\},$$

and analogously for other combinations. Using Lemma 4.1 it is easy to see that

$$
\begin{aligned}
&\mathbb{P}(T_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathbb{P}(T_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \cdots = \frac{\tau_{n-1}^2\sigma_{n-1}}{\tau_n} = \frac{1}{18},\\
&\mathbb{P}(S_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathbb{P}(S_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \cdots = \frac{\tau_{n-1}\sigma_{n-1}^2}{\sigma_n} = \frac{1}{30},\\
&\mathbb{P}(S_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathbb{P}(S_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \cdots = \frac{\tau_{n-1}^2\rho_{n-1}}{\sigma_n} = \frac{1}{30},\\
&\mathbb{P}(R_n \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathbb{P}(R_n \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \cdots = \frac{\sigma_{n-1}^3}{\rho_n} = \frac{1}{50},\\
&\mathbb{P}(R_n \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathbb{P}(R_n \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \cdots = \frac{\tau_{n-1}\sigma_{n-1}\rho_{n-1}}{\rho_n} = \frac{1}{50},
\end{aligned}
\tag{5.1}
$$

where dots indicate combinations in the same "group" (group sizes are 18, 21, 9, 14, and 36, see Figure 4.2). Of course, analogous results also hold for T_n^1, T_n^2, S_n^1, S_n^2. Furthermore, note that

$$\mathbb{P}(\pi_2(T_n^3) \in \cdot \mid T_n^3 \in [\blacktriangle,\blacktriangle,\blacktriangle]) = \mathsf{Unif}\,\mathcal{S}_{n-1}^1,$$

and analogously for other combinations and restrictions. Using this fact we obtain the following result, which relates uniform spanning trees on Sierpiński graphs to a multi-type Galton-Watson process:

Proposition 5.2. *Let $\mathcal{U}_\infty$ be one of $\mathcal{T}_\infty$, $\mathcal{T}_\infty^i$, $\mathcal{S}_\infty^i$, $\mathcal{R}_\infty$, and let U_∞ be the corresponding random object.*

(1) *The random tree*

$$\boldsymbol{\chi}(U_\infty) = (\chi_w(U_\infty))_{w\in\mathbb{W}^*}$$

is a labelled multi-type Galton-Watson tree with labels in $\mathbb{W}^*$ and types in $\mathcal{C}$. The type distribution of the root depends on the specific choice for $\mathcal{U}_\infty$ and is given by $\mathsf{Unif}\{\chi_\varnothing(f) : f \in \mathcal{U}_\infty\}$. The set of individuals is deterministic and equal to $\mathbb{W}^*$. Each individual has three children with suffixes $1, 2, 3$. For $x \in \mathcal{C}$ set

$$\mathcal{D}(x) = \big\{(\chi_1(f), \chi_2(f), \chi_3(f)) : f \in \mathcal{Q}_1, \chi_\varnothing(f) = x\big\} \subseteq \mathcal{C}^3.$$

Then, by Equation (5.1), the offspring distribution of an individual of type x is given by $\mathsf{Unif}\,\mathcal{D}(x)$, that is,

$$\mathbb{P}((\chi_{w1}(U_\infty), \chi_{w2}(U_\infty), \chi_{w3}(U_\infty)) \in \cdot \mid \chi_w(U_\infty) = x) = \mathsf{Unif}\,\mathcal{D}(x). \tag{5.2}$$

(2) *$(\boldsymbol{\chi}_n^\#(U_\infty))_{n\geq 0}$ is a multi-type Galton-Watson process with seven types, which is non-singular, positively regular, and supercritical. The type distribution of the root is given by the uniform distribution $\mathsf{Unif}\{\nu(\chi_\varnothing(f)) : f \in \mathcal{U}_\infty\}$. The offspring generating function is easily computed by means of Equation (5.2):*

using the abbreviation $s = \frac{1}{3}(z_1 + z_2 + z_3)$, *we have*

$$\begin{aligned}\boldsymbol{f}(\boldsymbol{z}) = \Big(& \tfrac{1}{2}s^2(z_5 + z_6),\ \tfrac{1}{2}s^2(z_4 + z_6),\ \tfrac{1}{2}s^2(z_4 + z_5), \\ & \tfrac{1}{10}s\big(3z_4^2 + 2z_4(z_5 + z_6) + 3sz_7\big), \\ & \tfrac{1}{10}s\big(3z_5^2 + 2z_5(z_4 + z_6) + 3sz_7\big), \\ & \tfrac{1}{10}s\big(3z_6^2 + 2z_6(z_4 + z_5) + 3sz_7\big), \\ & \tfrac{1}{25}\big(z_4^2 z_5 + z_4 z_5^2 + z_4^2 z_6 + z_4 z_6^2 + z_5^2 z_6 + z_5 z_6^2 \\ & \qquad + z_4 z_5 z_6 + 6s(z_4 + z_5 + z_6)z_7\big)\Big).\end{aligned}$$

Its mean matrix $\boldsymbol{M}$ *is given by*

$$\boldsymbol{M} = \frac{1}{150}\begin{pmatrix} 100 & 100 & 100 & 0 & 75 & 75 & 0 \\ 100 & 100 & 100 & 75 & 0 & 75 & 0 \\ 100 & 100 & 100 & 75 & 75 & 0 & 0 \\ 65 & 65 & 65 & 150 & 30 & 30 & 45 \\ 65 & 65 & 65 & 30 & 150 & 30 & 45 \\ 65 & 65 & 65 & 30 & 30 & 150 & 45 \\ 36 & 36 & 36 & 78 & 78 & 78 & 108 \end{pmatrix}.$$

The dominating eigenvalue of $\boldsymbol{M}$ *is equal to* 3. *The corresponding right and left eigenvectors are*

$$\boldsymbol{v}_R = (1,1,1,1,1,1,1)^t \qquad \text{and} \qquad \boldsymbol{v}_L = \tfrac{1}{288}(53,53,53,38,38,38,15),$$

respectively. $\boldsymbol{v}_R$ *and* $\boldsymbol{v}_L$ *are normalized so that* $\boldsymbol{v}_L \cdot \boldsymbol{v}_R = 1$ *and* $\|\boldsymbol{v}_L\|_1 = 1$.

(3) *Since* $\|\boldsymbol{\chi}_n^{\#}(U_\infty)\|_1 = \boldsymbol{\chi}_n^{\#}(U_\infty) \cdot \boldsymbol{v}_R = 3^n$, *it follows that*

$$\lim_{n\to\infty} 3^{-n}\boldsymbol{\chi}_n^{\#}(U_\infty) = \boldsymbol{v}_L$$

almost surely, see Mode (1971, Theorem 1.8.3).

Remark 5.3. In order to sample a uniform spanning tree on G_n, we may simulate the n-th generation of a labelled multi-type Galton-Watson tree as described above. In this process, we have to choose one of ▲, ▲, ▲ with equal probability as the type for the ancestor ∅ of the tree. It is possible to postpone this choice from the beginning to the n-th generation. To this end, collapse the three types ▲, ▲, ▲ into one type ▲. This yields again a labelled multi-type Galton-Watson tree, but now with five types {▲, ▲, ▲, ▲, ▲}. In order to obtain a uniform spanning tree on G_n, consider the n-th generation of this simplified labelled multi-type Galton-Watson tree and replace each occurrence of ▲ independently by one of ▲, ▲, ▲ with equal probability. This modified n-th generation describes a spanning tree on G_n, whose distribution is uniform. Figure 5.4 shows an example of a randomly generated spanning tree on G_5.

Remark 5.4. Suppose λ is a parameter of spanning trees in G_n, and we are interested in the behaviour of $\lambda(T_n)$ as $n \to \infty$. When $\lambda(T_n)$ is a functional of $\boldsymbol{\chi}_n^{\#}(T_\infty)$, say $\lambda(T_n) = h(\boldsymbol{\chi}_n^{\#}(T_\infty))$ for some linear function h, then $3^{-n}\lambda(T_n) \to h(\boldsymbol{v}_L)$ almost surely. Of course, this generalizes to positive homogeneous functions h. As a simple example consider the number of n-parts with i connected components. For

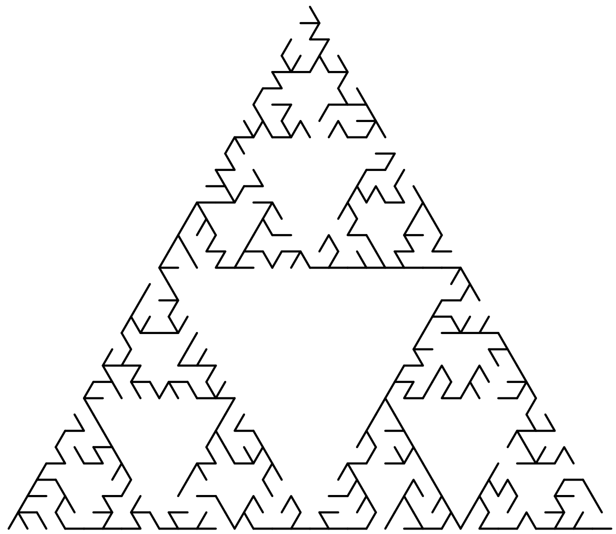


FIGURE 5.4. A randomly generated spanning tree on G_5.

$f \in \mathcal{Q}_\infty$, $i \in \{1,2,3\}$, and $n \geq 0$, let us denote this quantity by $c^{\#}_{i,n}(f) = |\{w \in \mathbb{W}^n : c(\chi_w(f)) = i\}|$. Then

$$\boldsymbol{c}^{\#}_n(f) = \left(c^{\#}_{1,n}(f), c^{\#}_{2,n}(f), c^{\#}_{3,n}(f)\right) = \boldsymbol{\chi}^{\#}_n(f) \cdot \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}^t$$

and therefore $3^{-n}\boldsymbol{c}^{\#}_n(U_\infty) \to \frac{1}{96}(53, 38, 5)$ almost surely as $n \to \infty$ if U_∞ is one of T_∞, T^i_∞, S^i_∞, R_∞ for $i \in \{1,2,3\}$. Note that, due to symmetry, $(\boldsymbol{c}^{\#}_n(U_\infty))_{n\geq 0}$ is a multi-type Galton-Watson process in its own right.

A straightforward calculation shows that the variance of $\boldsymbol{\chi}^{\#}_n$ and $\boldsymbol{c}^{\#}_n$ is of order 3^n, so that Chebyshev's inequality yields

$$\mathbb{P}(\|\boldsymbol{\chi}^{\#}_n(U_\infty) - 3^n\boldsymbol{v}_L\|_1 \geq \alpha^n) \ll 3^n\alpha^{-2n} \tag{5.3}$$

for any $\alpha \in (\sqrt{3}, 3)$, and an analogous inequality for $\boldsymbol{c}^{\#}_n$.

In the following we study two quantities of a random spanning forest of G_n, which need more work as the previous remark does not apply directly: The first quantity are the component sizes in S^1_n, S^2_n, S^3_n, R_n. In this case it turns out that components can be described using an augmented labelled multi-type Galton-Watson tree. Secondly, we study the degree distribution in T_n. Here the recursive description of uniform spanning trees in G_n (see Figure 4.2 and Proposition 5.2) and the rapid decay of tail probabilities given by (5.3) is used.

5.1. *Component sizes.* Spanning trees only have one component, but for random spanning forests S^i_n or R_n, the sizes (number of vertices or edges) of the components are interesting random variables. Let us briefly explain how their limiting distribution can be obtained.

First, we need some notation. For a non-empty subset B of VG_0, let f be an element of $\mathcal{Q}_\infty$, and assume that B is the vertex set of the union of some connected components of $\mathsf{Tr}^\infty_0 f$. Write $C_n(f, B)$ to denote the union of those components of $\mathsf{Tr}_n^\infty f$ having non-empty intersection with B. For example, if $f \in S_\infty^1$, then $C_n(f, \{u_1\})$ is the component of $\mathsf{Tr}_n^\infty f$ that contains u_1, $C_n(f, \{u_2, u_3\})$ is the component that contains u_2 and u_3, and $C_n(f, \{u_1, u_2, u_3\})$ is the entire spanning forest $\mathsf{Tr}_n^\infty f$.

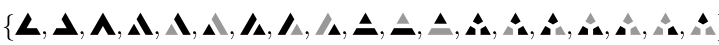
We are interested in the size of $C_n(f, B)$, which unfortunately is not a linear functional of $\boldsymbol{\chi}_n^\#(f)$. However, it is possible to define a subtree of the Galton-Watson-tree $\boldsymbol{\chi}(f)$ that encodes f and to add extra information to the types $\mathcal{C}$ that records the evolution of the components in $C_n(f, B)$. If f is randomly chosen, the resulting subtree with augmented types describes another labelled multi-type Galton-Watson tree, as will be shown in the following. For $n \geq 0$, let

$$\hat{W}_n(f, B) = \{w \in \mathbb{W}^n \,:\, C_n(f, B) \cap \psi_w(G_0) \neq \varnothing\}$$

be the set of those words $w \in \mathbb{W}^n$ for which the corresponding n-part $\psi_w(G_0)$ of the Sierpiński gasket has non-empty intersection with $C_n(f, B)$. Their union

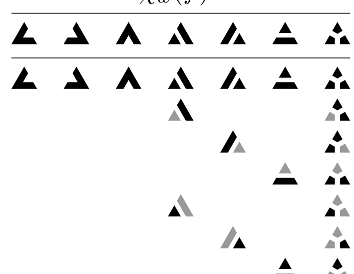
$$\hat{W}(f, B) = \bigcup_{n \geq 0} \hat{W}_n(f, B)$$

induces a subtree of $\mathbb{W}^*$, and each word in $\hat{W}(f, B)$ has one, two or three children. For $w \in \hat{W}(f, B)$, write $\hat{\kappa}_w(f, B)$ to denote the vertex set $V(\pi_w(\psi_w(G_0) \cap C_n(f, B)))$ (in words: the vertices of the n-part $\psi_w(G_0)$ that are in common components with vertices of B, projected back to G_0). To each $w \in \hat{W}_n(f, B)$, we assign one of the following nineteen types

$$\hat{\mathcal{C}} = \{$$  $$\}$$

encoding two pieces of information: $\chi_w(f)$ (structure of the restriction of f to the respective n-part) and $\hat{\kappa}_w(f, B)$ (black parts indicate which of the corner vertices are in common components with elements of B). We denote this assignment by $\hat{\chi}_w(f, B)$, see Table 5.4 for a precise definition of $\hat{\chi}_w(f, B)$ in terms of $\chi_w(f)$ and $\hat{\kappa}_w(f, B)$.

TABLE 5.4. The type $\hat{\chi}_w(f, B)$, given $\chi_w(f)$ and $\hat{\kappa}_w(f, B)$.



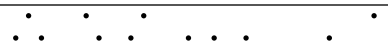
$\hat{\kappa}_w(f, B)$	$\chi_w(f)$
$\{u_1, u_2, u_3\}$	
$\{u_2, u_3\}$	
$\{u_1, u_3\}$	
$\{u_1, u_2\}$	
$\{u_1\}$	
$\{u_2\}$	
$\{u_3\}$	

Finally set $\hat{\boldsymbol{\chi}}(f, B) = (\hat{\chi}_w(f, B))_{w \in \hat{W}(f,B)}$. It is easy to see that it is possible to reconstruct the graph $C_n(f, B)$ from $\hat{\boldsymbol{\chi}}(f, B)$: formally,

$$C_n(f, B) = \bigcup_{w \in \hat{W}_n(f,B)} \psi_w(\hat{\eta}(\hat{\chi}_w(f, B))),$$

where $\hat{\eta}$ is given in Table 5.5.

TABLE 5.5. The mappings $\hat{\eta}$ and $\hat{c}$.

x												
$\hat{\eta}(x)$												
$\hat{c}(x)$	1	1	1	2	4	5	2	4	5	2	4	5
x												
$\hat{\eta}(x)$												
$\hat{c}(x)$	3	6	6	6	7	7	7					

Now let us define $\hat{c}(x)$ for $x \in \hat{\mathcal{C}}$ as in Table 5.5. For $i \in \{1, \ldots, 7\}$ and $n \geq 0$ set $\hat{c}^{\#}_{i,n}(f, B) = |\{w \in \hat{W}_n(f, B) \,:\, \hat{c}(\hat{\chi}_w(f, B)) = i\}|$ and

$$\hat{\boldsymbol{c}}^{\#}_n(f, B) = \big(\hat{c}^{\#}_{1,n}(f, B), \ldots, \hat{c}^{\#}_{7,n}(f, B)\big).$$

The vector $\hat{\boldsymbol{c}}^{\#}_n(f, B)$ counts the number of words in $\hat{W}(f, B)$ of given type up to symmetry. Note that the number of edges in $C_n(f, B)$ can be determined from $\hat{\boldsymbol{c}}^{\#}_n(f, B)$: it is given by

$$|EC_n(f, B)| = \hat{\boldsymbol{c}}^{\#}_n(f, B) \cdot (2, 1, 0, 1, 0, 0, 0)^t,$$

and the number of vertices in $C_n(f, B)$ satisfies

$$1 \leq |VC_n(f, B)| - |EC_n(f, B)| \leq 3,$$

the precise value of the difference depending on the type of f and the set B. Now let U_∞ be one of $T_\infty, T^i_\infty, S^i_\infty, R_\infty$ $(i \in \{1, 2, 3\})$, and choose $B \subseteq VG_0$ so that B is the vertex set of the union of some components of $\mathsf{Tr}^{\infty}_0 U_\infty$. Then $\hat{\boldsymbol{\chi}}(U_\infty, B)$ is a labelled multi-type Galton-Watson tree with types in $\hat{\mathcal{C}}$, and $(\hat{\boldsymbol{c}}^{\#}_n(U_\infty, B))_{n\geq 0}$ is a multi-type Galton-Watson process with seven types. The offspring generating function $\hat{\boldsymbol{f}}(\boldsymbol{z})$ of the process is given by

$$\begin{aligned}
\hat{\boldsymbol{f}}(\boldsymbol{z}) = \Big(& z_1^2 z_2,\ \tfrac{7}{10} z_1 z_2^2 + \tfrac{3}{10} z_1^2 z_3,\ \tfrac{7}{25} z_2^3 + \tfrac{18}{25} z_1 z_2 z_3, \\
& \tfrac{2}{10} z_1 z_4 z_5 + \tfrac{4}{10} z_1 z_2 z_4 + \tfrac{1}{10} z_4^2 + \tfrac{3}{10} z_1^2 z_6, \\
& \tfrac{2}{10} z_4 z_5 + \tfrac{4}{10} z_5 + \tfrac{1}{10} z_1 z_5^2 + \tfrac{3}{10} z_7, \\
& \tfrac{3}{25} z_2^2 z_4 + \tfrac{2}{25} z_2 z_4 z_5 + \tfrac{1}{25} z_4 z_5^2 + \tfrac{1}{25} z_5^2 + \tfrac{6}{25} z_1 z_2 z_6 + \tfrac{3}{25} z_5 z_7 \\
& \qquad + \tfrac{3}{25} z_1 z_3 z_4 + \tfrac{3}{25} z_4 z_6 + \tfrac{3}{25} z_1 z_5 z_6, \\
& \tfrac{6}{25} z_5 + \tfrac{1}{25} z_2 z_4^2 + \tfrac{2}{25} z_4 z_5 + \tfrac{1}{25} z_4^2 z_5 + \tfrac{6}{25} z_7 + \tfrac{3}{25} z_1 z_4 z_6 \\
& \qquad + \tfrac{3}{25} z_1 z_5 z_7 + \tfrac{3}{25} z_4 z_7 \Big),
\end{aligned}$$

and the mean matrix is given by

$$\hat{\boldsymbol{M}} = \frac{1}{50} \cdot \begin{pmatrix} 100 & 50 & 0 & 0 & 0 & 0 & 0 \\ 65 & 70 & 15 & 0 & 0 & 0 & 0 \\ 36 & 78 & 36 & 0 & 0 & 0 & 0 \\ 60 & 20 & 0 & 40 & 10 & 15 & 0 \\ 5 & 0 & 0 & 10 & 40 & 0 & 15 \\ 24 & 28 & 6 & 24 & 24 & 24 & 6 \\ 12 & 2 & 0 & 24 & 24 & 6 & 24 \end{pmatrix}.$$

Hence the multi-type Galton-Watson process is non-singular, but not positively regular, as the mean matrix is reducible. The dominating eigenvalue is 3, which belongs to the 3×3 block of $\hat{\boldsymbol{M}}$ in the upper left corner. It has multiplicity 1, and the corresponding right and left eigenvectors are

$$\hat{\boldsymbol{v}}_R = \left(1, 1, 1, \tfrac{5}{6}, \tfrac{1}{6}, \tfrac{2}{3}, \tfrac{1}{3}\right)^t, \qquad \hat{\boldsymbol{v}}_L = \tfrac{1}{96} \cdot (53, 38, 5, 0, 0, 0, 0),$$

respectively. $\hat{\boldsymbol{v}}_R$ and $\hat{\boldsymbol{v}}_L$ are normalized so that $\hat{\boldsymbol{v}}_L \cdot \hat{\boldsymbol{v}}_R = 1$ and $\|\hat{\boldsymbol{v}}_L\|_1 = 1$. Intuitively, the fact that only the first three entries in $\hat{\boldsymbol{v}}_L$ are nonzero (and that $\hat{\boldsymbol{M}}$ is dominated by the upper left 3×3-block) can be explained by the fact that n-parts of types such as , , etc. (some vertices belong to $C_n(f, B)$, others do not) can only occur at the "borders" between the components of the forest $\mathsf{Tr}_n^\infty f$, which only make up a very small part of the entire graph G_n.

For every choice of the boundary vertices B, there is a non-negative random variable $\hat{\theta}(U_\infty, B)$ such that

$$3^{-n} \hat{\boldsymbol{c}}_n^{\#}(U_\infty, B) \to \hat{\boldsymbol{v}}_L \hat{\theta}(U_\infty, B)$$

holds almost surely as $n \to \infty$, see Mode (1971, Theorem 2.4.1). By symmetry, there are eight different limit distributions, one for each of the following groups:

$$\begin{gathered}
\big\{\hat{\theta}(T_\infty, \{u_1, u_2, u_3\})\big\}, \\
\big\{\hat{\theta}(T_\infty^1, \{u_1, u_2, u_3\}), \hat{\theta}(T_\infty^2, \{u_1, u_2, u_3\}), \hat{\theta}(T_\infty^3, \{u_1, u_2, u_3\})\big\}, \\
\big\{\hat{\theta}(S_\infty^1, \{u_1, u_2, u_3\}), \hat{\theta}(S_\infty^2, \{u_1, u_2, u_3\}), \hat{\theta}(S_\infty^3, \{u_1, u_2, u_3\})\big\}, \\
\big\{\hat{\theta}(R_\infty, \{u_1, u_2, u_3\})\big\}, \\
\big\{\hat{\theta}(S_\infty^1, \{u_2, u_3\}), \hat{\theta}(S_\infty^2, \{u_1, u_3\}), \hat{\theta}(S_\infty^3, \{u_1, u_2\})\big\}, \\
\big\{\hat{\theta}(S_\infty^1, \{u_1\}), \hat{\theta}(S_\infty^2, \{u_2\}), \hat{\theta}(S_\infty^3, \{u_3\})\big\}, \\
\big\{\hat{\theta}(R_\infty, \{u_2, u_3\}), \hat{\theta}(R_\infty, \{u_1, u_3\}), \hat{\theta}(R_\infty, \{u_1, u_2\})\big\}, \\
\big\{\hat{\theta}(R_\infty, \{u_1\}), \hat{\theta}(R_\infty, \{u_2\}), \hat{\theta}(R_\infty, \{u_3\})\big\}.
\end{gathered}$$

Let us write $\hat{\theta}_i$ ($i \in \{0, \ldots, 7\}$) for a random variable having the same distribution as a random variable of the respective group above. Of course, $\hat{\theta}_0, \ldots, \hat{\theta}_3$ (the cases when $B = \{u_1, u_2, u_3\}$) are almost surely constant, i.e.

$$\hat{\theta}_0 = \hat{\theta}_1 = \hat{\theta}_2 = \hat{\theta}_3 = 1$$

almost surely. The remaining variables $\hat{\theta}_4, \ldots, \hat{\theta}_7$ have continuous densities, and

$$\mathbb{E}(\hat{\theta}_i) = \hat{v}_{i,R}$$

for $i \in \{4, \ldots, 7\}$, where $\hat{v}_{i,R}$ is the i-coordinate of $\hat{\boldsymbol{v}}_R$. Note also that $1 - \hat{\theta}_4$ and $\hat{\theta}_5$ have the same distribution, and the same holds for $1 - \hat{\theta}_6$ and $\hat{\theta}_7$. The limits of the renormalized component sizes can be expressed in terms of these random variables. To be precise,

$$\lim_{n \to \infty} 3^{-n} |VC_n(U_\infty, B)| = \lim_{n \to \infty} 3^{-n} |EC_n(U_\infty, B)| = \tfrac{3}{2} \hat{\theta}(U_\infty, B)$$

almost surely. In particular, the component $C_n(S_\infty^1, \{u_2, u_3\})$ is on average approximately five times larger than the complementary component $C_n(S_\infty^1, \{u_1\})$, since $\hat{v}_{4,R} = \frac{5}{6} = 5\hat{v}_{5,R}$.

5.2. *Degree distribution.* The distribution of the vertex degrees in a random spanning tree of the Sierpiński graph G_n was studied at length in the recent paper by Chang and Chen (2010). In particular, they determined the precise probability distribution of the degree of a given vertex, and determined the average proportion of the number of vertices of given degree as $n \to \infty$. Here we provide a somewhat different approach to this problem with the advantage that it also allows us to prove almost sure convergence of this proportion to a limit.

The number of vertices with a certain degree in a random spanning tree T_n is again not a simple functional of the types. In fact, the degree distribution of a vertex $v \in VG_n$ depends not only on n, but also on the level of the vertex v itself: by the level of a vertex, we mean the smallest k such that $v \in VG_k$. Let us first consider the degree distribution of the corner vertices. By symmetry, it is obviously sufficient to consider one of them. Let $\boldsymbol{d}_n(h)$ be the vector of the probabilities that the degree $\deg_{U_n} u_1$ of the lower-left corner vertex u_1 in a random spanning forest U_n is equal to $h \in \{0,1,2\}$ for $U_n \in \{T_n^1, T_n^2, T_n^3, S_n^1, S_n^2, S_n^3, R_n\}$. The entries are denoted by $d_n(\blacktriangle, h), d_n(\blacktriangle, h)$, etc. Thus $d_n(\blacktriangle, h) = \mathbb{P}(\deg_{T_n^1} u_1 = h)$, and the other entries are defined analogously. Then it is obvious that

$$\begin{aligned}
\boldsymbol{d}_0(0) &= (0,0,0,1,0,0,1)^t,\\
\boldsymbol{d}_0(1) &= (0,1,1,0,1,1,0)^t,\\
\boldsymbol{d}_0(2) &= (1,0,0,0,0,0,0)^t.
\end{aligned}$$

Moreover, we deduce from the recursive structure (see Figure 4.2) that $\boldsymbol{d}_n(h) = \boldsymbol{D}\boldsymbol{d}_{n-1}(h)$, where $\boldsymbol{D}$ is the matrix

$$\boldsymbol{D} = \frac{1}{150}\begin{pmatrix}
50 & 50 & 50 & 0 & 0 & 0 & 0\\
25 & 25 & 25 & 0 & 0 & 75 & 0\\
25 & 25 & 25 & 0 & 75 & 0 & 0\\
5 & 5 & 5 & 60 & 15 & 15 & 45\\
30 & 30 & 30 & 0 & 45 & 15 & 0\\
30 & 30 & 30 & 0 & 15 & 45 & 0\\
12 & 12 & 12 & 36 & 21 & 21 & 36
\end{pmatrix}.$$

This matrix has eigenvalues $1, \frac{3}{5}, \frac{1}{5}, \frac{1}{15}, \frac{1}{25}, 0, 0$, and we easily find that

$$\begin{aligned}
\boldsymbol{d}_n(0) &= \tfrac{11}{28} \cdot \left(\tfrac{3}{5}\right)^n \cdot (0,0,0,3,0,0,2)^t - \tfrac{1}{28} \cdot \left(\tfrac{1}{25}\right)^n \cdot (0,0,0,5,0,0,-6)^t,\\
\boldsymbol{d}_n(1) &= \tfrac{11}{14} \cdot (1,1,1,1,1,1,1)^t - \tfrac{2}{7} \cdot \left(\tfrac{3}{5}\right)^n \cdot (0,0,0,3,0,0,2)^t\\
&\qquad + \tfrac{1}{14} \cdot \left(\tfrac{1}{15}\right)^n \cdot (-25,10,10,-4,3,3,3)^t + \tfrac{1}{14} \cdot \left(\tfrac{1}{25}\right)^n \cdot (0,0,0,5,0,0,-6)^t,\\
\boldsymbol{d}_n(2) &= \tfrac{3}{14} \cdot (1,1,1,1,1,1,1)^t - \tfrac{3}{28} \cdot \left(\tfrac{3}{5}\right)^n \cdot (0,0,0,3,0,0,2)^t\\
&\qquad - \tfrac{1}{14} \cdot \left(\tfrac{1}{15}\right)^n \cdot (-25,10,10,-4,3,3,3)^t - \tfrac{1}{28} \cdot \left(\tfrac{1}{25}\right)^n \cdot (0,0,0,5,0,0,-6)^t
\end{aligned} \tag{5.4}$$

for $n \geq 1$. In particular, we see that the degree of a corner vertex is 1 in a random spanning tree of G_n with probability tending to $\frac{11}{14}$, and the degree is 2 with probability tending to $\frac{3}{14}$.

If now $v \in VG_n$ is a vertex of level $k > 0$, then there is a unique copy H of G_{n-k+1} in G_n such that v is the midpoint of one of its sides. The degree distribution of v in a random spanning tree T_n now only depends on k and the type of the restriction of T_n to H. For example, if v is the midpoint of the horizontal side of H, and the restriction is of type $\blacktriangle$, then the probability that v has degree h in T_n is

$$\frac{1}{6}\sum_{\ell=0}^{h}\big(d_{n-k}(\blacktriangle,\ell)+d_{n-k}(\blacktriangle,\ell)+d_{n-k}(\blacktriangle,\ell)\big)d_{n-k}(\blacktriangle,h-\ell)$$
$$+\frac{1}{18}\sum_{\ell=0}^{h}\big(d_{n-k}(\blacktriangle,\ell)+d_{n-k}(\blacktriangle,\ell)+d_{n-k}(\blacktriangle,\ell)\big)$$
$$\times\big(d_{n-k}(\blacktriangle,h-\ell)+d_{n-k}(\blacktriangle,h-\ell)+d_{n-k}(\blacktriangle,h-\ell)\big),$$

where we set $d_n(\cdot,h)=0$ if $h>2$. It follows immediately that for any fixed $k>0$ (or even more generally, if $n-k\to\infty$), the probabilities of the possible degrees $1,2,3,4$ of a level-k vertex converge to

$$0,\quad \frac{121}{196},\quad \frac{33}{98},\quad \frac{9}{196},$$

respectively. Intuitively, this means that leaves typically only occur at high levels.

Let now $W_n(h)$ denote the number of vertices of degree h in a random spanning tree T_n. We prove that $3^{-n}W_n(h)\to w(h)$ almost surely, where

$$w(1)=\frac{10957}{26976},\quad w(2)=\frac{6626035}{9090912},\quad w(3)=\frac{2943139}{9090912},\quad w(4)=\frac{124895}{3030304}.$$

Fix some $\alpha\in(\sqrt{3},3)$. For any $r\ge 0$, the number of copies of G_{r+1} occurring in G_n is 3^{n-r-1}. By (5.3), the number of such copies which have type $\blacktriangle$, $\blacktriangle$ or $\blacktriangle$ is $\frac{53}{96}\cdot 3^{n-r-1}+O(\alpha^{n-r-1})$ with probability $1-O((3/\alpha^2)^{n-r-1})$. The same is true for the types $\blacktriangle$, $\blacktriangle$, $\blacktriangle$ and type $\blacktriangle$, with the constant $\frac{53}{96}$ replaced by $\frac{19}{48}$ and $\frac{5}{96}$, respectively. Now the distribution of the degrees of the midpoints in each of the copies of G_{r+1} only depends on the type, and the different copies are pairwise independent. Let $m_r(\blacktriangle,h)$ be the expectation of the random variable that counts how many of the three "midpoints"

$$\tfrac{1}{2}(u_2+u_3),\qquad \tfrac{1}{2}(u_3+u_1),\qquad \tfrac{1}{2}(u_1+u_2)$$

have degree h in a random spanning forest of type $\blacktriangle$ in G_r, and define $m_r(\blacktriangle,h)$, etc. analogously. By symmetry,

$$m_r(\blacktriangle,h)=m_r(\blacktriangle,h)=m_r(\blacktriangle,h),$$
$$m_r(\blacktriangle,h)=m_r(\blacktriangle,h)=m_r(\blacktriangle,h).$$

By independence and another application of Chebyshev's inequality, we find that the total number of vertices of degree h among all level-$(n-r)$ vertices in a random spanning tree T_n is

$$3^{n-r-1}\big(\tfrac{53}{96}m_{r+1}(\blacktriangle,h)+\tfrac{19}{48}m_{r+1}(\blacktriangle,h)+\tfrac{5}{96}m_{r+1}(\blacktriangle,h)\big)+O(\alpha^{n-r-1})$$

for any $r\ge 0$ with probability $1-O((3/\alpha^2)^{n-r-1})$. Since there are only $O(3^{n/2})$ vertices at levels $\le n/2$, we can safely ignore them, and we obtain that the total number of vertices of degree h in a random spanning tree T_n is

$$W_n(h)=\sum_{r=0}^{\lfloor n/2\rfloor}3^{n-r-1}\big(\tfrac{53}{96}m_{r+1}(\blacktriangle,h)+\tfrac{19}{48}m_{r+1}(\blacktriangle,h)+\tfrac{5}{96}m_{r+1}(\blacktriangle,h)\big)+O(\alpha^n)$$
$$=3^n\sum_{r=0}^{\infty}3^{-r-1}\big(\tfrac{53}{96}m_{r+1}(\blacktriangle,h)+\tfrac{19}{48}m_{r+1}(\blacktriangle,h)+\tfrac{5}{96}m_{r+1}(\blacktriangle,h)\big)+O(\alpha^n)$$

with probability $1-O((3/\alpha^2)^{n/2})$, from which almost sure convergence of $3^{-n}W_n(h)$ follows immediately. It remains to find the values of the constants. Let us for instance determine $m_{r+1}(◣, 1)$:

$$\begin{aligned} m_{r+1}(◣, 1) = \tfrac{2}{9}\big(d_r(◣, 1) + d_r(◢, 1) + d_r(\blacktriangle, 1)\big)\big(d_r(◣, 0) + d_r(◢, 0) + d_r(\blacktriangle, 0)\big) \\ + \tfrac{1}{3}\big(d_r(◣, 1) + d_r(◢, 1) + d_r(\blacktriangle, 1)\big)\big(d_r(◭, 0) + d_r(◮, 0)\big) \\ + \tfrac{1}{3}\big(d_r(◣, 0) + d_r(◢, 0) + d_r(\blacktriangle, 0)\big)\big(d_r(◭, 1) + d_r(◮, 1)\big) \end{aligned}$$

by the same argument that was used earlier to determine the probabilities of the different degrees. Using (5.4) we find

$$m_{r+1}(◣, 1) = \tfrac{1}{1176} \cdot 375^{-r} \cdot (33 \cdot 15^r - 5)^2,$$

and thus

$$\sum_{r=0}^{\infty} 3^{-r-1} m_{r+1}(◣, 1) = \frac{49595}{166352}.$$

All other sums are obtained similarly. It follows that the proportion of vertices of degree $1, 2, 3, 4$ in a random spanning tree T_n converges almost surely to

$$\frac{10957}{40464} \approx 0.270784, \qquad \frac{6626035}{13636368} \approx 0.485909,$$
$$\frac{2943139}{13636368} \approx 0.215830, \qquad \frac{124895}{4545456} \approx 0.0274769,$$

respectively. These constants were already determined by Chang and Chen (2010) as the limits of the mean values, but our arguments show that we even have almost sure convergence.

6. Loop-erased random walk on Sierpiński graphs

This section is devoted to the analysis of loop-erased random walks on Sierpiński graphs and their limit process. Let us first recall some definitions, see for instance Lawler and Limic (2010). Let G be a finite and connected graph. The *(chronological) loop erasure* of a walk $x = (x_0, \ldots, x_n)$ in G yields a new walk $\mathsf{LE}(x)$ which is defined as follows:

- Set $\iota(0) = \max\{j \le n : x_j = x_0\}$.
- If $\iota(k) < n$, then set $\iota(k+1) = \max\{j \le n : x_j = x_{\iota(k)+1}\}$, otherwise set $\iota(k+1) = n$.
- If $K = \min\{k : \iota(k) = n\}$, then $\mathsf{LE}(x) = (x_{\iota(0)}, \ldots, x_{\iota(K)})$.

It is clear from the definition that $\mathsf{LE}(x)$ is self-avoiding.

Simple random walk $(X_n)_{n\ge 0}$ on a finite and connected graph G is a Markov chain with state space VG and transition probabilities $p(x, y)$ from vertex x to vertex y given by

$$p(x,y) = \begin{cases} \frac{1}{\deg x} & \text{if } x \text{ and } y \text{ are adjacent,} \\ 0 & \text{otherwise.} \end{cases}$$

For any $B \subseteq VG$, the *hitting time* $\mathsf{h}(B)$ is given by

$$\mathsf{h}(B) = \inf\{n : X_n \in B\}.$$

Since G is finite and connected, the hitting time $\mathsf{h}(B)$ is almost surely finite. Fix a vertex $x \in VG$ and some set $B \subseteq VG$ with $x \notin B$ and consider simple random walk

$(X_n)_{n\geq 0}$ starting at x. The random self-avoiding walk $\mathsf{LE}((X_n)_{0\leq n\leq \mathsf{h}(B)})$ is called *loop-erased random walk* from x to B. Figure 6.5 shows instances of loop-erased random walks from one corner vertex to another on G_5 and G_8, respectively. The aim of this and the following section is to study some of the properties of loop-erased walk on G_n and its limit process.

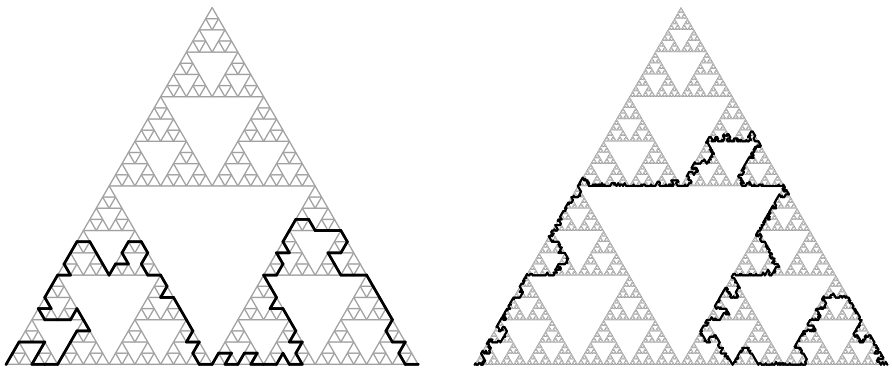


FIGURE 6.5. Instances of loop-erased random walk on G_5 (left) and G_8 (right).

Uniform spanning trees and loop-erased random walk are strongly connected concepts. A particular application of this connection is *Wilson's algorithm* (see Wilson, 1996), which is an efficient method for sampling uniform spanning trees of a graph G. Fix some ordering of the vertex set VG, and let $\{(X_n^x)_{n\geq 0} : x \in VG\}$ be a family of independent simple random walks on G, where $(X_n^x)_{n\geq 0}$ starts at x. We define a sequence $T_0, T_1, \ldots$ of random subtrees of G as follows:

- T_0 consists of the least vertex (according to the selected ordering) in G only.
- If T_k does not contain all vertices of G, let x be the least vertex in $VG \setminus VT_k$ and define
$$T_{k+1} = T_k \cup \mathsf{LE}\big((X_n^x)_{0\leq n\leq \mathsf{h}(VT_k)}\big).$$
If T_k is already spanning, then set $T_{k+1} = T_k$.

By construction there is a minimal (random) index K (at most $|VG|$) such that $T_K = T_{K+1}$. Then T_K is a uniform spanning tree of G. This idea can be reversed: suppose that T is a uniform spanning tree of G, and fix two vertices $x, y \in VG$. The random self-avoiding walk xTy turns out to have precisely the same distribution as a loop-erased random walk from x to y: this is easy to see from Wilson's algorithm if we assume that x and y are the least and second-least vertices in our ordering.

In the following we use this connection to study loop-erased random walk on Sierpiński graphs G_n in more detail: For example, if T is a uniformly chosen spanning tree on G_n, then $u_1 T_n u_2$ is a loop-erased random walk in G_n from u_1 to u_2. The description of T_∞ as a labelled multi-type Galton-Watson tree can be extended to describe the evolution of loop-erased random walks $u_1 T_0 u_2, u_1 T_1 u_2, \ldots$ by a labelled multi-type Galton-Watson tree with twelve types, which capture not only the structure of the spanning tree, but also the unique path between two corner vertices.

The set $\bar{\mathcal{C}} = \{$  $\}$ encodes the twelve possible types (in a rather obvious way). Fix $k \in \{1,2,3\}$ and let v, v' be the two vertices in VG_0 different from u_k. Let f be an element in $\mathcal{Q}_\infty$, so that v, v' are in the same component of the spanning forest $\mathsf{Tr}_0^\infty f$. Then v, v' are in the same component of $\mathsf{Tr}_n^\infty f$ for any $n \geq 0$. For $n \geq 0$ consider those n-parts of G_n which contain at least one edge of the self-avoiding walk $v(\mathsf{Tr}_n^\infty f)v'$, and let

$$W_n(f,k) = \left\{w \in \mathbb{W}^n \,:\, E(v(\mathsf{Tr}_n^\infty f)v') \cap \psi_w(EG_0) \neq \varnothing\right\}$$

be the addresses of these n-parts. Notice that $W_n(f,k)$ is naturally ordered by the fact that $v(\mathsf{Tr}_n^\infty f)v'$ walks along the n-parts $\psi_w(G_0)$ with $w \in W_n(f,k)$. Furthermore,

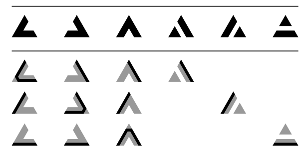
$$W(f,k) = \bigcup_{n\geq 0} W_n(f,k)$$

induces a subtree of $\mathbb{W}^*$, where each word in $W(f,k)$ has two or three children. Of course, $\chi_w(f) \in \{\ ,\ ,\ ,\ ,\ ,\ \}$ for any word $w \in W(f,k)$ (the walk has to enter and leave an n-part at a corner, which is only possible if at least two of the corners are connected). Moreover, for any $w \in W_n(f,k)$, the restriction of $E(v(\mathsf{Tr}_n^\infty f)v')$ to $\psi_w(EG_0)$ consists of one edge $e = \{x, x'\}$ or two incident edges $e = \{x, y\}$ and $e' = \{y, x'\}$ for some $x, x', y \in \psi_w(VG_0)$. Define $\bar{\kappa}_w(f,k)$ to be the unique $i \in \{1,2,3\}$ such that $\psi_w(u_i) \neq x, x'$. We encode the two bits of information given by $\chi_w(f)$ and $\bar{\kappa}_w(f,k)$ by one of the twelve types in $\bar{\mathcal{C}}$ in a natural way. Write $\bar{\chi}_w(f,k)$ to denote this type of the n-part $\psi_w(G_0)$ induced by f and k, and set

$$\bar{\boldsymbol{\chi}}(f,k) = (\bar{\chi}_w(f,k))_{w\in W(f,k)}.$$

For example, $\bar{\chi}_w(f,k) =$ if $\chi_w(f) =$ and $\bar{\kappa}_w(f,k) = 1$. Other types are assigned accordingly, see Table 6.6.

TABLE 6.6. The type $\bar{\chi}_w(f,k)$, given $\chi_w(f)$ and $\bar{\kappa}_w(f,k)$.

$\bar{\kappa}_w(f,k)$	$\chi_w(f)$
1	
2	
3	

In order to reconstruct the self-avoiding walk $v(\mathsf{Tr}_n^\infty f)v'$ from $\bar{\boldsymbol{\chi}}(f,k)$, let $\bar{\eta}$ be the map from $\bar{\mathcal{C}}$ to the set of subgraphs of G_0 defined in Table 6.7. Then

$$v(\mathsf{Tr}_n^\infty f)v' = \bigcup_{w\in W_n(f,k)} \psi_w(\bar{\eta}(\bar{\chi}_w(f,k))).$$

It is noteworthy that in general $\bar{\boldsymbol{\chi}}(f,k)$ contains more information than all the self-avoiding walks $v(\mathsf{Tr}_n^\infty f)v'$ for $n \geq 0$ (since it also contains additional structural information on the underlying spanning tree).

Last but not least, let $\bar{\nu}$ be the bijection from $\bar{\mathcal{C}}$ to $\{1,\dots,12\}$ given by Table 6.7. In analogy to the previous section, we define the type-counting functions $\bar{\chi}^{\#}_{i,n}(f,k) = |\{w \in W_n(f,k) \,:\, \bar{\nu}(\bar{\chi}_w(f,k)) = i\}|$ and

$$\bar{\boldsymbol{\chi}}^{\#}_n(f,k) = \left(\bar{\chi}^{\#}_{1,n}(f,k), \dots, \bar{\chi}^{\#}_{12,n}(f,k)\right)$$

for $i \in \{1,\dots,12\}$ and $n \geq 0$.

Table 6.7. The mappings $\bar\eta$ and $\bar\nu$.

x												
$\bar\eta(x)$												
$\bar\nu(x)$	1	2	3	4	5	6	7	8	9	10	11	12

Proposition 6.1. *Let $\mathcal{U}_\infty$ be one of $\mathcal{T}_\infty$, $\mathcal{T}_\infty^i$, or $\mathcal{S}_\infty^i$ for $i \in \{1,2,3\}$, and let U_∞ be the corresponding random object. Let $k \in \{1,2,3\}$, and assume that $\mathsf{Tr}_0^\infty U_\infty$ connects the two vertices in $VG_0 \setminus \{u_k\}$.*

(1) *The random tree*

$$\bar{\boldsymbol{\chi}}(U_\infty, k) = (\bar\chi_w(U_\infty, k))_{w \in W(U_\infty, k)}$$

is a labelled multi-type Galton-Watson tree with labels in $\mathbb{W}^*$ and types in $\bar{\mathcal{C}}$. The type distribution of the root is given by $\mathsf{Unif}\{\bar\chi_\varnothing(f,k) : f \in \mathcal{U}_\infty\}$. Its offspring generation is given in Table 6.8.

(2) *$(\bar{\boldsymbol{\chi}}_n^\#(U_\infty, k))_{n\geq 0}$ is a multi-type Galton-Watson process with twelve types, which is non-singular, positively regular, and supercritical. Using the abbreviations $s_1 = \frac{1}{3}(z_1 + z_2 + z_7)$, $s_2 = \frac{1}{3}(z_3 + z_4 + z_8)$, and $s_3 = \frac{1}{3}(z_5 + z_6 + z_9)$, the offspring generating function is given by*

$$\begin{aligned}
\bar{\boldsymbol{f}}(\boldsymbol{z}) = \Big(&\tfrac{1}{2}s_1(s_1 + z_{10}),\ \tfrac{1}{2}s_1(s_1 + z_{10}),\\
&\tfrac{1}{2}s_2(s_2 + z_{11}),\ \tfrac{1}{2}s_2(s_2 + z_{11}),\\
&\tfrac{1}{2}s_3(s_3 + z_{12}),\ \tfrac{1}{2}s_3(s_3 + z_{12}),\\
&\tfrac{1}{2}s_1(s_3 z_{11} + s_2 z_{12}),\ \tfrac{1}{2}s_2(s_3 z_{10} + s_1 z_{12}),\ \tfrac{1}{2}s_3(s_2 z_{10} + s_1 z_{11}),\\
&\tfrac{1}{10}\big(3s_1^2 + 4s_1 z_{10} + z_{10}(z_{10} + s_3 z_{11} + s_2 z_{12})\big),\\
&\tfrac{1}{10}\big(3s_2^2 + 4s_2 z_{11} + z_{11}(s_3 z_{10} + z_{11} + s_1 z_{12})\big),\\
&\tfrac{1}{10}\big(3s_3^2 + 4s_3 z_{12} + z_{12}(s_2 z_{10} + s_1 z_{11} + z_{12})\big)\Big).
\end{aligned}$$

Its mean matrix $\bar{\boldsymbol{M}}$ is

$$\bar{\boldsymbol{M}} = \frac{1}{30}\begin{pmatrix}
15 & 15 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 15 & 0 & 0\\
15 & 15 & 0 & 0 & 0 & 0 & 15 & 0 & 0 & 15 & 0 & 0\\
0 & 0 & 15 & 15 & 0 & 0 & 0 & 15 & 0 & 0 & 15 & 0\\
0 & 0 & 15 & 15 & 0 & 0 & 0 & 15 & 0 & 0 & 15 & 0\\
0 & 0 & 0 & 0 & 15 & 15 & 0 & 0 & 15 & 0 & 0 & 15\\
0 & 0 & 0 & 0 & 15 & 15 & 0 & 0 & 15 & 0 & 0 & 15\\
10 & 10 & 5 & 5 & 5 & 5 & 10 & 5 & 5 & 0 & 15 & 15\\
5 & 5 & 10 & 10 & 5 & 5 & 5 & 10 & 5 & 15 & 0 & 15\\
5 & 5 & 5 & 5 & 10 & 10 & 5 & 5 & 10 & 15 & 15 & 0\\
10 & 10 & 1 & 1 & 1 & 1 & 10 & 1 & 1 & 24 & 3 & 3\\
1 & 1 & 10 & 10 & 1 & 1 & 1 & 10 & 1 & 3 & 24 & 3\\
1 & 1 & 1 & 1 & 10 & 10 & 1 & 1 & 10 & 3 & 3 & 24
\end{pmatrix},$$

whose dominating eigenvalue $\bar{\alpha}$ *is* $\frac{4}{3} + \frac{1}{15}\sqrt{205} \approx 2.287855$. *The corresponding right and left eigenvectors are*

$$\bar{\boldsymbol{v}}_R = (a_1, a_1, a_1, a_1, a_1, a_1, a_2, a_2, a_2, a_3, a_3, a_3)^t,$$
$$\bar{\boldsymbol{v}}_L = (a_4, a_4, a_4, a_4, a_4, a_4, a_4, a_4, a_4, a_5, a_5, a_5),$$

where

$$a_1 = \tfrac{11}{26} + \tfrac{17}{533}\sqrt{205}, \qquad a_2 = \tfrac{17}{26} + \tfrac{49}{1066}\sqrt{205}, \qquad a_3 = \tfrac{1}{2} + \tfrac{13}{410}\sqrt{205},$$
$$a_4 = \tfrac{1}{18}\sqrt{205} - \tfrac{13}{18}, \qquad a_5 = \tfrac{5}{2} - \tfrac{1}{6}\sqrt{205}.$$

The vectors $\bar{\boldsymbol{v}}_R$ *and* $\bar{\boldsymbol{v}}_L$ *are normalized so that* $\bar{\boldsymbol{v}}_L \cdot \bar{\boldsymbol{v}}_R = 1$ *and* $\|\bar{\boldsymbol{v}}_L\|_1 = 1$.

(3) *There is a non-negative random variable* $\bar{\theta}(U_\infty, k)$ *such that*

$$\bar{\alpha}^{-n}\bar{\boldsymbol{\chi}}^{\#}_n(U_\infty, k) \to \bar{\boldsymbol{v}}_L\bar{\theta}(U_\infty, k)$$

almost surely. The distribution of $\bar{\theta}(U_\infty, k)$ *has a continuous density function, which is strictly positive on the set of positive reals and zero elsewhere. In particular,* $\bar{\theta}(U_\infty, k)$ *is almost surely positive. By symmetry, there are four different limit distributions, one for each of the following groups:*

$$\{\bar{\theta}(T_\infty, 1), \bar{\theta}(T_\infty, 2), \bar{\theta}(T_\infty, 3)\},$$
$$\{\bar{\theta}(T^1_\infty, 2), \bar{\theta}(T^1_\infty, 3), \bar{\theta}(T^2_\infty, 1), \bar{\theta}(T^2_\infty, 3), \bar{\theta}(T^3_\infty, 1), \bar{\theta}(T^3_\infty, 2)\},$$
$$\{\bar{\theta}(T^1_\infty, 1), \bar{\theta}(T^2_\infty, 2), \bar{\theta}(T^3_\infty, 3)\},$$
$$\{\bar{\theta}(S^1_\infty, 1), \bar{\theta}(S^2_\infty, 2), \bar{\theta}(S^3_\infty, 3)\}.$$

We write $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ *for random variables having the same distribution as a random variable in the respective group (ordered as above). Their expected values are* $\mathbb{E}(\bar{\theta}_0) = \frac{2}{3}a_1 + \frac{1}{3}a_2$, $\mathbb{E}(\bar{\theta}_1) = a_1$, $\mathbb{E}(\bar{\theta}_2) = a_2$ *and* $\mathbb{E}(\bar{\theta}_3) = a_3$, *respectively. Moreover,* $\mathbb{P}_{\bar{\theta}_0} = \frac{2}{3}\mathbb{P}_{\bar{\theta}_1} + \frac{1}{3}\mathbb{P}_{\bar{\theta}_2}$.

Proof: The first part of this result follows from Proposition 5.2. The second is a consequence of the first: the details are not difficult to verify. For the last part, see Mode (1971, Theorem 1.8.2 and Theorem 1.9.1). □

Remark 6.2. Similar to Remark 5.3, we can collapse three groups of types into new types:

- ▲, ▲, ▲ become ▲,
- ▲, ▲, ▲ become ▲,
- ▲, ▲, ▲ become ▲.

Fix again some $k \in \{1, 2, 3\}$, and let $f \in \mathcal{Q}_\infty$ be such that the vertices in $VG_0 \setminus \{u_k\}$ are in the same component of f. Now for $w \in W(f, k)$, set

$$\tilde{\chi}_w(f, k) = \begin{cases} \blacktriangle & \text{if } \bar{\chi}_w(f,k) \in \{\blacktriangle, \blacktriangle, \blacktriangle\}, \\ \blacktriangle & \text{if } \bar{\chi}_w(f,k) \in \{\blacktriangle, \blacktriangle, \blacktriangle\}, \\ \blacktriangle & \text{if } \bar{\chi}_w(f,k) \in \{\blacktriangle, \blacktriangle, \blacktriangle\}, \\ \bar{\chi}_w(f,k) & \text{otherwise,} \end{cases}$$

and $\tilde{\boldsymbol{\chi}}(f, k) = (\tilde{\chi}_w(f, k))_{w \in W(f,k)}$. If U_∞ is now one of T_∞, T^i_∞, S^i_∞ for $i \in \{1, 2, 3\}$, so that the vertices in $VG_0 \setminus \{u_k\}$ are in the same component of $\mathsf{Tr}^\infty_0 U_\infty$, then the random tree $\tilde{\boldsymbol{\chi}}(U_\infty, k)$ is a labelled multi-type Galton-Watson tree with types in $\{\blacktriangle, \blacktriangle, \blacktriangle, \blacktriangle, \blacktriangle, \blacktriangle\}$.

TABLE 6.8. Offspring generation of $\bar{\boldsymbol{\chi}}(U_\infty, k)$ for three types. The remaining types are obtained by symmetry taking suffixes into account.

Type	Offspring types with suffixes $(1,2)$ or $(1,2,3)$	Probability
	(,), (,), (,)	$\frac{1}{6}$
	(,), (,), (,), (,), (,), (,), (,), (,), (,)	$\frac{1}{18}$
	(, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (, ,)	$\frac{1}{18}$
	(, ,), (, ,), (, ,), (, ,), (, ,), (, ,), (,), (,), (,), (,), (,), (,), (,), (,), (,)	$\frac{1}{30}$
	(,), (,), (,), (,), (,), (,)	$\frac{1}{15}$
	(,)	$\frac{1}{10}$

In order to sample a loop-erased random walk in G_n from u_1 to u_2, we can simulate the n-th generation of $\bar{\boldsymbol{\chi}}(T_\infty, 3)$. At first we have to choose one of , , with equal probability as the type of the ancestor $\varnothing$. As in Remark 5.3 we may postpone this choice to the n-th generation. To do so, consider the n-th generation of the simplified tree $\tilde{\boldsymbol{\chi}}(T_\infty, 3)$. Independently replace each occurrence of

- by one of , , ,
- by one of , , ,
- by one of , , ,

always with equal probabilities. Then the modified n-th generation of $\tilde{\boldsymbol{\chi}}(T_\infty, 3)$ describes a loop-erased random walk in G_n from u_1 to u_2.

Remark 6.3. We set

$$\bar{c}(x) = \begin{cases} 1 & \text{if } x \in \{\ ,\ ,\ ,\ ,\ ,\ \}, \\ 2 & \text{if } x \in \{\ ,\ ,\ \}, \\ 3 & \text{if } x \in \{\ ,\ ,\ \}, \end{cases}$$

and once again, we introduce type counters: for $i \in \{1,2,3\}$ and $n \geq 0$, define $\bar{c}^{\#}_{i,n}(f,k) = |\{w \in W_n(f,k) : \bar{c}(\bar{\chi}_w(f)) = i\}|$ and

$$\bar{\boldsymbol{c}}^{\#}_n(f,k) = \big(\bar{c}^{\#}_{1,n}(f,k), \bar{c}^{\#}_{2,n}(f,k), \bar{c}^{\#}_{3,n}(f,k)\big),$$
$$\tilde{\boldsymbol{c}}^{\#}_n(f,k) = \big(\bar{c}^{\#}_{1,n}(f,k) + \bar{c}^{\#}_{2,n}(f,k), \bar{c}^{\#}_{3,n}(f,k)\big).$$

Then $\bar{\boldsymbol{c}}^{\#}_n(f,k)$ and $\tilde{\boldsymbol{c}}^{\#}_n(f,k)$ count the occurrences of types up to symmetry in the n-th generation of $\bar{\boldsymbol{\chi}}(f,k)$ and $\tilde{\boldsymbol{\chi}}(f,k)$, respectively.

For a random object U_∞ (one of T_∞, T^i_∞, S^i_∞) and suitable k, $(\bar{\boldsymbol{c}}^{\#}_n(U_\infty,k))_{n\geq 0}$ and $(\tilde{\boldsymbol{c}}^{\#}_n(U_\infty,k))_{n\geq 0}$ are multi-type Galton-Watson processes with offspring generating functions

$$\bar{\boldsymbol{g}}(z_1,z_2,z_3) = \big(\tfrac{1}{2}s(s+z_3),\ s^2 z_3,\ \tfrac{3}{10}s^2 + \tfrac{1}{5}sz_3(2+z_3) + \tfrac{1}{10}z_3^2\big), \tag{6.1}$$

where $s = \frac{2}{3}z_1 + \frac{1}{3}z_2$, and

$$\tilde{\boldsymbol{g}}(z_1,z_2) = \big(\tfrac{1}{3}z_1(z_1+z_2+z_1z_2),\ \tfrac{3}{10}z_1^2 + \tfrac{1}{5}z_1z_2(2+z_2) + \tfrac{1}{10}z_2^2\big), \tag{6.2}$$

respectively. If we set

$$\boldsymbol{\Sigma}(z_1,z_2,z_3) = \big(\mathsf{PGF}(\bar{\boldsymbol{\chi}}^{\#}_0(T_\infty,k),\boldsymbol{z}), \mathsf{PGF}(\bar{\boldsymbol{\chi}}^{\#}_0(S^k_\infty,k),\boldsymbol{z})\big) = \big(\tfrac{2}{3}z_1 + \tfrac{1}{3}z_2, z_3\big), \tag{6.3}$$

then $\boldsymbol{\Sigma} \circ \bar{\boldsymbol{g}} = \tilde{\boldsymbol{g}} \circ \boldsymbol{\Sigma}$. Note also that $\bar{\boldsymbol{c}}^{\#}_n(U_\infty,k)$ and $\tilde{\boldsymbol{c}}^{\#}_n(U_\infty,k)$ depend linearly on $\bar{\boldsymbol{\chi}}^{\#}_n(U_\infty,k)$, hence Proposition 6.1 implies

$$\bar{\alpha}^{-n}\bar{\boldsymbol{c}}^{\#}_n(T_\infty,k) \to (6a_4,3a_4,3a_5)\bar{\theta}(T_\infty,k), \quad \bar{\alpha}^{-n}\tilde{\boldsymbol{c}}^{\#}_n(T_\infty,k) \to (9a_4,3a_5)\bar{\theta}(T_\infty,k),$$
$$\bar{\alpha}^{-n}\bar{\boldsymbol{c}}^{\#}_n(S^k_\infty,k) \to (6a_4,3a_4,3a_5)\bar{\theta}(S^k_\infty,k), \quad \bar{\alpha}^{-n}\tilde{\boldsymbol{c}}^{\#}_n(S^k_\infty,k) \to (9a_4,3a_5)\bar{\theta}(S^k_\infty,k)$$

almost surely.

Remark 6.4. Using the previous remark, it is possible to describe the distribution of $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$. Let

$$\bar{\boldsymbol{\varphi}}(z) = (\mathbb{E}(e^{z\bar{\theta}_1}), \mathbb{E}(e^{z\bar{\theta}_2}), \mathbb{E}(e^{z\bar{\theta}_3})) \qquad \text{and} \qquad \tilde{\boldsymbol{\varphi}}(z) = (\mathbb{E}(e^{z\bar{\theta}_0}), \mathbb{E}(e^{z\bar{\theta}_3}))$$

be the moment generating functions of $(\bar{\theta}_1,\bar{\theta}_2,\bar{\theta}_3)$ and $(\bar{\theta}_0,\bar{\theta}_3)$, respectively. These two functions exist at least for $z \in \mathbb{C}$ with $\mathrm{Re}(z) \leq 0$. Furthermore, it is well known that

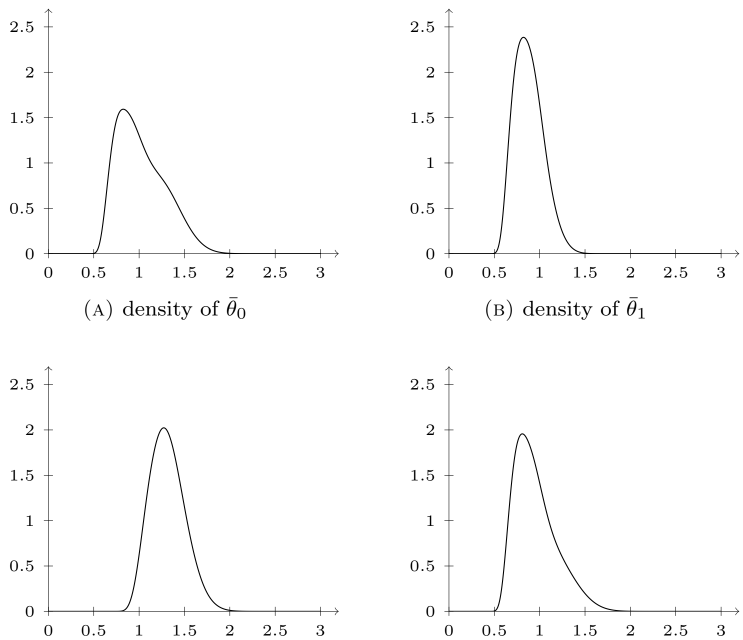
$$\bar{\boldsymbol{\varphi}}(\bar{\alpha}z) = \bar{\boldsymbol{g}}(\bar{\boldsymbol{\varphi}}(z)) \qquad \text{and} \qquad \tilde{\boldsymbol{\varphi}}(\bar{\alpha}z) = \tilde{\boldsymbol{g}}(\tilde{\boldsymbol{\varphi}}(z)).$$

holds whenever both sides are finite, see for instance Mode (1971, Theorem 1.8.1). Since $\bar{\boldsymbol{g}}$ and $\tilde{\boldsymbol{g}}$ are both polynomials, the moment generating functions $\bar{\boldsymbol{\varphi}}$ and $\tilde{\boldsymbol{\varphi}}$ exist for all $z \in \mathbb{C}$ and are entire functions, see Poincaré (1890). Furthermore, by iterating the offspring generating function it is possible to approximate the densities of $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$, see Figure 6.6.

In the following lemma, we prove some estimates for the moment generating functions of $\bar{\theta}_0, \ldots, \bar{\theta}_3$, which lead to estimates for the tails of the distributions. Let us remark that there exist general results concerning tail probabilities (see for instance Jones, 2004), but our situation does not satisfy the necessary conditions of these results. Thus we follow the arguments of Barlow and Perkins (1988, Proposition 3.1) and Kumagai (1993, Proposition 4.2). Let the constants $\bar{\gamma}_\ell$ and $\bar{\gamma}_r$ be defined by

$$\bar{\gamma}_\ell = \frac{\log 2}{\log \bar{\alpha}} \approx 0.837524 \qquad \text{and} \qquad \bar{\gamma}_r = \frac{\log 3}{\log \bar{\alpha}} \approx 1.32744.$$

Thus $\bar{\gamma}_\ell/(1-\bar{\gamma}_\ell) \approx 5.154759$ and $\bar{\gamma}_r/(\bar{\gamma}_r - 1) \approx 4.053954$. These constants play an important role in the following lemma:



(A) density of $\bar{\theta}_0$ (B) density of $\bar{\theta}_1$

(C) density of $\bar{\theta}_2$ (D) density of $\bar{\theta}_3$

FIGURE 6.6. A plot of the densities of $\bar{\theta}_i$ for $i \in \{0,1,2,3\}$. The densities are approximated by $n = 7$ iterations of the offspring generating function $\bar{g}$.

Lemma 6.5. *There are constants $C_{1,\ell}, C_{2,\ell} > 0$ such that*

$$e^{-C_{1,\ell}|z|^{\bar{\gamma}_\ell}} \le \mathbb{E}(e^{z\bar{\theta}_i}) \le e^{-C_{2,\ell}|z|^{\bar{\gamma}_\ell}} \quad (i \in \{0,1,2,3\})$$

for all $z \le -1$. The upper bounds also hold for $z \in \mathbb{C}$ with $\operatorname{Re} z \le 0$ and $|z| \ge 1$ (after taking absolute values). Analogously, there are constants $C_{1,r}, C_{2,r} > 0$ such that

$$e^{C_{1,r} z^{\bar{\gamma}_r}} \le \mathbb{E}(e^{z\bar{\theta}_i}) \le e^{C_{2,r} z^{\bar{\gamma}_r}} \quad (i \in \{0,1,2,3\})$$

for all sufficiently large $z \ge 0$ (for instance if $\mathbb{E}(e^{z\bar{\theta}_i}) \ge 4$ for $i \in \{1,2,3,4\}$). As a consequence the following statements hold:

- *There are constants $C_{3,\ell}, C_{4,\ell}, C_{5,\ell}, C_{6,\ell} > 0$ such that*
  $$C_{3,\ell} \exp(-C_{4,\ell} s^{-\bar{\gamma}_\ell/(1-\bar{\gamma}_\ell)}) \le \mathbb{P}(\bar{\theta}_i \le s) \le C_{5,\ell} \exp(-C_{6,\ell} s^{-\bar{\gamma}_\ell/(1-\bar{\gamma}_\ell)})$$
  for all $s \ge 0$ and all $i \in \{0,1,2,3\}$.
- *There are constants $C_{3,r}, C_{4,r}, C_{5,r}, C_{6,r} > 0$ such that*
  $$C_{3,r} \exp(-C_{4,r} s^{\bar{\gamma}_r/(\bar{\gamma}_r-1)}) \le \mathbb{P}(\bar{\theta}_i \ge s) \le C_{5,r} \exp(-C_{6,r} s^{\bar{\gamma}_r/(\bar{\gamma}_r-1)})$$
  for all $s \ge 0$ and all $i \in \{0,1,2,3\}$.
- *The random variables $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3$ have densities in C^∞.*

Proof: Set $\mathbb{C}_- = \{z \in \mathbb{C} : \operatorname{Re} z \le 0\}$. The random variables $\bar{\theta}_0, \bar{\theta}_1, \bar{\theta}_2, \bar{\theta}_3 \ge 0$ have positive densities on $(0, \infty)$. Thus $0 < |\mathbb{E}(e^{z\bar{\theta}_i})| < 1$ for all $z \in \mathbb{C}_- \setminus \{0\}$ and for all $i \in \{0,1,2,3\}$.

We start with the upper bounds of the left tail. Set $M(z) = \max\{|\mathbb{E}(e^{z\bar{\theta}_i})| : i \in \{0,1,2,3\}\}$. Then $M(\bar{\alpha}z) \leq M(z)^2$ for all $z \in \mathbb{C}_-$ using the functions $\bar{g}$ and $\tilde{g}$. Set $H(z) = -|z|^{-\bar{\gamma}_\ell} \log M(z)$, so that $H(\bar{\alpha}z) \geq H(z)$ for all $z \in \mathbb{C}_-$. Due to continuity there is a constant $C_{2,\ell} > 0$ such that $H(z) \geq C_{2,\ell}$ for all $z \in \mathbb{C}_-$ with $1 \leq |z| \leq \bar{\alpha}$. This implies $H(z) \geq C_{2,\ell}$ for all $z \in \mathbb{C}_-$ with $|z| \geq 1$ and thus $|\mathbb{E}(e^{z\bar{\theta}_i})| \leq e^{-C_{2,\ell}|z|^{\bar{\gamma}_\ell}}$ for all $z \in \mathbb{C}_-$ with $|z| \geq 1$ and $i \in \{0,1,2,3\}$.

For the lower bounds of the left tail set $m(z) = \min\{\mathbb{E}(e^{z\bar{\theta}_i}) : i \in \{0,1,3\}\}$, so that $m(\bar{\alpha}z) \geq \frac{1}{10}m(z)^2$ for all $z \leq 0$. If we set $h(z) = -|z|^{-\bar{\gamma}_\ell} \log m(z)$, then

$$h(\bar{\alpha}z) \leq \tfrac{1}{2}|z|^{-\bar{\gamma}_\ell} \log 10 + h(z)$$

for all $z \leq 0$. For $n \geq 0$ this implies

$$h(\bar{\alpha}^n z) \leq \big((\tfrac{1}{2})^1 + \cdots + (\tfrac{1}{2})^n\big)|z|^{-\bar{\gamma}_\ell} \log 10 + h(z) \leq |z|^{-\bar{\gamma}_\ell} \log 10 + h(z).$$

As before, there is a constant $C_{1,\ell} > 0$ such that $|z|^{-\bar{\gamma}_\ell} \log 10 + h(z) \leq C_{1,\ell}$ for all $-\bar{\alpha} \leq z \leq -1$. This implies $h(z) \leq C_{1,\ell}$ for all $z \leq -1$ and so $\mathbb{E}(e^{z\bar{\theta}_i}) \geq e^{-C_{1,\ell}|z|^{\bar{\gamma}_\ell}}$ for all $z \leq 0$ and $i \in \{0,1,3\}$. If $i = 2$, notice that

$$\bar{g}_2(z_1, z_2, z_3) \geq \tfrac{4}{9} z_1^2 z_3$$

for all $z_1, z_2, z_3 \geq 0$. Hence, using the lower bounds above,

$$\mathbb{E}(e^{\bar{\alpha}z\bar{\theta}_2}) \geq \tfrac{4}{9} e^{-3C_{1,\ell}|z|^{\bar{\gamma}_\ell}}$$

for all $z \leq -1$. By a suitable modification of $C_{1,\ell}$ we get the lower bound for the case $i = 2$.

The proof of the bounds for the right tail is very similar to the proof for the left tail, hence we omit the details.

For the remaining statements, see Barlow and Perkins (1988, Proposition 3.2, Lemma 3.4) and Bingham et al. (1987, Corollary 4.12.8). □

Analogous to Remark 5.4 it is easy to describe the limit behaviour of any parameter of loop-erased random walk in G_n from u_1 to u_2 that is a functional of $\bar{\boldsymbol{\chi}}_n^\#(T_\infty)$. As a simple example we consider the length of loop-erased random walk in G_n from u_1 to u_2, which is given by the distance $d_{T_n}(u_1, u_2)$, where d_{T_n} is the graph metric of the tree T_n. We remark that a similar derivation of the expectations below is given by Dhar and Dhar (1997), Hattori and Mizuno (2014).

Corollary 6.6. *If $n \geq 0$, then the probability generating functions of $d_{T_n}(u_1, u_2)$ and $d_{S_n^3}(u_1, u_2)$ are given by, with $\tilde{\boldsymbol{g}}$, $\bar{\boldsymbol{g}}$ and $\boldsymbol{\Sigma}$ as defined in (6.1), (6.2), (6.3),*

$$\big(\mathsf{PGF}(d_{T_n}(u_1,u_2), z), \mathsf{PGF}(d_{S_n^3}(u_1,u_2), z)\big) = \boldsymbol{\Sigma}(\bar{\boldsymbol{g}}^n(z, z^2, z)) = \tilde{\boldsymbol{g}}^n\big(\tfrac{2}{3}z + \tfrac{1}{3}z^2, z\big)$$

and the expectations are

$$\begin{pmatrix} \mathbb{E}(d_{T_n}(u_1,u_2)) \\ \mathbb{E}(d_{S_n^3}(u_1,u_2)) \end{pmatrix} = \begin{pmatrix} \frac{2}{3} + \frac{5}{123}\sqrt{205} & \frac{2}{3} - \frac{5}{123}\sqrt{205} \\ \frac{1}{2} + \frac{19}{410}\sqrt{205} & \frac{1}{2} - \frac{19}{410}\sqrt{205} \end{pmatrix} \cdot \begin{pmatrix} \big(\frac{4}{3} + \frac{1}{15}\sqrt{205}\,\big)^n \\ \big(\frac{4}{3} - \frac{1}{15}\sqrt{205}\,\big)^n \end{pmatrix}.$$

Furthermore,

$$\bar{\alpha}^{-n} d_{T_n}(u_1,u_2) \to \tfrac{1}{6}(\sqrt{205} - 7)\bar{\theta}(T_\infty, 3), \quad \bar{\alpha}^{-n} d_{S_n^3}(u_1,u_2) \to \tfrac{1}{6}(\sqrt{205} - 7)\bar{\theta}(S_\infty, 3)$$

almost surely as $n \to \infty$.

Proof: By the description using Galton-Watson trees, see Proposition 6.1 and Remark 6.3, we infer that

$$d_{T_n}(u_1,u_2) = \bar{\boldsymbol{c}}_n^{\#}(T_\infty,3)\cdot(1,2,1)^t,$$
$$d_{S_n^3}(u_1,u_2) = \bar{\boldsymbol{c}}_n^{\#}(S_\infty^3,3)\cdot(1,2,1)^t.$$

This implies the statement, since $(6a_4,3a_4,3a_5)\cdot(1,2,1)^t = \frac{1}{6}(\sqrt{205}-7)$. □

7. Convergence of loop-erased random walk

Let C be the set of continuous curves $\gamma\colon[0,\infty]\to K$ with $\gamma(0)=u_1$ and $\gamma(\infty)=u_2$ and set $d_C(\gamma,\delta)=\sup\{\|\gamma(t)-\delta(t)\|_2 : t\in[0,\infty]\}$ for $\gamma,\delta\in C$. Then (C,d_C) is a complete separable metric space. For $\gamma\in C$ set

$$\mathsf{h}(\gamma)=\inf\{t : \gamma(s)=u_2 \text{ for all } s\geq t\}\in(0,\infty].$$

A curve $\gamma\in C$ is called *self-avoiding* if $\gamma(s)\neq\gamma(t)$ for $0\leq s<t\leq\mathsf{h}(\gamma)$. Fix some curve γ in C and some integer $n\geq 0$. Then there is a unique integer $m\geq 1$ and two unique sequences

$$0=t_0<\cdots<t_m=\mathsf{h}(\gamma)$$

and $w_1,\ldots,w_m\in\mathbb{W}^n$ with the following properties:

- The curve γ walks along the n-parts $\psi_{w_j}(K)$: $\gamma([t_{j-1},t_j])\subseteq\psi_{w_j}(K)$ and $\gamma([t_{j-1},t_j])\cap\psi_{w_j}(K\setminus VG_0)\neq\varnothing$ for all $1\leq j\leq m$.
- The quantity t_j is the exit time of γ from $\psi_{w_j}(K)$: $t_j=\inf\{s>t_{j-1} : \gamma(s)\notin\psi_{w_j}(K)\}$ for all $1\leq j\leq m-1$.

As a consequence, the intersection of $\psi_{w_{j-1}}(K)$ and $\psi_{w_j}(K)$ consists of one point only, which is equal to $\gamma(t_{j-1})\in VG_n$. We write $\varDelta_n(\gamma)$ to denote the number m of n-parts traversed, $\mathsf{t}_{j,n}(\gamma)$ to denote the time t_j, and we set

$$W_n(\gamma)=(w_1,\ldots,w_m).$$

Last but not least set $\mathsf{s}_{j,n}(\gamma)=\mathsf{t}_{j,n}(\gamma)-\mathsf{t}_{j-1,n}(\gamma)$, which is the time spent in the n-part $\psi_{w_j}(K)$. It should be stressed, that $\big(\mathsf{t}_{j,n}(\gamma)\big)_{j=0,\ldots,\varDelta_n(\gamma)}$ are in general not equal to the consecutive hitting times on the set VG_n, as it might happen, that the curve γ enters the part $\psi_{w_j}(K)$ at $\psi_{w_j}(u_1)$, visits $\psi_{w_j}(u_3)$ without leaving $\psi_{w_j}(K)$, and leaves at $\psi_{w_j}(u_2)$.

By linear interpolation and constant extension we can associate to any walk $x=(x_0,\ldots,x_r)$ in G_n a curve $\mathsf{LI}(x)\colon[0,\infty]\to K$ as follows:

- Linear interpolation: set $\mathsf{LI}(x)(t)=(k+1-t)x_k+(t-k)x_{k+1}$ if $k\in\{0,\ldots,r-1\}$ and $k\leq t<k+1$.
- Constant extension: set $\mathsf{LI}(x)(t)=x_r$ for $t\geq r$.

If $\lambda>0$, write $\mathsf{LI}(x,\lambda)$ for the curve with rescaled time, i.e., $\mathsf{LI}(x,\lambda)(t)=\mathsf{LI}(x)(\lambda t)$. Note that $\mathsf{LI}(x,\lambda)\in C$ if $x_0=u_1$ and $x_r=u_2$.

Remark 7.1. Let $t\in\mathcal{T}_\infty$ and set $\gamma_n=\mathsf{LI}(u_1(\mathsf{Tr}_n^\infty t)u_2)\in C$ for $n\geq 0$. If $m\geq n$, then the number $\varDelta_n(\gamma_m)$ of n-parts visited by γ_m is given by

$$\varDelta_n(\gamma_m)=\bar{\boldsymbol{c}}_n^{\#}(t,3)\cdot(1,1,1)^t,$$

since $\bar{\boldsymbol{c}}_n^{\#}(t,3)$ counts the n-parts on the unique path from u_1 to u_2 by their type. Moreover, the words in $W_n(\gamma_m)$ associated to the n-parts visited by γ_m and the labels $W_n(t,3)$ of the n-th generation of the tree $\bar{\boldsymbol{\chi}}(t,3)$ are equal, if the natural ordering of $W_n(t,3)$ is used. Finally, the length of the self-avoiding walk $u_1(\mathsf{Tr}_n^\infty\, t)u_2$ is given by

$$\mathsf{h}(\gamma_n) = d_{\mathsf{Tr}_n^\infty\, t}(u_1, u_2) = \bar{\boldsymbol{c}}_n^{\#}(t,3) \cdot (1,2,1)^t,$$

since types  contribute 2 to the length while all other types contribute 1. If $m \geq n$ and $0 \leq j \leq \Delta_n(\gamma_n)$, then

$$\gamma_m(\mathsf{t}_{j,n}(\gamma_m)) = \gamma_n(\mathsf{t}_{j,n}(\gamma_n)) \in VG_n.$$

Let $x_{j,n} = \gamma_n(\mathsf{t}_{j,n}(\gamma_n)) \in VG_n$ and $W_n(\gamma_n) = (w_1, \ldots, w_{\Delta_n(\gamma_n)})$. It follows that, for any $0 \leq i < j \leq \Delta_n(\gamma_n)$,

- the vertices $x_{i,n}$ and $x_{j,n}$ are not the same,
- at $x_{j-1,n}$ the self-avoiding walk $u_1(\mathsf{Tr}_m^\infty\, t)u_2$ enters the n-part $\psi_{w_j}(K)$ and at $x_{j,n}$ it leaves this n-part,
- the quantity $\mathsf{s}_{j,n}(\gamma_m)$ is the length of the self-avoiding walk $u_1(\mathsf{Tr}_m^\infty\, t)u_2$ restricted to the segment from $x_{j-1,n}$ to $x_{j,n}$, i.e., it is equal to the number of edges of this walk inside the n-part $\psi_{w_j}(K)$:

  $$\mathsf{s}_{j,n}(\gamma_m) = d_{\mathsf{Tr}_m^\infty\, t}(x_{j-1,n}, x_{j,n}) = \bar{\boldsymbol{c}}_{m-n}^{\#}(\pi_{w_j}(t), \bar{\kappa}_{w_j}(t,3)) \cdot (1,2,1)^t.$$

The results of Section 6 indicate that $\mathsf{LI}(u_1 T_n u_2, \bar{\alpha}^n)$ converges almost surely for $n \to \infty$. The proof of this fact closely follows the arguments of Barlow and Perkins (1988), Hattori and Hattori (1991) and Kumagai (1993). In the first two references uni-type Galton-Watson processes are used, whereas in the last reference a Galton-Watson process with four types is used.

A pair $(W, (b_w)_{w\in W})$ with $W \subseteq \mathbb{W}^n$ and $b_w \in \bar{\mathcal{C}}$ is called *admissible* of length n if there is an element $t \in \mathcal{T}_\infty$ such that $W = W_n(t,3)$ and $b_w = \bar{\chi}_w(t)$ for $w \in W$. Notice that W inherits the natural ordering from $W_n(t,3)$. An admissible pair $(W, (b_w)_{w\in W})$ completely describes the self-avoiding walk connecting u_1 and u_2 in the spanning tree $\mathsf{Tr}_n^\infty\, t$ for some $t \in \mathcal{T}_\infty$. Loosely speaking, the following lemma states that conditioning on the n-th level, i.e. conditioning on $W_n(T_\infty,3) = W$ and $(\bar{\chi}_w(T_\infty,3))_{w\in W} = (b_w)_{w\in W}$ for some admissible pair $(W,(b_w)_{w\in W})$, the refinements in different n-parts are conditionally independent and for each n-part the refinement yields again a multi-type Galton-Watson tree.

Lemma 7.2. *Let $(W, (b_w)_{w\in W})$ be an admissible pair of length n. Then, under $\mathbb{P}(\,\cdot \mid W_n(T_\infty,3) = W, (\bar{\chi}_w(T_\infty,3))_{w\in W} = (b_w)_{w\in W})$, the following holds:*

- *The random trees $\bar{\boldsymbol{\chi}}(\pi_w(T_\infty), \bar{\kappa}_w(T_\infty,3))$ for $w \in W$ are independent labelled multi-type Galton-Watson trees with labels in $\mathbb{W}^*$ and types in $\bar{\mathcal{C}}$ as described in Proposition 6.1.*
- *For $w \in W$, $\bar{\alpha}^{-n}\bar{\boldsymbol{\chi}}_n^{\#}(\pi_w(T_\infty), \bar{\kappa}_w(T_\infty,3))$ converges almost surely to $\bar{\boldsymbol{v}}_L\bar{\theta}(w)$ for some non-negative random variable $\bar{\theta}(w)$, which has the same distribution as $\bar{\theta}_{\bar{c}(b_w)}$. In particular, $\bar{\theta}(w)$ is almost surely positive. The random variables $\bar{\theta}(w)$ for $w \in W$ are independent.*
- *We have $\Delta_n(\mathsf{LI}(u_1T_nu_2)) = |W|$ almost surely. Let $(w_1, w_2, \ldots)$ be the natural ordering of W and let $m \geq n$. Then the random variables*

  $$\mathsf{s}_{j,n}(\mathsf{LI}(u_1T_mu_2, \bar{\alpha}^m)) = \bar{\alpha}^{-m}\bar{\boldsymbol{c}}_{m-n}^{\#}(\pi_{w_j}(T_\infty), \bar{\kappa}_{w_j}(T_\infty,3)) \cdot (1,2,1)^t$$

  for $1 \leq j \leq |W|$ are independent and

  $$\mathsf{s}_{j,n}(\mathsf{LI}(u_1T_mu_2, \bar{\alpha}^m)) \to \tfrac{1}{6}(\sqrt{205} - 7)\bar{\alpha}^{-n}\bar{\theta}(w_j)$$

  almost surely as $m \to \infty$.

Proof: The first two parts are consequences of Proposition 6.1. The third part follows from the first and the second and from Remark 7.1. □

In the following we set $X_n = \mathsf{LI}(u_1 T_n u_2, \bar{\alpha}^n)$, so that $X_n\colon \Omega \to C$ is a random element in C and

$$X_n(\mathsf{t}_{j,n}(X_n)) = X_m(\mathsf{t}_{j,n}(X_m))$$

for all $m \geq n$. Define

$$\Omega' = \Big\{\omega \in \Omega \,:\, \lim_{m\to\infty} \mathsf{s}_{j,n}(X_m) \in (0,\infty) \text{ for } n \geq 0, 1 \leq j \leq \Delta_n(X_n)\Big\}.$$

Then using Lemma 7.2 we conclude that $\mathbb{P}(\Omega') = 1$. Fix some $\omega \in \Omega'$. For $n \geq 0$ and $1 \leq j \leq \Delta_n(X_n)$ set

$$\mathsf{S}_{j,n} = \lim_{m\to\infty} \mathsf{s}_{j,n}(X_m).$$

It follows that

$$\lim_{m\to\infty} \mathsf{t}_{j,n}(X_m) = \lim_{m\to\infty} \sum_{1\leq k\leq j} \mathsf{s}_{k,n}(X_m) = \sum_{1\leq k\leq j} \mathsf{S}_{k,n} \in (0,\infty).$$

We write $\mathsf{T}_{j,n}$ to denote this limit. Lastly, note that

$$\mathsf{h}(X_m) = \mathsf{t}_{1,0}(X_m) = \mathsf{t}_{\Delta_n(X_n),n}(X_m) \qquad \text{and thus} \qquad \mathsf{T}_{1,0} = \mathsf{T}_{\Delta_n(X_n),n}.$$

Theorem 7.3. *On Ω' the curve X_n converges uniformly as $n \to \infty$ to a limit curve X in C, which satisfies the following properties:*

- *$X(\mathsf{T}_{j,n}) = X_n(\mathsf{t}_{j,n}(X_n)) \in VG_n$ for all $n \geq 0$ and $0 \leq j \leq \Delta_n(X_n)$.*
- *If $W_n(X_n) = \{w_1, \dots, w_{\Delta_n(X_n)}\}$, then*

$$X(\mathsf{T}_{i,n}) \neq X(\mathsf{T}_{j,n}), \qquad X(\mathsf{T}_{j-1,n}), X(\mathsf{T}_{j,n}) \in \psi_{w_j}(VG_0),$$
$$X([\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n}]) \subseteq \psi_{w_j}(K), \qquad X([\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n}]) \cap \psi_{w_j}(K \setminus VG_0) \neq \varnothing$$

  for all $0 \leq i < j \leq \Delta_n(X_n)$. Hence $\Delta_n(X) = \Delta_n(X_n)$ and $W_n(X) = W_n(X_n)$ for all $n \geq 0$.

Proof: We closely follow the arguments of Hattori and Hattori (1991). Fix $\omega \in \Omega'$. We will show that X_n converges uniformly in $[0,\infty]$.

Let $n \geq 1$ be a non-negative integer. Then $\Delta_n(X_n) \geq 2$. By Definition of Ω' we have

$$a = \min\{\mathsf{S}_{j,n} \,:\, 1 \leq j \leq \Delta_n(X_n)\} > 0.$$

Hence there is a positive integer $M = M(n,\omega)$ with $M \geq n$ such that

$$\max\{|\mathsf{t}_{j,n}(X_m) - \mathsf{T}_{j,n}| \,:\, 0 \leq j \leq \Delta_n(X_n)\} \leq a$$

for all $m \geq M$. For convenience set $\mathsf{T}_{\Delta_n(X_n)+1,n} = \mathsf{t}_{\Delta_n(X_n)+1,n}(X_m) = \infty$ for all $0 \leq n \leq m$. Now consider $t \in [0,\infty]$. There is an integer j with $1 \leq j \leq \Delta_n(X_n)+1$ such that $\mathsf{T}_{j-1,n} \leq t \leq \mathsf{T}_{j,n}$. Let $m \geq M$ and distinguish the following cases:

- $1 < j < \Delta_n(X_n)$: We infer that

$$\mathsf{t}_{j-2,n}(X_m) \leq \mathsf{T}_{j-2,n} + a \leq \mathsf{T}_{j-2,n} + \mathsf{S}_{j-1,n} = \mathsf{T}_{j-1,n} \leq t,$$
$$t \leq \mathsf{T}_{j,n} = \mathsf{T}_{j+1,n} - \mathsf{S}_{j+1,n} \leq \mathsf{T}_{j+1,n} - a \leq \mathsf{t}_{j+1,n}(X_m).$$

  Since $X_m([\mathsf{t}_{j-2,n}(X_m), \mathsf{t}_{j+1,n}(X_m)]) \subseteq \psi_{w_1}(K) \cup \psi_{w_2}(K) \cup \psi_{w_3}(K)$ for some $w_1, w_2, w_3 \in \mathbb{W}^n$ with $\psi_{w_1}(K) \cap \psi_{w_2}(K) = \{X_m(\mathsf{t}_{j-1,n}(X_m))\}$ and $\psi_{w_2}(K) \cap \psi_{w_3}(K) = \{X_m(\mathsf{t}_{j,n}(X_m))\}$, we obtain

$$\|X_m(t) - X_m(\mathsf{t}_{j-1,n}(X_m))\|_2 \leq 2^{1-n}.$$

- $j = 1$: It follows as before that $0 \le t \le \mathsf{t}_{2,n}(X_m)$ for all $m \ge M$. Hence
$$\|X_m(t) - X_m(\mathsf{t}_{0,n}(X_m))\|_2 \le 2^{1-n}.$$
- $j = \Delta_n(X_n)$: Again, $\mathsf{t}_{\Delta_n(X_n)-2,n}(X_m) \le t \le \mathsf{t}_{\Delta_n(X_n)+1,n}$ and therefore
$$\|X_m(t) - X_m(\mathsf{t}_{\Delta_n(X_n)-1,n}(X_m))\|_2 \le 2^{-n}.$$
- $j = \Delta_n(X_n) + 1$: Then $\mathsf{t}_{\Delta_n(X_n)-1,n}(X_m) \le t \le \mathsf{t}_{\Delta_n(X_n)+1,n}$ and
$$\|X_m(t) - X_m(\mathsf{t}_{\Delta_n(X_n),n}(X_m))\|_2 \le 2^{-n}.$$

In any case we have
$$\|X_m(t) - X_m(\mathsf{t}_{j-1,n}(X_m))\|_2 \le 2^{1-n}$$
for $m \ge M$. Now let $m_1, m_2 \ge M$. Since $X_{m_1}(\mathsf{t}_{j-1,n}(X_{m_1})) = X_{m_2}(\mathsf{t}_{j-1,n}(X_{m_2}))$, the estimate above implies
$$\begin{aligned}&\|X_{m_1}(t) - X_{m_2}(t)\|_2\\ &\le \|X_{m_1}(t) - X_{m_1}(\mathsf{t}_{j-1,n}(X_{m_1}))\|_2 + \|X_{m_2}(t) - X_{m_2}(\mathsf{t}_{j-1,n}(X_{m_2}))\|_2 \le 2^{2-n}.\end{aligned}$$
As $X_m(0) = u_1$ and $X_m(\infty) = u_2$, we have proved that X_n converges uniformly to a limit curve X in C.

The first property listed in Theorem 7.3 follows from the fact that, for all $m \ge n$, $X_m(\mathsf{t}_{j,n}(X_m)) = X_n(\mathsf{t}_{j,n}(X_n))$ and $\mathsf{t}_{j,n}(X_m) \to \mathsf{T}_{j,n}$, $X_m \to X$ uniformly as $m \to \infty$.

In order to show the second property let t be in $(\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n})$. Then, for sufficiently large m, $t \in (\mathsf{t}_{j-1,n}(X_m), \mathsf{t}_{j,n}(X_m))$. Due to Remark 7.1 we have $X_m(t) \in \psi_{w_j}(K)$ for all sufficiently large m. As $X_m(t) \to X(t)$ it follows that $X(t) \in \psi_{w_j}(K)$. Thus Remark 7.1 and the first property imply the second. $\square$

Let γ be a curve in C, w be a word in $\mathbb{W}^*$, and ι be a letter in $\mathbb{W}$. We say that γ has a *peak* of type ι in the n-part $\psi_w(K)$ if there are $t_1 < t_2$ such that

- $\gamma([t_1, t_2]) \subseteq \psi_w(K)$,
- $\gamma(t_1) \ne \gamma(t_2)$ and $\gamma(t_1), \gamma(t_2) \in \psi_w(VG_0 \setminus \{u_\iota\})$,
- $\gamma((t_1, t_2)) \cap \psi_w(VG_0) = \{\psi_w(u_\iota)\}$.

Intuitively speaking, this means that the curve passes through one of the corners of the n-part $\psi_w(K)$ without moving on to the adjacent part.

Lemma 7.4. *Almost surely, the limit curve X has no peaks. In particular,*
$$X([0,\infty]) \cap VG_n = \{u_1 = X(\mathsf{T}_{0,n}), X(\mathsf{T}_{1,n}), \ldots, X(\mathsf{T}_{\Delta_n(X_n),n}) = u_2\}$$
almost surely for all $n \ge 0$. If $i, j \in \{1, \ldots, \Delta_n(X_n)\}$ with $i < j - 1$, then
$$X([\mathsf{T}_{i-1,n}, \mathsf{T}_{i,n}]) \cap X([\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n}]) = \varnothing$$
almost surely. Finally, $|X([0,\infty]) \cap \psi_w(VG_0)| \le 2$ for all $w \in \mathbb{W}^*$ almost surely.

Proof: If $\iota \in \mathbb{W}$ and $n \ge 0$, we write $\iota^n \in \mathbb{W}^n$ for the n-fold repetition of the letter ι and define $x(\iota) \in \bar{\mathcal{C}}$ by

$x(\iota) =$
 if $\iota = 1$,
 if $\iota = 2$,
 if $\iota = 3$.

Let w be a word in $\mathbb{W}^n$ for some $n \geq 0$. For $m \geq n$ write $A_m = A_m(w,\iota)$ to denote the event

$$A_m = \{w\iota^{m-n} \in W_m(T_\infty, 3), \bar{\chi}_{w\iota^{m-n}}(T_\infty, 3) = x(\iota)\}.$$

Then, for any $m \geq n$, $A_m \supseteq A_{m+1}$ and

$$\mathbb{P}(A_{m+1} \mid A_m) = \tfrac{6}{18} = \tfrac{1}{3}$$

as one can see easily by inspection of Table 6.8. Hence

$$\mathbb{P}(A_m) = \left(\tfrac{1}{3}\right)^{m-n} \mathbb{P}(A_n).$$

Since

$$\{X \text{ has a peak of type } \iota \text{ in } \psi_w(K)\} = \bigcap_{m \geq n} A_m,$$

we infer that

$$\mathbb{P}(X \text{ has a peak of type } \iota \text{ in } \psi_w(K)) = 0.$$

This yields

$$\mathbb{P}(X \text{ has a peak}) \leq \sum_{w \in \mathbb{W}^*} \sum_{\iota \in \mathbb{W}} \mathbb{P}(X \text{ has a peak of type } \iota \text{ in } \psi_w(K)) = 0.$$

In order to show the last assertion of the lemma, let w be a word in $\mathbb{W}^n$. For $i \in \{1,2,3\}$ let w_i be the word in $\mathbb{W}^n$ (if it exists) for which $w_i \neq w$ and $\psi_w(K) \cap \psi_{w_i}(K) = \{\psi_w(u_i)\}$. If $|X([0,\infty]) \cap \psi_w(VG_0)| = 3$ for some $w \in \mathbb{W}^*$, then X has a peak in one of the parts $\psi_w(K)$, $\psi_{w_1}(K)$, $\psi_{w_2}(K)$, $\psi_{w_3}(K)$. Thus

$$\mathbb{P}(|X([0,\infty]) \cap \psi_w(VG_0)| = 3 \text{ for some } w \in \mathbb{W}^*) \leq \mathbb{P}(X \text{ has a peak}) = 0.$$

Consider two indices $i, j \in \{1, \ldots, \mathit{\Delta}_n(X_n)\}$ with $i < j-1$. Then $X([\mathsf{T}_{i-1,n}, \mathsf{T}_{i,n}])$ and $X([\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n}])$ are contained in distinct n-parts of K and, furthermore,

$$\{X(\mathsf{T}_{i-1,n}), X(\mathsf{T}_{i,n})\} \cap \{X(\mathsf{T}_{j-1,n}), X(\mathsf{T}_{j,n})\} = \varnothing.$$

Hence peaks in both n-parts are the only possibility for a non-empty intersection. However, this has probability 0. $\square$

On $\mathit{\Omega}'$ define $\mathsf{S}_{*,n}$ for $n \geq 0$ by

$$\mathsf{S}_{*,n} = \max\{\mathsf{S}_{j,n} \,:\, 1 \leq j \leq \mathit{\Delta}_n(X_n)\}.$$

Then $\mathsf{S}_{*,0} = \mathsf{S}_{1,0} = \mathsf{h}(X)$ and $\mathsf{S}_{*,n+1} \leq \mathsf{S}_{*,n}$ for all $n \geq 0$. Therefore the limit $\lim_{n\to\infty} \mathsf{S}_{*,n}$ exists and is finite on $\mathit{\Omega}'$.

Lemma 7.5. $\mathsf{S}_{*,n} \to 0$ *almost surely as* $n \to \infty$.

Proof: If $(W, (b_w)_{w\in W})$ is admissible, then write $A(W, (b_w)_{w\in W})$ to denote the event

$$A(W, (b_w)_{w\in W}) = \{W_n(T_\infty, 3) = W, (\bar{\chi}_w(T_\infty, 3))_{w\in W} = (b_w)_{w\in W}\}.$$

Let $\epsilon > 0$, then

$$\begin{aligned}\mathbb{P}(\mathsf{S}_{*,n} \geq \epsilon) &= \mathbb{P}(\mathsf{S}_{j,n} \geq \epsilon \text{ for some } 1 \leq j \leq \mathit{\Delta}_n(X_n)) \\ &\leq \sum_{1 \leq j \leq \mathit{\Delta}_n(X_n)} \mathbb{P}(\mathsf{S}_{j,n} \geq \epsilon) \\ &= \sum_{(W,(b_w)_{w\in W})} \mathbb{P}(A(W, (b_w)_{w\in W})) \sum_{1\leq j\leq |W|} \mathbb{P}(\mathsf{S}_{j,n} \geq \epsilon \mid A(W, (b_w)_{w\in W})),\end{aligned}$$

where the sum is taken over all admissible pairs. For sake of notation set $c = \frac{1}{6}(\sqrt{205}-7)$. If $(W,(b_w)_{w\in W})$ is admissible, then, under $\mathbb{P}(\,\cdot \mid A(W,(b_w)_{w\in W}))$, the random variable $\mathsf{S}_{j,n}$ has the same distribution as $c\bar{\alpha}^{-n}\bar{\theta}_i$ for some $i \in \{1,2,3\}$, see Lemma 7.2. For $s \in \mathbb{R}$ set

$$M(s) = \max\{\mathbb{E}(e^{s\bar{\theta}_1}), \mathbb{E}(e^{s\bar{\theta}_2}), \mathbb{E}(e^{s\bar{\theta}_3})\}.$$

Fix some $s > 0$; then $M(cs)$ is finite due to Remark 6.4. Applying Markov's inequality yields

$$\mathbb{P}(c\bar{\alpha}^{-n}\bar{\theta}_i \geq \epsilon) = \mathbb{P}(e^{cs\bar{\theta}_i} \geq e^{s\epsilon\bar{\alpha}^n}) \leq e^{-s\epsilon\bar{\alpha}^n} M(cs)$$

for all $i \in \{1,2,3\}$. Hence we obtain

$$\begin{aligned}
\mathbb{P}(\mathsf{S}_{*,n} \geq \epsilon) &\leq \sum_{(W,(b_w)_{w\in W})} \mathbb{P}(A(W,(b_w)_{w\in W}))|W| e^{-s\epsilon\bar{\alpha}^n} M(cs) \\
&= e^{-s\epsilon\bar{\alpha}^n} M(cs) \sum_{(W,(b_w)_{w\in W})} \mathbb{P}(A(W,(b_w)_{w\in W}))|W| \\
&= e^{-s\epsilon\bar{\alpha}^n} M(cs)\, \mathbb{E}(\Delta_n(X_n))
\end{aligned}$$

using Lemma 7.2 once again. Since $\Delta_n(X_n) = \bar{\boldsymbol{c}}_n^{\#}(T_\infty, 3)\cdot(1,1,1)^t$, a short computation shows that

$$\mathbb{E}(\Delta_n(X_n)) = \big(\tfrac{1}{2}+\tfrac{3}{82}\sqrt{205}\,\big)\cdot\big(\tfrac{4}{3}+\tfrac{1}{15}\sqrt{205}\,\big)^n + \big(\tfrac{1}{2}-\tfrac{3}{82}\sqrt{205}\,\big)\cdot\big(\tfrac{4}{3}-\tfrac{1}{15}\sqrt{205}\,\big)^n \leq 3\bar{\alpha}^n$$

for all $n \geq 0$. Therefore

$$\mathbb{P}(\mathsf{S}_{*,n} \geq \epsilon) \leq 3\bar{\alpha}^n e^{-s\epsilon\bar{\alpha}^n} M(cs)$$

for all $n \geq 0$. By monotonicity

$$\mathbb{P}\Big(\lim_{n\to\infty} \mathsf{S}_{*,n} \geq \epsilon\Big) = 0$$

and, as $\epsilon > 0$ is arbitrary, $\mathsf{S}_{*,n} \to 0$ almost surely. □

Let Ω'' be the set of all $\omega \in \Omega'$ with the property that the assertions of Lemma 7.4 and Lemma 7.5 hold. Then $\mathbb{P}(\Omega'') = 1$. Using the previous preparations we are now able to prove that the curve X is almost surely self-avoiding and that the random times $\mathsf{T}_{j,n}$ are almost surely equal to the consecutive hitting times on the set VG_n.

Theorem 7.6. *On Ω'' the following holds:*

- *The limit curve X is self-avoiding.*
- *For any $1 \leq j \leq \Delta_n(X_n)$,*

$$\mathsf{T}_{j,n} = \mathsf{t}_{j,n}(X) = \inf\{t > \mathsf{t}_{j-1,n}(X) \,:\, X(t) \in VG_n\}.$$

Proof: Fix $\omega \in \Omega''$ and consider times $0 \leq t_1 < t_2 \leq \mathsf{h}(X)$. By Lemma 7.5 there is an integer $n \geq 0$ such that $\mathsf{S}_{*,n} < \frac{1}{3}(t_2 - t_1)$. Thus there are indices $i,j \in \{1,\ldots,\Delta_n(X_n)-1\}$ with $i < j-1$ such that $t_1 \in [\mathsf{T}_{i-1,n}, \mathsf{T}_{i,n}]$ and $t_2 \in [\mathsf{T}_{j-1,n}, \mathsf{T}_{j,n}]$. Since $i < j-1$, Lemma 7.4 implies that $X(t_1) \neq X(t_2)$, which proves that X is self-avoiding.

The second statement follows immediately using the first statement, Lemma 7.4, and Theorem 7.3. □

Remark 7.7. For $\omega \in \Omega''$, the topological closure of the discrete set

$$\mathsf{T} = \{\mathsf{T}_{j,n} \,:\, n \geq 0, 0 \leq j \leq \Delta_n(X_n)\}$$

contains the interval $[0, \mathsf{h}(X)]$. Hence X is the continuous extension of

$$\mathsf{T} \to K, \quad \mathsf{T}_{j,n} \mapsto X_n(\mathsf{t}_{j,n}(X_n)).$$

Remark 7.8. The map

$$\mathcal{T}_n \to C, \quad t \mapsto \mathsf{LI}(u_1 t u_2)$$

is not one-to-one. However, it is possible to use this map and the law of the labelled multi-type Galton-Watson tree of Proposition 6.1 to describe the law of the process X.

We use the following lemma as a partial substitute for the missing Markov property in order to prove some properties of the process $(X(t))_{t\geq 0}$.

Lemma 7.9. *For any $n \in \mathbb{N}_0$, the following holds:*

- *If $t \geq s$ and $\|X(s)\|_2 \geq 2^{-n}$, then $\|X(t)\|_2 \geq 2^{-n}$.*
- *If $t \geq s$, then $\|X(t)\|_2 \geq \frac{1}{2}\|X(s)\|_2$.*
- *On Ω'' we have*

$$\{\|X(t)\|_2 \geq 2^{-n}\} = \{\sup\{\|X(s)\|_2 \,:\, s \leq t\} \geq 2^{-n}\} = \{\mathsf{T}_{1,n} \leq t\} = \{\mathsf{S}_{1,n} \leq t\}.$$

Proof: The first statement is a simple consequence of the geometry of K and implies the second. For the third one note that on Ω'' the curve X is self-avoiding, has no peaks and $\mathsf{T}_{1,n} = \mathsf{S}_{1,n}$ is the hitting time of $\{2^{-n}u_2, 2^{-n}u_3\} = \psi_{11\cdots 1}(\{u_2, u_3\})$, where $11\cdots 1$ is the word of length n whose letters are all equal to 1. For $n \geq 1$, this implies that the first hitting time of $\{2^{-n}u_2, 2^{-n}u_3\} = \psi_{11\cdots 1}(\{u_2, u_3\})$ is equal to the last exit time of the set $2^{-n}K = \psi_{11\cdots 1}(K)$. This implies the statement. $\square$

Theorem 7.10. *The following holds:*

(1) *There are $C_{7,\ell}, C_{8,\ell} > 0$ such that for all $s, t \in [0, \infty)$ and all $\delta \in [0, 1]$,*

$$C_{3,\ell} \exp(-C_{7,\ell}(\delta t^{-\bar{\gamma}_\ell})^{1/(1-\bar{\gamma}_\ell)}) \leq \mathbb{P}(\|X(t)\|_2 \geq \delta)$$

and

$$\begin{aligned}\mathbb{P}(\|X(s+t) - X(s)\|_2 \geq \delta) &\leq \mathbb{P}(\sup\{\|X(s+u) - X(s)\|_2 \,:\, 0 \leq u \leq t\} \geq \delta) \\ &\leq C_{5,\ell} \exp(-C_{8,\ell}(\delta t^{-\bar{\gamma}_\ell})^{1/(1-\bar{\gamma}_\ell)}).\end{aligned}$$

(2) *There are $C_{7,r}, C_{8,r} > 0$ such that for all $t \in [0, \infty)$ and all $\delta \in [0, 1]$,*

$$\begin{aligned}C_{3,r} \exp(-C_{7,r}(\delta^{-1/\bar{\gamma}_\ell} t)^{\bar{\gamma}_r/(\bar{\gamma}_r - 1)}) &\leq \mathbb{P}(\sup\{\|X(u)\|_2 \,:\, 0 \leq u \leq t\} \leq \delta) \\ &\leq \mathbb{P}(\|X(t)\|_2 \leq \delta)\end{aligned}$$

and

$$\mathbb{P}(\|X(t)\|_2 \leq \delta) \leq C_{5,r} \exp(-C_{8,r}(\delta^{-1/\bar{\gamma}_\ell} t)^{\bar{\gamma}_r/(\bar{\gamma}_r - 1)}).$$

(3) *For any $p > 0$, there exist constants $C_9(p), C_{10}(p) > 0$ such that for all $s \in [0, \infty)$ and all $t \in [0, 1]$,*

$$C_9(p)\, t^{p\bar{\gamma}_\ell} \leq \mathbb{E}(\|X(t)\|_2^p) \qquad \textit{and} \qquad \mathbb{E}(\|X(s+t) - X(s)\|_2^p) \leq C_{10}(p)\, t^{p\bar{\gamma}_\ell}.$$

(4) *There are constants* $C_{11}, C_{12} > 0$ *such that for all* $s \in [0,\infty)$

$$\limsup_{t\searrow 0} \frac{\|X(s+t)-X(s)\|_2}{t^{\bar{\gamma}_\ell}(\log\log(1/t))^{1-\bar{\gamma}_\ell}} \le C_{11} \qquad \textit{and}$$

$$\liminf_{t\searrow 0} \frac{\|X(t)\|_2}{t^{\bar{\gamma}_\ell}(\log\log(1/t))^{-\bar{\gamma}_\ell(1-1/\bar{\gamma}_r)}} \ge C_{12}$$

hold almost surely. Note that $1-\bar{\gamma}_\ell \approx 0.162475 > 0$ *and* $-\bar{\gamma}_\ell(1-1/\bar{\gamma}_r) \approx -0.206594 < 0$.

(5) *The Hausdorff dimension* $\dim_H X([0,\infty])$ *of the path* $X([0,\infty])$ *almost surely satisfies*

$$\dim_H X([0,\infty]) = \frac{1}{\bar{\gamma}_\ell} = \frac{\log\bar{\alpha}}{\log 2} \approx 1.193995.$$

Proof: In order to prove the first statement choose $n \in \mathbb{N}$ such that $2^{-n} \le \delta \le 2^{-(n-1)}$. Then, using Lemma 7.9,

$$\mathbb{P}(\mathsf{S}_{1,n-1} \le t) \le \mathbb{P}(\|X(t)\|_2 \ge \delta)$$

and

$$\mathbb{P}(\sup\{\|X(s+u)-X(s)\|_2 \,:\, 0 \le u \le t\} \ge \delta) \\ \le \mathbb{P}(\mathsf{T}_{j-1,n+1} \ge s, \mathsf{S}_{j,n+1} \le t \text{ for some } j \ge 1).$$

By conditioning as in Lemma 7.2, the distribution of $\mathsf{S}_{j,m}$ is equal to the distribution of $\frac{1}{6}(\sqrt{205}-7)\bar{\alpha}^{-m}\bar{\theta}_i$ for some $i \in \{1,2,3\}$ and $\mathsf{S}_{j,m}$ is independent of $\mathsf{T}_{j-1,m}$. Hence the bounds on the tail probability follow from Lemma 6.5. More or less the same arguments yield the second statement. By integrating the bounds of the first statement we get the bounds on $\mathbb{E}(\|X(t)\|_2^p)$ and $\mathbb{E}(\|X(s+t)-X(s)\|_2^p)$, respectively. The fourth statement follows by the usual Borel-Cantelli argument. The path $X([0,\infty])$ is the limit set of a random recursive construction with multiple types. Thus the formula for the Hausdorff dimension follows from Hattori (2000, Theorem 3.8), where such random sets are studied in general. □

Remark 7.11. The properties (1), (3), (4) proved above are slightly weaker forms of analogous results of Barlow and Perkins (1988, Theorem 4.3, Corollary 4.4, Theorem 4.7) for Brownian motion and Kumagai (1993, Theorem 4.5, Corollary 4.6, Theorem 4.8) for more general diffusion processes that contain Brownian motion as a special case, respectively. In several cases the statements of the previous theorem are formulated for the special increment $X(t) = X(t) - X(0)$ and not for a general increment $X(s+t) - X(s)$, which is, we only consider the starting time $s = 0$. One reason for the weaker statements is the lack of the Markov property. Another difficulty in the general case lies in the fact that parts of the curve that lie in different k-parts of the Sierpiński gasket K can still be close to each other near the vertices where these k-parts are connected. At the corner (time $s = 0$), this cannot happen. It seems plausible, however, that the strong forms of the cited statements also hold in our case. Fortunately, the formula for the Hausdorff dimension does not rely on the first four properties, but only on the fact that the path $X([0,\infty])$ is the limit set of a specific random recursive construction with multiple types.

Remark 7.12. We note that all we have proved in this section remains true if we replace T_n by S_n^3. In particular, $\mathsf{LI}(u_1 S_n^3 u_2, \bar{\alpha}^n)$ converges almost surely in (C, d_C) to a limit curve and the results of 7.3–7.10 hold with S_n^3 in place of T_n.

8. Limit of the tree metric

Consider a generic $\omega \in \Omega$. Then $T_n(\omega)$ is a spanning tree on G_n and it is the trace $\mathsf{Tr}_n^m T_m(\omega)$ for all $m \geq n$. Let u, v be two vertices in VG_n for some $n \geq 0$. Their distance $d_{T_m(\omega)}(u, v)$ with respect to the spanning tree $T_m(\omega)$ is well-defined and Corollary 6.6 indicates that $\bar{\alpha}^{-m} d_{T_m(\omega)}(u, v)$ converges for $m \to \infty$, where $\bar{\alpha} = \frac{4}{3} + \frac{1}{15}\sqrt{205}$ is the dominating eigenvalue of Proposition 6.1. If this limit exists for all u, v in the countable set

$$V_* = \bigcup_{n \geq 0} VG_n = \bigcup_{w \in \mathbb{W}^*} \psi_w(VG_0),$$

and it is positive whenever $u \neq v$, then the limit defines a metric $d_{*,\omega}$ on V_*:

$$d_{*,\omega}(u, v) = \lim_{m \to \infty} \bar{\alpha}^{-m} d_{T_m(\omega)}(u, v)$$

for all $u, v \in V_*$. In the following we show that $d_{*,\omega}$ exists for almost all $\omega \in \Omega$ and yields a random metric d_* on V_*. Let $\mathbb{M}(V_*)$ be the set of all metrics on V_*. We equip $\mathbb{M}(V_*)$ with the σ-algebra $\mathcal{M}(V_*)$ which is induced by the mappings

$$\mathbb{M}(V_*) \to \mathbb{R}, \quad d \mapsto d(u, v)$$

for $u, v \in V_*$. We recall some notions from metric theory, see for instance Chiswell (2001). A metric space (X, d) is 0-*hyperbolic* if

$$d(u, v) + d(x, y) \leq \max\{d(u, x) + d(v, y), d(u, y) + d(v, x)\}$$

holds for all $u, v, x, y \in X$ (*four point condition*). A *metric segment* in (X, d) is the image of an isometric embedding $[a, b] \to X$ for some $a, b \in \mathbb{R}$. Finally, (X, d) is called an $\mathbb{R}$-*tree* if, for any $x, y \in X$, there is a unique arc connecting x, y and this arc is a metric segment. We note that (X, d) is an $\mathbb{R}$-tree if and only if (X, d) is connected and 0-hyperbolic, see Chiswell (2001, Lemma 2.4.13).

Theorem 8.1. *For almost all $\omega \in \Omega$ the limit*

$$d_{*,\omega}(u, v) = \lim_{m \to \infty} \bar{\alpha}^{-m} d_{T_m(\omega)}(u, v)$$

exists for all $u, v \in V_*$ and yields a metric $d_{*,\omega}$ on the set V_*, such that $(V_*, d_{*,\omega})$ is a 0-hyperbolic and totally bounded metric space. Thus, for a suitable subset $\Omega''' \subseteq \Omega$ of probability 1,

$$\Omega''' \to \mathbb{M}(V_*), \quad \omega \mapsto d_{*,\omega}$$

is a random metric in $(\mathbb{M}(V_*), \mathcal{M}(V_*))$. Furthermore, for $\omega \in \Omega'''$ the Cauchy completion of $(V_*, d_{*,\omega})$ is a compact $\mathbb{R}$-tree.

Proof: For $x, y \in VG_0$ and $w \in \mathbb{W}^n$, define $\Omega(w, x, y)$ to be the set of all $\omega \in \Omega$ such that, whenever x, y are connected in the restriction $\pi_w(T_n(\omega))$, the curve $\mathsf{LI}(x\pi_w(T_m(\omega))y, \bar{\alpha}^m)$ converges in (C, d_C) as $m \to \infty$, $m \geq n$, and the assertions of Theorems 7.3–7.6 hold. The usual conditioning argument shows that $\mathbb{P}(\Omega(w, x, y)) = 1$ for all $w \in \mathbb{W}^*$ and all $x, y \in VG_0$. Thus

$$\Omega''' = \bigcap_{w \in \mathbb{W}^*} \bigcap_{x, y \in VG_0} \Omega(w, x, y)$$

has probability 1. Fix an element $\omega \in \Omega'''$. Then for all $u, v \in V_*$ the limit

$$d_{*,\omega}(u, v) = \lim_{m \to \infty} \bar{\alpha}^{-m} d_{T_m(\omega)}(u, v)$$

exists and is an element of $[0, \infty)$. By construction of Ω''', we have $d_{*,\omega}(u, v) > 0$ for all $u, v \in V_*$, $u \neq v$, which are neighbours in G_n for some n. Hence $d_{*,\omega}(u, v) > 0$ for all $u, v \in V_*$, $u \neq v$. Furthermore, as $d_{T_m(\omega)}$ is the graph metric of the tree $T_m(\omega)$, it satisfies the triangle inequality and the four point condition. Thus the limit $d_{*,\omega}$ also satisfies the triangle inequality and the four point condition. Altogether we have proved that $(V_*, d_{*,\omega})$ is a 0-hyperbolic metric space if $\omega \in \Omega'''$. For $x, y \in VG_0$ and $w \in \mathbb{W}^n$ define $A(w, x, y)$ to be the set of all $\omega \in \Omega'''$, such that, whenever x, y are connected in the restriction $\pi_w(T_n(\omega))$, then $d_{*,\omega}(\psi_w(x), \psi_w(y)) \leq 2^{-n}$. Using the Borel-Cantelli lemma together with the bounds of Lemma 6.5, we see that

$$A_n = \bigcap_{w \in \mathbb{W}^n} \bigcap_{x,y \in VG_0} A(w, x, y)$$

holds eventually with probability 1. Hence, for $\omega \in \Omega'''$, there is an $N = N(\omega)$ such that $\omega \in A_n$ for all $n \geq N$. Fix some $n \geq N$. For $x \in VG_n$ let $C_x = C_x(\omega)$ be the set of all $y \in VG_m$ $(m \geq n)$, such that all vertices v on the path $xT_m(\omega)y$ satisfy $\|v - x\|_2 \leq 2^{-n}$. If $y \in C_x \cap VG_n$, then $d_{*,\omega}(x, y) \leq 2^{-n}$. If $y \in C_x \setminus VG_n$, then we can find $x = x_n, x_{n+1}, \ldots, x_m = y$, such that $x_k \in VG_k$ and x_{k-1}, x_k are either identical or neighbours in $T_k(\omega)$. Thus

$$d_{*,\omega}(x, y) \leq \sum_{k=n+1}^{m} d_{*,\omega}(x_{k-1}, x_k) \leq \sum_{k=n+1}^{m} 2^{-k} \leq 2^{-n}.$$

Thus, if $B_{*,\omega}(x, 2^{-n})$ denotes the ball of radius 2^{-n} centered at x with respect to $d_{*,\omega}$, then $C_x \subseteq B_{*,\omega}(x, 2^{-n})$. Hence

$$V_* = \bigcup_{x \in VG_n} C_x = \bigcup_{x \in VG_n} B_{*,\omega}(x, 2^{-n}),$$

which means that $(V_*, d_{*,\omega})$ is totally bounded. To check measurability we note that $\omega \mapsto d_{T_m(\omega)}(u, v)$ is measurable for fixed $u, v \in V_*$ (if m is sufficiently large). Thus the limit $\omega \mapsto d_{*,\omega}(u, v)$ is measurable, too. By definition of $\mathcal{M}(V_*)$, this implies measurability of $\omega \mapsto d_{*,\omega}$.

In order to prove that the Cauchy completion $(\check{V}_{*,\omega}, \check{d}_{*,\omega})$ of $(V_*, d_{*,\omega})$ for $\omega \in \Omega'''$ is an $\mathbb{R}$-tree, it is sufficient to show that the completion is connected, as 0-hyperbolicity is preserved by completion, see Chiswell (2001, Lemma 2.2.11). We show that the completion contains a path from u_1 to any x. Let $x_1, x_2, \ldots$ be a Cauchy sequence in V_* with $x_n \to x$. Denote by $\alpha_n \colon [0, \infty] \to K$ the limit curve of $\mathsf{LI}(u_1 T_m(\omega) x_n, \bar{\alpha}^m)$ as $m \to \infty$, which exists by construction of Ω'''. Then $D_n = \alpha_n^{-1}(V_*)$ is a dense subset of $[0, \infty]$ by Lemma 7.5. Note that $t = d_{*,\omega}(u_1, \alpha_n(t))$ for all $t \in D_n$ such that $t \leq \min\{s : \alpha_n(s) = x_n\}$. Therefore the restriction $\alpha_n \colon D_n \to V_*$ is continuous with respect to $d_{*,\omega}$ and thus has a continuous extension $\beta_n \colon [0, \infty] \to \check{V}_{*,\omega}$. Set $s_0 = 0$ and

$$s_n = \max\{t \in [0, \infty] \,:\, \beta_k = \beta_n \text{ on } [0, t] \text{ for all } k \geq n\}.$$

Then we have $s_0 \leq s_1 \leq \cdots$ and $\beta_n(s_n) \to x$, and

$$\beta \colon [0, \infty] \to \check{V}_{*,\omega}, \quad \beta(t) = \begin{cases} u_1 & \text{if } t = 0, \\ \beta_n(t) & \text{if } s_{n-1} < t \leq s_n, \\ x & \text{otherwise,} \end{cases}$$

is a continuous curve connecting u_1 and x (whose image is a metric segment). Finally, $(\check{V}_{*,\omega}, \check{d}_{*,\omega})$ is compact for $\omega \in \Omega'''$, since it is the completion of the totally bounded metric space $(V_*, d_{*,\omega})$. □

Let ω be an element of the set Ω''' defined in the previous proof and let $(\check{V}_{*,\omega}, \check{d}_{*,\omega})$ be the Cauchy completion of $(V_*, d_{*,\omega})$. Consider an element $x \in \check{V}_{*,\omega}$. Suppose that $x_1, x_2, \dots$ is a Cauchy sequence in $(V_*, d_{*,\omega})$, such that $x_n \to x$ with respect to $\check{d}_{*,\omega}$. Then it is easy to see that $x_1, x_2, \dots$ is also a Cauchy sequence in $(V_*, \|\cdot\|_2)$ and thus has a limit in $(K, \|\cdot\|_2)$, which does not depend on the specific Cauchy sequence but only on $x \in \check{V}_{*,\omega}$. We write $\xi_\omega(x)$ to denote this limit in $(K, \|\cdot\|_2)$. Then $\xi_\omega \colon \check{V}_{*,\omega} \to K$ is a well-defined, continuous map, such that the restriction $\xi_\omega|_{V_*}$ to V_* is the identity.

Lemma 8.2. *Let ω be in Ω'''. Then $1 \leq |\xi_\omega^{-1}(x)| \leq 4$ for all $x \in V_*$ and $1 \leq |\xi_\omega^{-1}(x)| \leq 3$ for all $x \in K \setminus V_*$.*

Proof: For every point $x \in K$ we can find a sequence in V_* that converges to this point in $(K, \|\cdot\|_2)$ and which is Cauchy in $(V_*, d_{*,\omega})$. Thus the map ξ_ω is surjective, whence $|\xi_\omega^{-1}(x)| \geq 1$. As in the previous proof every sequence $x_1, x_2, \dots \in V_*$ converging to a point in $\xi_\omega^{-1}(x)$ in $(V_*, d_{*,\omega})$ yields a metric segment connecting u_1 and that point. Using the geometry of the Sierpiński gasket it is easy to see that there are at most four (respectively three if $x \notin V_*$) distinct metric segments joining u_1 and a point in $\xi_\omega^{-1}(x)$. This proves the claim. □

Theorem 8.3. *Let ω be an element of Ω'''. Then the hitting time $\mathsf{h}(X(\omega))$ of the limit curve $X(\omega)$ in u_2 is equal to the distance $d_{*,\omega}(u_1, u_2)$. Furthermore, if $\gamma_\omega \colon [0, d_{*,\omega}(u_1, u_2)] \to \check{V}_{*,\omega}$ is the unique isometric embedding with $\gamma_\omega(0) = u_1$ and $\gamma_\omega(d_{*,\omega}(u_1, u_2)) = u_2$, then*

$$X(t, \omega) = \xi_\omega(\gamma_\omega(t))$$

for all $t \in [0, d_{*,\omega}(u_1, u_2)]$.

Proof: The statement is a consequence of the definition of the limit curve $X(\omega)$ and the limit metric $d_{*,\omega}$, see Theorem 7.3 and Theorem 8.1. □

For $\omega \in \Omega'''$ define $A(\omega)$ to be the set $\{x \in K : |\xi_\omega^{-1}(x)| > 1\}$. These are points that "can be reached from two (or more) different directions". To understand how this happens, it is useful to consider spanning forests with two components: given for instance some $f \in \mathcal{S}^1_\infty$, every element v of V_* can be associated uniquely to one of the components: $v \in V(G_n)$ for some n, and v either belongs to the same component as u_1 in $\mathsf{Tr}_m^\infty f$ for all $m \geq n$ or to the same component as u_2 and u_3, again for all $m \geq n$. There are, however, some points in the completion K that can be reached as limits from both sides; they form the so-called "interface". In a spanning tree, there is only one component, but the same phenomenon can occur at higher levels, within certain n-parts on which the spanning tree induces a spanning forest with more than one component.

In the following we give a description of $A(\omega)$ in terms of Galton-Watson trees and show that the Hausdorff dimension $\dim_H A(\omega)$ is strictly less than 1 for almost all ω. For $f \in \mathcal{Q}_\infty$ and $n \geq 0$ let $\check{W}_n(f)$ be the set of all $w \in \mathbb{W}^n$, such that

$\psi_w(VG_0)$ contains vertices of two distinct components of $\mathsf{Tr}_n^\infty f$. The union

$$\check{W}(f) = \bigcup_{n\geq 0} \check{W}_n(f)$$

induces a subtree of $\mathbb{W}^*$. On a single n-part $\psi_w(VG_0)$ with $w \in \check{W}_n(f)$ we always observe one of the following possibilities:

- The restriction $\pi_w(\mathsf{Tr}_n^\infty f)$ has two components and these two components belong to two distinct components of $\mathsf{Tr}_n^\infty f$. In this case we set $\check{\chi}_w(f) = \chi_w(f) \in \{$▲, ▲, ▲$\}$.
- The restriction $\pi_w(\mathsf{Tr}_n^\infty f)$ has three components and two of them belong to the same component of $\mathsf{Tr}_n^\infty f$. In this case we define $\check{\chi}_w(f) \in \{$▲, ▲, ▲$\}$ depending on which two of the three components in $\pi_w(\mathsf{Tr}_n^\infty f)$ belong to the same component of $\mathsf{Tr}_n^\infty f$.
- The restriction $\pi_w(\mathsf{Tr}_n^\infty f)$ has three components and these three components belong to three distinct components of $\mathsf{Tr}_n^\infty f$. In this case we set $\check{\chi}_w(f) = \chi_w(f) =$ ▲.

Let $\check{\mathcal{C}} = \{$▲, ▲, ▲, ▲, ▲, ▲, ▲$\}$ and set

$$\check{\boldsymbol{\chi}}(f) = (\check{\chi}_w(f))_{w\in\check{W}(f)}.$$

As in Section 5.1 it is easy to see that $\check{\boldsymbol{\chi}}(U_\infty)$ is a labelled multi-type Galton-Watson tree with types in $\check{\mathcal{C}}$, where U_∞ is one of $S_\infty^1, S_\infty^2, S_\infty^3, R_\infty$. The associated counting process $(\check{\boldsymbol{c}}^\#(U_\infty))_{n\geq 0}$, which counts type occurrences in one generation up to symmetry, is a multi-type Galton-Watson process with three types, offspring generating function

$$\begin{aligned}\check{\boldsymbol{g}}(\boldsymbol{z}) = \Big(&\tfrac{1}{10}(4z_1 + 3z_1^2 + 3z_2),\\ &\tfrac{1}{25}(6z_1 + 3z_1^2 + z_1^3 + 6z_2 + 9z_1z_2),\\ &\tfrac{1}{25}z_1(3z_1 + 4z_1^2 + 9z_2 + 9z_3)\Big)\end{aligned}$$

and mean matrix

$$\check{\boldsymbol{M}} = \frac{1}{50} \cdot \begin{pmatrix} 50 & 15 & 0 \\ 48 & 30 & 0 \\ 72 & 18 & 18 \end{pmatrix}.$$

This mean matrix has the dominating eigenvalue $\check{\alpha} = \frac{3}{5}\bar{\alpha} = \frac{4}{5} + \frac{1}{25}\sqrt{205} \approx 1.372712$. Define

$$I(f) = \bigcap_{n\geq 0} \bigcup_{w\in\check{W}_n(f)} \psi_w(K).$$

Then $I(f)$ is the limit set of the component boundaries and

$$\dim_H I(U_\infty) \leq \frac{\log\check{\alpha}}{\log 2} = \frac{\log\bar{\alpha}}{\log 2} - \frac{\log\frac{5}{3}}{\log 2} \approx 0.457029$$

holds almost surely using a result by Tsujii (1991, Proposition 3.9). It seems that other results on the Hausdorff dimension do not apply to this specific random recursive construction, so that we only obtain an upper bound. Of course, $I(T_\infty) = \varnothing$ and so $\dim_H I(T_\infty) = 0$.

Proposition 8.4. *For $\omega \in \Omega'''$ we have*

$$A(\omega) = \bigcup_{w \in \mathbb{W}^*} \psi_w(I(\pi_w(T_\infty(\omega))))$$

and thus

$$\dim_H A(\omega) \leq \frac{\log \breve{\alpha}}{\log 2} = \frac{\log \bar{\alpha}}{\log 2} - \frac{\log \frac{5}{3}}{\log 2} \approx 0.457029$$

for almost all ω.

Proof: Note that $A(\omega)$ contains $\psi_w(I(\pi_w(T_\infty(\omega))))$ for all $w \in \mathbb{W}^*$. On the other hand, if $x \in A(\omega)$, then $\xi_\omega^{-1}(x)$ contains at least two distinct points in $\check{V}_{*,\omega}$, say x_1 and x_2. Denote by $\overline{u_1x_1}$ (respectively $\overline{u_1x_2}$) the metric segment connecting u_1 and x_1 (respectively x_2). Then there is a word $w \in \mathbb{W}^*$ such that $x \in \psi_w(K)$ and

$$\overline{u_1x_1} \cap \overline{u_1x_2} \cap \xi_\omega^{-1}(\psi_w(K)) = \varnothing.$$

This implies that $x \in \psi_w(I(\pi_w(T_\infty(\omega))))$. The usual conditioning argument shows that

$$\dim_H \psi_w(I(\pi_w(T_\infty(\omega)))) \leq \frac{\log \breve{\alpha}}{\log 2}$$

for almost all ω. As $\mathbb{W}^*$ is a countable set and the Hausdorff dimension behaves nicely under countable unions the claim follows. □

Remark 8.5. Note the occurrence of the constant $\frac{5}{3}$, which is the *resistance scaling factor* of the Sierpiński gasket. It also occurs prominently in the formula for the number of spanning trees (see Teufl and Wagner (2011b) for the connection between resistance scaling and the number of spanning trees): if we regard G_n as an electrical network, where each edge represents a unit resistor, then the effective resistance between two of the boundary vertices u_1, u_2, u_3 is $\frac{2}{3} \cdot (\frac{5}{3})^n$. There is a simple heuristic explanation why the identity

$$\log \breve{\alpha} = \log \bar{\alpha} - \log \tfrac{5}{3}$$

must hold: it is well known (cf. Bollobás, 1998, p. 44, Theorem 1) that the effective resistance between two vertices equals the number of *thickets*, i.e., spanning forests with two components each containing one of the two vertices, divided by the number of spanning trees. For every spanning tree of G_n, one can obtain a thicket by removing an edge from the unique path between u_1 and u_2; conversely, we can turn a thicket into a spanning tree by inserting an edge that connects the two components at the interface. The identity now follows (at least heuristically) from a simple double-counting argument.

9. Other self-similar graphs

The same ideas apply to other self-similar graphs as well: it was shown by Teufl and Wagner (2011a) that the recursions for counting spanning trees and forests in self-similar sequences of graphs have simple explicit solutions as for the Sierpiński graphs if the number of "boundary" vertices is two (as for example in the case of the graphs associated with the modified Koch curve, see Figure 9.7) or three (as for the Sierpiński graphs), provided that the automorphism group acts with either full symmetry or like the alternating group on the set of boundary vertices. For two boundary vertices, this technical condition is always satisfied. The explicit counting formulae guarantee that the projections will still be measure-preserving, and all other arguments can be carried out in the same way as in the previous sections.

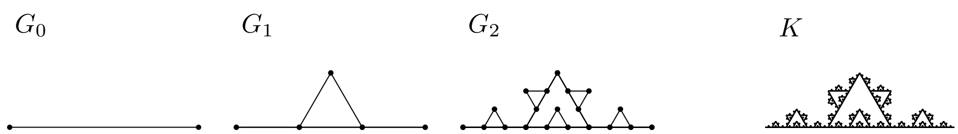


FIGURE 9.7. The modified Koch curve.

For two boundary vertices, the rescaling factor is precisely the average length of loop-erased random walk from one boundary vertex to the other in G_1 (the initial graph G_0 being a single edge), which is always a rational number. For example, for the sequence of graphs in Figure 9.7, the rescaling constant is $\frac{10}{3}$ (in other words, the length of loop-erased random walk from one boundary vertex of G_n to the other grows like $(\frac{10}{3})^n$). It follows that the Hausdorff dimension of the limit curve is almost surely $\log(\frac{10}{3})/\log 3 \approx 1.095903274$ in this example. As a second example, consider the Sierpiński graphs with two subdivisions on each edge in Figure 9.8: in this case, we find that the rescaling factor is $\frac{1}{735}(1431 + \sqrt{1669656})$ (it is a priori clear that it has to be algebraic of degree ≤ 2, being an eigenvalue of a 2×2-matrix with rational entries), giving us a Hausdorff dimension of ≈ 1.192117286 for the limit curve of loop-erased random walk.

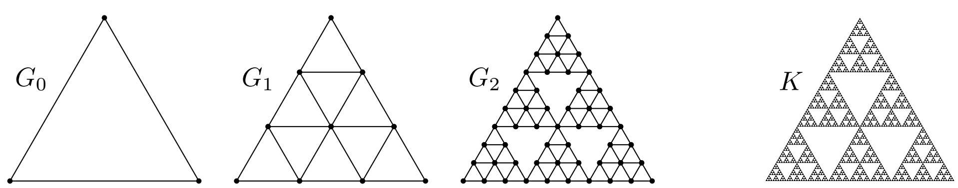


FIGURE 9.8. Sierpiński graphs with two subdivisions.

If the number of boundary vertices is four or more (which happens, for instance, for the higher-dimensional analogues of the Sierpiński graphs), then more different types of spanning forests have to be considered, and there are generally no exact counting formulae. However, asymptotic formulae should hold in such cases, making the projections "asymptotically measure-preserving", so that analogous results hold in such cases. The details might be quite intricate though, and new geometric phenomena arise as well: for instance, with four boundary vertices, it becomes possible that a loop-erased random walk on G_n enters and leaves some of the copies of G_k ($k < n$) more than once, which is not possible in the case of Sierpiński graphs that we considered.

References

J.A. Anema. Counting spanning trees on fractal graphs (2012). Preprint, http://arxiv.org/abs/1211.7341.

M.T. Barlow and E.A. Perkins. Brownian motion on the Sierpiński gasket. *Probab. Theory Related Fields* **79** (4), 543–623 (1988). MR966175.

N.H. Bingham, C.M. Goldie and J.L. Teugels. *Regular variation*, volume 27 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge (1987). ISBN 0-521-30787-2. MR898871.

B. Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York (1998). ISBN 0-387-98488-7. MR1633290.

R. Burton and R. Pemantle. Local characteristics, entropy and limit theorems for spanning trees and domino tilings via transfer-impedances. *Ann. Probab.* **21** (3), 1329–1371 (1993). MR1235419.

S.-C. Chang and L.-C. Chen. Structure of spanning trees on the two-dimensional Sierpinski gasket. *Discrete Math. Theor. Comput. Sci.* **12** (3), 151–176 (2010). MR2786472.

S.-C. Chang, L.-C. Chen and W.-S. Yang. Spanning trees on the Sierpinski gasket. *J. Stat. Phys.* **126** (3), 649–667 (2007). MR2294471.

I. Chiswell. *Introduction to Λ-trees.* World Scientific Publishing Co. Inc., River Edge, NJ (2001).

D. Dhar and A. Dhar. Distribution of sizes of erased loops for loop-erased random walks. *Phys. Rev. E (3)* **55** (3), R2093–R2096 (1997). DOI: 10.1103/PhysRevE.55.R2093.

M. Fukushima and T. Shima. On a spectral analysis for the Sierpiński gasket. *Potential Anal.* **1** (1), 1–35 (1992). MR1245223.

B.M. Hambly, K. Hattori and T. Hattori. Self-repelling walk on the Sierpiński gasket. *Probab. Theory Related Fields* **124** (1), 1–25 (2002). MR1929809.

K. Hattori. Exact Hausdorff dimension of self-avoiding processes on the multi-dimensional Sierpinski gasket. *J. Math. Sci. Univ. Tokyo* **7** (1), 57–98 (2000). MR1749981.

K. Hattori and T. Hattori. Self-avoiding process on the Sierpiński gasket. *Probab. Theory Related Fields* **88** (4), 405–428 (1991). MR1105711.

K. Hattori, T. Hattori and S. Kusuoka. Self-avoiding paths on the pre-Sierpiński gasket. *Probab. Theory Related Fields* **84** (1), 1–26 (1990). MR1027817.

K. Hattori, T. Hattori and S. Kusuoka. Self-avoiding paths on the three-dimensional Sierpiński gasket. *Publ. Res. Inst. Math. Sci.* **29** (3), 455–509 (1993). MR1233746.

K. Hattori and M. Mizuno. Loop-erased random walk on the Sierpinski gasket. *Stochastic Process. Appl.* **124** (1), 566–585 (2014). MR3131305.

T. Hattori and S. Kusuoka. The exponent for the mean square displacement of self-avoiding random walk on the Sierpiński gasket. *Probab. Theory Related Fields* **93** (3), 273–284 (1992). MR1180701.

O.D. Jones. Large deviations for supercritical multitype branching processes. *J. Appl. Probab.* **41** (3), 703–720 (2004). MR2074818.

T. Kumagai. Construction and some properties of a class of non-symmetric diffusion processes on the Sierpiński gasket. In *Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990)*, volume 283 of *Pitman Res. Notes Math. Ser.*, pages 219–247. Longman Sci. Tech., Harlow (1993). MR1354157.

G.F. Lawler and V. Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2010). ISBN 978-0-521-51918-2. MR2677157.

G.F. Lawler, O. Schramm and W. Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.* **32** (1B), 939–995

(2004). MR2044671.

S.S. Manna, D. Dhar and S.N. Majumdar. Spanning trees in two dimensions. *Phys. Rev. A (3)* **46** (10), R4471–R4474 (1992). DOI: 10.1103/PhysRevA.46.R4471.

C.J. Mode. *Multitype branching processes. Theory and applications.* Modern Analytic and Computational Methods in Science and Mathematics, No. 34. American Elsevier Publishing Co., Inc., New York (1971). MR0279901.

H. Poincaré. Sur une classe nouvelle de transcendantes uniformes. *J. Math. Pures Appl. (4)* **6**, 313–365 (1890). Œuvres, Tome IV, Paris, 1950, 537–582. http://gallica.bnf.fr/ark:/12148/bpt6k107454k/f316.

O. Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.* **118**, 221–288 (2000). MR1776084.

T. Shima. On eigenvalue problems for the random walks on the Sierpiński pre-gaskets. *Japan J. Indust. Appl. Math.* **8** (1), 127–141 (1991). MR1093832.

W. Sierpiński. Sur une courbe dont tout point est un point de ramification. *C. R. Acad. Sci. Paris* **160**, 302–305 (1915). http://gallica.bnf.fr/ark:/12148/bpt6k31131/f3.image.

E. Teufl. Uniform spanning trees and loop-erased random walks on Sierpinski graphs (2011). Talk at the 4th Cornell Conference on Analysis, Probability, and Mathematical Physics on Fractals. http://homepages.uconn.edu/fractals/cornell4/.

E. Teufl and S. Wagner. The number of spanning trees in self-similar graphs. *Ann. Comb.* **15** (2), 355–380 (2011a). MR2813521.

E. Teufl and S. Wagner. Resistance scaling and the number of spanning trees in self-similar lattices. *J. Stat. Phys.* **142** (4), 879–897 (2011b). MR2773791.

E. Teufl and S.G. Wagner. The number of spanning trees of finite Sierpinski graphs. In *Fourth Colloquium on Mathematics and Computer Science*, Discrete Math. Theor. Comput. Sci. Proc., AG, pages 411–414. Assoc. Discrete Math. Theor. Comput. Sci., Nancy (2006). http://www.dmtcs.org/dmtcs-ojs/index.php/proceedings/article/view/dmAG0135.

Y. Tsujii. Markov-self-similar sets. *Hiroshima Math. J.* **21** (3), 491–519 (1991). MR1148991.

D.B. Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pages 296–303. ACM, New York (1996). MR1427525.