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Fixed points of multivariate smoothing transforms with scalar weights

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Abstract. Given a sequence $(C_1, \ldots, C_d, T_1, T_2, \ldots)$ of real-valued random variables with $N := \#\{j \geq 1 : T_j \neq 0\} < \infty$ almost surely, there is an associated smoothing transformation which maps a distribution P on $\mathbb{R}^d$ to the distribution of $\sum_{j \geq 1} T_j \mathbf{X}^{(j)} + \mathbf{C}$ where $\mathbf{C} = (C_1, \ldots, C_d)$ and $(\mathbf{X}^{(j)})_{j \geq 1}$ is a sequence of independent random vectors with distribution P independent of $(C_1, \ldots, C_d, T_1, T_2, \ldots)$. We are interested in the fixed points of this mapping. By improving on the techniques developed in Alsmeyer et al. (2012) and Alsmeyer and Meiners (2013), we determine the set of all fixed points under weak assumptions on $(C_1, \ldots, C_d, T_1, T_2, \ldots)$. In contrast to earlier studies, this includes the most intricate case when the T_j take both positive and negative values with positive probability. In this case, in some situations, the set of fixed points is a subset of the corresponding set when the T_j are replaced by their absolute values, while in other situations, additional solutions arise.

1. Introduction

For a given $d \in \mathbb{N}$ and a given sequence $(\mathbf{C}, T) = ((C_1, \ldots, C_d), (T_j)_{j \geq 1})$ where $C_1, \ldots, C_d$, $T_1, T_2, \ldots$ are real-valued random variables with $N := \#\{j \geq 1 : T_j \neq 0\} < \infty$ almost surely, consider the mapping on the set of probability measures on $\mathbb{R}^d$ that maps a distribution P to the law of the random variable $\sum_{j \geq 1} T_j \mathbf{X}^{(j)} + \mathbf{C}$

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where $(\mathbf{X}^{(j)})_{j\geq 1}$ is a sequence of independent random vectors with distribution P independent of $(\mathbf{C}, T)$. P is a fixed point of this mapping iff, with $\mathbf{X}$ denoting a random variable with distribution P,

$$\mathbf{X} \stackrel{\text{law}}{=} \sum_{j\geq 1} T_j \mathbf{X}^{(j)} + \mathbf{C}. \tag{1.1}$$

In this paper we identify all solutions to (1.1) under suitable assumptions.

Due to the appearance of the distributional fixed-point equation (1.1) in various applications such as interacting particle systems Durrett and Liggett (1983), branching random walks Biggins (1977); Biggins and Kyprianou (1997), analysis of algorithms Neininger and Rüschendorf (2004); Rösler (1991); Volkovich and Litvak (2010), and kinetic gas theory Bassetti et al. (2011), there is a large body of papers dealing with it in different settings.

The articles Alsmeyer et al. (2012); Alsmeyer and Meiners (2012, 2013); Biggins (1977); Biggins and Kyprianou (1997, 2005); Durrett and Liggett (1983); Iksanov (2004); Liu (1998) treat the case $d = 1$ in which we rewrite (1.1) as

$$X \stackrel{\text{law}}{=} \sum_{j\geq 1} T_j X^{(j)} + C. \tag{1.2}$$

In all these references it is assumed that $T_j \geq 0$ a.s. for all $j \geq 1$. The most comprehensive result is provided in Alsmeyer and Meiners (2013). There, under mild assumptions on the sequence $(C, T_1, T_2, \ldots)$, which include the existence of an $\alpha \in (0, 2]$ such that $\mathbb{E}[\sum_{j\geq 1} T_j^\alpha] = 1$, it is shown that there exists a pair (W^*, W) of random variables on a specified probability space such that W^* is a particular (*endogenous*[1]) solution to (1.2) and W is a nonnegative solution to the tilted homogeneous equation $W \stackrel{\text{law}}{=} \sum_{j\geq 1} T_j^\alpha W^{(j)}$ where the $W^{(j)}$ are i.i.d. copies of W independent of $(C, T_1, T_2, \ldots)$. Furthermore, a distribution P on $\mathbb{R}$ is a solution to (1.2) if and only if it is the law of a random variable of the form

$$W^* + W^{1/\alpha} Y_\alpha \tag{1.3}$$

where Y_α is a strictly α-stable random variable independent of (W^*, W).[2] This result constitutes an almost complete solution of the fixed-point problem in dimension one leaving open only the case when the T_j take positive and negative values with positive probability. In a setup including the latter case, which will be called the case of weights with mixed signs hereafter, we derive the analogue of (1.3) thereby completing the picture in dimension one under mild assumptions. It is worth pointing out here that while one could guess at first glance that (1.3) carries over to the case of weights with mixed signs with the additional restriction that Y_α should be symmetric α-stable rather than strictly α-stable, an earlier work Alsmeyer (2006) dealing with (1.2) in the particular case of deterministic weights T_j, $j \geq 1$ suggests that this is not always the case. Indeed, if, for instance, $1 < \alpha < 2$, $\mathbb{E}[\sum_{j\geq 1} |T_j|^\alpha] = 1$ and $\sum_{j\geq 1} T_j = 1$ a.s., it is readily checked that addition of a constant to any solution again gives a solution which cannot be expressed as in (1.3). In fact, this is not the only situation in which additional solutions arise and these are typically not constants but limits of certain martingales not appearing in the case of nonnegative weights T_j, $j \geq 1$.

[1] See Section 3.5 for the definition of *endogeny*.

[2] For convenience, random variables with degenerate laws are assumed strictly 1-stable.

Our setup is a mixture of the one- and the multi-dimensional setting in the sense that we consider probability distributions on $\mathbb{R}^d$ while the weights T_j, $j \geq 1$ are scalars. Among others, this allows us to deal with versions of (1.1) for stochastic processes which can be understood as generalized equations of stability for stochastic processes. Earlier papers that are concerned with finding all fixed points of the smoothing transform in a multivariate setting are Bassetti and Matthes (2014); Mentemeier (2015+). On the one hand, the setup in these references is more general since there the T_j, $j \geq 1$ are $d \times d$ matrices rather than scalars. On the other hand, they are less general since they cover the case $\alpha = 2$ only Bassetti and Matthes (2014) (where the definition of α is a suitable extension of the definition given above) or the case of matrices T_j with nonnegative entries and solutions $\mathbf{X}$ with nonnegative components only Mentemeier (2015+).

We continue the introduction with a more detailed description of two applications, namely, kinetic models and stable distributions.

Kinetic models. Motivated by questions coming from kinetic gas theory and economics, Bassetti *et al.* Bassetti and Ladelli (2012); Bassetti et al. (2011); Matthes and Toscani (2008) consider a certain evolution equation for time-dependent probability distributions on $\mathbb{R}$ and investigate convergence of its solutions to stationary distributions, *i.e.*, distributions that are invariant under the dynamics of the evolution equation. It turns out that the stationary distributions are fixed points of a smoothing transform associated to a vector (C, T), where $C = 0$ and N is a fixed integer ≥ 2.

The classical Kac equation Kac (1956) is a particular case of the equation studied in Bassetti and Ladelli (2012); Bassetti et al. (2011) and the corresponding stationary distributions are fixed points of the smoothing transform with $d = 1$, $C = 0$, $N = 2$, $T_1 = \sin(\Theta)$, $T_2 = \cos(\Theta)$ where Θ is a random angle uniformly distributed over $[0, 2\pi]$. The corresponding fixed-point equation is

$$X \overset{\text{law}}{=} \sin(\Theta)X^{(1)} + \cos(\Theta)X^{(2)}. \tag{1.4}$$

Another particular case covered in Bassetti and Ladelli (2012); Bassetti et al. (2011) are inelastic Kac models Pulvirenti and Toscani (2004). The stationary distributions which correspond to these equations satisfy the analogous fixed-point equation

$$X \overset{\text{law}}{=} \sin(\Theta)|\sin(\Theta)|^{\beta-1}X^{(1)} + \cos(\Theta)|\cos(\Theta)|^{\beta-1}X^{(2)} \tag{1.5}$$

where Θ is as above and $\beta > 1$ is a parameter.

Generalized equations of stability. A distribution P on $\mathbb{R}^d$ is called *stable* iff there exists an $\alpha \in (0, 2]$ such that for every $n \in \mathbb{N}$ there is a $\mathbf{c}_n \in \mathbb{R}^d$ with

$$\mathbf{X} \overset{\text{law}}{=} n^{-1/\alpha} \sum_{j=1}^{n} \mathbf{X}^{(j)} + \mathbf{c}_n \tag{1.6}$$

where $\mathbf{X}$ has distribution P and $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}, \ldots$ is a sequence of i.i.d. copies of $\mathbf{X}$, see Samorodnitsky and Taqqu (1994, Corollary 2.1.3). α is called the *index of stability* and P is called α*-stable*.

Clearly, stable distributions are fixed points of certain smoothing transforms. For instance, given a random variable $\mathbf{X}$ satisfying (1.6) for all $n \in \mathbb{N}$, one can choose a random variable N with support $\subseteq \mathbb{N}$ and then define $T_1 = \ldots = T_n = n^{-1/\alpha}$, $T_j = 0$ for $j > n$ and $\mathbf{C} = \mathbf{c}_n$ on $\{N = n\}$, $n \in \mathbb{N}$. Then $\mathbf{X}$ satisfies (1.1).

Hence, fixed-point equations of smoothing transforms can be considered as generalized equations of stability; some authors call fixed points of smoothing transforms "stable by random weighted mean" Liu (2001). It is worth pointing out that the form of (the characteristic functions of) strictly stable distributions can be deduced from our main result, Theorem 2.4, the proof of which can be considered as a generalization of the classical derivation of the form of stable laws given by Gnedenko and Kolmogorov (1968).

2. Main results

2.1. *Assumptions.* Without loss of generality for the results considered here, we assume that

$$N \;=\; \sup\{j \geq 1 : T_j \neq 0\} \;=\; \sum_{j \geq 1} \mathbb{1}_{\{T_j \neq 0\}} \;\in\; \mathbb{N}_0. \tag{2.1}$$

Also, we define the function

$$m : [0,\infty) \to [0,\infty], \quad \gamma \mapsto \mathbb{E}\Big[\sum_{j=1}^{N} |T_j|^{\gamma}\Big]. \tag{2.2}$$

Naturally, assumptions on $(\mathbf{C}, T)$ are needed in order to solve (1.1). Throughout the paper, the following assumptions will be in force:

$$\mathbb{P}(T_j \in \{0\} \cup \{\pm r^n : n \in \mathbb{Z}\} \text{ for all } j \geq 1) < 1 \quad \text{for all } r \geq 1. \tag{A1}$$

$$m(0) = \mathbb{E}[N] \in (1,\infty]. \tag{A2}$$

$$m(\alpha) = 1 \text{ for some } \alpha > 0 \text{ and } m(\vartheta) > 1 \text{ for all } \vartheta \in [0,\alpha). \tag{A3}$$

We briefly discuss the assumptions (A1)-(A3)[3] beginning with (A1). With $\mathbb{R}^*$ denoting the multiplicative group $(\mathbb{R} \setminus \{0\}, \times)$, let

$$\mathbb{G}(T) \;:=\; \bigcap \big\{G : G \text{ is a closed multiplicative subgroup of } \mathbb{R}^* \\ \text{satisfying } \mathbb{P}(T_j \in G \text{ for } j = 1,\ldots,N) = 1\big\}.$$

$\mathbb{G}(T)$ is the closed multiplicative subgroup $\subset \mathbb{R}^*$ generated by the nonzero T_j. There are seven possibilities: (C1) $\mathbb{G}(T) = \mathbb{R}^*$, (C2) $\mathbb{G}(T) = \mathbb{R}_> := (0,\infty)$, (D1) $\mathbb{G}(T) = r^{\mathbb{Z}} \cup -r^{\mathbb{Z}}$ for some $r > 1$, (D2) $\mathbb{G}(T) = r^{\mathbb{Z}}$ for some $r > 1$, (D3) $\mathbb{G}(T) = (-r)^{\mathbb{Z}}$ for some $r > 1$, (S1) $\mathbb{G}(T) = \{1,-1\}$, and (S2) $\mathbb{G}(T) = \{1\}$. (A1) can be reformulated as: Either (C1) or (C2) holds. For the results considered, the cases (S1) and (S2) are simple and it is no restriction to rule them out (see Alsmeyer (2006, Proposition 3.1) in case $d = 1$; the case $d \geq 2$ can be treated by considering marginals). Although the cases (D1)-(D3) in which the T_j generate a (non-trivial) discrete group could have been treated along the lines of this paper, they are ruled out for convenience since they create the need for extensive notation and case distinction. Caliebe (2003, Lemma 2) showed that only simple cases are eliminated when assuming (A2). (A3) is natural in view of earlier studies of fixed points of the smoothing transform, see *e.g.* Alsmeyer (2006, Proposition 5.1) and Alsmeyer and

[3] Although (A3) implies (A2), we use both to keep the presentation consistent with earlier works.

Meiners (2012, Theorem 6.1 and Example 6.4). We refer to α as the *characteristic index (of T)*.

Define

$$p := \mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha \mathbb{1}_{\{T_j>0\}}\bigg] \quad \text{and} \quad q := \mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha \mathbb{1}_{\{T_j<0\}}\bigg]. \tag{2.3}$$

(A3) implies that $0 \le p, q \le 1$ and $p + q = 1$. At some places it will be necessary to distinguish the following cases:

$$\underline{\text{Case I}}\text{: } p = 1, q = 0. \quad \underline{\text{Case II}}\text{: } p = 0, q = 1. \quad \underline{\text{Case III}}\text{: } 0 < p, q < 1. \tag{2.4}$$

Case I corresponds to $\mathbb{G}(T) = \mathbb{R}_>$. Cases II and III correspond to $\mathbb{G}(T) = \mathbb{R}^*$. In dimension $d = 1$, Case I is covered by the results in Alsmeyer and Meiners (2013) while Case II can be lifted from these results. Case III is genuinely new.

In our main results, we additionally assume the following condition to be satisfied:

$$\text{(A4a) or (A4b) holds,} \tag{A4}$$

where

$$\mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha \log(|T_j|)\bigg] \in (-\infty, 0) \text{ and } \mathbb{E}\bigg[\bigg(\sum_{j=1}^{N} |T_j|^\alpha\bigg) \log^+\bigg(\sum_{j=1}^{N} |T_j|^\alpha\bigg)\bigg] < \infty; \tag{A4a}$$

$$\text{there exists some } \theta \in [0, \alpha) \text{ satisfying } m(\theta) \ < \ \infty. \tag{A4b}$$

Further, in Case III when $\alpha = 1$, we need the assumption

$$\mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha \delta_{-\log(|T_j|)}(\cdot)\bigg] \text{ is spread-out}, \quad \mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha (\log^-(|T_j|))^2\bigg] < \infty,$$
$$\mathbb{E}\bigg[\sum_{j=1}^{N} |T_j|^\alpha \log(|T_j|)\bigg] \in (-\infty, 0) \quad \text{and} \quad \mathbb{E}\bigg[h_3\bigg(\sum_{j=1}^{N} |T_j|^\alpha\bigg)\bigg] < \infty \tag{A5}$$

for $h_3(x) := x(\log^+(x))^3 \log^+(\log^+(x))$. (A5) is stronger than (A4a). The last assumption that will occasionally show up is

$$|T_j| < 1 \text{ a.s. for all } j \ge 1. \tag{A6}$$

However, (A6) will not be assumed in the main theorems since by a stopping line technique, the general case can be reduced to cases in which (A6) holds. It will be stated explicitly whenever at least one of the conditions (A4a), (A4b), (A5) or (A6) is assumed to hold.

2.2. *Notation and background.* In order to state our results, we introduce the underlying probability space and some notation that comes with it.

Let $\mathbb{V} := \bigcup_{n\ge 0} \mathbb{N}^n$ denote the infinite Ulam-Harris tree where $\mathbb{N}^0 := \{\varnothing\}$. We use the standard Ulam-Harris notation, which means that we abbreviate $v = (v_1, \ldots, v_n) \in \mathbb{V}$ by $v_1 \ldots v_n$. vw is short for $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ when $w = (w_1, \ldots, w_m) \in \mathbb{N}^m$. We make use of standard terminology from branching processes and call the $v \in \mathbb{V}$ *(potential) individuals* and say that v is a *member of the nth generation* if $v \in \mathbb{N}^n$. We write $|v| = n$ if $v \in \mathbb{N}^n$ and define $v|_k$ to be the restriction of v to its first k components if $k \leq |v|$ and $v|_k = v$, otherwise. In particular, $v|_0 = \varnothing$. $v|_k$ will be called the *ancestor of* v *in the* k*th generation*.

Assume a family $(\mathbf{C}(v), T(v))_{v\in\mathbb{V}} = (C_1(v), \ldots, C_d(v), T_1(v), T_2(v), \ldots)_{v\in\mathbb{V}}$ of i.i.d. copies of the sequence $(\mathbf{C}, T) = (\mathbf{C}, T_1, T_2, \ldots)$ is given on a fixed probability space $(\Omega, \mathcal{A}, \mathbb{P})$ that also carries all further random variables we will be working with. For notational convenience, we assume that

$$(C_1(\varnothing), \ldots, C_d(\varnothing), T_1(\varnothing), T_2(\varnothing), \ldots) \;=\; (C_1, \ldots, C_d, T_1, T_2, \ldots).$$

Throughout the paper, we let

$$\mathcal{A}_n \;:=\; \sigma((\mathbf{C}(v), T(v)) : |v| < n), \qquad n \geq 0 \tag{2.5}$$

be the σ-algebra of all family histories before the nth generation and define $\mathcal{A}_\infty := \sigma(\mathcal{A}_n : n \geq 0)$.

Using the family $(\mathbf{C}(v), T(v))_{v\in\mathbb{V}}$, we define a Galton-Watson branching process as follows. Let $N(v) := \sup\{j \geq 1 : T_j(v) \neq 0\}$ so that the $N(v)$, $v \in \mathbb{V}$ are i.i.d. copies of N. Put $\mathcal{G}_0 := \{\varnothing\}$ and, recursively,

$$\mathcal{G}_{n+1} \;:=\; \{vj \in \mathbb{N}^{n+1} : v \in \mathcal{G}_n, 1 \leq j \leq N(v)\}, \qquad n \in \mathbb{N}_0. \tag{2.6}$$

Let $\mathcal{G} := \bigcup_{n\geq 0} \mathcal{G}_n$ and $N_n := |\mathcal{G}_n|$, $n \geq 0$. Then $(N_n)_{n\geq 0}$ is a Galton-Watson process. $\mathbb{E}[N] > 1$ guarantees supercriticality and hence $\mathbb{P}(\mathsf{S}) > 0$ where

$$\mathsf{S} \;:=\; \{N_n > 0 \text{ for all } n \geq 0\}$$

is the survival set. Further, we define multiplicative weights $L(v)$, $v \in \mathbb{V}$ as follows. For $v = v_1 \ldots v_n \in \mathbb{V}$, let

$$L(v) \;:=\; \prod_{k=1}^{n} T_{v_k}(v|_{k-1}). \tag{2.7}$$

Then the family $\mathbf{L} := (L(v))_{v\in\mathbb{V}}$ is called *weighted branching process*. It can be used to iterate (1.1). Let $(\mathbf{X}^{(v)})_{v\in\mathbb{V}}$ be a family of i.i.d. random variables defined on $(\Omega, \mathcal{A}, \mathbb{P})$ independent of the family $(\mathbf{C}(v), T(v))_{v\in\mathbb{V}}$. For convenience, let $\mathbf{X}^{(\varnothing)} =: \mathbf{X}$. If the distribution of $\mathbf{X}$ is a solution to (1.1), then, for $n \in \mathbb{N}_0$,

$$\mathbf{X} \stackrel{\text{law}}{=} \sum_{|v|=n} L(v)\mathbf{X}^{(v)} + \sum_{|v|<n} L(v)\mathbf{C}(v). \tag{2.8}$$

An important special case of (1.1) is the homogeneous equation

$$\mathbf{X} \stackrel{\text{law}}{=} \sum_{j\geq 1} T_j \mathbf{X}^{(j)} \tag{2.9}$$

in which $\mathbf{C} = \mathbf{0} = (0, \ldots, 0) \in \mathbb{R}^d$ a.s. Iteration of (2.9) leads to

$$\mathbf{X} \stackrel{\text{law}}{=} \sum_{|v|=n} L(v)\mathbf{X}^{(v)}. \tag{2.10}$$

Finally, for $u \in \mathbb{V}$ and a function $\Psi = \Psi((\mathbf{C}(v), T(v))_{v\in\mathbb{V}})$ of the weighted branching process, let $[\Psi]_u$ be defined as $\Psi((\mathbf{C}(uv), T(uv))_{v\in\mathbb{V}})$, that is, the same function but applied to the weighted branching process rooted in u. The $[\cdot]_u$, $u \in \mathbb{V}$ are called *shift operators*.

2.3. *Existence of solutions to* (1.1)*nd related equations.* Under certain assumptions on $(\mathbf{C}, T)$, a solution to (1.1) can be constructed as a function of the weighted branching process $(L(v))_{v\in\mathbb{V}}$. Let $\mathbf{W}_0^* := \mathbf{0}$ and

$$\mathbf{W}_n^* \;:=\; \sum_{|v|<n} L(v)\mathbf{C}(v), \quad n \in \mathbb{N}. \tag{2.11}$$

$\mathbf{W}_n^*$ is well-defined since a.s. $\{|v| < n\}$ has only finitely many members v with $L(v) \neq 0$. Whenever $\mathbf{W}_n^*$ converges in probability to a finite limit as $n \to \infty$, we set

$$\mathbf{W}^* \;:=\; \lim_{n\to\infty} \mathbf{W}_n^* \tag{2.12}$$

and note that $\mathbf{W}^*$ defines a solution to (1.1). Indeed, if $\mathbf{W}_n^* \to \mathbf{W}^*$ in probability as $n \to \infty$, then also $[\mathbf{W}_n^*]_j \to [\mathbf{W}^*]_j$ in probability as $n \to \infty$. By standard arguments, there is a (deterministic) sequence $n_k \uparrow \infty$ such that $[\mathbf{W}_{n_k}^*]_j \to [\mathbf{W}^*]_j$ a.s. for all $j \geq 1$. Since $N < \infty$ a.s., this yields

$$\lim_{k\to\infty} \mathbf{W}_{n_k+1}^* \;=\; \lim_{k\to\infty} \sum_{j=1}^{N} T_j[\mathbf{W}_{n_k}^*]_j + \mathbf{C} \;=\; \sum_{j=1}^{N} T_j[\mathbf{W}^*]_j + \mathbf{C} \quad \text{a.s.}$$

and hence,

$$\mathbf{W}^* \;=\; \sum_{j=1}^{N} T_j[\mathbf{W}^*]_j + \mathbf{C} \quad \text{a.s.} \tag{2.13}$$

because $\mathbf{W}_{n_k+1}^* \to \mathbf{W}^*$ in probability.

The following proposition provides sufficient conditions for $\mathbf{W}_n^*$ to converge in probability.

Proposition 2.1. *Assume that* (A1)-(A3) *hold. Each of the following conditions is sufficient for* $\mathbf{W}_n^*$ *to converge in probability.*

(i) *For some* $0 < \beta \leq 1$, $m(\beta) < 1$ *and* $\mathbb{E}[|C_j|^\beta] < \infty$ *for all* $j = 1, \ldots, d$.

(ii) *For some* $\beta > 1$, $\sup_{n\geq 0} \mathbb{E}[|\mathbf{W}_n^*|^\beta] < \infty$ *and either* $T_j \geq 0$ *a.s. for all* $j \in \mathbb{N}$ *or* $\mathbb{E}[C] = 0$.

(iii) $0 < \alpha < 1$, $\mathbb{E}[\sum_{j\geq 1} |T_j|^\alpha \log(|T_j|)]$ *exists and equals* 0, *and, for some* $\delta > 0$, $\mathbb{E}[|C_j|^{\alpha+\delta}] < \infty$ *for* $j = 1, \ldots, d$.

For the most part, the proposition is known. Details along with the relevant references are given at the end of Section 3.5.

Of major importance in this paper are the solutions to the one-dimensional tilted homogeneous fixed-point equation

$$W \;\stackrel{\text{law}}{=}\; \sum_{j\geq 1} |T_j|^\alpha W^{(j)} \tag{2.14}$$

where W is a finite, nonnegative random variable and the $W^{(j)}$, $j \geq 1$ are i.i.d. copies of W independent of the sequence $(T_1, T_2, \ldots)$. Equation (2.14) (for nonnegative random variables) is equivalent to the functional equation

$$f(t) \;=\; \mathbb{E}\Big[\prod_{j\geq 1} f(|T_j|^\alpha t)\Big] \quad \text{for all } t \geq 0 \tag{2.15}$$

where f denotes the Laplace transform of W. (2.14) and (2.15) have been studied extensively in the literature and the results that are important for the purposes of this paper are summarized in the following proposition.

Proposition 2.2. *Assume that* (A1)–(A4) *hold. Then*

(a) *there is a Laplace transform φ of a probability distribution on $[0,\infty)$ such that $\varphi(1) < 1$ and φ solves* (2.15)*;*

(b) *every other Laplace transform $\hat{\varphi}$ of a probability distribution on $[0,\infty)$ solving* (2.15) *is of the form $\hat{\varphi}(t) = \varphi(ct)$, $t \geq 0$ for some $c \geq 0$;*

(c) *$1 - \varphi(t)$ is regularly varying of index 1 at 0;*

(d) *a (nonnegative, finite) random variable W solving* (2.14) *with Laplace transform φ can be constructed explicitly on $(\Omega, \mathcal{A}, \mathbb{P})$ via*

$$W \;:=\; \lim_{n\to\infty} \sum_{|v|=n} 1 - \varphi(|L(v)|^{\alpha}) \quad a.s. \tag{2.16}$$

Source: (a), (b) and (c) are known. A unified treatment and references are given in Alsmeyer et al. (2012, Theorem 3.1). (d) is contained in Alsmeyer et al. (2012, Theorem 6.2(a)). □

Throughout the paper, we denote by φ the Laplace transform introduced in Proposition 2.2(a) and by W the random variable defined in (2.16). By Proposition 2.2(c), $D(t) := t^{-1}(1-\varphi(t))$ is slowly varying at 0. If D has a finite limit at 0, then, by scaling, we assume this limit to be 1. Equivalently, if W is integrable, we assume $\mathbb{E}[W] = 1$. In this case, W is the limit of the additive martingale (sometimes called Biggins' martingale) in the branching random walk based on the point process $\sum_{j=1}^{N} \delta_{-\log(|T_j|^{\alpha})}$, namely, $W = \lim_{n\to\infty} W_n$ a.s. where

$$W_n \;=\; \sum_{|v|=n} |L(v)|^{\alpha}, \qquad n \in \mathbb{N}_0. \tag{2.17}$$

As indicated in the introduction, for certain parameter constellations, another random variable plays an important role here. Define

$$Z_n \;:=\; \sum_{|v|=n} L(v), \quad n \in \mathbb{N}_0. \tag{2.18}$$

Let $Z := \lim_{n\to\infty} Z_n$ if the limit exists in the a.s. sense and $Z = 0$, otherwise. The question of when $(Z_n)_{n\geq 0}$ is a.s. convergent is nontrivial.

Theorem 2.3. *Assume that* (A1)-(A4) *are true. Then the following assertions hold.*

(a) *If $0 < \alpha < 1$, then $Z_n \to 0$ a.s. as $n \to \infty$.*

(b) *If $\alpha > 1$, then Z_n converges a.s. and $\mathbb{P}(\lim_{n\to\infty} Z_n = 0) < 1$ iff $\mathbb{E}[Z_1] = 1$ and Z_n converges in $\mathcal{L}^{\beta}$ for some/all $1 < \beta < \alpha$. Further, for these to be true $(Z_n)_{n\geq 0}$ must be a martingale.*

(c) *If $\alpha = 2$ and* (A4a) *holds or $\alpha > 2$, then Z_n converges a.s. iff $Z_1 = 1$ a.s.*

Here are simple sufficient conditions for part (b) of the theorem: if $1 < \alpha < 2$, $\mathbb{E}[Z_1] = 1$, $\mathbb{E}[|Z_1|^{\beta}] < \infty$ and $m(\beta) < 1$ for some $\beta > \alpha$, then $(Z_n)_{n\geq 0}$ is an $\mathcal{L}^{\beta}$-bounded martingale which converges in $\mathcal{L}^{\beta}$ and a.s. The assertion follows in the, by now, standard way via an application of the Topchiĭ-Vatutin inequality for martingales. We omit further details which can be found on p. 182 in Alsmeyer and Kuhlbusch (2010) and in Rosler et al. (2000).

If $\alpha = 1$, the behaviour of $(Z_n)_{n\geq 0}$ is irrelevant for us. However, for completeness, we mention that if $\mathbb{E}[Z_1] = 1$ or $\mathbb{E}[Z_1] = -1$, then $(Z_n)_{n\geq 0} = (W_n)_{n\geq 0}$ or $(Z_n)_{n\geq 0} = ((-1)^n W_n)_{n\geq 0}$, respectively. Criteria for $(W_n)_{n\geq 0}$ to have a nontrivial limit can be found in Alsmeyer and Iksanov (2009); Biggins (1977); Lyons (1997). If $\mathbb{E}[Z_1] \in (-1,1)$, then, under suitable assumptions, $Z_n \to 0$ a.s. We refrain from providing any details.

Theorem 2.3 will be proved in Section 4.4.

2.4. *Multivariate fixed points.* Most of the analysis concerning the equations (1.1) and (2.9) will be carried out in terms of Fourier transforms of solutions. Indeed, (1.1) and (2.9) are equivalent to

$$\phi(\mathbf{t}) \;=\; \mathbb{E}\bigg[e^{\mathrm{i}\langle \mathbf{t},\mathbf{C}\rangle} \prod_{j\geq 1} \phi(T_j\mathbf{t})\bigg] \quad \text{for all } \mathbf{t}\in\mathbb{R}^d, \tag{2.19}$$

and

$$\phi(\mathbf{t}) \;=\; \mathbb{E}\bigg[\prod_{j\geq 1} \phi(T_j\mathbf{t})\bigg] \quad \text{for all } \mathbf{t}\in\mathbb{R}^d, \tag{2.20}$$

respectively. Here, $\langle\cdot,\cdot\rangle$ denotes the standard scalar product in $\mathbb{R}^d$ and i the imaginary unit. Let $\mathfrak{F}$ denote the set of Fourier transforms of probability distributions on $\mathbb{R}^d$ and

$$\mathcal{S}(\mathfrak{F})(\mathbf{C}) \;:=\; \{\phi\in\mathfrak{F} : \phi \text{ solves } (2.19)\}. \tag{2.21}$$

Further, let $\mathcal{S}(\mathfrak{F}) := \mathcal{S}(\mathfrak{F})(\mathbf{0})$, that is,

$$\mathcal{S}(\mathfrak{F}) \;:=\; \{\phi\in\mathfrak{F} : \phi \text{ solves } (2.20)\}. \tag{2.22}$$

The dependence of $\mathcal{S}(\mathfrak{F})(\mathbf{C})$ on $\mathbf{C}$ is made explicit in the notation since at some points we will compare $\mathcal{S}(\mathfrak{F})(\mathbf{C})$ and $\mathcal{S}(\mathfrak{F})(\mathbf{0})$. The dependence of $\mathcal{S}(\mathfrak{F})(\mathbf{C})$ and $\mathcal{S}(\mathfrak{F})$ on T is not made explicit because T is kept fixed throughout.

Henceforth, let $\mathbb{S}^{d-1} = \{\mathbf{x}\in\mathbb{R}^d : |\mathbf{x}| = 1\}$ denote the unit sphere $\subseteq \mathbb{R}^d$.

Theorem 2.4. *Assume (A1)-(A4) and that $\mathbf{W}_n^* \to \mathbf{W}^*$ in probability*[4] *as $n\to\infty$. Further, recall the definitions of W and Z from (2.16) and (2.18), respectively.*

(a) *Let $0<\alpha<1$.*

(a1) *Let $\mathbb{G}(T) = \mathbb{R}_>$. Then $\mathcal{S}(\mathfrak{F})$ consists of the ϕ of the form*

$$\phi(\mathbf{t}) = \mathbb{E}\bigg[\exp\bigg(\mathrm{i}\langle\mathbf{W}^*,\mathbf{t}\rangle - W\int|\langle\mathbf{t},\mathbf{s}\rangle|^\alpha\Big[1-\mathrm{i}\,\mathrm{sign}(\langle\mathbf{t},\mathbf{s}\rangle)\tan\Big(\frac{\pi\alpha}{2}\Big)\Big]\,\sigma(\mathrm{d}\mathbf{s})\bigg)\bigg] \tag{2.23}$$

where σ is a finite measure on the Borel σ-field of $\mathbb{S}^{d-1}$.

(a2) *Let $\mathbb{G}(T) = \mathbb{R}^*$. Then $\mathcal{S}(\mathfrak{F})$ consists of the ϕ of the form*

$$\phi(\mathbf{t}) = \mathbb{E}\bigg[\exp\bigg(\mathrm{i}\langle\mathbf{W}^*,\mathbf{t}\rangle - W\int|\langle\mathbf{t},\mathbf{s}\rangle|^\alpha\,\sigma(\mathrm{d}\mathbf{s})\bigg)\bigg] \tag{2.24}$$

where σ is a symmetric finite measure on $\mathbb{S}^{d-1}$, i.e., $\sigma(B) = \sigma(-B)$ for Borel sets $B\subseteq\mathbb{S}^{d-1}$.

(b) *Let $\alpha = 1$.*

[4] When $\mathbf{C} = \mathbf{0}$ a.s., then $\mathbf{W}_n^* \to \mathbf{W}^* = \mathbf{0}$ a.s. as $n\to\infty$.

(b1) *Let* $\mathbb{G}(T) = \mathbb{R}_>$ *and assume that* $\mathbb{E}[\sum_{j\geq 1} |T_j|(\log^-(|T_j|))^2] < \infty$. *Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form*

$$\phi(\mathbf{t}) = \mathbb{E}\Big[\exp\Big(\mathrm{i}\langle \mathbf{W}^* + W\mathbf{a},\mathbf{t}\rangle - W\int|\langle \mathbf{t},\mathbf{s}\rangle|\sigma(\mathrm{d}s) - \mathrm{i}W\frac{2}{\pi}\int\langle \mathbf{t},\mathbf{s}\rangle\log(|\langle \mathbf{t},\mathbf{s}\rangle|)\sigma(\mathrm{d}\mathbf{s})\Big)\Big] \quad (2.25)$$

where $\mathbf{a} \in \mathbb{R}^d$ *and* σ *is a finite measure on* $\mathbb{S}^{d-1}$ *with* $\int s_k\, \sigma(\mathrm{d}\mathbf{s}) = 0$, $k = 1, \ldots, d$.

(b2) *Let* $\mathbb{G}(T) = \mathbb{R}^*$ *and assume that* $\mathbb{E}[\sum_{j\geq 1} |T_j|(\log^-(|T_j|))^2] < \infty$ *holds in Case II and that* (A5) *holds in Case III. Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form*

$$\phi(\mathbf{t}) = \mathbb{E}\left[\exp\left(\mathrm{i}\langle \mathbf{W}^*,\mathbf{t}\rangle - W\int|\langle \mathbf{t},\mathbf{s}\rangle|\,\sigma(\mathrm{d}\mathbf{s})\right)\right] \quad (2.26)$$

where σ *is a symmetric finite measure on* $\mathbb{S}^{d-1}$.

(c) *Let* $1 < \alpha < 2$.

(c1) *Let* $\mathbb{G}(T) = \mathbb{R}_>$. *Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form given in* (2.23)

$$\phi(\mathbf{t}) = \mathbb{E}\Big[\exp\Big(\mathrm{i}\langle \mathbf{W}^*,\mathbf{t}\rangle - W\int|\langle \mathbf{t},\mathbf{s}\rangle|^\alpha\Big(1-\mathrm{i}\,\mathrm{sign}(\langle \mathbf{t},\mathbf{s}\rangle)\tan\Big(\frac{\pi\alpha}{2}\Big)\Big)\,\sigma(\mathrm{d}\mathbf{s})\Big)\Big] \quad (2.27)$$

where σ *is a finite measure on* $\mathbb{S}^{d-1}$.

(c2) *Let* $\mathbb{G}(T) = \mathbb{R}^*$. *Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form*

$$\phi(\mathbf{t}) = \mathbb{E}\Big[\exp\Big(\mathrm{i}\langle \mathbf{W}^*+Z\mathbf{a},\mathbf{t}\rangle - W\int|\langle \mathbf{t},\mathbf{s}\rangle|^\alpha\,\sigma(\mathrm{d}\mathbf{s})\Big)\Big] \quad (2.28)$$

where $\mathbf{a} \in \mathbb{R}^d$, σ *is a symmetric finite measure on* $\mathbb{S}^{d-1}$, *and* $Z := \lim_{n\to\infty} Z_n$ *if this limit exists in the a.s. sense, and* $Z = 0$, *otherwise.*

(d) *Let* $\alpha = 2$. *Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form*

$$\phi(\mathbf{t}) \;=\; \mathbb{E}\left[\exp\left(\mathrm{i}\langle \mathbf{W}^* + Z\mathbf{a},\mathbf{t}\rangle - W\frac{\mathbf{t}\Sigma\mathbf{t}^\mathsf{T}}{2}\right)\right] \quad (2.29)$$

where $\mathbf{a} \in \mathbb{R}^d$, Σ *is a symmetric positive semi-definite (possibly zero)* $d \times d$ *matrix and* $\mathbf{t}^\mathsf{T}$ *is the transpose of* $\mathbf{t} = (t_1, \ldots, t_d)$, *and* $Z := \lim_{n\to\infty} Z_n$ *if this limit exists in the a.s. sense, and* $Z = 0$, *otherwise.*

(e) *Let* $\alpha > 2$. *Then* $\mathcal{S}(\mathfrak{F})$ *consists of the* ϕ *of the form*

$$\phi(\mathbf{t}) \;=\; \mathbb{E}[\exp(\mathrm{i}\langle \mathbf{W}^* + \mathbf{a},\mathbf{t}\rangle)], \quad (2.30)$$

where $\mathbf{a} \in \mathbb{R}^d$. *Furthermore,* $\mathbf{a} = \mathbf{0}$ *if* $\mathbb{P}(Z_1 = 1) < 1$.

Theorem 2.4 can be restated as follows. When the assumptions of the theorem hold, a distribution P on $\mathbb{R}^d$ is a solution to (1.1) if and only if it is the law of a random variable of the form

$$\mathbf{W}^* + Z\mathbf{a} + W^{1/\alpha}\mathbf{Y}_\alpha \quad (2.31)$$

where $\mathbf{W}^*$ is the special (endogenous[5]) solution to the inhomogeneous equation, Z is a special (endogenous) solution to the one-dimensional homogeneous equation (which vanishes in most cases, but can be nontrivial when $\alpha > 1$), $\mathbf{a} \in \mathbb{R}^d$, W is a special (endogenous) nonnegative solution to the tilted equation (2.14), and $\mathbf{Y}_\alpha$ is a strictly α-stable (symmetric α-stable if $\mathbb{G}(T) = \mathbb{R}^*$) random vector independent of $(\mathbf{C}, T)$.[6] Hence, the solutions are scale mixtures of strictly (symmetric if $\mathbb{G}(T) = \mathbb{R}^*$) stable distributions with a random shift. Theorem 2.4 in particular provides a deep insight into the structure of all fixed points since stable distributions (see *e.g.* Samorodnitsky and Taqqu (1994) and the references therein) and the random variables $\mathbf{W}^*$, W, and Z are well understood. For instance, the tail behavior of solutions of the form (2.31) can be derived from the tail behavior of $\mathbf{W}^*$, W, Z, and $\mathbf{Y}_\alpha$. The tail behavior of stable random variables is known, the tail behavior of W has been intensively investigated over the last decades, see *e.g.* Alsmeyer and Iksanov (2009); Alsmeyer and Kuhlbusch (2010); Biggins (1979); Biggins and Kyprianou (2005); Buraczewski (2009); Buraczewski et al. (2013); Durrett and Liggett (1983); Iksanov (2004); Iksanov and Polotskiy (2006); Iksanov and Rösler (2006); Jelenković and Olvera-Cravioto (2012b); Liang and Liu (2011); Liu (1998, 2000). The tail behavior of $\mathbf{W}^*$ has been investigated by several authors in the recent past Alsmeyer et al. (2013); Buraczewski et al. (2013); Buraczewski and Kolesko (2014); Jelenković and Olvera-Cravioto (2012a,b). Some of the cited papers concern the one-dimensional case only. However, since the T_j are scalars here, the tail behavior of $\mathbf{W}^*$ can be reduced to the behavior of its one-dimensional components and thus the results apply. The tail behavior of Z has been analysed in Alsmeyer et al. (2013).

2.5. *Univariate fixed points.* Corollary 2.5 given next, together with Theorems 2.1 and 2.2 of Alsmeyer and Meiners (2013), provides a reasonably full description of the one-dimensional fixed points of the homogeneous smoothing transforms in the case $\mathbb{G}(T) = \mathbb{R}^*$.

Corollary 2.5. *Let $d = 1$, $C = 0$ and $\mathbb{G}(T) = \mathbb{R}^*$. Assume that (A1)-(A4) hold true. If $\alpha = 1$, additionally assume that $\mathbb{E}[\sum_{j\geq 1} |T_j|(\log^-(|T_j|))^2] < \infty$ in Case II and (A5) in Case III. Then $\mathcal{S}(\mathfrak{F})$ is composed of the ϕ of the form*

$$\phi(t) = \begin{cases} \mathbb{E}[\exp(-W|\sigma t|^\alpha)], & 0 < \alpha < 1, \\ \mathbb{E}[\exp(-W|\sigma t|)], & \alpha = 1, \\ \mathbb{E}[\exp(\mathrm{i}aZt - W|\sigma t|^\alpha)], & 1 < \alpha < 2, \\ \mathbb{E}[\exp(\mathrm{i}aZt - W(\sigma t)^2)], & \alpha = 2, \end{cases} \tag{2.32}$$

where $Z = \lim_{n\to\infty} Z_n$ if the limit exists in the a.s. sense, and $Z = 0$, otherwise. Further, σ ranges over $[0, \infty)$ and a ranges over $\mathbb{R}$. When $\alpha > 2$, $\mathcal{S}(\mathfrak{F})$ only contains the Fourier transform of δ_0 (the constant function 1) unless $Z_1 = 1$ a.s., in which case $\mathcal{S}(\mathfrak{F}) = \{t \mapsto \exp(\mathrm{i}at) : a \in \mathbb{R}\}$.

[5] See Section 3.5 for the definition of *endogeny*.

[6] For convenience, random variables degenerate at $\mathbf{0}$ are considered strictly α-stable and random variables with degenerate laws are assumed strictly 1-stable here.

The Kac caricature revisited. As an application of Corollary 2.5, we discuss equations (1.4) and (1.5). In this context $d = 1$, $C = 0$ and $T_1 = \sin(\Theta)|\sin(\Theta)|^{\beta-1}$, $T_2 = \cos(\Theta)|\cos(\Theta)|^{\beta-1}$ and $T_j = 0$ for all $j \geq 3$ where Θ is uniformly distributed on $[0, 2\pi]$. Further, $\beta = 1$ in the case of (1.4) and $\beta > 1$ in the case of (1.5). In order to apply Corollary 2.5, we have to check whether (A1)-(A4) and, when $\alpha = 1$, (A5) hold (note that we are in Case III).

Since Θ has a continuous distribution, (A1) and the spread-out property in (A5) hold. Further, for $\alpha = 2/\beta$ and $\vartheta \in [0, \alpha)$,

$$|T_1|^\alpha + |T_2|^\alpha \;=\; |\sin(\Theta)|^2 + |\cos(\Theta)|^2 \;=\; 1 \quad \text{and} \quad |T_1|^\vartheta + |T_2|^\vartheta \;>\; 1 \quad \text{a.s.}$$

Therefore, (A3) (hence (A2)) holds with $\alpha = 2/\beta$ and $W = 1$. The latter almost immediately implies (A4a). Moreover, since $|\sin(\Theta)| < 1$ and $|\cos(\Theta)| < 1$ a.s., m is finite and strictly decreasing on $[0, \infty)$, in particular the second condition in (A5) holds (since m is the Laplace transform of a suitable finite measure on $[0, \infty)$, it has finite second derivative everywhere on $(0, \infty)$). Further, when $\alpha = 1$ (*i.e.* $\beta = 2$), the last condition in (A5) is trivially fulfilled since $|T_1| + |T_2| = 1$. Finally, observe that $\mathbb{E}[Z_1] = 0$ which allows us to conclude from Theorem 2.3(b) that $Z = 0$ whenever $\alpha \in (1, 2]$.

Now Corollary 2.5 yields

Corollary 2.6. *The solutions to (1.4) are precisely the centered normal distributions, while the solutions to (1.5) are precisely the symmetric $2/\beta$-stable distributions.*

2.6. *The functional equation of the smoothing transform.* For appropriate functions f, call

$$f(t) \;=\; \mathbb{E}\Big[\prod_{j \geq 1} f(T_j t)\Big] \qquad \text{for all } t \tag{2.33}$$

the functional equation of the smoothing transform. Understanding its properties is the key to solving (2.9). (2.33) has been studied extensively in the literature especially when f is the Laplace transform of a probability distribution on $[0, \infty)$. The latest reference is Alsmeyer et al. (2012) where $T_j \geq 0$ a.s., $j \in \mathbb{N}$, and decreasing functions $f : [0, \infty) \to [0, 1]$ are considered. Necessitated by the fact that we permit the random coefficients T_j, $j \in \mathbb{N}$ in the main equations to take negative values with positive probability, we need a two-sided version of this functional equation. We shall determine all solutions to (2.33) within the class $\mathcal{M}$ of functions $f : \mathbb{R} \to [0, 1]$ that satisfy the following properties:

(i) $f(0) = 1$ and f is continuous at 0;
(ii) f is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$.

A precise description of $\mathcal{S}(\mathcal{M})$ which is the set of members of $\mathcal{M}$ that satisfy (2.33) is given in the following theorem.

Theorem 2.7. *Assume that (A1)–(A4) hold true and let $d = 1$. Then the set $\mathcal{S}(\mathcal{M})$ is given by the functions of the form*

$$f(t) \;=\; \begin{cases} \mathbb{E}[\exp(-W c_1 t^\alpha)] & \text{for } t \geq 0, \\ \mathbb{E}[\exp(-W c_{-1} |t|^\alpha)] & \text{for } t \leq 0 \end{cases} \tag{2.34}$$

where $c_1, c_{-1} \geq 0$ are constants and $c_1 = c_{-1}$ if $\mathbb{G}(T) = \mathbb{R}^*$.

This theorem is Theorem 2.2 of Alsmeyer et al. (2012) in case where all T_j are nonnegative. In Section 4.2, we prove the extension to the case where the T_j take negative values with positive probability.

The rest of the paper is structured as follows. The proof of our main result, Theorem 2.4 splits into two parts, the direct part and the converse part. The direct part is to verify that the Fourier transforms given in (2.23)-(2.30) are actually members of $\mathcal{S}(\mathfrak{F})$; this is done in Section 4.1. The converse part is to show that any $\phi \in \mathcal{S}(\mathfrak{F})$ is of the form as stated in the theorem. This requires considerable efforts and relies heavily on the properties of the weighted branching process introduced in Section 2.2. The results on this branching process which we need in the proofs of our main results are provided in Section 3. In Section 4, we first solve the functional equation of the smoothing transform in the case $\mathbb{G}(T) = \mathbb{R}^*$ (Section 4.2). Theorem 2.3 is proved in Section 4.4. The homogeneous equation (2.9) is solved in Section 4.5, while the converse part of Theorem 2.4 is proved in Section 4.6.

The scheme of the proofs follows that in Alsmeyer et al. (2012); Alsmeyer and Meiners (2012, 2013). Repetitions cannot be avoided entirely and short arguments from the cited sources are occasionally repeated to make the paper at hand more self-contained. However, we omit proofs when identical arguments could have been given and provide only sketches of proofs when the degree of similarity is high.

3. Branching processes

In this section we provide all concepts and tools from the theory of branching processes that will be needed in the proofs of our main results.

3.1. *Weighted branching and the branching random walk.* Using the weighted branching process $(L(v))_{v\in\mathbb{V}}$ we define a related *branching random walk* $(\mathcal{Z}_n)_{n\geq 0}$ by

$$\mathcal{Z}_n \ := \ \sum_{v\in\mathcal{G}_n} \delta_{S(v)} \tag{3.1}$$

where $S(v) := -\log(|L(v)|)$, $v \in \mathbb{V}$ and $\mathcal{G}_n$ is the set of individuals residing in the nth generation, see (2.6). By μ we denote the intensity measure of the point process $\mathcal{Z} := \mathcal{Z}_1$, *i.e.*, $\mu(B) := \mathbb{E}[\mathcal{Z}(B)]$ for Borel sets $B \subseteq \mathbb{R}$. m (defined in (2.2)) is the Laplace transform of μ, that is, for $\gamma \in \mathbb{R}$,

$$m(\gamma) \ = \ \int e^{-\gamma x}\,\mu(\mathrm{d}x) \ = \ \mathbb{E}\Bigg[\sum_{j=1}^{N} e^{-\gamma S(v)}\Bigg].$$

By nonnegativity, m is well-defined on $\mathbb{R}$ but may assume the value $+\infty$. (A3) guarantees $m(\alpha) = 1$. This enables us to use a classical exponential change of measure. To be more precise, let $(S_n)_{n\geq 0}$ denote a random walk starting at 0 with increment distribution $\mathbb{P}(S_1 \in \mathrm{d}x) := \mu_\alpha(\mathrm{d}x) := e^{-\alpha x}\mu(\mathrm{d}x)$. It is well known (see *e.g.* Biggins and Kyprianou (1997, Lemma 4.1)) that then, for any given $n \in \mathbb{N}_0$, the distribution of S_n is given by

$$\mathbb{P}(S_n \in B) \ = \ \mathbb{E}\Bigg[\sum_{|v|=n} |L(v)|^\alpha \mathbb{1}_B(S(v))\Bigg], \quad B \subset \mathbb{R} \text{ Borel}. \tag{3.2}$$

3.2. *Auxiliary facts about weighted branching processes.*

Lemma 3.1. *If* (A1)–(A3) *hold, then* $\inf_{|v|=n} S(v) \to \infty$ *a.s. on* $\mathcal{S}$ *as* $n \to \infty$. *Equivalently,* $\sup_{|v|=n} |L(v)| \to 0$ *a.s. as* $n \to \infty$.

Source: This is Biggins (1998, Theorem 3). □

The following lemma will be used to reduce Case II to Case I.

Lemma 3.2. *Let the sequence* T *satisfy* (A1)–(A3). *Then so does the sequence* $(L(v))_{|v|=2}$. *If* (A4a) *or* (A4b) *holds for* T, *then* (A4a) *or* (A4b), *respectively, holds for* $(L(v))_{|v|=2}$. *If, moreover,* $\mathbb{E}[\sum_{j\geq 1} |T_j|^\alpha (\log^-(|T_j|))^2] < \infty$, *then the same holds for the sequence* $(L(v))_{|v|=2}$.

Proof: Throughout the proof we assume that $(T_j)_{j\geq 1}$ satisfies $m(\alpha) = 1$ which is the first part of (A3). Then $\mathbb{E}[\sum_{|v|=2} L(v)^\alpha] = 1$ which is the first part of (A3) for $(L(v))_{|v|=2}$. We shall use (3.2) to translate statements for $(T_j)_{j\geq 1}$ and $(L(v))_{|v|=2}$ into equivalent but easier ones for S_1 and S_2. (A1) for $(T_j)_{j\geq 1}$ corresponds to S_1 being nonlattice. But if S_1 is nonlattice, so is S_2. The second part of (A3) for $(T_j)_{j\geq 1}$ corresponds to $\mathbb{E}[e^{\vartheta S_1}] > 1$ which implies $\mathbb{E}[e^{\vartheta S_2}] > 1$. The same argument applies to (A4b). The first condition in (A4a) for $(T_j)_{j\geq 1}$, $m'(\alpha) \in (-\infty, 0)$, translates into $\mathbb{E}[S_1] \in (0, \infty)$. This implies $\mathbb{E}[S_2] = 2\mathbb{E}[S_1] \in (0, \infty)$ which is the first condition in (A4a) for $(L(v))_{|v|=2}$. As to the second condition in (A4a), notice that validity of (A4a) for $(T_j)_{j\geq 1}$ in combination with Biggins' theorem Lyons (1997) implies that $W_n \to W$ as $n \to \infty$ in mean. Then $\sum_{|v|=2n} |L(v)|^\alpha$ also converges in mean to W. Using the converse implication in Biggins' theorem gives that $(L(v))_{|v|=2}$ satisfies the second condition in (A4a) as well. Finally, $\mathbb{E}[\sum_{j\geq 1} |T_j|^\alpha (\log^-(|T_j|))^2] < \infty$ translates via (3.2) into $\mathbb{E}[(S_1^+)^2] < \infty$. Then $\mathbb{E}[(S_2^+)^2] \leq \mathbb{E}[(S_1^+ + (S_2 - S_1)^+)^2] < \infty$. □

3.3. *Multiplicative martingales and infinite divisibility.* We shall investigate the functional equation

$$f(\mathbf{t}) \;=\; \mathbb{E}\Big[\prod_{j\geq 1} f(T_j \mathbf{t})\Big], \qquad \mathbf{t} \in \mathbb{R}^d \tag{3.3}$$

within the set $\mathfrak{F}$ of Fourier transforms of probability distributions on $\mathbb{R}^d$ and, for technical reasons, for $d = 1$ within the class $\mathcal{M}$ introduced in Section 2.6. In order to treat the functions of $\mathfrak{F}$ and $\mathcal{M}$ simultaneously, we introduce the class $\mathcal{B}$ of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ satisfying $\sup_{\mathbf{t}\in\mathbb{R}^d} |f(\mathbf{t})| = 1$ and $f(0) = 1$. Then $\mathfrak{F} \subseteq \mathcal{B}$ and, when $d = 1$, $\mathcal{M} \subseteq \mathcal{B}$. By $\mathcal{S}(\mathcal{B})$ we denote the class of $f \in \mathcal{B}$ satisfying (3.3).

For an $f \in \mathcal{S}(\mathcal{B})$, we define the corresponding multiplicative martingale

$$M_n(\mathbf{t}) \;:=\; M_n(\mathbf{t}, \mathbf{L}) \;:=\; \prod_{|v|=n} f(L(v)\mathbf{t}), \qquad n \in \mathbb{N}_0,\; \mathbf{t} \in \mathbb{R}^d. \tag{3.4}$$

The notion *multiplicative martingale* is justified by the following lemma.

Lemma 3.3. *Let* $f \in \mathcal{S}(\mathcal{B})$ *and* $\mathbf{t} \in \mathbb{R}^d$. *Then* $(M_n(\mathbf{t}))_{n\geq 0}$ *is a bounded martingale w.r.t.* $(\mathcal{A}_n)_{n\geq 0}$ *and thus converges a.s. and in mean to a random variable* $M(\mathbf{t}) := M(\mathbf{t}, \mathbf{L})$ *satisfying*

$$\mathbb{E}[M(\mathbf{t})] \;=\; f(\mathbf{t}). \tag{3.5}$$

Source: Minor modifications in the proof of Biggins and Kyprianou (1997, Theorem 3.1) yield the result. □

Lemma 3.4. *Given $f \in \mathcal{S}(\mathcal{B})$, let M denote the limit of the associated multiplicative martingales. Then, for every $\mathbf{t} \in \mathbb{R}^d$,*

$$M(\mathbf{t}) \;=\; \prod_{|v|=n} [M]_v(L(v)\mathbf{t}) \quad a.s. \tag{3.6}$$

The identity holds for all $\mathbf{t} \in \mathbb{R}^d$ simultaneously a.s. if $f \in \mathcal{S}(\mathfrak{F})$.

Proof: For $n \in \mathbb{N}_0$, we have $|\{|v| = n\}| < \infty$ a.s., and hence

$$\begin{aligned} M(\mathbf{t}) &= \lim_{k\to\infty} \prod_{|v|=n} \prod_{|w|=k} f(L(vw)\mathbf{t}) \\ &= \prod_{|v|=n} \lim_{k\to\infty} \prod_{|w|=k} f([L(w)]_v L(v)\mathbf{t}) \;=\; \prod_{|v|=n} [M]_v(L(v)\mathbf{t}) \end{aligned}$$

for every $\mathbf{t} \in \mathbb{R}^d$ a.s. For $f \in \mathcal{S}(\mathfrak{F})$, by standard arguments, the identity holds for all $\mathbf{t} \in \mathbb{R}^d$ simultaneously a.s. □

Before we state our next result, we remind the reader that a measure ν on the Borel sets of $\mathbb{R}^d$ is called a *Lévy measure* if $\int (1 \wedge |\mathbf{x}|^2)\, \nu(\mathrm{d}\mathbf{x}) < \infty$, see *e.g.* Kallenberg (2002, p. 290). In particular, any Lévy measure assigns finite mass to sets of the form $\{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| \geq \varepsilon\}$, $\varepsilon > 0$.

Proposition 3.5. *Let $\phi \in \mathcal{S}(\mathfrak{F})$ with associated multiplicative martingales $(\Phi_n(\mathbf{t}))_{n\geq 0}$ and martingale limit $\Phi(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$. Then, a.s. as $n \to \infty$, $(\Phi_n)_{n\geq 0}$ converges pointwise to a random characteristic function Φ of the form $\Phi = \exp(\Psi)$ with*

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle - \frac{\mathbf{t}\boldsymbol{\Sigma}\mathbf{t}^{\mathsf{T}}}{2} + \int \left(e^{\mathrm{i}\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{\mathrm{i}\langle \mathbf{t},\mathbf{x}\rangle}{1+|\mathbf{x}|^2} \right) \nu(\mathrm{d}\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^d, \tag{3.7}$$

where $\mathbf{W}$ is an $\mathbb{R}^d$ valued $\mathbf{L}$-measurable random variable, $\boldsymbol{\Sigma}$ is an $\mathbf{L}$-measurable random positive semi-definite $d \times d$ matrix, and ν is an $\mathbf{L}$-measurable random Lévy measure on $\mathbb{R}^d$. Moreover,

$$\mathbb{E}[\Phi(\mathbf{t})] \;=\; \phi(\mathbf{t}) \quad \textit{for all } \mathbf{t} \in \mathbb{R}^d. \tag{3.8}$$

This proposition is the d-dimensional version of Theorem 1 in Caliebe (2003) and can be proved analogously.[7] Therefore, we refrain from giving further details.

Now pick some $f \in \mathcal{S}(\mathcal{B})$. The proof of Lemma 3.4 applies and gives the counterpart of (3.6)

$$M(\mathbf{t}) \;=\; \prod_{v \in \mathcal{T}_u} [M]_v(L(v)\mathbf{t}) \quad \text{a.s.} \tag{3.9}$$

for $\mathcal{T}_u := \{v \in \mathcal{G} : S(v) > u,\, S(v|_k) \leq u \text{ for } 0 < k < |v|\}$, $u \geq 0$. Taking expectations reveals that f also solves the functional equation with the weight sequence $(L(v))_{v\in\mathcal{T}_u}$ instead of the sequence $(T_j)_{j\geq 1}$. Further, when $f \in \mathcal{S}(\mathfrak{F})$, the

[7] The proof of Theorem 1 in Caliebe (2003) contains an inaccuracy that needs to be corrected. Retaining the notation of the cited paper, we think that it cannot be excluded that the set of continuity points $\mathcal{C}$ of the function $F(l)$ appearing in the proof of Theorem 1 in Caliebe (2003) depends on l. In the cited proof, this dependence is ignored when the limit $\lim_{u\to\infty, u\in\mathcal{C}}$ appears outside the expectation on p. 386. However, this problem can be overcome by using a slightly more careful argument.

proofs of Lemma 8.7(b) in Alsmeyer et al. (2012) and Lemma 4.4 in Alsmeyer and Meiners (2013) carry over to the present situation and yield

$$M(\mathbf{t}) \;=\; \lim_{u\to\infty} \prod_{v\in\mathcal{T}_u} f(L(v)\mathbf{t}) \;=\; \lim_{u\to\infty} M_{\mathcal{T}_u}(\mathbf{t}) \quad \text{for all } \mathbf{t} \text{ in } \mathbb{R}^d \text{ a.s.} \tag{3.10}$$

where $M_{\mathcal{T}_u}(\mathbf{t}) := \prod_{v\in\mathcal{T}_u} f(L(v)\mathbf{t})$. This formula allows us to derive useful representations for the random Lévy triplet of the limit Φ of the multiplicative martingale corresponding to a given $\phi \in \mathcal{S}(\mathfrak{F})$. Denote by $\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$ the one-point compactification of $\mathbb{R}^d$.

Lemma 3.6. *Let* $\mathbf{X}$ *be a solution to* (2.9) *with characteristic function* ϕ *and* d*-dimensional distribution (function)* F*. Let further* $(\mathbf{W}, \mathbf{\Sigma}, \nu)$ *be the random Lévy triplet of the limit* Φ *of the multiplicative martingale corresponding to* ϕ*, see Proposition 3.5. Then*

$$\sum_{v\in\mathcal{T}_u} F(\cdot/L(v)) \;\stackrel{\mathrm{v}}{\to}\; \nu \quad \text{as } u\to\infty \text{ a.s.} \tag{3.11}$$

where $\stackrel{\mathrm{v}}{\to}$ *denotes vague convergence on* $\overline{\mathbb{R}^d}\setminus\{\mathbf{0}\}$*. Further, for any* $h>0$ *with* $\nu(\{|\mathbf{x}|=h\})=0$ *a.s., the limit*

$$\mathbf{W}(h) \;:=\; \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{\{|\mathbf{x}|\le h/|L(v)|\}} \mathbf{x}\, F(\mathrm{d}\mathbf{x}) \tag{3.12}$$

exists a.s. and

$$\mathbf{W} \;=\; \mathbf{W}(h) + \int\limits_{\{h<|\mathbf{x}|\le 1\}} \mathbf{x}\,\nu(\mathrm{d}\mathbf{x}) + \int\limits_{\{|\mathbf{x}|>1\}} \frac{\mathbf{x}}{1+|\mathbf{x}|^2}\,\nu(\mathrm{d}\mathbf{x}) - \int\limits_{\{|\mathbf{x}|\le 1\}} \frac{\mathbf{x}|\mathbf{x}|^2}{1+|\mathbf{x}|^2}\,\nu(\mathrm{d}\mathbf{x}) \quad a.s. \tag{3.13}$$

where $\int_{\{h<|\mathbf{x}|\le 1\}} = -\int_{\{1<|\mathbf{x}|\le h\}}$ *when* $h>1$.

Proof: First, notice that by (3.10), for fixed $\mathbf{t}\in\mathbb{R}^d$, we have

$$\Phi_{\mathcal{T}_u}(\mathbf{t}) \;:=\; \prod_{v\in\mathcal{T}_u} \phi(L(v)\mathbf{t}) \;\to\; \Phi(\mathbf{t}) \;=\; \lim_{n\to\infty} \prod_{|v|=n} \phi(L(v)\mathbf{t}) \quad \text{a.s.}$$

along any fixed sequence $u\uparrow\infty$. $\Phi_{\mathcal{T}_u}(\mathbf{t})$ is a uniformly integrable martingale in u with right-continuous paths and therefore the convergence holds outside a $\mathbb{P}$-null set for all sequences $u\uparrow\infty$. Using the a.s. continuity of Φ on $\mathbb{R}^d$ (see Proposition 3.5), standard arguments show that the convergence holds for all $\mathbf{t}\in\mathbb{R}^d$ and all sequences $u\uparrow\infty$ on an event of probability one, *cf.* the proof Alsmeyer and Meiners (2013, Lemma 4.4). On this event, one can use the theory of triangular arrays as in the proof of Proposition 3.5 to infer that Φ has a representation $\Phi=\exp(\Psi)$ with Ψ as in (3.7). Additionally, Theorem 15.28(i) and (iii) in Kallenberg (2002) give (3.11) and (3.12), respectively. Note that the integrand of (3.7) being $(e^{\mathrm{i}\langle\mathbf{t},\mathbf{x}\rangle}-1-\mathrm{i}\langle\mathbf{t},\mathbf{x}\rangle/(1+|\mathbf{x}|^2))$ rather than $(e^{\mathrm{i}\langle\mathbf{t},\mathbf{x}\rangle}-1-\mathrm{i}\langle\mathbf{t},\mathbf{x}\rangle\mathbb{1}_{\{|\mathbf{x}|\le 1\}})$ as it is in Kallenberg (2002) (see *e.g.* Corollary 15.8 in the cited reference) does not affect ν but it does influence $\mathbf{W}$. The integrals $\int_{\{|\mathbf{x}|>1\}} \mathbf{x}/(1+|\mathbf{x}|^2)\nu(\mathrm{d}\mathbf{x})$ and $\int_{\{|\mathbf{x}|\le 1\}} \mathbf{x}|\mathbf{x}|^2/(1+|\mathbf{x}|^2)\nu(\mathrm{d}\mathbf{x})$ appearing in (3.13) are the corresponding compensation. □

3.4. *The embedded BRW with positive steps only.* In this section, an embedding technique, invented in Biggins and Kyprianou (2005), is explained. This approach is used to reduce cases in which (A6) does not hold to cases where it does.

Let $\mathcal{G}_0^> := \{\varnothing\}$, and, for $n \in \mathbb{N}$,

$$\mathcal{G}_n^> := \{vw \in \mathcal{G} : v \in \mathcal{G}_{n-1}^>, S(vw) > S(v) \geq S(vw|_k) \text{ for all } |v| < k < |vw|\}.$$

For $n \in \mathbb{N}_0$, $\mathcal{G}_n^>$ is called the nth strictly increasing ladder line. The sequence $(\mathcal{G}_n^>)_{n\geq 0}$ contains precisely those individuals v the positions of which are strict records in the random walk $S(\varnothing), S(v|_1), \ldots, S(v)$. Using the $\mathcal{G}_n^>$, we can define the nth generation point process of the embedded BRW of strictly increasing ladder heights by

$$\mathcal{Z}_n^> \;:=\; \sum_{v \in \mathcal{G}_n^>} \delta_{S(v)}. \tag{3.14}$$

$(\mathcal{Z}_n^>)_{n\geq 0}$ is a branching random walk with positive steps only. Let $T^> := (L(v))_{v\in\mathcal{G}_1^>}$ and denote by $\mathbb{G}(T^>)$ the closed multiplicative subgroup generated by $T^>$. The following result states that the point process $\mathcal{Z}^> := \mathcal{Z}_1^>$ inherits the assumptions (A1)-(A5) from $\mathcal{Z}$ and that also the closed multiplicative groups generated by T and $T^>$ coincide. We write $\mu_\alpha^>$ for the measure defined by

$$\mu_\alpha^>(B) \;:=\; \mathbb{E}\bigg[\sum_{v\in\mathcal{G}_1^>} e^{-\alpha S(v)}\delta_{S(v))}(B)\bigg], \qquad B \subseteq \mathbb{R}_\geq \;\text{ Borel.}$$

Proposition 3.7. *Assume* (A1)-(A3). *The following assertions hold.*

(a) $\mathbb{P}(|\mathcal{G}_1^>| < \infty) = 1$.
(b) $\mathcal{Z}^>$ *satisfies* (A1)-(A3) *where* (A3) *holds with the same* α *as for* $\mathcal{Z}$.
(c) *If* $\mathcal{Z}$ *further satisfies* (A4a) *or* (A4b), *then the same holds true for* $\mathcal{Z}^>$, *respectively.*
(d) *If* $\mathcal{Z}$ *satisfies* (A5), *then so does* $\mathcal{Z}^>$.
(e) *Let* $\mathbb{G}(\mathcal{Z})$ *be the minimal closed additive subgroup* G *of* $\mathbb{R}$ *such that* $\mathcal{Z}(\mathbb{R}\setminus G) = 0$ *a.s. and define* $\mathbb{G}(\mathcal{Z}^>)$ *analogously in terms of* $\mathcal{Z}^>$ *instead of* $\mathcal{Z}$. *Then* $\mathbb{G}(\mathcal{Z}^>) = \mathbb{G}(\mathcal{Z}) = \mathbb{R}$.
(f) $\mathbb{G}(T^>) = \mathbb{G}(T)$.

Remark 3.8. Notice that assertion (f) in Proposition 3.7 is the best one can get. For instance, one cannot conclude that if T has mixed signs (Case III), then so has $T^>$. Indeed, if T_1 is a Bernoulli random variable with success probability p and $T_2 = -U$ for a random variable U which is uniformly distributed on $(0,1)$, then all members of $\mathcal{G}_1^>$ have negative weights, that is, $T^>$ has negative signs only (Case II).

Proof of Proposition 3.7: Assertions (a), (b), (c) and (e) can be formulated in terms of the $|L(v)|$, $v \in \mathbb{V}$ only and, therefore, follow from Alsmeyer et al. (2012, Lemma 9.1) and Alsmeyer and Meiners (2013, Proposition 3.2).

It remains to prove (d) and (f). For the proof of (d) assume that $\mathcal{Z}$ satisfies (A5). By (c), $\mathcal{Z}^>$ also satisfies (A4a) and, in particular, the third condition in (A5). Further, the first condition in (A5) says that μ_α, the distribution of S_1, is spread-out. We have to check that then $\mu_\alpha^>$ is also spread-out. It can be checked (see *e.g.* Biggins and Kyprianou (2005)) that $\mu_\alpha^>$ is the distribution of S_σ for $\sigma = \inf\{n \geq 0 : S_n > 0\}$. Hence Lemma 1 in Araman and Glynn (2006) (or Corollaries 1 and 3 of Alsmeyer (2002)) shows that the distribution of S_σ is also spread-out.

That the second condition in (A5) carries over is Alsmeyer and Meiners (2013, Proposition 3.2(d)). The final condition of (A5) is that $\mathbb{E}[h_3(W_1)] < \infty$ where $h_n(x) = x(\log^+(x))^n \log^+(\log^+(x))$, $n = 2, 3$. In view of the validity of (A4a), Theorem 1.4 in Alsmeyer and Iksanov (2009) yields $\mathbb{E}[h_2(W)] < \infty$. Now notice that W is not only the limit of the martingale $(W_n)_{n\geq 0}$ but also of the martingale $W_n^> = \sum_{v\in\mathcal{G}_n^>} |L(v)|^\alpha$, $n \in \mathbb{N}_0$, see *e.g.* Proposition 5.1 in Alsmeyer and Kuhlbusch (2010). The converse implication of the cited theorem then implies that $\mathbb{E}[h_3(W_1^>)] < \infty$.

Regarding the proof of (f) we infer from (e) that $-\log(\mathbb{G}(|T|)) = \mathbb{G}(\mathcal{Z}) = \mathbb{G}(\mathcal{Z}^>) = -\log(\mathbb{G}(|T^>|))$ where $|T| = (|T_j|)_{j\geq 1}$ and $|T^>| = (|L(v)|)_{v\in\mathcal{G}_1^>}$. Thus, by (A1), $\mathbb{G}(|T^>|) = \mathbb{G}(|T|) = \mathbb{R}_>$. If $\mathbb{G}(T) = \mathbb{R}_>$, then $T = |T|$ and $T^> = |T^>|$ a.s. and thus $\mathbb{G}(|T^>|) = \mathbb{R}_>$ as well. It remains to show that if $\mathbb{G}(T) = \mathbb{R}^*$, then $\mathbb{G}(T^>) = \mathbb{R}^*$ as well. To this end, it is enough to show that $\mathbb{G}(T^>) \cap (-\infty, 0) \neq \emptyset$. If $\mathbb{P}(T_j \in (-1, 0)) > 0$ for some $j \geq 1$, then $\mathbb{P}(j \in \mathcal{G}_1^> \text{ and } T_j < 0) > 0$. Assume now $\mathbb{P}(T_j \in (-1, 0)) = 0$ for all $j \geq 1$. Since $\mathbb{G}(T) = \mathbb{R}^*$ there is an $x \geq 1$ such that $-x \in \mathrm{supp}(T_j)$ for some $j \geq 1$ where $\mathrm{supp}(X)$ denotes the support (of the law) of a random variable X. By (A3), we have $m(\alpha) = 1 < m(\beta)$ for all $\beta \in [0, \alpha)$. This implies that for some $k \geq 1$, $\mathbb{P}(|T_k| \in (0, 1)) > 0$ and, moreover, $\mathbb{P}(T_k \in (0, 1)) > 0$ since $\mathbb{P}(T_k \in (-1, 0)) = 0$. Thus, for some $y \in (0, 1)$, we have $y \in \mathrm{supp}(T_k)$. Let m be the minimal positive integer such that $xy^m < 1$. Then $-xy^m \in \mathbb{G}(T^>)$. $\square$

3.5. *Endogenous fixed points.* The concept of *endogeny*, introduced in Aldous and Bandyopadhyay (2005, Definition 7), is important for the problems considered here. For the purposes of this paper, it is enough to study endogeny in dimension $d = 1$.

Suppose that $W^{(v)}$, $v \in \mathbb{V}$ is a family of random variables such that, for each fixed $n \in \mathbb{N}_0$, the $W^{(v)}$, $|v| = n$ are i.i.d. and independent of $\mathcal{A}_n$. Further suppose that

$$W^{(v)} \;=\; \sum_{j\geq 1} T_j(v) W^{(vj)} \quad \text{a.s.} \tag{3.15}$$

for all $v \in \mathbb{V}$. Then the family $(W^{(v)})_{v\in\mathbb{V}}$ is called a *recursive tree process*, the family $(T(v))_{v\in\mathbb{V}}$ *innovations process* of the recursive tree process. The recursive tree process $(W^{(v)})_{v\in\mathbb{V}}$ is called nonnegative if the $W^{(v)}$, $v \in \mathbb{V}$ are all nonnegative, it is called *invariant* if all its marginal distributions in all generations are identical. There is a one-to-one correspondence between the solutions to (2.9) (in dimension $d = 1$) and recursive tree processes $(W^{(v)})_{v\in\mathbb{V}}$ as above, see Lemma 6 in Aldous and Bandyopadhyay (2005). An invariant recursive tree process $(W^{(v)})_{v\in\mathbb{V}}$ is *endogenous* if $W^{(\varnothing)}$ is measurable w.r.t. the innovations process $(T(v))_{v\in\mathbb{V}}$.

Definition 3.9 (*cf.* Definition 8.2 in Alsmeyer et al. (2012))**.**

- A distribution is called *endogenous* (w.r.t. the sequence $(T_j)_{j\geq 1}$) if it is the marginal distribution of an endogenous recursive tree process with innovations process $(T(v))_{v\in\mathbb{V}}$.
- A random variable W is called *endogenous fixed point (w.r.t. $(T_j)_{j\geq 1}$)* if there exists an endogenous recursive tree process with innovations process $(T(v))_{v\in\mathbb{V}}$ such that $W = W^{(\varnothing)}$ a.s.

A random variable W is called non-null when $\mathbb{P}(W \neq 0) > 0$. $W = 0$ is an endogenous fixed point. Of course, the main interest is in non-null endogenous fixed points W. An endogenous recursive tree process $(W^{(v)})_{v\in\mathbb{V}}$ will be called non-null when $W^{(\varnothing)}$ is non-null.

Endogenous fixed points have been introduced in a slightly different way in Alsmeyer and Meiners (2013, Definition 4.6). In Alsmeyer and Meiners (2013, Definition 4.6), a random variable W (or its distribution) is called endogenous (w.r.t. to $(T(v))_{v\in\mathbb{V}}$) if W is measurable w.r.t. $(L(v))_{v\in\mathbb{V}}$ and if

$$W = \sum_{|v|=n} L(v)[W]_v \quad \text{a.s.} \tag{3.16}$$

for all $n \in \mathbb{N}_0$. It is immediate that $([W]_v)_{v\in\mathbb{V}}$ then defines an endogenous recursive tree process. Therefore, Definition 4.6 in Alsmeyer and Meiners (2013) is (seemingly) stronger than the original definition of endogeny. The next lemma shows that the two definitions are equivalent.

Lemma 3.10. *Let (A1)-(A4) hold and let $(W^{(v)})_{v\in\mathbb{V}}$ be an endogenous recursive tree process with innovations process $(T(v))_{v\in\mathbb{V}}$. Then $W^{(v)} = [W^{(\varnothing)}]_v$ a.s. for all $v \in \mathbb{V}$ and (3.16) holds.*

The arguments in the following proof are basically contained in Alsmeyer et al. (2012, Proposition 6.4).

Proof: For $u \in \mathbb{V}$ and $n \in \mathbb{N}_0$, (3.15) implies $W^{(u)} = \sum_{|v|=n}[L(v)]_u W^{(uv)}$ a.s., which together with the martingale convergence theorem yields

$$\begin{aligned}
\exp(\mathrm{i}tW^{(u)}) &= \lim_{n\to\infty} \mathbb{E}[\exp(\mathrm{i}tW^{(u)})|\mathcal{F}_{|u|+n}] \\
&= \lim_{n\to\infty} \mathbb{E}\Big[\exp\Big(\mathrm{i}t \sum_{|v|=n} [L(v)]_u W^{(uv)}\Big)|\mathcal{F}_{|u|+n}\Big] \\
&= \lim_{n\to\infty} \prod_{|v|=n} \phi([L(v)]_u t) \;=\; [\Phi(t)]_u \quad \text{a.s.}
\end{aligned} \tag{3.17}$$

where ϕ denotes the Fourier transform of $W^{(\varnothing)}$ and $\Phi(t)$ denotes the a.s. limit of the multiplicative martingale $\prod_{|v|=n} \phi(L(v)t)$ as $n \to \infty$. The left-hand side in (3.17) is continuous in t. The right-hand side is continuous in t a.s. by Proposition 3.5. Therefore, (3.17) holds simultaneously for all $t \in \mathbb{R}$ a.s. In particular, $\exp(\mathrm{i}tW^{(\varnothing)}) = \Phi(t)$ for all $t \in \mathbb{R}$ a.s. Thus, $\exp(\mathrm{i}tW^{(u)}) = [\Phi(t)]_u = \exp(\mathrm{i}t[W^{(\varnothing)}]_u)$ for all $t \in \mathbb{R}$ a.s. This implies $W^{(u)} = [W^{(\varnothing)}]_u$ a.s. $\square$

Justified by Lemma 3.10 we shall henceforth use (3.16) as the definition of endogeny. Theorem 6.2 in Alsmeyer et al. (2012) gives (almost) complete information about nonnegative endogenous fixed points in the case when the T_j, $j \geq 1$ are nonnegative. This result has been generalized in Alsmeyer and Meiners (2013), Theorems 4.12 and 4.13. Adapted to the present situation, all these findings are summarized in the following proposition.

Proposition 3.11. *Assume that (A1)-(A4) hold true. Then*

(a) *there is a nonnegative non-null endogenous fixed point W w.r.t. the sequence $(|T_j|^\alpha)_{j\geq 1}$ given by (2.16). Any other nonnegative endogenous fixed point w.r.t. $(|T_j|^\alpha)_{j\geq 1}$ is of the form cW for some $c \geq 0$.*

(b) *W defined in (2.16) further satisfies*

$$W = \lim_{t\to\infty} D\left(e^{-\alpha t}\right) \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} = \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D\big(e^{-\alpha S(v)}\big) \tag{3.18}$$

$$= \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} \int_{\{|x|<e^{\alpha S(v)}\}} x\,\mathbb{P}(W\in \mathrm{d}x) \quad a.s. \tag{3.19}$$

where D is as defined on p. 76.

(c) *There are no non-null endogenous fixed points w.r.t. $(|T_j|^{\beta})_{j\geq 1}$ for $\beta\neq\alpha$.*

(d) *If, additionally, $\mathbb{E}[\sum_{j\geq 1}|T_j|^{\alpha}(\log^-(|T_j|))^2]<\infty$, then any endogenous fixed point w.r.t. $(|T_j|^{\alpha})_{j\geq 1}$ is of the form cW for some $c\in\mathbb{R}$, where W is given by (2.16).*

Proof: (a) is Theorem 6.2(a) in Alsmeyer et al. (2012) (and partially already stated in Proposition 2.2). (3.18) is (11.8) in Alsmeyer et al. (2012), (3.19) is (4.39) in Alsmeyer and Meiners (2013). (c) is Theorem 6.2(b) in Alsmeyer et al. (2012) in the case of nonnegative (or nonpositive) recursive tree processes and Theorem 4.12 in Alsmeyer and Meiners (2013) in the general case. (d) is Theorem 4.13 in Alsmeyer and Meiners (2013). □

In the case of weights with mixed signs there may be endogenous fixed points other than those described in Proposition 3.11. Theorem 3.12 given next states that under (A1)-(A4) these fixed points are always a deterministic constant times Z, the limit of $Z_n = \sum_{|v|=n} L(v)$, $n\in\mathbb{N}_0$.

Theorem 3.12. *Suppose (A1)-(A4). Then the following assertions hold.*

(a) *If $\alpha<1$, there are no non-null endogenous fixed points w.r.t. T.*

(b) *Let $\alpha=1$ and assume that $\mathbb{E}[\sum_{j\geq 1}|T_j|(\log^-(|T_j|))^2]<\infty$ holds in Cases I and II and (A5) holds in Case III. Then the endogenous fixed points are precisely of the form cW a.s., $c\in\mathbb{R}$ in Case I, while in Cases II and III, there are no non-null endogenous fixed points w.r.t. T.*

(c) *If $\alpha>1$, the following assertions are equivalent.*
   (i) *There is a non-null endogenous fixed point w.r.t. T.*
   (ii) *Z_n converges a.s. and $\mathbb{P}(\lim_{n\to\infty} Z_n=0)<1$.*
   (iii) *$\mathbb{E}[Z_1]=1$ and $(Z_n)_{n\geq 0}$ converges in $\mathcal{L}^{\beta}$ for some/all $1<\beta<\alpha$.*

   If either of the conditions (i)-(iii) is satisfied, then $(Z_n)_{n\geq 0}$ is a uniformly integrable martingale. In particular, Z_n converges a.s. and in mean to some random variable Z with $\mathbb{E}[Z]=1$ which is an endogenous fixed point w.r.t. T. Any other endogenous fixed point is of the form cZ for some $c\in\mathbb{R}$.

The proof of this result is postponed until Section 4.4.

We finish the section on endogenous fixed points with the proof of Proposition 2.1 which establishes the existence of $\mathbf{W}^*$. If well-defined, the latter random variable can be viewed as an endogenous inhomogeneous fixed point.

Proof of Proposition 2.1: By using the Cramér-Wold device we can and do assume that $d=1$. We shall write W_n^* for $\mathbf{W}_n^*$.
(i) It suffices to show that the infinite series $\sum_{v\in\mathbb{V}}|L(v)||C(v)|$ converges a.s. which is, of course, the case if the sum has a finite moment of order β. The latter follows

easily (see, for instance, Jelenković and Olvera-Cravioto (2012a, Lemma 4.1) or Alsmeyer and Meiners (2013, Proposition 5.4)).
(ii) The assumption entails $\mathbb{E}[|W_1^*|^\beta] = \mathbb{E}[|C|^\beta] < \infty$ and thereupon $-\infty < \mathbb{E}[C] < \infty$. If $T_j \geq 0$ a.s. for all $j \geq 1$, then sufficiency follows from Alsmeyer and Meiners (2013, Proposition 5.4). Thus, assume that $\mathbb{E}[C] = 0$. Then $(W_n^*)_{n\geq 0}$ is a martingale w.r.t. $(\mathcal{A}_n)_{n\geq 0}$, which is $\mathcal{L}^\beta$-bounded by assumption and, hence, a.s. convergent.
(iii) This is the first part of Theorem 1.1 in Buraczewski and Kolesko (2014). □

3.6. *A multitype branching process and homogeneous stopping lines.* In this section we assume that (A1)–(A4) and (A6) hold and that we are in the case of weights with mixed signs (Case III). Because of the latter assumption, when defining the branching random walk $(\mathcal{Z}_n)_{n\geq 0}$ from $(L(v))_{v\in\mathbb{V}}$, information is partially lost since each position $S(v)$ is defined in terms of the absolute value $|L(v)|$ of the corresponding weight $L(v)$, $v \in \mathbb{V}$. This loss of information can be compensated by keeping track of the sign of $L(v)$. Define

$$\tau(v) \ := \ \begin{cases} 1 & \text{if } L(v) > 0, \\ -1 & \text{if } L(v) < 0 \end{cases}$$

for $v \in \mathcal{G}$. For the sake of completeness, let $\tau(v) = 0$ when $L(v) = 0$. The positions $S(v)$, $v \in \mathbb{V}$ together with the signs $\tau(v)$, $v \in \mathbb{V}$ define a multitype general branching process with type space $\{1, -1\}$.

Define $M(\gamma) := (\mu_\gamma^{k,\ell}(\mathbb{R}))_{k,\ell=1,-1}$ where

$$\mu_\gamma^{k,\ell}(\cdot) \ := \ \mathbb{E}\Big[\sum_{j\geq 1: \mathrm{sign}(T_j)=k\ell} |T_j|^\gamma \delta_{S(j)}(\cdot)\Big].$$

Note that $M(\gamma)$ is nonnegative and symmetric since $\mu_\gamma^{k,\ell}(\cdot)$ depends on the product $k\ell$ only. For $\gamma = \alpha$, $p = \mathbb{E}[\sum_{j\geq 1} T_j^\alpha \mathbb{1}_{\{T_j>0\}}] = \mu_\alpha^{1,1}(\mathbb{R}) = \mu_\alpha^{-1,-1}(\mathbb{R})$ and $q = 1-p = \mathbb{E}[\sum_{j\geq 1}|T_j|^\alpha \mathbb{1}_{\{T_j<0\}}] = \mu_\alpha^{1,-1}(\mathbb{R}) = \mu_\alpha^{-1,1}(\mathbb{R})$. Therefore,

$$M(\alpha) = \begin{pmatrix} p & q \\ q & p \end{pmatrix}$$

where $0 < p, q < 1$ since we are in Case III. Next, we establish that the general branching process possesses the following properties.

(i) For all $h > 0$ and all $h_1, h_{-1} \in [0, h)$ either $\mu_\alpha^{1,1}(\mathbb{R} \setminus h\mathbb{Z}) > 0$, $\mu_\alpha^{1,-1}(\mathbb{R} \setminus (h_{-1} - h_1 + h\mathbb{Z})) > 0$ or $\mu_\alpha^{-1,1}(\mathbb{R} \setminus (h_1 - h_{-1} + h\mathbb{Z})) > 0$. Further, $M(\alpha)$ is irreducible.

(ii) Either $M(0)$ has finite entries only and Perron-Frobenius eigenvalue $\rho > 1$ or $M(0)$ has an infinite entry.

(iii) $M(\alpha)$ has eigenvalues 1 and $2p-1$ (with right eigenvectors $(1,1)^\mathsf{T}$ and $(1,-1)^\mathsf{T}$, respectively). 1 is the Perron-Frobenius eigenvalue of $M(\alpha)$.

(iv) The first moments of $\mu_\alpha^{k,\ell}$ are finite and positive for all $k, \ell \in \{1, -1\}$.

(i)–(iv) correspond to assumptions (A1)–(A4) in Iksanov and Meiners (2015) and will justify the applications of the limit theorems of the cited paper.

Proof of the validity of (i)–(iv): (i) $M(\alpha)$ is irreducible because all its entries are positive. Now assume for a contradiction that for some $h > 0$ and some $h_1, h_{-1} \in [0, h)$, $\mu_\alpha^{1,1}(\mathbb{R} \setminus h\mathbb{Z}) = \mu_\alpha^{1,-1}(\mathbb{R} \setminus (h_{-1} - h_1 + h\mathbb{Z})) = \mu_\alpha^{-1,1}(\mathbb{R} \setminus (h_1 - h_{-1} + h\mathbb{Z})) = 0$.

Since $\mu_\alpha^{1,-1} = \mu_\alpha^{-1,1}$ is nonzero, this implies $(h_{-1} - h_1 + h\mathbb{Z}) \cap (h_1 - h_{-1} + h\mathbb{Z}) \neq \emptyset$. Hence, there are $m, n \in \mathbb{Z}$ such that $h_{-1} - h_1 + hm = h_1 - h_{-1} + hn$, equivalently, $2(h_{-1} - h_1) = h(n-m)$. Thus, $h_{-1} - h_1$ belongs to the lattice $\frac{h}{2}\mathbb{Z}$. This contradicts (A1). While (iii) can be verified by elementary calculations, (ii) is an immediate consequence of (iii) for $M(0)$ has strictly larger entries than $M(\alpha)$ which has Perron-Frobenius eigenvalue 1. (iv) follows from (A4). □

Recall that $\mathcal{T}_t = \{v \in \mathcal{G} : S(v) > t \text{ but } S(v|_k) \leq t \text{ for all } 0 \leq k < |v|\}$, $t \geq 0$.

Proposition 3.13. *Assume that (A1)-(A3) and (A6) hold and that we are in Case III, i.e., $0 < p, q < 1$. Further, let $h : [0, \infty) \to (0, \infty)$ be a càdlàg function such that $h(t) \leq Ct^\gamma$ for all sufficiently large t and some $C > 0$, $\gamma \geq 0$.*

(a) *Suppose that (A4a) holds and that $\mathbb{E}\big[\sum_{k=1}^N |T_k|^\alpha S(k)^{1+\gamma}\big] < \infty$. Then, for $\beta = \alpha$, $j = 1, -1$, any $\varepsilon > 0$ and all sufficiently large c, the following convergence in probability holds as $t \to \infty$ on the survival set S*

$$\frac{\sum_{v \in \mathcal{T}_t : S(v) \leq t+c, \tau(v)=j} e^{-\beta(S(v)-t)} h(S(v)-t)}{\sum_{v \in \mathcal{T}_t} e^{-\alpha(S(v)-t)} h(S(v)-t)} \to \frac{1}{2} - \varepsilon(c) \geq \frac{1}{2} - \varepsilon. \tag{3.20}$$

(b) *Suppose that (A4b) holds. Then the convergence in (3.20) holds in the a.s. sense for all $\beta \geq \theta$ (with θ defined in (A4b)) and sufficiently large c that may depend on β.*

If one chooses $c = \infty$ (i.e., if one drops the condition $S(v) \leq t + c$), the result holds with $\varepsilon(\infty) = 0$.

Proof: (a) and (b) can be deduced from general results on convergence of multi-type branching processes, namely, Theorems 2.1 and 2.4 in Iksanov and Meiners (2015). The basic assumptions (A1)–(A4) of the cited article are fulfilled for these coincide with (i)-(iv) here. Assumption (A5) and Condition 2.2 in Iksanov and Meiners (2015) correspond to (A4a) and (A4b) here, respectively. Further, for fixed $j \in \{1, -1\}$, the numerator in (3.20) is $Z^\phi(t) = \sum_{v \in \mathcal{G}} [\phi]_v(t - S(v))$ for

$$\phi(t) = \sum_{k=1}^N e^{-\beta(S(k)-t)} h(S(k)-t) \mathbb{1}_{\{t < S(k) \leq t+c,\, \tau(k) = \tau(\varnothing) j\}},$$

while the denominator is of the form $Z^\psi(t)$ with

$$\psi(t) = \sum_{k=1}^N e^{-\alpha(S(k)-t)} h(S(k)-t) \mathbb{1}_{\{\tau(k) = \tau(\varnothing) j\}}.$$

The verification of the remaining conditions of Theorems 2.1 and 2.4 in Iksanov and Meiners (2015) is routine and can be carried out as in the proof of Proposition 9.3 in Alsmeyer et al. (2012).

The last statement follows from the same proof if one replaces c in the definition of ϕ by $+\infty$. □

The final result in this section is on the asymptotic behaviour of $\sum_{v \in \mathcal{T}_t} L(v)$ in Case III when $\alpha = 1$:

Lemma 3.14. *Assume that (A1)-(A6) hold, that $\alpha = 1$ and that $0 < p, q < 1$. Then*

$$t \sum_{v \in \mathcal{T}_t} L(v) \to 0 \quad \text{as } t \to \infty \text{ in probability.} \tag{3.21}$$

Proof: We first explain how (3.21) follows from Theorem 2.14 in Iksanov and Meiners (2015) and then check that the assumptions of the cited theorem are satisfied in the given situation.

Notice that, for $t \geq 0$, $\sum_{v \in \mathcal{T}_t} L(v) = e^{-t} Z^{\phi^1}(t) - e^{-t} Z^{\phi^{-1}}(t)$, where

$$Z^{\phi^j}(t) = \sum_{v \in \mathcal{G}} [\phi^j]_v(t - S(v)) \quad \text{and} \quad \phi^j(t) = e^t \sum_{|v|=1} e^{-S(v)} \mathbb{1}_{[0,S(v))}(t) \mathbb{1}_{\{\tau(v)=\tau(\varnothing)j\}},$$

$j = 1, -1$. Now (3.21) is a consequence of

$$\begin{aligned} t\Big|\sum_{v \in \mathcal{T}_t} L(v)\Big| &= t\big|e^{-t} Z^{\phi^1}(t) - e^{-t} Z^{\phi^{-1}}(t)\big| \\ &\leq t\Big|e^{-t} Z^{\phi^1}(t) - \frac{W}{2}\Big| + t\Big|e^{-t} Z^{\phi^{-1}}(t) - \frac{W}{2}\Big| \ \to \ 0 \end{aligned}$$

in $\mathbb{P}$-probability as $t \to \infty$, where the limit relation is guaranteed by Theorem 2.14 in Iksanov and Meiners (2015).

It remains to justify the application of Theorem 2.14 in Iksanov and Meiners (2015). The fact that assumptions (A1)–(A4) of Iksanov and Meiners (2015) hold has been verified at the beginning of the present subsection (see conditions (i)-(iv)). Our condition (A6) is the standing assumption in Iksanov and Meiners (2015). Condition 2.8 of Iksanov and Meiners (2015) follows from the last condition in (A5), namely, $\mathbb{E}[h_3(W_1)] < \infty$ where W_1 is as in (2.17). Condition 2.9 is a consequence of the first condition in (A5). (2.12) in Iksanov and Meiners (2015) (with $\delta = 1$) is the second condition of (A5). In order to check that Condition 2.13 in Iksanov and Meiners (2015) holds, first notice that $\mathbb{E}[\phi^j(s)] \leq e^t \mathbb{P}(S_1 > t)$ for all $t \geq 0$ and $j = 1, -1$. Here, (A5) implies that $\mathbb{E}[S_1^2] < \infty$ and, therefore,

$$t \int_t^\infty e^{-s} \mathbb{E}[\phi^j(s)] \, \mathrm{d}s \ \leq \ \int_t^\infty s \, \mathbb{P}(S_1 > s) \, \mathrm{d}s \ \to \ 0 \quad \text{as } t \to \infty.$$

Similarly,

$$t \sup_{s \geq t} e^{-s} \mathbb{E}[\phi^j(s)] \ \leq \ t \, \mathbb{P}(S_1 > t) \ \to \ 0 \quad \text{as } t \to \infty.$$

This implies validity of Condition 2.13.

Left with checking that (2.10) in Iksanov and Meiners (2015) holds, observe that $e^{-t} Z^{\phi^j}(t) \leq \sum_{v \in \mathcal{T}_t} e^{-S(v)} =: W(t)$ for $j = 1, -1$. Hence, it suffices to verify that

$$\mathbb{E}\Big[h_2\Big(\sup_{t \geq 0} W(t)\Big)\Big] \ < \ \infty \tag{3.22}$$

where $h_n(x) = x(\log^+(x))^n \log^+(\log^+(x))$, $n = 2, 3$. To this end, we invoke Lemma 8.1 of Alsmeyer and Kuhlbusch (2010) (which is an extension of an observation in Biggins (1979)). The cited lemma gives that for any $0 < a < 1$, there is a finite constant $C(a) > 0$ such that

$$\mathbb{P}(W > at) \ \geq \ C(a) \mathbb{P}\Big(\sup_{s \geq 0} W(s) > t\Big) \quad \text{for all } t > 1.$$

In view of this inequality and since h_2 is regularly varying at $+\infty$, we conclude that for $\mathbb{E}[h_2(\sup_{s \geq 0} W(s))] < \infty$ to hold it is sufficient that $\mathbb{E}[h_2(W)] < \infty$. Now $\mathbb{E}[h_3(W_1)] < \infty$ by (A5) which implies $\mathbb{E}[h_2(W)] < \infty$ according to Proposition 4.2 in Iksanov and Meiners (2015). $\square$

4. Proofs of the main results

4.1. *Proof of the direct part of Theorem 2.4.*

Proof of Theorem 2.4 (direct part): We only give (a sketch of) the proof in the case $\alpha = 1$ and $\mathbb{G}(T) = \mathbb{R}_>$. The other cases can be treated analogously. Let ϕ be as in (2.25), *i.e.*,

$$\phi(\mathbf{t}) = \mathbb{E}\Big[\exp\Big(\mathrm{i}\langle \mathbf{W}^* + W\mathbf{a}, \mathbf{t}\rangle - W\int |\langle \mathbf{t},\mathbf{s}\rangle|\,\sigma(\mathrm{d}s) - \mathrm{i}W\frac{2}{\pi}\int \langle \mathbf{t},\mathbf{s}\rangle \log(|\langle \mathbf{t},\mathbf{s}\rangle|)\,\sigma(\mathrm{d}\mathbf{s})\Big)\Big]$$

for some $\mathbf{a} \in \mathbb{R}^d$ and a finite measure σ on $\mathbb{S}^{d-1}$ with $\int s_k\,\sigma(\mathrm{d}\mathbf{s}) = 0$ for $k = 1, \ldots, d$. Using $\mathbf{W}^* = \sum_{j\geq 1} T_j[\mathbf{W}^*]_j + \mathbf{C}$ a.s. and $W = \sum_{j\geq 1} T_j[W]_j$ a.s., see (2.13) and (3.16), we obtain

$$\begin{aligned}
&\mathrm{i}\langle \mathbf{W}^* + W\mathbf{a}, \mathbf{t}\rangle - W\int |\langle \mathbf{t},\mathbf{s}\rangle|\,\sigma(\mathrm{d}s) - \mathrm{i}W\frac{2}{\pi}\int \langle \mathbf{t},\mathbf{s}\rangle \log(|\langle \mathbf{t},\mathbf{s}\rangle|)\,\sigma(\mathrm{d}\mathbf{s}) \\
&\quad = \mathrm{i}\langle \mathbf{C}, \mathbf{t}\rangle + \mathrm{i}\sum_{j\geq 1}\langle [\mathbf{W}^*]_j + [W]_j\mathbf{a}, T_j\mathbf{t}\rangle \\
&\qquad - \sum_{j\geq 1}[W]_j\int |\langle T_j\mathbf{t},\mathbf{s}\rangle|\,\sigma(\mathrm{d}s) - \mathrm{i}\sum_{j\geq 1}[W]_j\frac{2}{\pi}\int \langle T_j\mathbf{t},\mathbf{s}\rangle \log(|\langle \mathbf{t},\mathbf{s}\rangle|)\,\sigma(\mathrm{d}\mathbf{s}). \qquad (4.1)
\end{aligned}$$

Further, since $\int s_k\,\sigma(\mathrm{d}\mathbf{s}) = 0$ for $k = 1, \ldots, d$, we have

$$\int \langle T_j\mathbf{t},\mathbf{s}\rangle \log(|\langle \mathbf{t},\mathbf{s}\rangle|)\,\sigma(\mathrm{d}\mathbf{s}) \;=\; \int \langle T_j\mathbf{t},\mathbf{s}\rangle \log(|\langle T_j\mathbf{t},\mathbf{s}\rangle|)\,\sigma(\mathrm{d}\mathbf{s})$$

for all $j \geq 1$ with $T_j > 0$. Substituting this in (4.1), passing to exponential functions, taking expectations on both sides and then using that the pairs $([\mathbf{W}^*]_j, [W]_j)$, $j \geq 1$ are i.i.d. copies of $(\mathbf{W}^*, W)$ independent of $(\mathbf{C}, T)$ one can check that ϕ satisfies (2.19). □

4.2. *Solving the functional equation in $\mathcal{M}$.*

Theorem 4.1. *Assume that (A1)–(A4) hold true and let $d = 1$. Let $f \in \mathcal{S}(\mathcal{M})$ and denote the limit of the corresponding multiplicative martingale by M. Then there are constants $c_1, c_{-1} \geq 0$ such that*

$$M(t) \;=\; \begin{cases} \exp(-Wc_1 t^\alpha) & \textit{for } t \geq 0, \\ \exp(-Wc_{-1}|t|^\alpha) & \textit{for } t \leq 0 \end{cases} \quad \textit{a.s.} \qquad (4.2)$$

Furthermore, if $\mathbb{G}(T) = \mathbb{R}^*$, then $c_1 = c_{-1}$.

We first prove Theorem 4.1 in Cases I and II (see (2.4)). Case III needs some preparatory work and will be settled at the end of this section.

Proof of Theorem 4.1: Case I: The statement is a consequence of Theorem 8.3 in Alsmeyer et al. (2012).
Case II: For $f \in \mathcal{S}(\mathcal{M})$, iteration of (2.33) in terms of the weighted branching model gives

$$f(t) \;=\; \mathbb{E}\Big[\prod_{|v|=2} f(L(v)t)\Big], \qquad t \in \mathbb{R}. \qquad (4.3)$$

By Lemma 3.2, $(L(v))_{|v|=2}$ satisfies (A1)–(A4). Further, the endogenous fixed point W is (by uniqueness) the endogenous fixed point for $(|L(v)|^\alpha)_{|v|=2}$. Since in Case

II all T_j, $j \in \mathbb{N}$ are a.s. nonpositive, all $L(v)$, $|v| = 2$ are a.s. nonnegative. This allows us to invoke the conclusion of the already settled Case I to infer that (4.2) holds with constants $c_1, c_{-1} \geq 0$. Using (3.6) for $n = 1$ and $t > 0$ we get

$$\begin{aligned} \exp(-Wc_1 t^\alpha) &= M(t) = \prod_{j\geq 1} [M]_j(T_j t) = \prod_{j \geq 1} \exp(-[W]_j c_{-1} |T_j t|^\alpha) \\ &= \exp\bigg(-c_{-1} \sum_{j\geq 1} |T_j|^\alpha [W]_j t^\alpha\bigg) = \exp(-Wc_{-1}t^\alpha) \quad \text{a.s.} \end{aligned}$$

In particular, $c_1 = c_{-1}$. □

Assuming that Case III prevails, *i.e.*, $0 < p, q < 1$, we prove four lemmas. While Lemmas 4.2 and 4.5 are principal and will be used in the proof of (the remaining part of) Theorem 4.1, Lemmas 4.3 and 4.4 are auxiliary and will be used in the proof of Lemma 4.5.

Lemma 4.2. *Let $f \in \mathcal{S}(\mathcal{M})$. If $f(t) = 1$ for some $t \neq 0$, then $f(u) = 1$ for all $u \in \mathbb{R}$.*

Proof: Let $t \neq 0$ with $f(t) = 1$, w.l.o.g. $t > 0$. We have

$$1 = f(t) = \mathbb{E}\bigg[\prod_{j\geq 1} f(T_j t)\bigg].$$

Since all factors on the right-hand side of this equation are bounded from above by 1, they must all equal 1 a.s. In particular, since $\mathbb{P}(T_j < 0) > 0$ for some j (see Proposition 3.7(e)), there is some $t' < 0$ with $f(t') = 1$. Let $s := \min\{t, |t'|\}$. Then, since f is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$, we have $f(u) = 1$ for all $|u| \leq s$. Now pick an arbitrary $u \in \mathbb{R}$, $|u| > s$ and let $\tau := \inf\{n \geq 1 : \sup_{|v|=n} |L(v)u| \leq s\}$. Then $\tau < \infty$ a.s. by Lemma 3.1. Since $(\prod_{|v|=n} f(L(v)u))_{n\geq 0}$ is a bounded martingale, the optional stopping theorem gives

$$f(u) = \mathbb{E}\bigg[\prod_{|v|=\tau} f(L(v)u)\bigg] = 1.$$

This completes the proof since u was arbitrary with $|u| > s$. □

Recall that $D(t) := t^{-1}(1 - \varphi(t))$ and define $D_\alpha(t) := \frac{1-f(t)}{|t|^\alpha}$ for $t \neq 0$ and

$$K_\mathrm{l} := \liminf_{t\to\infty} \frac{D_\alpha(e^{-t}) \vee D_\alpha(-e^{-t})}{D(e^{-\alpha t})}, \quad \text{and} \quad K_\mathrm{u}^\pm := \limsup_{t\to\infty} \frac{D_\alpha(\pm e^{-t})}{D(e^{-\alpha t})}.$$

Further, put $K_\mathrm{u} := K_\mathrm{u}^+ \vee K_\mathrm{u}^-$.

Lemma 4.3. *Assume that (A1)-(A4) and (A6) hold, and let $f \in \mathcal{S}(\mathcal{M})$ with $f(t) < 1$ for some (hence all) $t \neq 0$. Then*

$$0 < K_\mathrm{l} \leq K_\mathrm{u} < \infty.$$

Proof of Lemma 4.3: The proof of this lemma is an extension of the proof of Lemma 11.5 in Alsmeyer et al. (2012). Though the basic idea is identical, modifications are needed at several places.

Due to the convexity of φ, D is nonincreasing, and therefore

$$\begin{aligned}
&\sum_{v\in\mathcal{T}_t:\tau(v)=j} e^{-\alpha S(v)} D(e^{-\alpha S(v)})\mathbb{1}_{\{S(v)\le t+c\}} \\
&\qquad\qquad \ge \frac{\sum_{v\in\mathcal{T}_t:\tau(v)=j} e^{-\alpha S(v)}\mathbb{1}_{\{S(v)-t\le c\}}}{\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}} D(e^{-\alpha t}) \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}
\end{aligned}$$

for $j=1,-1$. By Proposition 3.13 with $h=1$, the ratio tends to something $\ge \frac{1}{2}-\varepsilon$ in probability on S for given $\varepsilon>0$ when c is chosen sufficiently large. The second converges to W a.s. on S by (3.18). Further,

$$\begin{aligned}
\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(L(v)) \ &\ge \sum_{v\in\mathcal{T}_t:\tau(v)=j} e^{-\alpha S(v)} D_\alpha(je^{-S(v)})\mathbb{1}_{\{S(v)\le t+c\}} \\
&\ge\ e^{-\alpha c} D_\alpha(je^{-(t+c)}) \sum_{v\in\mathcal{T}_t:\tau(v)=j} e^{-\alpha S(v)}\mathbb{1}_{\{S(v)\le t+c\}} \\
&\ge\ e^{-\alpha c}\frac{D_\alpha(je^{-(t+c)})}{D(e^{-\alpha(t+c)})} \sum_{v\in\mathcal{T}_t:\tau(v)=j} e^{-\alpha S(v)} D(e^{-\alpha S(v)})\mathbb{1}_{\{S(v)\le t+c\}}.
\end{aligned}$$

For $j=1$, passing to the limit $t\to\infty$ along an appropriate subsequence gives

$$-\log(M(1))\ \ge\ e^{-\alpha c}K_{\mathrm{u}}^{+}\Big(\frac{1}{2}-\varepsilon\Big)W \quad \text{a.s.}$$

where the convergence of the left-hand side follows from taking logarithms in (3.10), *cf.* Alsmeyer et al. (2012, Lemma 8.7(c)). Now one can argue literally as in the proof of Lemma 11.5 in Alsmeyer et al. (2012) to conclude that $K_{\mathrm{u}}^{+}<\infty$. $K_{\mathrm{u}}^{-}<\infty$ follows by choosing $j=-1$ in the argument above.

In order to conclude that $K_{\mathrm{l}}>0$, we derive an upper bound for $-\log(M(1))$

$$\begin{aligned}
\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(L(v)) &\le e^{\alpha c}\big(D_\alpha(e^{-t})\vee D_\alpha(-e^{-t})\big)\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)}\mathbb{1}_{\{S(v)\le t+c\}} \\
&\quad + \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(L(v))\mathbb{1}_{\{S(v)>t+c\}} \\
&\le e^{\alpha c}\frac{D_\alpha(e^{-t})\vee D_\alpha(-e^{-t})}{D(e^{-\alpha t})}\sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D(e^{-\alpha S(v)})\mathbb{1}_{\{S(v)\le t+c\}} \\
&\quad + \sum_{v\in\mathcal{T}_t} e^{-\alpha S(v)} D_\alpha(L(v))\mathbb{1}_{\{S(v)>t+c\}}.
\end{aligned}$$

Now letting $t\to\infty$ along an appropriate subsequence and using Proposition 3.13, we obtain that

$$-\log(M(1))\ \le\ e^{\alpha c}K_{\mathrm{l}}W + K_{\mathrm{u}}\varepsilon W.$$

Hence, $K_{\mathrm{l}}=0$ would imply $M(1)=1$ a.s., in particular, $f(1)=\mathbb{E}[M(1)]=1$ which is a contradiction by Lemma 4.2. □

Lemma 4.4. *Suppose that* (A1)–(A4) *and* (A6) *hold, and let $f\in\mathcal{S}(\mathcal{M})$ with $f(t)<1$ for some $t\neq 0$. Let $(t_n')_{n\ge 1}$ be a sequence of non-zero reals tending to* 0. *Then there are a subsequence $(t_{n_k}')_{k\ge 1}$ and a function $g:[-1,1]\to[0,\infty)$ which is* *decreasing on* $[-1,0]$*, increasing on* $[0,1]$*, and satisfies* $g(0)=0$ *and* $g(1)=1$ *such that*

$$\frac{1-f(zt_k)}{1-f(t_k)} \underset{k\to\infty}{\longrightarrow} g(z) \quad \textit{for all } z\in[-1,1] \tag{4.4}$$

where $(t_k)_{k\geq 1}=(t'_{n_k})_{k\geq 1}$ *or* $(t_k)_{k\geq 1}=(-t'_{n_k})_{k\geq 1}$*. The sequence* $(t_k)_{k\geq 1}$ *can be chosen such that*

$$\liminf_{k\to\infty}(1-f(t_k))/(1-\varphi(|t_k|^{\alpha})) \;\geq\; K_{\mathrm{l}} \;>\; 0 \tag{4.5}$$

with φ *as defined in Proposition 2.2(a).*

Proof: From Lemma 4.2 we infer that $1-f(t)>0$ for all $t\neq 0$. Thus, the ratio in (4.4) is well-defined. Recalling that $f(t)$ is nonincreasing for $t\geq 0$ and nondecreasing for $t<0$ we conclude that, for $z\in[0,1]$, $(1-f(zt))/(1-f(t))\leq 1$, while for $z\in[-1,0]$, $(1-f(zt))/(1-f(t))\leq(1-f(-t))/(1-f(t))$. The problem here is that at this point we do not know whether the latter ratio is bounded as $t\to 0$. However, according to Lemma 4.3

$$\liminf_{n\to\infty}\frac{(1-f(t'_n))\vee(1-f(-t'_n))}{1-\varphi(|t'_n|^{\alpha})} \;\geq\; K_{\mathrm{l}} \;>\; 0.$$

Hence, there is a subsequence of either $(t'_n)_{n\geq 1}$ or $(-t'_n)_{n\geq 1}$ which, for convenience, we again denote by $(t'_n)_{n\geq 1}$ such that

$$\liminf_{n\to\infty}\frac{1-f(t'_n)}{1-\varphi(|t'_n|^{\alpha})} \;\geq\; K_{\mathrm{l}} \;>\; 0.$$

Another appeal to Lemma 4.3 gives

$$\limsup_{n\to\infty}\frac{1-f(-t'_n)}{1-f(t'_n)} \;=\; \limsup_{n\to\infty}\frac{1-f(-t'_n)}{1-\varphi(|t'_n|^{\alpha})}\,\frac{1-\varphi(|t'_n|^{\alpha})}{1-f(t'_n)} \;\leq\; \frac{K_{\mathrm{u}}}{K_{\mathrm{l}}} \;<\; \infty.$$

Hence, the selection principle enables us to choose a subsequence $(t_n)_{n\geq 1}$ of $(t'_n)_{n\geq 1}$ along which convergence in (4.4) holds for each $z\in[-1,1]$ (details of the selection argument can be found in Alsmeyer et al. (2012, Lemma 11.2)). The resulting limit g satisfies $g(0)=0$ and $g(1)=1$. From the construction, it is clear that (4.5) holds. □

Lemma 4.5. *Suppose that* (A1)*–*(A4) *hold and let* $f\in\mathcal{S}(\mathcal{M})$ *with* $f(t)<1$ *for some* $t\neq 0$*. Then*

$$\lim_{t\to 0}\frac{1-f(zt)}{1-f(t)} \;=\; |z|^{\alpha} \quad \textit{for all } z\in\mathbb{R}. \tag{4.6}$$

Proof: Taking expectations in (3.9) for $u=0$ reveals that $f\in\mathcal{S}(\mathcal{M})$ satisfies (2.33) with T replaced by $T^{>}=(L(v))_{v\in\mathcal{G}_1^{>}}$. Furthermore, Proposition 3.7 ensures that the validity of (A1)-(A4) for T carries over to $T^{>}$ with the same characteristic exponent α. Since $|L(v)|<1$ a.s. for all $v\in\mathcal{G}_1^{>}$ we can and do assume until the end of proof that assumptions (A1)-(A4) and (A6) hold.

As in Alsmeyer et al. (2012); Biggins and Kyprianou (1997); Iksanov (2004), the basic equation is the following rearrangement of (2.33)

$$\frac{1-f(zt_n)}{|z|^{\alpha}(1-f(t_n))} \;=\; \mathbb{E}\Big[\sum_{j\geq 1}|T_j|^{\alpha}\frac{1-f(zT_jt_n)}{|zT_j|^{\alpha}(1-f(t_n))}\prod_{k<j}f(zT_kt_n)\Big] \tag{4.7}$$

for $z \in [-1,1]$ and $(t_n)_{n\geq 1}$ as in Lemma 4.4. The idea is to take the limit as $n \to \infty$ and then interchange limit and expectation. To justify the interchange, we use the dominated convergence theorem. To this end, we need to bound the ratios

$$|T_j|^\alpha \frac{1-f(zT_jt_n)}{|zT_j|^\alpha(1-f(t_n))} = |T_j|^\alpha \frac{1-f(zT_jt_n)}{1-\varphi(|zT_jt_n|^\alpha)} \frac{1-\varphi(|zT_jt_n|^\alpha)}{|zT_j|^\alpha(1-\varphi(|t_n|^\alpha))} \frac{1-\varphi(|t_n|^\alpha)}{1-f(t_n)}. \tag{4.8}$$

By Lemma 4.3, for all sufficiently large n,

$$\frac{1-f(zT_jt_n)}{1-\varphi(|zT_jt_n|^\alpha)} \;\leq\; K_{\mathrm{u}}+1 \quad \text{for all } j\geq 1 \text{ and } z \in [-1,1] \text{ a.s.}$$

Since $(t_n)_{n\geq 1}$ is chosen such that (4.5) holds, for all sufficiently large n,

$$\frac{1-\varphi(|t_n|^\alpha)}{1-f(t_n)} \;\leq\; K_{\mathrm{l}}^{-1}+1.$$

Finally, when (A4a) holds, then $D(t) = t^{-1}(1-\varphi(t)) \to 1$ as $t \to \infty$. This implies that the second ratio on the right-hand side of (4.8) remains bounded uniformly in z for all $j \geq 1$ a.s. as $n \to \infty$. If (A4b) holds, $D(t)$ is slowly varying at 0 and, using a Potter bound Bingham et al. (1989, Theorem 1.5.6(a)), one infers that, for an appropriate constant $K>0$ and all sufficiently large n,

$$\frac{1-\varphi(|zT_jt_n|^\alpha)}{|zT_j|^\alpha(1-\varphi(|t_n|^\alpha))} \;=\; \frac{D(|zT_jt_n|^\alpha)}{D(|t_n|^\alpha)} \;\leq\; K|zT_j|^{\theta-\alpha} \quad \text{for all } j\geq 1,\ z\in[-1,1] \text{ a.s.}$$

where θ comes from (A4b). Consequently, the dominated convergence theorem applies and letting $n \to \infty$ in (4.7) gives

$$g(z)/|z|^\alpha \;=\; \mathbb{E}\Big[\sum_{j\geq 1} |T_j|^\alpha \frac{g(zT_j)}{|zT_j|^\alpha}\Big], \qquad z \in [-1,1]$$

with g defined in (4.4). For $x \geq 0$, define $h_1(x) := e^{\alpha x} g(e^{-x})$ and $h_{-1}(x) := e^{\alpha x} g(-e^{-x})$. h_1 and h_{-1} satisfy the following system of Choquet-Deny type functional equations

$$h_1(x) \;=\; \int h_1(x+y)\mu_\alpha^+(\mathrm{d}y) + \int h_{-1}(x+y)\,\mu_\alpha^-(\mathrm{d}y), \quad x\geq 0, \tag{4.9}$$

$$h_{-1}(x) \;=\; \int h_{-1}(x+y)\mu_\alpha^+(\mathrm{d}y) + \int h_1(x+y)\,\mu_\alpha^-(\mathrm{d}y), \quad x\geq 0, \tag{4.10}$$

where

$$\mu_\alpha^\pm(B) \;=\; \mathbb{E}\Big[\sum_{j\geq 1} \mathbb{1}_{\{\pm T_j>0\}}|T_j|^\alpha \mathbb{1}_{\{S(j)\in B\}}\Big], \quad B \subseteq [0,\infty) \text{ Borel.}$$

By (A6), μ_α^+ and μ_α^- are concentrated on $\mathbb{R}_>$ and $\mu_\alpha^+(\mathbb{R}_>) + \mu_\alpha^-(\mathbb{R}_>) = 1$. By Lemma 4.4, g is bounded and, hence, h_1 and h_{-1} are locally bounded on $[0,\infty)$. Now use that $1 = h_1(0)$ in (4.9) to obtain that $h_j(y_0) \geq 1$ for some $j \in \{1,-1\}$ and some $y_0 > 0$. Then, since g is nonincreasing on $[-1,0]$ and nondecreasing on $[0,1]$, $h_j(y) > 0$ for all $y \in [0, y_0]$.

The desired conclusions can be drawn from Ramachandran et al. (1988, Theorem 1), but it requires less additional arguments to invoke the general Corollary 4.2.3 in Rao and Shanbhag (1994). Unfortunately, we do not know at this point that the functions h_j, $j = 1,-1$ are continuous which is one of the assumptions of Chapter 4 in Rao and Shanbhag (1994). On the other hand, as pointed out right after (3.1.1) in Rao and Shanbhag (1994), this problem can be overcome by considering

$$H_j^{(k)}(x) \;=\; k\int_0^{1/k} h_j(x+y)\,\mathrm{d}y, \qquad j=-1,1,\; k\in\mathbb{N}.$$

Since the h_j are nonnegative, so are the $H_j^{(k)}$. Further, since one of the h_j are strictly positive on $[0,y_0]$, the corresponding $H_j^{(k)}$ is strictly positive on $[0,y_0)$ as well. Local boundedness of the h_j implies continuity of the $H_j^{(k)}$. For fixed $k\in\mathbb{N}$ and $j=1,-1$, using the definition of $H_j^{(k)}$, (4.9) or (4.10), respectively, and Fubini's theorem, one can conclude that

$$H_j^{(k)}(x) \;=\; \int H_j^{(k)}(x+y)\mu_\alpha^+(\mathrm{d}y) + \int H_{-j}^{(k)}(x+y)\mu_\alpha^-(\mathrm{d}y).$$

Thus, for fixed k, $H_1^{(k)}$ and $H_{-1}^{(k)}$ satisfy the same system of equations (4.9) and (4.10). From Corollary 4.2.3 in Rao and Shanbhag (1994) we now infer that there are product-measurable processes $(\xi_j(x))_{x\geq 0}$, $j=1,-1$ with

(i) $H_j^{(k)}(x) = H_j^{(k)}(0)\mathbb{E}[\xi_j(x)] < \infty$, $x\geq 0$;
(ii) $\xi_j(x+y)=\xi_j(x)\xi_j(y)$ for all $x,y\geq 0$;
(iii) $\int \xi_j(x)\,\mu_\alpha(\mathrm{d}x)=1$ (pathwise).

(ii) together with the product-measurability of ξ_j implies that $\xi_j(x)=e^{\alpha_j x}$ for all $x\geq 0$ for some random variable α_j. Then condition (iii) becomes $\int e^{\alpha_j x}\mu_\alpha(\mathrm{d}x)=1$ (pathwise) which can be rewritten as $\varphi_{\mu_\alpha}(\alpha_j)=0$ (pathwise) for the Laplace transform φ_{μ_α} of μ_α. By (A6), φ_{μ_α} is strictly decreasing and hence $\alpha_j=0$ (pathwise). From (i) we therefore conclude $H_j^{(k)}(x) = H_j^{(k)}(0)$, $j=1,-1$. Since h_j is locally bounded and has only countably many discontinuities, $H_j^{(k)}(x)\to h_j(x)$ for (Lebesgue-)almost all x in $[0,\infty)$. From the fact that the $H_j^{(k)}$ are constant, we infer that the h_j are constant (Lebesgue-)a.e. This in combination with the fact that $e^{-\alpha x}h_j(x)=g(je^{-x})$ is monotone implies that h_j is constant on $(0,\infty)$, $h_j(x)=c_j$ for all $x\geq 0$, say, $j=1,-1$. From $H_j^{(k)}>0$ on $[0,y)$ for some y we further conclude $c_j>0$, $j=1,-1$. Now (4.9) for $x>0$ can be rewritten as $c_1=pc_1+qc_{-1}$. Since $0<p,q<1$ by assumption, we conclude $c_{-1}=c_1=:c$. Finally, (4.9) for $x=0$ yields $1=c$.

By now we have shown that for any sequence $(t_n')_{n\geq 0}$ in $\mathbb{R}\setminus\{0\}$ with $t_n'\to 0$ there is a subsequence $(t_{n_k}')_{k\geq 0}$ such that

$$\frac{1-f(zt_k)}{1-f(t_k)} \underset{k\to\infty}{\longrightarrow} |z|^\alpha \quad \text{for } |z|\leq 1 \tag{4.11}$$

for $(t_k)_{k\geq 1}=(t_{n_k}')_{k\geq 1}$ or $(t_k)_{k\geq 1}=(-t_{n_k}')_{k\geq 1}$. Replacing z by $-z$ in the formula above, we see that the same limiting relation holds for the sequence $(-t_k)_{k\geq 1}$ so that every sequence tending to 0 has a subsequence along which (4.11) holds. This implies (4.6). $\square$

Proof of Theorem 4.1: Case III: Let $f\in\mathcal{S}(\mathcal{M})$. If $f(t)=1$ for some $t\neq 0$, then $f(u)=1$ for all $u\in\mathbb{R}$ by Lemma 4.2. In this case $M(t)=1$ for all $t\in\mathbb{R}$, and (4.2) holds with $c_1=c_{-1}=0$. Assume now that $f(t)\neq 1$ for all $t\neq 0$. Using Lemma

4.5 and arguing as in the proof of Alsmeyer et al. (2012, Lemma 8.8) we conclude

$$-\log(M(t)) \;=\; |t|^{\alpha}(-\log(M(1)))$$

for any $t \neq 0$ and

$$-\log(M(1)) \;=\; \sum_{|v|=n} |L(v)|^{\alpha}[-\log(M(1))]_v \quad \text{a.s. for all } n \in \mathbb{N}_0.$$

Since f takes values in $[0,1]$, we have $0 \leq M(1) \leq 1$ a.s. Moreover, by the dominated convergence theorem,

$$1 = f(0) = \lim_{t\to 0} f(t) = \lim_{t\to 0} \mathbb{E}[M(t)] = \mathbb{E}\big[\lim_{t\to 0} M(1)^{|t|^{\alpha}}\big] = \mathbb{P}(M(1) > 0).$$

Consequently, $0 < M(1) \leq 1$ a.s. Since $f(t) \neq 1$ for $t \neq 0$ we infer $\mathbb{P}(M(1) = 1) < 1$. Therefore, $-\log(M(1))$ is a nonnegative, non-null endogenous fixed point of the smoothing transform with weights $|T_j|^{\alpha}$. From Proposition 3.11(a), we infer the existence of a constant $c > 0$ such that $-\log(M(1)) = cW$. Consequently, $M(t) = \exp(-Wc|t|^{\alpha})$ a.s. for all $t \in \mathbb{R}$. □

4.3. *Determining ν and $\mathbf{\Sigma}$.*

Lemma 4.6. *Suppose that (A1)-(A4) hold. Let $(\mathbf{W}, \mathbf{\Sigma}, \nu)$ be the random Lévy triplet which appears in (3.7).*

(a) *There exists a finite, deterministic measure σ on the Borel subsets of $\mathbb{S}^{d-1}$ such that*

$$\nu(A) \;=\; W \iint_{\mathbb{S}^{d-1}\times(0,\infty)} \mathbb{1}_A(rs) r^{-(1+\alpha)}\,\sigma(\mathrm{d}s)\,\mathrm{d}r \tag{4.12}$$

for all Borel sets $A \subseteq \mathbb{R}^d$ a.s. Furthermore, σ is symmetric, i.e., $\sigma(B) = \sigma(-B)$ for all Borel sets $B \subseteq \mathbb{S}^{d-1}$ if $\mathbb{G}(T) = \mathbb{R}^*$. $\alpha \geq 2$ implies $\sigma = 0$ (and, thus, $\nu = 0$ a.s.).

(b) *If $\alpha \neq 2$, then $\mathbf{\Sigma} = 0$ a.s. If $\alpha = 2$, then there is a deterministic symmetric positive semi-definite (possibly zero) matrix Σ with $\mathbf{\Sigma} = W\Sigma$ a.s.*

Proof: By (3.6) and (3.7),

$$\begin{aligned}
&\mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle - \frac{\mathbf{t}\mathbf{\Sigma}\mathbf{t}^{\mathsf{T}}}{2} + \int \left(e^{\mathrm{i}\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{\mathrm{i}\langle \mathbf{t},\mathbf{x}\rangle}{1+|\mathbf{x}|^2}\right)\nu(\mathrm{d}\mathbf{x})\\
&\quad= \mathrm{i}\sum_{|v|=n} L(v)\langle [\mathbf{W}]_v, \mathbf{t}\rangle \;-\; \frac{\sum_{|v|=n} L(v)^2 \mathbf{t}[\mathbf{\Sigma}]_v\mathbf{t}^{\mathsf{T}}}{2}\\
&\qquad+ \sum_{|v|=n} \int \left(e^{\mathrm{i}L(v)\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{\mathrm{i}L(v)\langle \mathbf{t},\mathbf{x}\rangle}{1+|\mathbf{x}|^2}\right)[\nu]_v(\mathrm{d}\mathbf{x})\\
&\quad= \mathrm{i}\sum_{|v|=n} L(v)\left(\langle [\mathbf{W}]_v, \mathbf{t}\rangle + \int \left[\frac{\langle \mathbf{t},\mathbf{x}\rangle}{1+L(v)^2|\mathbf{x}|^2} - \frac{\langle \mathbf{t},\mathbf{x}\rangle}{1+|\mathbf{x}|^2}\right][\nu]_v(\mathrm{d}x)\right)\\
&\qquad- \frac{\sum_{|v|=n} L(v)^2 \mathbf{t}[\mathbf{\Sigma}]_v\mathbf{t}^{\mathsf{T}}}{2}\\
&\qquad+ \sum_{|v|=n} \int \left(e^{\mathrm{i}L(v)\langle \mathbf{t},\mathbf{x}\rangle} - 1 - \frac{\mathrm{i}L(v)\langle \mathbf{t},\mathbf{x}\rangle}{1+L(v)^2|\mathbf{x}|^2}\right)[\nu]_v(\mathrm{d}\mathbf{x}), \quad \mathbf{t} \in \mathbb{R}^d.
\end{aligned}$$

The uniqueness of the Lévy triplet implies

$$\mathbf{\Sigma} \;=\; \sum_{|v|=n} L(v)^2 [\mathbf{\Sigma}]_v, \tag{4.13}$$

$$\int g(\mathbf{x})\,\nu(\mathrm{d}\mathbf{x}) \;=\; \sum_{|v|=n} \int g(L(v)\mathbf{x})\,[\nu]_v(\mathrm{d}\mathbf{x}) \tag{4.14}$$

a.s. for all $n \in \mathbb{N}_0$ and all non-negative Borel-measurable functions g on $\mathbb{R}^d$.
(a) Let $I_r(B) = \{\mathbf{x} \in \mathbb{R}^d : r \leq |\mathbf{x}|, \mathbf{x}/|\mathbf{x}| \in B\}$ where $r > 0$ and B is a Borel subset of $\mathbb{S}^{d-1}$. Define $I_r(B)$ for $r < 0$ as $I_{|r|}(-B)$. Since ν is a (random) Lévy measure, $\nu(I_r(B)) < \infty$ a.s. for all $r \neq 0$ and all B. When choosing $g = \mathbb{1}_{I_r(B)}$, (4.14) becomes

$$\nu(I_r(B)) \;=\; \sum_{|v|=n} [\nu]_v(I_{L(v)^{-1}r}(B)) \tag{4.15}$$

a.s. for all $n \in \mathbb{N}_0$. For fixed B define

$$f_B(r) \;:=\; \begin{cases} 1 & \text{if } r = 0, \\ \mathbb{E}[\exp(-\nu(I_{r^{-1}}(B)))] & \text{if } r \neq 0. \end{cases}$$

Then, by (4.15) and the independence of $(L(v))_{|v|=n}$ and $([\nu]_v)_{|v|=n}$,

$$f_B(r) \;=\; \mathbb{E}\Bigg[\exp\Bigg(-\sum_{|v|=n}[\nu]_v(I_{L(v)^{-1}r^{-1}}(B))\Bigg)\Bigg] \;=\; \mathbb{E}\Bigg[\prod_{|v|=n} f_B(L(v)r)\Bigg]$$

for all $r \in \mathbb{R}$. Further, f_B is nondecreasing on $(-\infty, 0]$ and nonincreasing on $[0, \infty)$. Since $I_{r^{-1}}(B) \downarrow \emptyset$ as $r \uparrow 0$ or $r \downarrow 0$, f_B is continuous at 0, and we conclude that $f_B \in \mathcal{S}(\mathcal{M})$. From Theorem 4.1, we infer that the limit M_B of the multiplicative martingales associated with f_B is of the form

$$M_B(r) \;=\; \begin{cases} \exp(-W\sigma(B)\alpha^{-1}r^{\alpha}) & \text{for } r \geq 0, \\ \exp(-W\sigma(-B)\alpha^{-1}|r|^{\alpha}) & \text{for } r \leq 0 \end{cases} \quad \text{a.s.},$$

where $\sigma(B)$ and $\sigma(-B)$ are nonnegative constants (depending on B but not on r) and $\sigma(B) = \sigma(-B)$ if $\mathbb{G}(T) = \mathbb{R}^*$. On the other hand, by an argument as in Alsmeyer and Meiners (2013, Lemma 4.8), we infer that $M_B(r) = \exp(-\nu(I_{r^{-1}}(B)))$ for all $r \in \mathbb{R}$ a.s. and thus

$$\nu(I_r(B)) \;=\; \begin{cases} W\sigma(B)\alpha^{-1}r^{-\alpha} & \text{for } r > 0, \\ W\sigma(B)\alpha^{-1}|r|^{-\alpha} & \text{for } r < 0 \end{cases} \quad \text{a.s.} \tag{4.16}$$

Let $\mathcal{D} := \{[\mathbf{a}, \mathbf{b}) \cap \mathbb{S}^{d-1} : \mathbf{a}, \mathbf{b} \in \mathbb{Q}^d\}$ where $[\mathbf{a}, \mathbf{b}) = \{\mathbf{x} \in \mathbb{R}^d : a_k \leq x_k < b_k \text{ for } k = 1, \ldots, d\}$. $\mathcal{D}$ is a countable generator of the Borel sets on $\mathbb{S}^{d-1}$. (4.16) holds for all $B \in \mathcal{D}$ and all $r \in \mathbb{Q}$ simultaneously, a.s. From (4.16) one infers (since $\mathbb{P}(W > 0) > 0$) that σ is a content on $\mathcal{D}$ (that is, σ is nonnegative, finitely additive and $\sigma(\emptyset) = 0$). It is even σ-additive on $\mathcal{D}$ (whenever the countable union of disjoint sets from $\mathcal{D}$ is again in $\mathcal{D}$). Thus, there is a unique continuation of σ to a measure on the Borel sets on $\mathbb{S}^{d-1}$. For ease of notation, this measure will again be denoted by σ. By uniqueness, (4.16) holds for all Borel sets $B \subseteq \mathbb{S}^{d-1}$ and $r \in \mathbb{Q}$ simultaneously, a.s. and, by standard arguments, extends to all $r \in \mathbb{R}$ as well. Lemma 2.1 in Kuelbs (1973) now yields that (4.12) holds for all Borel sets $A \subseteq \mathbb{R}^d$ a.s.

(b) Turning to $\mathbf{\Sigma}$, we write $\mathbf{\Sigma} = (\mathbf{\Sigma}_{k\ell})_{k,\ell=1,\ldots,d}$. For every $k,\ell = 1,\ldots,d$, (4.13) implies

$$\mathbf{\Sigma}_{k\ell} = \sum_{|v|=n} L(v)^2[\mathbf{\Sigma}_{k\ell}] \quad \text{a.s. for all } n \in \mathbb{N}_0, \tag{4.17}$$

that is, $\mathbf{\Sigma}_{k\ell}$ is an endogenous fixed point w.r.t. $(T_j^2)_{j\geq 1}$. If $\alpha \neq 2$, then $\mathbf{\Sigma}_{k\ell} = 0$ a.s. by Proposition 3.11(c). Suppose $\alpha = 2$. Since $\mathbf{\Sigma}_{kk} \geq 0$ a.s., we infer $\mathbf{\Sigma}_{kk} = W\Sigma_{kk}$ for some $\Sigma_{kk} \geq 0$ by Proposition 3.11(a). Further, the Cauchy-Schwarz inequality implies $-\mathbf{\Sigma}_{k\ell} + W\sqrt{\Sigma_{kk}\Sigma_{\ell\ell}} \geq 0$ a.s. Hence, (4.17) and Lemma 4.16 in Alsmeyer and Meiners (2013) imply $\mathbf{\Sigma}_{k\ell} = W\Sigma_{k\ell}$ for some $\Sigma_{k\ell} \in \mathbb{R}$. Consequently, $\mathbf{\Sigma} = W\Sigma$ for $\Sigma = (\Sigma_{k\ell})_{k,\ell=1,\ldots,d}$. Since $\mathbf{\Sigma}$ is symmetric and positive semi-definite, so is Σ. □

4.4. *The proofs of Theorems 2.3 and 3.12.* The key ingredient to the proofs of Theorems 2.3 and 3.12 is a bound on the tails of fixed points. This bound is provided by the following lemma.

Lemma 4.7. *Let $d = 1$ and assume that (A1)-(A4) hold. Let X be a solution to (2.9). Then*

(a) $\mathbb{P}(|X| > t) = O(1 - \varphi(t^{-\alpha}))$ *as* $t \to \infty$.
(b) $\mathbb{P}(|X| > t) = o(1 - \varphi(t^{-\alpha}))$ *as* $t \to \infty$ *if X is an endogenous fixed point.*

Proof: By (3.11) and (4.12), with F denoting the distribution of X,

$$\sum_{v\in\mathcal{T}_u} F(A/L(v)) \;\to\; W \iint \mathbb{1}_A(rs) r^{-(1+\alpha)}\,\sigma(\mathrm{d}s)\,\mathrm{d}r \quad \text{as } u \to \infty \text{ a.s.} \tag{4.18}$$

for every Borel set $A \subset \mathbb{R}$ that has a positive distance from 0 (since ν is continuous). Use the above formula for $A = \{|x| > 1\}$ and rewrite it in terms of $G(t) := t^{-\alpha}\mathbb{P}(|X| > t^{-1})$ for $t > 0$. This gives

$$\sum_{v\in\mathcal{T}_u} e^{-\alpha S(v)} G(e^{-S(v)}) \;\to\; W\frac{\sigma(\{1,-1\})}{\alpha} \quad \text{as } u \to \infty \text{ a.s.} \tag{4.19}$$

Now one can follow the arguments given in the proof of Lemma 4.9 in Alsmeyer and Meiners (2013) to conclude (a).

Finally, assume that X is endogenous, $X = X^{(\varnothing)}$, say, for the root value of an endogenous recursive tree process $(X^{(v)})_{v\in\mathbb{V}}$. Denote by Φ the limit of the multiplicative martingales associated with the Fourier transform ϕ of X. Then it is implicit in the proof of Lemma 3.10 that $\Phi(t) = \exp(\mathrm{i}tX^{(\varnothing)})$ a.s., that is, the Lévy measure in the random Lévy triplet of Φ vanishes a.s. Hence, the right-hand side of (4.18) vanishes a.s. (b) now follows by the same arguments as assertion (b) of Lemma 4.9 in Alsmeyer and Meiners (2013). □

Proof of Theorem 3.12: (a) Assume that $\alpha < 1$. Then one can argue as in the proof of Theorem 4.12 in Alsmeyer and Meiners (2013) (with $L(v)$ there replaced by $|L(v)|$ here) to infer that $X = 0$ a.s.
(b) Here $\alpha = 1$.
Case I in which $T_j \geq 0$ a.s., $j \in \mathbb{N}$. The result follows from Proposition 3.11(d).
Case II in which $T_j \leq 0$ a.s., $j \in \mathbb{N}$. If X is an endogenous fixed point w.r.t. T, then X is an endogenous fixed point w.r.t. $(L(v))_{|v|=2}$. We use Lemma 3.2 to reduce the problem to the already settled Case I where T_j, $j \in \mathbb{N}$ have to be replaced with

the nonnegative $L(v)$, $|v| = 2$. This allows us to conclude that $X = cW$ for some $c \in \mathbb{R}$. However, since

$$cW \ = \ X \ = \ \sum_{j \geq 0} T_j [X]_j \ = \ \sum_{j \geq 0} (-|T_j|) c [W]_j \ = \ -cW \quad \text{a.s.}$$

we necessarily have $c = 0$.

Case III. Using the embedding technique of Section 3.4, we can assume w.l.o.g. that (A6) holds in addition to (A1)–(A5). If after the embedding, we are in Case II rather than Case III, then $X = 0$ a.s., by what we have already shown. Therefore, the remaining problem is to conclude that $X = 0$ a.s. in Case III under the assumptions (A1)–(A6).

Let Φ denote the limit of the multiplicative martingales associated with the Fourier transform of X. From the proof of Lemma 3.10 we conclude that $\Phi(t) = \exp(\mathrm{i}tX)$ a.s. which together with (3.12) and (3.13) implies

$$X \ = \ \lim_{t \to \infty} \sum_{v \in \mathcal{T}_t} L(v) I(|L(v)|^{-1}) \quad \text{a.s. as } t \to \infty \tag{4.20}$$

where $I(t) := \int_{\{|x| \leq t\}} x \, F(\mathrm{d}x)$. Integration by parts gives

$$I(t) \ = \ \int_0^t \mathbb{P}(X > s) - \mathbb{P}(X < -s) \, \mathrm{d}s - t(\mathbb{P}(X > t) - \mathbb{P}(X < -t)). \tag{4.21}$$

The contribution of the second term to (4.20) is negligible, for

$$t|\mathbb{P}(X > t) - \mathbb{P}(X < -t)| = o(D(t^{-1})) \quad \text{as } t \to \infty \tag{4.22}$$

by Lemma 4.7(b) and

$$\sum_{v \in \mathcal{T}_t} |L(v)| D(|L(v)|^{-1}) \ \to \ W \quad \text{a.s.}$$

by (3.18). Hence,

$$X \ = \ \lim_{t \to \infty} \sum_{v \in \mathcal{T}_t} L(v) \int_0^{|L(v)|^{-1}} \big(\mathbb{P}(X > s) - \mathbb{P}(X < -s)\big) \, \mathrm{d}s \quad \text{a.s. as } t \to \infty. \tag{4.23}$$

One can replace the integral from 0 to $|L(v)|^{-1}$ above by the corresponding integral with $|L(v)|^{-1}$ replaced by e^t. To justify this, in view of (4.22), it is enough to check that

$$\limsup_{t \to \infty} \sum_{v \in \mathcal{T}_t} |L(v)| \int_{e^t}^{|L(v)|^{-1}} (1 - \varphi(s^{-1})) \, \mathrm{d}s \ \leq \ cW \quad \text{a.s.} \tag{4.24}$$

for some constant $c \geq 0$. This statement is derived in the proof of Theorem 4.13 in Alsmeyer and Meiners (2013) under the assumptions (A1)–(A3), (A4a) and $\mathbb{E}[\sum_{j \geq 1} |T_j| (\log^-(|T_j|))^2] < \infty$; see (4.41), (4.42) and the subsequent lines in the cited reference. Consequently, we arrive at the following representation of X

$$X \ = \ \lim_{t \to \infty} \sum_{v \in \mathcal{T}_t} L(v) \int_0^{e^t} \big(\mathbb{P}(X > s) - \mathbb{P}(X < -s)\big) \mathrm{d}s \quad \text{a.s.} \tag{4.25}$$

Now observe that (A5) implies (A4a), and that, under (A1)-(A4a), $t(1-\varphi(t^{-1})) \to 1$ as $t \to \infty$ because W is integrable as it is the a.s. limit of uniformly integrable martingale $(\sum_{|v|=n} |L(v)|)_{n\geq 0}$. In combination with (4.22) this yields $|\mathbb{P}(X > s) - \mathbb{P}(X < -s)| = o(s^{-1})$ as $s \to \infty$ and so, for every $\varepsilon > 0$ there is a t_0 such that

$$\left| \int_{e^{t_0}}^{e^t} \big(\mathbb{P}(X>s) - \mathbb{P}(X<-s)\big)\mathrm{d}s \right| \ \leq \ \varepsilon \int_{e^{t_0}}^{e^t} s^{-1}\mathrm{d}s \ = \ \varepsilon(t-t_0)$$

for all $t \geq t_0$. Lemma 3.14 thus implies $X = 0$ a.s. as claimed.
(c) Assume that $\alpha > 1$.
(i)⇒(iii): Let X be a non-null endogenous fixed point w.r.t. T. Then using Lemma 4.7(b) and recalling that according to Proposition 2.2(c) $1 - \varphi(t^{-\alpha})$ is regularly varying of index $-\alpha$ at ∞ we conclude that $\mathbb{E}[|X|^\beta] < \infty$ for all $\beta \in (0, \alpha)$. In particular, $\mathbb{E}[X|\mathcal{A}_n]$ is an $\mathcal{L}^\beta$-bounded martingale with limit (a.s. and in $\mathcal{L}^\beta$) X. Further, using that $\{|v| = n : L(v) \neq 0\}$ is a.s. finite for each $n \in \mathbb{N}_0$, we obtain

$$\mathbb{E}[X|\mathcal{A}_n] \ = \ \mathbb{E}\bigg[\sum_{|v|=n} L(v)[X]_v \bigg| \mathcal{A}_n \bigg] \ = \ Z_n \mathbb{E}[X] \quad \text{a.s.}$$

Hence $\mathbb{E}[Z_n] = 1$ for all $n \in \mathbb{N}_0$ and $X = \lim_{n\to\infty} Z_n \mathbb{E}[X]$ a.s. and in $\mathcal{L}^\beta$. In particular, $\mathbb{E}[Z_1] < \infty$ and $(Z_n)_{n\geq 0}$ converges in $\mathcal{L}^\beta$, which is (iii). As a further consequence, we obtain the uniqueness of the endogenous fixed point up to a real scaling factor.
(iii)⇒(ii): The implication follows because $(Z_n)_{n\geq 0}$ is a martingale, and convergence in $\mathcal{L}^\beta$ for some $\beta > 1$ implies uniform integrability.
(ii)⇒(i): For every $n \in \mathbb{N}_0$,

$$\begin{aligned} Z \ &= \ \lim_{k\to\infty} Z_{n+k} \ = \ \lim_{k\to\infty} \sum_{|v|=n} \sum_{|w|=k} L(v)[L(w)]_v \\ &= \ \sum_{|v|=n} L(v) \lim_{k\to\infty} \sum_{|w|=k} [L(w)]_v \ = \ \sum_{|v|=n} L(v)[Z]_v \quad \text{a.s.} \end{aligned} \tag{4.26}$$

This means that Z is an endogenous fixed point w.r.t. T which is non-null because $\mathbb{P}(Z = 0) < 1$ by assumption. □

Proof of Theorem 2.3: (a) is Lemma 4.14(a) in Alsmeyer and Meiners (2013); (b) is Theorem 3.12(c) of the present paper.
(c) Let $\alpha \geq 2$. If $Z_1 = 1$ a.s., then $Z_n = 1$ a.s. and the a.s. convergence to $Z = 1$ is trivial. Conversely, assume that $\mathbb{P}(Z_1 = 1) < 1$ and that $Z_n \to Z$ a.s. as $n \to \infty$. According to part (b) of the theorem, $(Z_n)_{n\geq 0}$ is a martingale, and $Z_n \to Z$ in $\mathcal{L}^\beta$ for all $\beta \in (1, \alpha)$. By an approach that is close to the one taken in Alsmeyer et al. (2009, Proof of Theorem 1.2) we shall show that this produces a contradiction. Pick some $\beta \in (1, 2)$ if $\alpha = 2$ and take $\beta = 2$ if $\alpha > 2$. For $Z_n \to Z$ in $\mathcal{L}^\beta$ to hold true it is necessary that $\mathbb{E}[|Z_1 - 1|^\beta] < \infty$. Then, using the lower bound in the Burkholder-Davis-Gundy inequality Chow and Teicher (1997, Theorem 11.3.1), we

infer that for some constant $c_\beta > 0$ we have

$$\begin{aligned}
\mathbb{E}[|Z-1|^\beta] &\geq c_\beta \mathbb{E}\Big[\Big(\sum_{n\geq 0}(Z_{n+1}-Z_n)^2\Big)^{\beta/2}\Big] \\
&= c_\beta \mathbb{E}\Big[\Big(\sum_{n\geq 0}\Big(\sum_{|v|=n} L(v)([Z_1]_v - 1)\Big)^2\Big)^{\beta/2}\Big] \\
&\geq c_\beta \mathbb{E}\Big[\Big(\sum_{n=0}^{m-1}\Big(\sum_{|v|=n} L(v)([Z_1]_v - 1)\Big)^2\Big)^{\beta/2}\Big] \qquad (4.27)
\end{aligned}$$

for every $m \in \mathbb{N}$. Since $\beta \in (1,2]$, the function $x \mapsto x^{\beta/2}$ ($x \geq 0$) is concave which implies $(x_1 + \ldots + x_m)^{\beta/2} \geq m^{\beta/2-1}(x_1^{\beta/2} + \ldots + x_m^{\beta/2})$ for any $x_1, \ldots, x_m \geq 0$. Plugging this estimate into (4.27) gives

$$\mathbb{E}[|Z-1|^\beta] \geq c_\beta m^{\beta/2-1} \sum_{n=0}^{m-1} \mathbb{E}\Big[\Big|\sum_{|v|=n} L(v)([Z_1]_v - 1)\Big|^\beta\Big].$$

Given $\mathcal{A}_n$, $\sum_{|v|=n} L(v)([Z_1]_v - 1)$ is a weighted sum of i.i.d. centered random variables and hence the terminal value of a martingale. Thus, we can again use the lower bound of the Burkholder-Davis-Gundy inequality Chow and Teicher (1997, Theorem 11.3.1) and then Jensen's inequality on $\{W_n(2) > 0\}$ where $W_n(\gamma) = \sum_{|v|=n} |L(v)|^\gamma$ to infer

$$\begin{aligned}
\mathbb{E}[|Z-1|^\beta] &\geq c_\beta^2 m^{\beta/2-1} \sum_{n=0}^{m-1} \mathbb{E}\Big[\Big(\sum_{|v|=n} L(v)^2([Z_1]_v - 1)^2\Big)^{\beta/2}\Big] \\
&= c_\beta^2 m^{\beta/2-1} \sum_{n=0}^{m-1} \mathbb{E}\Big[W_n(2)^{\beta/2}\Big(\sum_{|v|=n} \frac{L(v)^2}{W_n(2)}([Z_1]_v - 1)^2\Big)^{\beta/2}\Big] \\
&\geq c_\beta^2 m^{\beta/2-1} \sum_{n=0}^{m-1} \mathbb{E}\Big[W_n(2)^{\beta/2} \sum_{|v|=n} \frac{L(v)^2}{W_n(2)}([Z_1]_v - 1)^\beta\Big)\Big] \\
&= c_\beta^2 m^{\beta/2-1} \mathbb{E}[|Z_1-1|^\beta] \sum_{n=0}^{m-1} \mathbb{E}[W_n(2)^{\beta/2}].
\end{aligned}$$

To complete the proof it suffices to show

$$\lim_{m\to\infty} m^{\beta/2-1} \sum_{n=0}^{m-1} \mathbb{E}[W_n(2)^{\beta/2}] = \infty \qquad (4.28)$$

for the latter contradicts $\mathbb{E}[|Z-1|^\beta] < \infty$.
Case $\alpha > 2$: Recalling that $m(2) > 1$ in view of (A3) and that $\beta = 2$ we have $\mathbb{E}[W_n(2)^{\beta/2}] = \mathbb{E}[W_n(2)] = m(2)^n \to \infty$ and thereupon (4.28).
Case $\alpha = 2$: Since (A4a) is assumed, we infer $W_n(2) \to W$ a.s. and in $\mathcal{L}^1$. In particular $\mathbb{E}[W_n(2)^{\beta/2}] \to \mathbb{E}[W^{\beta/2}] > 0$ which entails (4.28). ☐

Remark 4.8. In Theorem 2.3(c) the case when $\alpha = 2$ and (A4a) fails remains a challenge. Here, some progress can be achieved once the asymptotics of $\mathbb{E}[W_n(2)^\gamma]$ as $n \to \infty$ has been understood, where $\gamma \in (0,1)$, which is a nontrivial problem since $\lim_{n\to\infty} W_n(2) = 0$ a.s. In the case when Seneta-Heyde scaling constants of the martingale $(W_n(2))_{n\geq 0}$ (*i.e.*, constants $c_n > 0$ such that $c_n^{-1} W_n(2)$ converges in distribution as $n \to \infty$ to a non-trivial limit) are available and when the sequence $((c_n^{-1} W_n(2))^\gamma)_{n\geq 0}$ is uniformly integrable, $\mathbb{E}[W_n(2)^\gamma]$ is of order c_n^γ which can then be used to conclude (4.28). While in general, Seneta-Heyde norming constants are either not available or not explicit Biggins and Kyprianou (1997), in the case when $m'(2) = 0$, called the *boundary case* by Biggins and Kyprianou Biggins and Kyprianou (2005), recent progress does allow us to obtain (4.28).

Indeed, under (A1)–(A3) and when $\alpha = 2$, $m'(2) = 0$, $\mathbb{E}[\sum_{|v|=1} S(v)^2 e^{-\alpha S(v)}] < \infty$, $\mathbb{E}[W_1(2)(\log^+(W_1(2)))^2] < \infty$, and $\mathbb{E}[\widetilde{W_1}(2) \log^+(\widetilde{W_1}(2))] < \infty$ where $\widetilde{W_1}(2)$ is defined as $\sum_{|v|=1} S(v)^+ e^{-2S(v)}$, $\sqrt{n} W_n$ converges in distribution to a non-trivial limit by Theorem 1.1 in Aïdékon and Shi (2014). Further, for $s > 0$,

$$\begin{aligned}
\mathbb{P}\Big(\min_{v\in\mathcal{G}} S(v) < -s\Big) &\leq \mathbb{E}\Big[\sum_{v\in\mathcal{G}} e^{-2S(v)} e^{2S(v)} \mathbb{1}_{\{S(v)<-s, S(v|k)\geq -s \text{ for } k<|v|\}}\Big] \\
&\leq e^{-2s}\mathbb{P}(S_n < -s \text{ for some } n \in \mathbb{N}_0) \leq e^{-2s}.
\end{aligned}$$

Consequently, for $t > 0$,

$$\begin{aligned}
\mathbb{P}(\sqrt{n} W_n(2) > t) &\leq e^{-2s} + \mathbb{P}\Big(\sqrt{n} \sum_{|v|=n} e^{-2S(v)} \mathbb{1}_{\{S(v|k)\geq -s \text{ for all } 0\leq k\leq |v|\}} > t\Big) \\
&\leq e^{-2s} + \frac{\sqrt{n}}{t}\mathbb{E}\Big[\sum_{|v|=n} e^{-2S(v)} \mathbb{1}_{\{S(v|k)\geq -s \text{ for all } 0\leq k\leq |v|\}}\Big] \\
&= e^{-2s} + \frac{\sqrt{n}}{t}\mathbb{P}\Big(\min_{0\leq k\leq n} S_k > -s\Big).
\end{aligned}$$

We know from Kozlov (1976) that $\mathbb{P}(\min_{0\leq k\leq n} S_k > -s) \leq C\frac{1+s}{\sqrt{n}}$ for all $n \geq 0$ for a constant C that does not depend on s and n. Therefore, choosing $s = t^\varepsilon$ for $\varepsilon > 0$, we conclude that

$$\sup_{n\geq 0} \mathbb{P}(\sqrt{n} W_n(2) > t) \leq e^{-2s} + C\frac{1+s}{t} = e^{-2t^\varepsilon} + C\frac{1+t^\varepsilon}{t} \leq \frac{C_\varepsilon}{t^{1-\varepsilon}}$$

for all $t \geq 1$ and a constant C_ε depending on ε but not on t. This implies uniform integrability of the sequence $(n^{\gamma/2} W_n(2)^\gamma)_{n\in\mathbb{N}}$ for all $\gamma \in (0,1)$.

In conclusion, under the stated assumptions, (4.28) holds and thus, $(Z_n)_{n\geq 0}$ has a non-trivial a.s. limit only when $Z_n = 1$ a.s. for all $n \in \mathbb{N}_0$.

4.5. *Solving the homogeneous equation in* $\mathbb{R}^d$.

Lemma 4.9. *Assume that* (A1)–(A4) *hold, that* $\alpha = 1$, *and that* $\mathbb{G}(T) = \mathbb{R}_>$. *Further, assume that* $\mathbb{E}[\sum_{j\geq 1} T_j (\log^-(T_j))^2] < \infty$. *Let* $\mathbf{X} = (X_1, \ldots, X_d)$ *be a solution to* (2.9) *with distribution function* F *and let* $\mathbf{W}(1) = (W(1)_1, \ldots, W(1)_d)$ *be defined by*

$$\mathbf{W}(1) = \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{\{|\mathbf{x}|\leq L(v)^{-1}\}} \mathbf{x}\, F(\mathrm{d}\mathbf{x}) \quad a.s.,$$

i.e., as in (3.12). *Then there is a finite constant* $K > 0$ *with* $\max_{j=1,\ldots,d} |W(1)_j| \leq KW$ *a.s.*

Proof: First of all, the existence of the limit that defines $\mathbf{W}(1)$ follows from Lemma 3.6. Fix $j \in \{1, \ldots, d\}$. Then, with F_j denoting the distribution of X_j,

$$\begin{aligned} W(1)_j &= \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{\{|x|\leq L(v)^{-1}\}} x\, F_j(\mathrm{d}x) \\ &= \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \bigg(\int_0^{L(v)^{-1}} (\mathbb{P}(X_j > x) - \mathbb{P}(X_j < -x))\,\mathrm{d}x \\ &\qquad - L(v)^{-1}(\mathbb{P}(X_j > L(v)^{-1}) - \mathbb{P}(X < -L(v)^{-1}))\bigg) \quad \text{a.s.} \end{aligned} \tag{4.29}$$

By Lemma 4.7, there is a finite constant $K_1 > 0$ such that

$$\mathbb{P}(|X_j| > t) \;\leq\; K_1(1-\varphi(t^{-1}) \;=\; K_1 t^{-1} D(t^{-1}) \quad \text{for all sufficiently large } t. \tag{4.30}$$

Therefore, by (3.18),

$$\limsup_{t\to\infty} \bigg| \sum_{v\in\mathcal{T}_t} (\mathbb{P}(X_j > L(v)^{-1}) - \mathbb{P}(X < -L(v)^{-1}))\bigg| \;\leq\; K_1 W \quad \text{a.s.}$$

It thus suffices to show that, for $I_j(t) := \int_0^t \big(\mathbb{P}(X_j > x) - \mathbb{P}(X_j < -x)\big)\,\mathrm{d}x$, $t \geq 0$,

$$\limsup_{t\to\infty} \bigg| \sum_{v\in\mathcal{T}_t} L(v) I_j\big(L(v)^{-1}\big)\bigg| \;\leq\; K_2 W \quad \text{a.s.} \tag{4.31}$$

for some finite constant $K_2 > 0$. We write $I_j\big(L(v)^{-1}\big) = I_j\big(L(v)^{-1}\big) - I_j(e^t) + I_j(e^t)$ and observe that by (4.30) and (4.24),

$$\limsup_{t\to\infty} \bigg| \sum_{v\in\mathcal{T}_t} L(v)\big(I_j\big(L(v)^{-1}\big) - I_j(e^t)\big)\bigg| \;\leq\; K_3 W \quad \text{a.s.} \tag{4.32}$$

for some finite constant $K_3 > 0$. It thus suffices to show that

$$\limsup_{t\to\infty} |I_j(e^t)| \sum_{v\in\mathcal{T}_t} L(v) \;\leq\; K_4 W \quad \text{a.s.} \tag{4.33}$$

If $\limsup_{t\to\infty} |I_j(e^t)|/D(e^{-t}) = \infty$, then using (3.18) we infer $|I_j(e^t)| \sum_{v\in\mathcal{T}_t} L(v) \to \infty$ as $t \to \infty$ a.s. on the survival set S. This implies $|W(1)_j| = \infty$ a.s. on S, thereby leading to a contradiction, for the absolute value of any other term that contributes to $W(1)_j$ is bounded by a constant times W a.s. Therefore, $\limsup_{t\to\infty} |I_j(e^t)|/D(e^{-t}) < \infty$ a.s. which together with (3.18) proves (4.33). $\square$

For the next theorem, recall that $Z := \lim_{n\to\infty} Z_n = \lim_{n\to\infty} \sum_{|v|=n} L(v)$ whenever the limit exists in the a.s. sense, and $Z = 0$, otherwise.

Theorem 4.10. *Assume that (A1)-(A4) hold. Let ϕ be the Fourier transform of a probability distribution on $\mathbb{R}^d$ solving (2.20), and let $\Phi = \exp(\Psi)$ be the limit of the multiplicative martingale corresponding to ϕ.*

(a) *Let $0 < \alpha < 1$. Then there exists a finite measure σ on $\mathbb{S}^{d-1}$ such that*

$$\Psi(\mathbf{t}) = -W \int |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha}\, \sigma(\mathrm{d}\mathbf{s}) + \mathrm{i} W \tan\big(\frac{\pi\alpha}{2}\big) \int \langle \mathbf{t}, \mathbf{s}\rangle |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha-1}\, \sigma(\mathrm{d}\mathbf{s}) \tag{4.34}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$. If $\mathbb{G}(T) = \mathbb{R}^*$, then σ is symmetric and the second integral in (4.34) vanishes.

(b) *Let* $\alpha = 1$.

(b1) Assume that Case I prevails and that $\mathbb{E}[\sum_{j\geq 1} |T_j|(\log^-(|T_j|))^2] < \infty$. *Then there exist an* $\mathbf{a} \in \mathbb{R}^d$ *and a finite measure* σ *on* $\mathbb{S}^{d-1}$ *with* $\int s_j \, \sigma(\mathrm{d}\mathbf{s}) = 0$ *for* $j = 1, \ldots, d$ *such that*

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}W\langle \mathbf{a}, \mathbf{t}\rangle - W\int |\langle \mathbf{t}, \mathbf{s}\rangle| \, \sigma(\mathrm{d}\mathbf{s}) - \mathrm{i}W\frac{2}{\pi}\int \langle \mathbf{t}, \mathbf{s}\rangle \log(|\langle \mathbf{t}, \mathbf{s}\rangle|) \, \sigma(\mathrm{d}\mathbf{s}) \tag{4.35}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$.

(b2) Assume that Case II or III prevails and $\mathbb{E}[\sum_{j\geq 1} |T_j|(\log^-(|T_j|))^2] < \infty$ *in Case II and* (A5) *in Case III. Then there exist a finite symmetric measure* σ *on* $\mathbb{S}^{d-1}$ *such that*

$$\Psi(\mathbf{t}) \;=\; -W\int |\langle \mathbf{t}, \mathbf{s}\rangle| \, \sigma(\mathrm{d}\mathbf{s}) \tag{4.36}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$.

(c) *Let* $1 < \alpha < 2$. *Then there exist an* $\mathbf{a} \in \mathbb{R}^d$ *and a finite measure* σ *on* $\mathbb{S}^{d-1}$ *such that*

$$\Psi(\mathbf{t}) = \mathrm{i}Z\langle \mathbf{a}, \mathbf{t}\rangle - W\int |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha} \, \sigma(\mathrm{d}\mathbf{s}) + \mathrm{i}W\tan\Big(\frac{\pi\alpha}{2}\Big)\int \langle \mathbf{t}, \mathbf{s}\rangle |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha-1} \, \sigma(\mathrm{d}\mathbf{s}) \tag{4.37}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$. *If* $\mathbb{G}(T) = \mathbb{R}^*$, *then* σ *is symmetric and the second integral in* (4.37) *vanishes.*

(d) *Let* $\alpha = 2$. *If* $\mathbb{E}[Z_1] = 1$, *then additionally assume that* (A4a) *holds. Then there exist an* $\mathbf{a} \in \mathbb{R}^d$ *and a symmetric positive semi-definite (possibly zero)* $d \times d$ *matrix* Σ *such that*

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \mathbf{a}, \mathbf{t}\rangle - W\frac{\mathbf{t}\Sigma\mathbf{t}^{\mathsf{T}}}{2} \tag{4.38}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$. *If* $\mathbb{P}(Z_1 = 1) < 1$, *then* $\mathbf{a} = \mathbf{0}$.

(e) *Let* $\alpha > 2$. *Then there is an* $\mathbf{a} \in \mathbb{R}^d$ *such that*

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \mathbf{a}, \mathbf{t}\rangle \tag{4.39}$$

a.s. for all $\mathbf{t} \in \mathbb{R}^d$. *If* $\mathbb{P}(Z_1 = 1) < 1$, *then* $\mathbf{a} = \mathbf{0}$.

Proof: We start by recalling that Ψ satisfies (3.7) by Proposition 3.5.
(e) If $\alpha > 2$, then, according to Lemma 4.6, $\mathbf{\Sigma} = 0$ a.s. and $\nu = 0$ a.s. (3.7) then simplifies to $\Psi(\mathbf{t}) = \mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle$ and (3.6) implies that $\mathbf{W} = \sum_{|v|=n} L(v)[\mathbf{W}]_v$ a.s. for all $n \in \mathbb{N}_0$. Hence, each component of $\mathbf{W}$ is an endogenous fixed point w.r.t. T. From Theorem 3.12(c) we know that non-null endogenous fixed points w.r.t. T exist only if $\mathbb{E}[Z_1] = 1$ and Z_n converges a.s. and in mean to Z and then each endogenous fixed point is a multiple of Z. Now if $Z_1 = 1$ a.s., then $Z = 1$ a.s. and we arrive at $\mathbf{W} = \mathbf{a}$ for some $\mathbf{a} \in \mathbb{R}^d$ which is equivalent to (4.39). If $\mathbb{P}(Z_1 = 1) \neq 1$, we conclude from Theorem 2.3(c) that $Z = 0$ a.s. Hence, (4.39) holds with $\mathbf{a} = \mathbf{0}$.
(d) Let $\alpha = 2$. By Lemma 4.6, Ψ is of the form

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle - W\frac{\mathbf{t}\Sigma\mathbf{t}^{\mathsf{T}}}{2} \quad \text{a.s.}$$

for a deterministic symmetric positive semi-definite matrix Σ. Since i and 1 are linearly independent, one again concludes from (3.6) that $\mathbf{W}$ is an endogenous fixed point w.r.t. T. The proof of the remaining part of assertion (d) proceeds as the corresponding part of the proof of part (e) with the exception that if $\mathbb{E}[Z_1] = 1$, (A4a) has to be assumed in order to apply Theorem 2.3(c).

We now turn to the case $0 < \alpha < 2$. By Lemma 4.6, $\mathbf{\Sigma} = 0$ a.s. and there exists a finite measure $\tilde{\sigma}$ on the Borel subsets of $\mathbb{S}^{d-1}$ such that

$$\nu(A) \;=\; W \iint_{\mathbb{S}^{d-1}\times(0,\infty)} \mathbb{1}_A(rs) r^{-(1+\alpha)}\,\mathrm{d}r\,\tilde{\sigma}(\mathrm{d}s) \tag{4.12}$$

for all Borel sets $A \subseteq \mathbb{R}^d$ a.s. Plugging this into (3.7), we conclude that Ψ is of the form

$$\begin{aligned}\Psi(\mathbf{t}) \;&=\; \mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle + W \iint \left(e^{\mathrm{i}r\langle \mathbf{t},\mathbf{s}\rangle} - 1 - \frac{\mathrm{i}r\langle \mathbf{t},\mathbf{s}\rangle}{1+r^2}\right)\frac{\mathrm{d}r}{r^{1+\alpha}}\,\tilde{\sigma}(\mathrm{d}\mathbf{s}) \\ &=\; \mathrm{i}\langle \mathbf{W}, \mathbf{t}\rangle + W \int I(\langle \mathbf{t},\mathbf{s}\rangle)\,\tilde{\sigma}(\mathrm{d}s), \quad \mathbf{t}\in\mathbb{R}^d, \end{aligned}\tag{4.40}$$

where

$$I(t) \;:=\; \int_0^\infty \left(e^{\mathrm{i}tr} - 1 - \frac{\mathrm{i}tr}{1+r^2}\right)\frac{\mathrm{d}r}{r^{1+\alpha}}, \quad t\in\mathbb{R}.$$

The value of $I(t)$ is known (see *e.g.* Gnedenko and Kolmogorov (1968, pp. 168))

$$I(t) \;=\; \begin{cases} \mathrm{i}ct - t^\alpha e^{-\frac{\pi \mathrm{i}\alpha}{2}}\frac{\Gamma(1-\alpha)}{\alpha}, & 0<\alpha<1, \\ \mathrm{i}ct - (\pi/2)t - \mathrm{i}t\log(t), & \alpha = 1, \\ \mathrm{i}ct + t^\alpha e^{-\frac{\pi \mathrm{i}}{2}\alpha}\frac{\Gamma(2-\alpha)}{\alpha(\alpha-1)}, & 1<\alpha<2, \end{cases}$$

for $t > 0$, where Γ denotes Euler's Gamma function and $c \in \mathbb{R}$ is a constant that depends only on α. Further, $I(-t) = \overline{I(t)}$, the complex conjugate of $I(t)$. Finally, observe that $\mathbf{s}_0 := \int \mathbf{s}\,\tilde{\sigma}(\mathrm{d}\mathbf{s}) \in \mathbb{R}^d$ since it is the integral of a function which is bounded on $\mathbb{S}^{d-1}$ w.r.t. to a finite measure.

(a) Let $0 < \alpha < 1$. Plugging in the corresponding value of $I(t)$ in (4.40) gives

$$\begin{aligned}\Psi(\mathbf{t}) &= \mathrm{i}\langle \mathbf{W} + Wc\mathbf{s}_0, \mathbf{t}\rangle \\ &\quad - W\frac{\Gamma(1-\alpha)}{\alpha}\left(e^{-\frac{\pi \mathrm{i}}{2}\alpha}\int_{\{\langle \mathbf{t},\mathbf{s}\rangle>0\}} \langle \mathbf{t},\mathbf{s}\rangle^\alpha\,\tilde{\sigma}(\mathrm{d}\mathbf{s}) + e^{\frac{\pi \mathrm{i}}{2}\alpha}\int_{\{\langle \mathbf{t},\mathbf{s}\rangle<0\}} |\langle \mathbf{t},\mathbf{s}\rangle|^\alpha\,\tilde{\sigma}(\mathrm{d}\mathbf{s})\right) \\ &= \mathrm{i}\langle \mathbf{W} + Wc\mathbf{s}_0, \mathbf{t}\rangle - \frac{W}{\alpha}\Gamma(1-\alpha)\cos\left(\frac{\pi\alpha}{2}\right)\int |\langle \mathbf{t},\mathbf{s}\rangle|^\alpha\,\tilde{\sigma}(\mathrm{d}\mathbf{s}) \\ &\quad + \mathrm{i}\frac{W}{\alpha}\Gamma(1-\alpha)\sin\left(\frac{\pi\alpha}{2}\right)\int \langle \mathbf{t},\mathbf{s}\rangle|\langle \mathbf{t},\mathbf{s}\rangle|^{\alpha-1}\,\tilde{\sigma}(\mathrm{d}\mathbf{s}).\end{aligned}$$

Now define $\sigma := \frac{\Gamma(1-\alpha)}{\alpha}\cos\left(\frac{\pi\alpha}{2}\right)\tilde{\sigma}$ and notice that $\cos(\frac{\pi\alpha}{2}) > 0$ since $0 < \alpha < 1$. Then we get

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle - W\int |\langle \mathbf{t},\mathbf{s}\rangle|^\alpha\,\sigma(\mathrm{d}\mathbf{s}) + \mathrm{i}W\tan\left(\frac{\pi\alpha}{2}\right)\int \langle \mathbf{t},\mathbf{s}\rangle|\langle \mathbf{t},\mathbf{s}\rangle|^{\alpha-1}\,\sigma(\mathrm{d}\mathbf{s}),$$

where $\tilde{\mathbf{W}} := \mathbf{W} + Wc\mathbf{s}_0$. Now, using linear independence of 1 and i and (3.6), we conclude that

$$\begin{aligned}&\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle + W\tan\left(\frac{\pi\alpha}{2}\right)\int \langle \mathbf{t},\mathbf{s}\rangle|\langle \mathbf{t},\mathbf{s}\rangle|^{\alpha-1}\,\sigma(\mathrm{d}\mathbf{s}) \\ &= \sum_{|v|=n} L(v)\langle [\tilde{\mathbf{W}}]_v, \mathbf{t}\rangle + \sum_{|v|=n} L(v)|L(v)|^{\alpha-1}[W]_v\tan\left(\frac{\pi\alpha}{2}\right)\int \langle \mathbf{t},\mathbf{s}\rangle|\langle \mathbf{t},\mathbf{s}\rangle|^{\alpha-1}\,\sigma(\mathrm{d}\mathbf{s})\end{aligned}$$

for all $\mathbf{t} \in \mathbb{R}^d$ and all $n \in \mathbb{N}_0$ a.s. For each $j = 1, \ldots, d$, evaluate the above equation at $\mathbf{t} = t\mathbf{e}_j$ for some arbitrary $t > 0$ and with $\mathbf{e}_j$ denoting the jth unit vector. Then divide by t and let $t \to \infty$. This gives that each coordinate of $\tilde{\mathbf{W}}$ is an endogenous fixed point w.r.t. T which, therefore, must vanish by Theorem 3.12(a). If $\mathbb{G}(T) = \mathbb{R}^*$, then σ is symmetric by Lemma 4.6 and the integral $\int \langle \mathbf{t}, \mathbf{s}\rangle |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha-1} \, \sigma(\mathrm{d}\mathbf{s})$ is equal to zero. The proof of (a) is thus complete.

(b) Let $\alpha = 1$. Again, we plug the corresponding value of $I(t)$ in (4.40). With $\tilde{\mathbf{W}} := \mathbf{W} + Wc\mathbf{s}_0$ and $\sigma := \frac{\pi}{2}\tilde{\sigma}$, this yields

$$\Psi(\mathbf{t}) \;=\; \mathrm{i}\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle - W \int |\langle \mathbf{t}, \mathbf{s}\rangle| \, \sigma(\mathrm{d}s) - \mathrm{i}W \frac{2}{\pi} \int \langle \mathbf{t}, \mathbf{s}\rangle \log(|\langle \mathbf{t}, \mathbf{s}\rangle|) \, \sigma(\mathrm{d}\mathbf{s}) \tag{4.41}$$

for all $\mathbf{t} \in \mathbb{R}^d$ a.s.

(b2) The measure σ is symmetric by Lemma 4.6 and the last integral in (4.41) vanishes because the integrand is odd. Now combine (3.6) and (4.41) and use the linear independence of 1 and i to conclude

$$\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle \;=\; \sum_{|v|=n} L(v) \langle [\tilde{\mathbf{W}}]_v, \mathbf{t}\rangle \tag{4.42}$$

for all $\mathbf{t} \in \mathbb{R}^d$ a.s. Choosing $\mathbf{t} = \mathbf{e}_j$ for $j = 1, \ldots, d$, we see that each coordinate of $\tilde{\mathbf{W}}$ is an endogenous fixed point w.r.t. T which must vanish a.s. by Theorem 3.12(b).

(b1) We show that $\int s_j \, \sigma(\mathrm{d}\mathbf{s}) = 0$ for $j = 1, \ldots, d$ or, equivalently, $\mathbf{s}_0 = \mathbf{0}$. To this end, use (3.6) and the linear independence of 1 and i to obtain that

$$\begin{aligned}
\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle - &W \frac{2}{\pi} \int \langle \mathbf{t}, \mathbf{s}\rangle \log(|\langle \mathbf{t}, \mathbf{s}\rangle|) \, \sigma(\mathrm{d}\mathbf{s}) \\
&= \sum_{|v|=n} L(v) \langle [\tilde{\mathbf{W}}]_v, \mathbf{t}\rangle - \sum_{|v|=n} L(v) \log(L(v)) [W]_v \frac{2}{\pi} \int \langle \mathbf{t}, \mathbf{s}\rangle \, \sigma(\mathrm{d}\mathbf{s}) \\
&\qquad - \sum_{|v|=n} L(v) [W]_v \frac{2}{\pi} \int \langle \mathbf{t}, \mathbf{s}\rangle \log(|\langle \mathbf{t}, \mathbf{s}\rangle|) \, \sigma(\mathrm{d}\mathbf{s})
\end{aligned} \tag{4.43}$$

a.s. for all $n \in \mathbb{N}_0$. Assume for a contradiction that for some $j \in \{1, \ldots, d\}$, we have $\int s_j \sigma(\mathrm{d}\mathbf{s}) \neq 0$. Then put $J(\mathbf{t}) := \int \langle \mathbf{t}, \mathbf{s}\rangle \log(|\langle \mathbf{t}, \mathbf{s}\rangle|) \, \sigma(\mathrm{d}\mathbf{s})$, $\mathbf{t} \in \mathbb{R}^d$. For $u \neq 0$, one has

$$J(u\mathbf{e}_j) = \int u s_j \log(|u s_j|) \, \sigma(\mathrm{d}\mathbf{s}) = u \log(|u|) \int s_j \, \sigma(\mathrm{d}\mathbf{s}) + u \int s_j \log(|s_j|) \, \sigma(\mathrm{d}\mathbf{s}).$$

Thus, $J(u\mathbf{e}_j) = 0$ iff

$$\log(|u|) \;=\; -\frac{\int s_j \log(|s_j|) \, \sigma(\mathrm{d}\mathbf{s})}{\int s_j \, \sigma(\mathrm{d}\mathbf{s})}.$$

Since we assume $\int s_j \sigma(\mathrm{d}\mathbf{s}) \neq 0$, one can choose $u \neq 0$ such that $J(u\mathbf{e}_j) = 0$. Evaluating (4.43) at $u\mathbf{e}_j$ and then dividing by $u \neq 0$ gives

$$\tilde{W}_j \;=\; \sum_{|v|=n} L(v) [\tilde{W}_j]_v - \sum_{|v|=n} L(v) \log(L(v)) [W]_v \frac{2}{\pi} \int s_j \, \sigma(\mathrm{d}\mathbf{s}),$$

where $\tilde{W}_j$ is the jth coordinate of $\tilde{\mathbf{W}}$. Using (3.13) we infer

$$\tilde{\mathbf{W}} = \mathbf{W} + Wc\mathbf{s}_0 = \mathbf{W}(1) + \int\limits_{\{|\mathbf{x}|>1\}} \frac{\mathbf{x}}{1+|\mathbf{x}|^2}\nu(\mathrm{d}\mathbf{x}) - \int\limits_{\{|\mathbf{x}|\leq 1\}} \frac{\mathbf{x}|\mathbf{x}|^2}{1+|\mathbf{x}|^2}\nu(\mathrm{d}\mathbf{x}) + Wc\mathbf{s}_0$$

a.s., where $\mathbf{W}(1) = \lim_{t\to\infty} \sum_{v\in\mathcal{T}_t} L(v) \int_{\{|\mathbf{x}|\leq|L(v)|^{-1}\}} \mathbf{x}\, F(\mathrm{d}\mathbf{x})$. Since we know from Lemma 4.6 that all randomness in ν comes from a scalar factor W, we conclude that $\tilde{\mathbf{W}} = \mathbf{W}(1)+\tilde{\mathbf{c}}W$ a.s. for some $\tilde{\mathbf{c}} \in \mathbb{R}^d$. From Lemma 4.9, we know that $|\tilde{W}_j| \leq KW$ a.s. for some $K \geq 0$. In the case $\int s_j\, \sigma(\mathrm{d}\mathbf{s}) < 0$ we use these observations to conclude

$$\begin{aligned} -KW \ \leq\ \tilde{W}_j \ &=\ \sum_{|v|=n} L(v)[\tilde{W}_j]_v - \sum_{|v|=n} L(v)\log(L(v))[W]_v \frac{2}{\pi}\int s_j\, \sigma(\mathrm{d}\mathbf{s}) \\ &\leq\ KW - \sum_{|v|=n} L(v)\log(L(v))[W]_v \frac{2}{\pi}\int s_j\, \sigma(\mathrm{d}\mathbf{s}) \ \to\ -\infty \end{aligned}$$

a.s. on S, the set of survival since $\sum_{|v|=n} L(v)[W]_v = W > 0$ a.s. on S and $\sup_{|v|=n} L(v) \to 0$ a.s. by Lemma 3.1. This is a contradiction. Analogously, one can produce a contradiction when $\int s_j\, \sigma(\mathrm{d}\mathbf{s}) > 0$. Consequently, $\int s_j \sigma(\mathrm{d}\mathbf{s}) = 0$ for $j = 1, \ldots, d$. Using this and the equation (3.16) for W in (4.43), we conclude that

$$\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle \ =\ \sum_{|v|=n} L(v)\langle [\tilde{\mathbf{W}}]_v, \mathbf{t}\rangle$$

a.s. for all $n \in \mathbb{N}_0$. Evaluating this equation at $\mathbf{t} = \mathbf{e}_j$ shows that $\tilde{W}_j$ is an endogenous fixed point w.r.t. T, hence $\tilde{W}_j = Wa_j$ a.s. by Theorem 3.12(b), $j = 1, \ldots, d$. Hence, $\tilde{\mathbf{W}} = W\mathbf{a}$ for $\mathbf{a} = (a_1, \ldots, a_d)$. The proof of (b) is complete.
(c) Let $1 < \alpha < 2$. Plugging the corresponding value of $I(t)$ in (4.40) and arguing as in the case $0 < \alpha < 1$ we infer

$$\Psi(\mathbf{t}) \ =\ \mathrm{i}\langle \tilde{\mathbf{W}}, \mathbf{t}\rangle - W\bigg(\int |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha}\, \sigma(\mathrm{d}\mathbf{s}) - \mathrm{i}\tan\big(\frac{\pi\alpha}{2}\big)\int \langle \mathbf{t}, \mathbf{s}\rangle^{\alpha} |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha-1}\, \sigma(\mathrm{d}\mathbf{s})\bigg)$$

where $\sigma := -\alpha^{-1}(\alpha-1)^{-1}\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2})\tilde{\sigma}$ (notice that $\cos(\frac{\pi\alpha}{2}) < 0$) and, as before, $\tilde{\mathbf{W}} := \mathbf{W} + c\mathbf{s}_0$. The equality $\tilde{\mathbf{W}} = \mathbf{a}Z$ for some $\mathbf{a} \in \mathbb{R}^d$ can be checked as in the proof of the corresponding assertion in the case $\alpha = 2$. If $\mathbb{G}(T) = \mathbb{R}^*$, then σ is symmetric by Lemma 4.6 and the integral $\int \langle \mathbf{t}, \mathbf{s}\rangle |\langle \mathbf{t}, \mathbf{s}\rangle|^{\alpha-1}\, \sigma(\mathrm{d}\mathbf{s})$ vanishes. □

4.6. *Proof of the converse part of Theorem 2.4.* From what we have already derived in the preceding sections, there is only a small step to go in order to prove the converse part of Theorem 2.4. The techniques needed to do this final step have been developed in Alsmeyer and Meiners (2012). Thus we shall only give a sketch of the proof.

Proof of Theorem 2.4 (converse part): Fix any $\phi \in \mathcal{S}(\mathfrak{F})$. Then define the corresponding multiplicative martingale by setting

$$M_n(\mathbf{t}) :=\ \exp\bigg(\mathrm{i}\sum_{|v|<n} L(v)\langle \mathbf{C}(v), \mathbf{t}\rangle\bigg) \prod_{|v|=n} \phi(L(v)\mathbf{t}) \ =:\ \exp(\mathrm{i}\langle \mathbf{W}_n^*, \mathbf{t}\rangle)\Phi_n(\mathbf{t}), \tag{4.44}$$

$\mathbf{t} \in \mathbb{R}^d$. From (2.19) one can deduce just as in the homogeneous case that, for fixed $\mathbf{t} \in \mathbb{R}^d$, $(M_n(\mathbf{t}))_{n\in\mathbb{N}_0}$ is a bounded martingale w.r.t. $(\mathcal{A}_n)_{n\in\mathbb{N}_0}$ and, thus, converges a.s. and in mean to a limit $M(\mathbf{t})$ with $\mathbb{E}[M(\mathbf{t})] = \phi(\mathbf{t})$. On the other hand, $\mathbf{W}_n^* \to \mathbf{W}^*$ in probability implies $\Phi_n(\mathbf{t}) \to \Phi(\mathbf{t}) := M(\mathbf{t})/\exp(\mathrm{i}\langle \mathbf{W}^*, \mathbf{t}\rangle)$ in probability as $n \to \infty$. Mimicking the proof of Theorem 4.2 in Alsmeyer and Meiners (2012), one can show that $\psi(\mathbf{t}) = \mathbb{E}[\Phi(\mathbf{t})]$ is a solution to (2.20) and that the $\Phi(\mathbf{t})$, $\mathbf{t} \in \mathbb{R}^d$ are the limits of the multiplicative martingales associated with ψ. Hence $\Phi(\mathbf{t}) = \exp(\Psi(\mathbf{t}))$ for some Ψ as in Theorem 4.10. Finally, $\phi(\mathbf{t}) = \mathbb{E}[M(\mathbf{t})] = \mathbb{E}[\exp(\mathrm{i}\langle \mathbf{W}^*, \mathbf{t}\rangle + \Psi(\mathbf{t}))]$, $\mathbf{t} \in \mathbb{R}^d$. The proof is complete. □

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Notation

$\mathbb{N}$	$\{1,2,3,\ldots\}$, set of natural numbers
$\mathbb{N}_0$	$\{0,1,2,3,\ldots\}$, set of natural numbers and zero
$\mathbb{R}^*$	$\mathbb{R}\setminus\{0\}$, multiplicative group of $\mathbb{R}$
$\mathbb{R}_\geq$	set of nonnegative real numbers
$\mathbb{R}_>$	set of positive real numbers
$\mathbb{S}^{d-1}$	d-dimensional unit sphere
$\mathbb{V}$	$\bigcup_{n\in\mathbb{N}_0}\mathbb{N}^n$, infinite Ulam-Harris tree
i	$\sqrt{-1}$, imaginary unit
$\mathcal{L}(\cdot)$	the law/distribution (of a random variable)
$\mathrm{sign}(\cdot)$	the sign (of a real number)
$\overset{\mathrm{law}}{=}$	equality in law
$\mathbf{C}$	$(C_1,\ldots,C_d)$, see (1.1)
T	$(T_j)_{j\geq1}$, see (1.1)
m	function that maps $[0,\infty)\ni\gamma\mapsto\mathbb{E}[\sum_j\lvert T_j\rvert^\gamma]$
α	minimal positive real with $m(\alpha)=1$, characteristic index, see (A3)
θ	$\theta<\alpha$ with $m(\theta)<\infty$, see (A4b)
p,q	$p=\mathbb{E}[\sum_j\lvert T_j\rvert^\alpha\mathbb{1}_{\{T_j>0\}}]$ and $q:=\mathbb{E}[\sum_j\lvert T_j\rvert^\alpha\mathbb{1}_{\{T_j<0\}}]$, see (2.3)
$\mathbb{G}(T)$	closed multiplicative subgroup $\subset\mathbb{R}^*$ generated by the nonzero T_j
$\mathcal{M}$, $\mathcal{S}(\mathcal{M})$	set of functions defined on p. 80, functions in $\mathcal{M}$ solving (2.33)
$\mathcal{B}$, $\mathcal{S}(\mathcal{B})$	$\{f:\mathbb{R}^d\to\mathbb{C}:f(0)=1\ \&\ \sup_{\mathbf{t}}\lvert f(\mathbf{t})\rvert\leq1\}$, functions in $\mathcal{B}$ solving (3.3)
$\mathfrak{F}$	set of Fourier transforms of probability measures on $\mathbb{R}^d$
$\mathcal{S}(\mathfrak{F})$	functions in $\mathfrak{F}$ solving (2.20)
$\mathcal{S}(\mathfrak{F})(\mathbf{C})$	functions in $\mathfrak{F}$ solving (2.19)
$\mathbf{C}(v)$	$(C_1(v),\ldots,C_d(v))$, independent copy of $\mathbf{C}$, see Section 2.2
$T(v)$	$(T_j(v))_{j\geq1}$, independent copy of T, see Section 2.2
$L(v)$	multiplicative weight, see Section 2.2
$\mathbf{L}$	$(L(v))_{v\in\mathbb{V}}$, family of multiplicative weights
$\mathcal{A}_n$, $\mathcal{A}_\infty$	$\sigma((\mathbf{C}(v),T(v)):\lvert v\rvert<n)$, $\sigma(\mathcal{A}_n:n\geq0)$
$S(v)$	$-\log\lvert L(v)\rvert$, position of particle v
$\mathbf{S}$	$(S(v))_{v\in\mathbb{V}}$, family of positions
$\tau(v)$	$\mathrm{sign}(L(v))$, type of particle v
$\mathcal{G}_n$	$\sum_{\lvert v\rvert=n}\mathbb{1}_{\{L(v)\neq0\}}$, number of nonzero weights in generation n
$\mathcal{G}$	$\bigcup_{n\geq0}\mathcal{G}_n$
N_n	$\lvert\mathcal{G}_n\rvert$, number of nth generation particles
S	$\{N_n>0$ for all $n\}$, survival set
$\mathcal{Z}_n$	$\sum_{v\in\mathcal{G}_n}\delta_{S(v)}$, point process of nth generation positions
$[\cdot]_u$	shift operator, see p. 74
$\mathbf{W}_n^*$, $\mathbf{W}^*$	$\sum_{\lvert v\rvert<n}L(v)\mathbf{C}(v)$, $\lim_{n\to\infty}\mathbf{W}_n^*$, see (2.11) and (2.12)
φ	Laplace transform solving (2.15)
W	random variable with Laplace transform φ, defined via (2.14)
D	function that maps $t\mapsto t^{-1}(1-\varphi(t))$
W_n	$\sum_{\lvert v\rvert=n}\lvert L(v)\rvert^\alpha$, additive martingale or Biggins' martingale, see (2.17)
Z_n, Z	$\sum_{\lvert v\rvert=n}L(v)$, $\lim_{n\to\infty}Z_n$ (if the lim exists), see (2.18)
$(S_n)_{n\geq0}$	associated random walk, defined via (3.2)
$M_n(\mathbf{t})$	$\prod_{\lvert v\rvert=n}f(L(v)\mathbf{t})$, multiplicative martingales, see (3.4)
$M(\mathbf{t})$	$\lim_{n\to\infty}M_n(\mathbf{t})$, limit of the multiplicative martingales
$(\mathbf{W},\mathbf{\Sigma},\nu)$	random Lévy triplet, see (3.7)

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