Fixation to Consensus on Tree-related Graphs

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Abstract. We study a continuous time Markov process whose state space consists of an assignment of $+1$ or -1 to each vertex of a graph G. The graphs that we treat are related to homogeneous trees of degree $K \geq 3$, such as finite or infinite stacks of such trees. The initial spin configuration is chosen from a Bernoulli product measure with density θ of $+1$ spins. The system evolves according to an agreement inducing dynamics: each vertex, at rate 1, changes its spin value to agree with the majority of its neighbors. We study the long time behavior of this system and prove that, if θ is close enough to 1, the system reaches fixation to consensus. The geometric percolation-type arguments introduced here may be of independent interest.

1. Introduction

In this work we study the long term behavior of continuous time Markov processes whose states assign either $+1$ or -1 (usually called a spin value) to each vertex x in a graph G. The graphs G we consider are related to homogeneous trees of degree K and include infinite stacks of homogeneous trees. These graphs will be specified in Section 2, where we will also discuss some earlier papers where such stacks of trees have been studied. The geometric and percolation theoretic methods we introduce to carry out our analysis (see especially Section 4 and Appendix B) are potentially of independent interest.

We denote by $\sigma_x(t)$ the value of the spin at vertex $x \in G$ at time $t \geq 0$. Starting from a random initial configuration $\sigma(0) = \{\sigma_x(0)\}_{x \in G}$ drawn from the independent Bernoulli product measure

$$\mu_\theta(\sigma_x(0) = +1) = \theta = 1 - \mu_\theta(\sigma_x(0) = -1), \tag{1.1}$$

the system then evolves in continuous time according to an agreement inducing dynamics: at rate 1, each vertex changes its value if it disagrees with more than half of its neighbors, and tosses a fair coin in the event of a tie. Our arguments and results easily extend to many other types of dynamics, as discussed in Remark 2.3 below; these include processes in discrete time, as in Kanoria and Montanari (2011), and different rules for tie-breaking.

Our process corresponds to the zero-temperature limit of Glauber dynamics for a stochastic Ising model with ferromagnetic nearest neighbor interactions and no external magnetic field (see, e.g., Nanda et al. (2000) or Krapivsky et al. (2010)). This process has been studied extensively in the physical and mathematical literature – primarily on graphs such as the hyper-lattice $\mathbb{Z}^d$ and the homogeneous tree of degree K, $\mathbb{T}_K$. A physical motivation is the behavior of a magnetic system following the extreme case of a deep quench, i.e., when a system has reached equilibrium at infinite temperature and is instantaneously reduced to zero temperature. For references on this and related problems see, e.g., Nanda et al. (2000) or Krapivsky et al. (2010). The main focus in the study of this model is the formation and evolution of boundaries delimiting same spin cluster domains: these domains shrink or grow or split or coalesce as their boundaries evolve. An interesting question is whether the system has a limiting configuration, or equivalently does every vertex eventually stop flipping? Whether

$$\lim_{t \to \infty} \sigma_x(t) \tag{1.2}$$

exists for almost every initial configuration, realization of the dynamics and for all $x \in G$ in the underlying graph depends on θ and on the structure of the underlying graph G. Nanda, Newman and Stein Nanda et al. (2000) investigated this question when $G = \mathbb{Z}^2$ and $\theta = \frac{1}{2}$ and found that in this case the limit does not exist, i.e., every vertex flips forever. Their work extended an old result of Arratia Arratia (1983), who showed the same on $\mathbb{Z}$ for $\theta \neq 0$ or 1. One important consequence of the methods of Nanda et al. (2000) is that $\sigma_x(\infty)$ does exist for almost every initial configuration, realization of the dynamics and every $x \in G$ if the graph is such that every vertex has an odd number of neighbors, such as for example $\mathbb{T}_K$ for K odd.

Another question of interest is whether sufficient bias in the initial configuration leads the system to reach consensus in the limit. I.e., does there exist $\theta_* \in (0, 1)$, such that for $\theta \geq \theta_*$,

$$\forall x \in G, \mathbb{P}_\theta(\exists T = T(\sigma(0), \omega, x) < \infty \text{ so that } \sigma_x(t) = +1 \text{ for } t \geq T) = 1. \tag{1.3}$$

We will refer to (1.3) as **fixation to consensus** (of +1). Kanoria and Montanari Kanoria and Montanari (2011) studied fixation to consensus on homogeneous trees of degree $K \geq 3$ for a process with synchronous time dynamics. Their process has the same update rules as ours, except that all vertices update simultaneously and at integer times $t \in \mathbb{N}$. For each K, Kanoria and Montanari defined the consensus threshold $\rho_*(K)$ to be the smallest bias in $\rho = 2\theta - 1$ such that the dynamics converges to the all $+1$ configuration, and proved upper and lower bounds for ρ_* as a function of K. Fixation to consensus was also investigated on $\mathbb{Z}^d$ for the asynchronous dynamics model. It was conjectured by Liggett Liggett (1985) that fixation to consensus holds there for all $\theta > \frac{1}{2}$. Fontes, Schonmann and Sidoravicius Fontes

et al. (2002) proved consensus for all $d \geq 2$ with θ_* strictly less than but very close to 1 and Morris Morris (2011) proved that $\theta_*(d) \to 1/2$ as $d \to \infty$.

In Howard (2000) Howard investigated the asynchronous dynamics in detail on $\mathbb{T}_3$ and showed how fixation takes place. On this tree graph, vertices fixate in spin chains (defined as doubly infinite paths of vertices of the same spin sign). Though no spin chains are present at time 0 when $\theta = 1/2$, Howard showed that for any $\epsilon > 0$, there are (almost surely) infinitely many distinct $+1$ and -1 spin chains at time ϵ. He also showed the existence of a phase transition in θ: there exists a critical $\theta_c \in (0, \frac{1}{2})$ such that if $\theta < \theta_c$, $+1$ spin chains do not form almost surely, whereas if $\theta > \theta_c$ they almost surely form in finite time. Our work is motivated by that of Howard, but for more general tree-related graphs.

2. Statements of Theorems

Let S^∞ denote a doubly infinite stack of homogeneous trees of degree $K \geq 3$, i.e., the graph with vertex set $\mathbb{T}_K \times \mathbb{Z}$ and edge set specified below. The main focus of this paper is proving fixation to consensus on S^∞ for the process started with an independent identically distributed initial configuration of parameter θ. Such infinite stacks of trees have been studied before in the context of Bernoulli percolation Grimmett and Newman (1990) and Ising models Newman and Wu (1990). More general nonamenable graphs have also been studied — see, e.g., Lyons (2000). One motivation for these studies is that as the parameter of the model varies, the behavior is sometimes like that on a simple homogeneous tree and sometimes like that on a simple amenable graph like $\mathbb{Z}^d$.

We express S^∞ as

$$S^\infty = \bigcup_{i=-\infty}^{\infty} S_i, \tag{2.1}$$

where $S_i = \mathbb{T}_K \times \{i\} = \{(u, i) : u \in \mathbb{T}_K\}$, and think of this as a decomposition of the infinite stack S^∞ into layers S_i. Let the edge set of S^∞, $\mathbb{E}^\infty$, be such that any two vertices $x, y \in S^\infty$ are connected by an edge $e_{xy} \in \mathbb{E}^\infty$ if and only if:

i $x = (u_x, k), y = (v_y, k) \in S_k$ for some k, and the corresponding u_x and v_y are adjacent vertices in $\mathbb{T}_K$; or

ii $x = (u_x, k)$ for some k and $y = (u_x, k + 1)$; or

iii $x = (u_x, k)$ for some k and $y = (u_x, k - 1)$.

For a more detailed description of the Markov process than the one given in Section 1, we associate to each vertex $x \in S^\infty$ a rate 1 Poisson process whose arrival times we think of as a sequence of clock rings at x. We will denote the arrival times of these Poisson processes by $\{\tau_{x,n}\}_{n=1,2,\ldots}$ and take the Poisson processes associated to different vertices to be mutually independent. We associate to the (x, n)'s independent Bernoulli($1/2$) random variables with values $+1$ or -1, which will represent the fair coin tosses to be used in the event of a tie. Let $\mathbb{P}_{\text{dyn}}$ be the probability measure for the realization of the dynamics (clock rings and tie-breaking coin tosses), and denote by $\mathbb{P}_\theta = \mu_\theta \times \mathbb{P}_{\text{dyn}}$ the joint probability measure on the space Ω of initial configurations $\sigma(0)$ and realizations of the dynamics; an element of Ω will be denoted ω.

The main result of this paper is the following theorem, which shows fixation to consensus for nontrivial θ; its proof is given in Section 4. Unlike Kanoria and

Montanari Kanoria and Montanari (2011), here we do not attempt to obtain good upper bounds on θ_* though we expect θ_* to approach $1/2$ with increasing degree K. We restrict ourselves to proving fixation to $+1$ for θ close enough to 1 with the standard majority update rule: when its clock rings, each vertex updates to agree with the majority of its neighbors or tosses a fair coin in the event of a tie. See Remark 2.3 below for other update rules to which our arguments and results apply.

Theorem 2.1. *Given $K \geq 3$, there exists $\theta_* < 1$ such that for $\theta > \theta_*$ the process on S^∞ fixates to consensus.*

The same fixation to consensus result holds for the following graphs, as stated in Theorem 2.2 below, whose proof is also given in Section 4:

- Homogeneous trees $\mathbb{T}_K$ of degree $K \geq 3$.
- Finite width stacks of homogeneous trees of degree $K \geq 3$ with free or periodic boundary conditions. These are graphs, which we will denote by S_f^l or S_p^l, with vertex set $\mathbb{T}_K \times \{0, 1, \ldots, l-1\}$ and edge set $\mathbb{E}_f$ or $\mathbb{E}_p$. $\mathbb{E}_f$ and $\mathbb{E}_p$ are defined similarly to the edge set $\mathbb{E}^\infty$ of S^∞: two vertices $x, y \in S_f^l$ are connected by an edge $e_{xy} \in \mathbb{E}_f$ if and only if either condition i above holds; or
  (1) $x, y \in S_k$ for $1 \leq k \leq l-2$ and either condition ii or iii holds; or
  (2) $x = (u_x, 0)$ and $y = (u_x, 1)$; or
  (3) $x = (u_x, l-1)$ and $y = (u_x, l-2)$.
      Any two vertices $x, y \in S_p^l$ are connected by an edge $e_{xy} \in \mathbb{E}_p$ if and only if either condition i holds; or
  (4) $x, y \in S_k$ for $1 \leq k \leq l-2$ and either condition ii or iii holds; or
  (5) $x = (u_x, 0)$ and $y = (u_x, 1)$ or $y = (u_x, l-1)$; or
  (6) $x = (u_x, l-1)$ and $y = (u_x, l-2)$ or $y = (u_x, 0)$.
- Semi-infinite stacks of homogeneous trees of degree $K \geq 3$ with free boundary conditions. These are graphs, which we will denoted by S^{semi}, with vertex set $\mathbb{T}_K \times \{0, 1, \ldots\}$ and edge set $\mathbb{E}^{\mathrm{semi}}$. Two vertices $x, y \in S^{\mathrm{semi}}$ are connected by an edge $e_{xy} \in \mathbb{E}^{\mathrm{semi}}$ if and only either condition i holds; or
  (8) $x, y \in S_k$ for $1 \leq k$ and either condition ii or iii holds; or
  (9) $x = (u_x, 0)$ and $y = (u_x, 1)$.

Theorem 2.2. *Fix $K \geq 3$ and $l \geq 2$ and let G be one of the following graphs: $\mathbb{T}_K$, S_f^l, S_p^l or S^{semi}. There exists $\theta_* < 1$ such that for $\theta > \theta_*$ the process on G fixates to consensus.*

Remark 2.3. Our results have natural extensions to other dynamics. Let N_0 be the maximum number of neighbors of a vertex in the graph G where G is any of the graphs of Theorem 2.2 or $G = S^\infty$; for some $M_0 > \frac{N_0}{2}$, we can change (arbitrarily) the update rules for those vertices whose number of $+1$ neighbors is strictly less than M_0, and the conclusions of Theorem 2.1 or 2.2 remain valid with the same proof. For large N_0, M_0 can be taken much larger than $\frac{N_0}{2}$. For example, on the infinite stack S^∞ of K-trees, $N_0 = K + 2$ and for $K \geq 5$, one can take $M_0 = K - 1 = N_0 - 3$ (as is readily seen from the proof of Theorem 2.1). A special case of this type of extension of our results is to modify the update rule in the event of a tie: e.g., instead of flipping a fair coin, flip a biased coin with any bias $p \in [0, 1]$ or do nothing. We can also change from two-valued spins to any fixed number q of spin values, say $1, 2, \ldots, q$. The initial configuration is given by the measure $\nu(x$ is assigned color i at time $0) = \epsilon_i$ where $i \in \{1, \ldots, q\}$ and $\sum_i \epsilon_i = 1$

and the updating is done via a majority rule, e.g., by a rule that respects majority agreement of neighbors on, say, color 1. We can then think of color 1 as the $+1$ spin from before, and the other $q - 1$ colors together representing the -1 spin. If ϵ_1 is close enough to 1, we again obtain fixation to $+1$ consensus. All our results also apply to the synchronous dynamics of Kanoria and Montanari (2011).

3. Preliminaries

In order to prove Theorem 2.1 we will show that if we take θ close enough to 1, then already at time 0 there are stable structures of $+1$ vertices, which are fixed for all time. We will choose these structures to be subsets (denoted $\mathcal{T}_i$) of the layers S_i in the decomposition of S^∞ such that they are stable with respect to the dynamics. We will define a set $\mathcal{T}$ as the union of $\mathcal{T}_i$ for all i, and show that for θ close enough to 1, the complement of $\mathcal{T}$ is a union of almost surely finite components.

3.1. A Set of Fixed Vertices in S^∞.

Definition 3.1. For i fixed, let $\mathcal{T}_i^{+,l}(t)$ be the union of all subgraphs H of S_i that are isomorphic to $\mathbb{T}_l$ with $\sigma_x(t) = +1, \forall x \in H$.

We point out that $\mathcal{T}_i^{+,K-1}(t)$ is stable for $K \geq 5$, since every $x \in \mathcal{T}_i^{+,K-1}(t)$ has $K - 1$ out of $K + 2$ neighbors of spin $+1$ and $K - 1 > \frac{K+2}{2}$ for $K \geq 5$. Not only is this set stable with respect to the dynamics on S^∞ as in Theorem 2.1, but it's also stable with respect to the dynamics on S_i and the other graphs of Theorem 2.2. Let $\mathcal{T}$ represent the union of $\mathcal{T}_i^{+,K-1}(0)$ across all levels S_i, i.e.,

$$\mathcal{T} = \bigcup_{j=-\infty}^{\infty} \mathcal{T}_i, \tag{3.1}$$

where for shorthand notation, $\mathcal{T}_i = \mathcal{T}_i^{+,K-1}(0)$.

If $K \leq 4$, $\mathcal{T}$ as defined above is not stable with respect to the dynamics. In these cases the argument will be changed somewhat as discussed in Section 4.

3.2. Asymmetric Site Percolation on $\mathbb{T}_K$.
The goal of this subsection is to state and prove a geometric probability estimate, Proposition 3.5, which concerns asymmetric site percolation on $\mathbb{T}_K$ distributed according to the product measure μ_θ with:

$$\mu_\theta(\sigma_x = +1) = \theta = 1 - \mu_\theta(\sigma_x = -1), \forall x \in \mathbb{T}_K. \tag{3.2}$$

This equals the distribution of $\sigma(0, \omega)$ restricted to the layers S_i, and therefore applies to these graphs as well. The statement and proof of Proposition 3.5 require a series of definitions. The first of these defines graphical subsets of $\mathbb{T}_K$, whereas the second concerns probabilistic events for subgraphs of $\mathbb{T}_K$ that have a specific orientation. Later, in the proof of Theorem 2.1 which is given in Section 4, Proposition 3.5 will be applied to certain subsets of S_i.

Definition 3.2. Certain rooted subtrees of $\mathbb{T}_K$ Let x, y in $\mathbb{T}_K$ be two adjacent vertices, and denote by $A_y[x]$ the connected component of x in $\mathbb{T}_K \backslash \{y\}$ – see Figure 3.1.
Let x, y, z be three adjacent vertices in $\mathbb{T}_K$, such that x and z are neighbors of y. Denote by $A_{x,z}[y]$ the connected component of y in $\mathbb{T}_K \backslash \{x \cup z\}$ – see Figure 3.2.
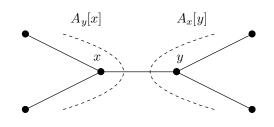
FIGURE 3.1. $A_y[x]$ and $A_x[y]$ are tree graphs whose roots have coordination number $K - 1$
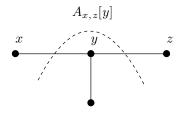

FIGURE 3.2. $A_{x,z}[y]$ is a tree graph whose root has coordination number $K - 2$

Definition 3.3. Random $(K - 1)$-ary trees of spin $+1$ with a certain orientation

Let T be a deterministic subtree of $\mathbb{T}_K$ with at least two vertices, and v be a leaf of T; i.e., v has a neighbor v' in T and $(K - 1)$ neighbors in $\mathbb{T}_K \setminus T$. Tree$^+[v]$ is the event that there is a subgraph H of $A_{v'}[v]$ isomorphic to $\mathbb{T}_{K-1}$ and containing v, such that $\sigma_u = +1, \forall u \in H$ – see Figure 3.3.

Let T be a deterministic subtree of $\mathbb{T}_K$ with at least five vertices, and v be a **2-point** of T (i.e., a vertex of T with exactly two neighbors in T) that is also **good** (i.e., both its neighbors in T are also 2-points of T). Let v', w be the two neighbors of v in T and let w' be w's other neighbor in T. Tree$^+[v, w]$ is the event that there is a subgraph H of $A_{v',w}[v] \cup A_{v,w'}[w]$ isomorphic to $\mathbb{T}_{K-1}$ and containing v and w, such that $\sigma_u = +1, \forall u \in H$; here $A_{v',w}[v] \cup A_{v,w'}[w]$ is the graph with vertex set $\mathbb{V}_{A_{v',w}[v]} \cup \mathbb{V}_{A_{v,w'}[w]}$ and edge set $\mathbb{E}^\infty_{A_{v',w}[v]} \cup \mathbb{E}^\infty_{A_{v,w'}[w]} \cup e_{vw}$ – see Figure 3.4.
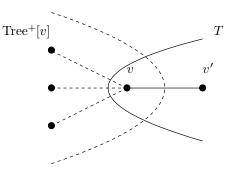
FIGURE 3.3. The event $\text{Tree}^+[v]$ asserts the existence of a random $(K-1)$-ary tree of spin $+1$ that contains a leaf, v, of T
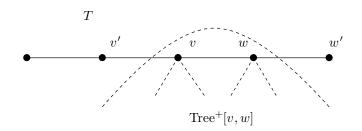
FIGURE 3.4. The event $\text{Tree}^+[v, w]$ asserts the existence of a random $(K-1)$-ary tree of spin $+1$ that contains a good 2-point, v, of T, and one of its neighbors, w

For distinct leaves v of T, the events $\{\text{Tree}^+[v]\}_{v \in T}$ are defined on disjoint subsets of $\mathbb{T}_K$, and are therefore independent; they are also identically distributed. The same is true for $\{\text{Tree}^+[v, w]\}_{v, w \in T}$ for disjoint pairs $\{v, w\}$. The following is essentially the same as Definition 3.1, with the only difference being that here we define the graph $\mathcal{T}^{+,l}$ on $\mathbb{T}_K$, whereas before we defined the same random graph on S_i.

Definition 3.4. Let $\mathcal{T}^{+,l}$ be the union of all subgraphs H of $\mathbb{T}_K$ that are isomorphic to $\mathbb{T}_l$ with $\sigma_x = +1, \forall x \in H$.

The next proposition estimates the probability that none of the vertices of a given set Λ belong to any random $(K-1)$-ary tree of spin $+1$ (see Definition 3.3). This proposition is a main ingredient in the proof of Theorem 2.1.

Proposition 3.5. *For any* $\lambda \in (0,1), \exists \theta_\lambda \in (0,1)$ *such that for* $\theta \geq \theta_\lambda$ *and any deterministic finite nonempty subset* Λ *of* $\mathbb{T}_K$,

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq \lambda^{|\Lambda|}. \tag{3.3}$$

Proof: Let T be the minimal spanning tree containing all the vertices of Λ. We will call the vertices of Λ the **special** vertices of T. Note that all the leaves of T are special vertices.

We first suppose $|\Lambda| \geq 2$; the simpler case $|\Lambda| = 1$ will be handled at the end of the proof. By the distinctness and disjointness results of Lemma C.1 from the Appendix, there exist constants $\epsilon_1, \epsilon_2 \in (0, \infty)$ depending only on K, such that for each such tree T, one or both of the following is valid:

a) there are at least $\epsilon_1|\Lambda|$ leaves v in T, with the events $\{\text{Tree}^+(v)\}$ mutually independent, and/or

b) there are at least $\frac{1}{2}\epsilon_2|\Lambda|$ edges having endpoints v, w in T with v a good special 2-point, and the events $\{\text{Tree}^+(v, w)\}_{v,w}$ mutually independent.

Let us first suppose that a) holds. We claim that, for v any leaf of T,

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq [1 - \mu_\theta(\text{Tree}^+[v])]^{\epsilon_1|\Lambda|}. \tag{3.4}$$

The claim follows from a string of inclusions. First,

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{v \in T, v \text{ leaf of } T} \{v \notin \mathcal{T}^{+,K-1}\}. \tag{3.5}$$

But if v is a leaf of T, then

$$\{v \notin \mathcal{T}^{+,K-1}\} \subseteq \text{Tree}^+[v]^c, \tag{3.6}$$

so that

$$\bigcap_{v \in T, v \text{ leaf of } T} \{v \notin \mathcal{T}^{+,K-1}\} \subseteq \bigcap_{v \in T, v \text{ leaf of } T} \text{Tree}^+[v]^c. \tag{3.7}$$

Labeling $\epsilon_1|\Lambda|$ of the leaves in a) as v_j, we restrict the above intersection to the leaves v_j of T, so that

$$\bigcap_{v \in T, v \text{ leaf of } T} \text{Tree}^+[v]^c \subseteq \bigcap_{j=1}^{\epsilon_1|\Lambda|} \text{Tree}^+[v_j]^c. \tag{3.8}$$

Since the events $\text{Tree}^+[v_j]$ are mutually independent,

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq \prod_{j=1}^{\epsilon_1|\Lambda|} \mu_\theta(\text{Tree}^+[v]^c), \tag{3.9}$$

implying the claim.

Alternatively, suppose that b) holds. Now we claim that

$$\mu_\theta(\Lambda \cap \mathcal{T}^{+,K-1} = \oslash) \leq [1 - \mu_\theta(\text{Tree}^+[v, w])]^{\frac{1}{2}\epsilon_2|\Lambda|}, \tag{3.10}$$

where v is a good special 2-point of T and w is one of v's neighbors. This claim also follows from a string of inclusions. First,

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{\substack{\{v,w\} \in T, v,w \text{ adj.} \\ \text{and } v \text{ is a good special 2-point of } T}} \{\{v, w\} \notin \mathcal{T}^{+,K-1}\}. \tag{3.11}$$

If $\{v, w\}$ are adjacent and v is a good special 2-point of T, then

$$\{\{v, w\} \notin \mathcal{T}^{+,K-1}\} \subseteq \text{Tree}^+[v, w]^c. \tag{3.12}$$

As with the proof of the previous claim, we label $\frac{1}{2}\epsilon_2|\Lambda|$ of the pairs of vertices given in b) as $\{v_j, w_j\}$. Then

$$\{\Lambda \cap \mathcal{T}^{+,K-1} = \oslash\} \subseteq \bigcap_{j=1}^{\frac{1}{2}\epsilon_2|\Lambda|} \text{Tree}^+[v_j, w_j]^c. \tag{3.13}$$

The second claim follows from the mutual independence of the events $\text{Tree}^+[v_j, w_j]$. The two claims imply (3.3) for $|\Lambda| \geq 2$ by taking

$$\lambda > \lambda^*(\theta) = \min\left\{(1 - \mu_\theta(\text{Tree}^+[v])^{\epsilon_1}, (1 - \mu_\theta(\text{Tree}^+[v, w])^{\frac{1}{2}\epsilon_2}\right\}, \tag{3.14}$$

and using Lemma A.5 of Appendix A.

If $|\Lambda| = 1$, suppose the only vertex in Λ is 0, a distinguished vertex. Then

$$\mu_\theta(0 \notin \mathcal{T}^{+,K-1}) \;=\; 1 - \mu_\theta\left(\bigcup_{j=1}^{K} \text{Tree}^+[a_j]\right) \tag{3.15}$$

$$\leq\; 1 - \mu_\theta(\text{Tree}^+[a_1]), \tag{3.16}$$

where $a_1, \ldots, a_K$ are the neighbors of 0 and $\text{Tree}^+[a_j]$ is defined as in Definition 3.3 with T the tree containing only vertices 0 and a_j. Then (3.3) follows in this case by taking $\lambda > \lambda^*(\theta) = 1 - \mu_\theta(\text{Tree}^+[a_1])$ and using Equation (A.4) and Lemma A.3. This completes the proof.

□

4. Main Results

We study the connected components of $S^\infty \setminus \mathcal{T}$ as a subgraph of S^∞, and show that if θ is close enough to 1 these connected components are finite almost surely. We will show that in each of these finite components, the subset of -1 vertices is eliminated in finite time leading to fixation of all vertices to $+1$.

Definition 4.1. For any $x \in S^\infty$, D_x is the connected component of x in $S^\infty \setminus \mathcal{T}$: D_x is the set of vertices $y \in S^\infty$ s.t. $x \overset{S^\infty \setminus \mathcal{T}}{\leftrightarrow} y$, i.e., there exists a path $(x_0 = x, x_1, \ldots, x_N = y)$ in S^∞ with every $x_j \notin \mathcal{T}$.

Proposition 4.2. *Given K, there exists $\theta_* < 1$ such that for $\theta > \theta_*$, $S^\infty \setminus \mathcal{T}$ is a union of almost surely finite connected components.*

Proof: It suffices to show that D_0 is finite almost surely, where 0 is a distinguished vertex in S^∞. Since $\mathbb{E}_\theta[|D_0|] < \infty$ implies $D_0 < \infty$ a.s., it suffices to show $\mathbb{E}_\theta[|D_0|] < \infty$.

Let γ_N represent any site self-avoiding path in S^∞ of length $|\gamma_N| = N \geq 0$ starting at 0, then by standard arguments

$$\mathbb{E}_\theta[|D_0|] \leq \sum_{N=0}^{\infty} \sum_{\gamma_N, |\gamma_N|=N} \mathbb{P}(\gamma_N \in D_0), \tag{4.1}$$

where by $\gamma_N \in D_0$ we mean that all the vertices of γ_N belong to D_0.

To show the sum is finite we need to bound $\mathbb{P}(\gamma_N \in D_0)$. Suppose the vertex set of γ_N is $\Lambda_1 \cup \ldots \cup \Lambda_J$, where for each $1 \leq i \leq J$, Λ_i is a nonempty subset of S_{l_i} for some $l_i \in \mathbb{Z}$ with the l_i distinct. We now apply Proposition 3.5 to Λ_i in each of the layers S_{l_i}, which are isomorphic to $\mathbb{T}_K$. This shows that for any $\lambda \in (0,1), \exists \theta_\lambda \in (0,1)$ such that for $\theta \geq \theta_\lambda$,

$$\mathbb{P}_\theta(\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash) \leq \lambda^{|\Lambda_i|}. \tag{4.2}$$

Since the Λ_i are subsets of distinct levels S_{l_i} of S^∞, the events $\{\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash\}$ are mutually independent. Therefore for $\theta \geq \theta_\lambda$,

$$\mathbb{P}_\theta(\gamma_N \in D_0) = \mathbb{P}_\theta\left(\{\Lambda_1 \cap \mathcal{T}_{l_1}^{+,K-1} = \oslash\} \cap \ldots \cap \{\Lambda_J \cap \mathcal{T}_{l_J}^{+,K-1} = \oslash\}\right) \tag{4.3}$$

$$= \prod_{i=1}^{J} \mathbb{P}_\theta\left(\{\Lambda_i \cap \mathcal{T}_{l_i}^{+,K-1} = \oslash\}\right) \tag{4.4}$$

$$\leq \lambda^{|\Lambda_1|+\ldots+|\Lambda_J|} \tag{4.5}$$

$$= \lambda^N. \tag{4.6}$$

Equation (4.1) and the above bound on $\mathbb{P}_\theta(\gamma_N \in D_0)$ imply

$$\mathbb{E}_\theta\left[|D_0|\right] \leq \sum_{N=0}^{\infty} \lambda^N \sum_{\gamma_N, |\gamma_N|=N} 1 \tag{4.7}$$

$$= \sum_{N=0}^{\infty} \rho(N)\lambda^N, \tag{4.8}$$

where $\rho(N)$ is the number of self-avoiding paths of length N starting at 0. It is easy to see that

$$\rho(N) \leq (K+2)(K+1)^{N-1}. \tag{4.9}$$

Thus

$$\mathbb{E}_\theta\left[|D_0|\right] \leq (K+2)\sum_{N=0}^{\infty}(K+1)^{N-1}\lambda^N. \tag{4.10}$$

The proof is finished by choosing $\theta_* = \theta_\lambda$ for $\lambda < \frac{1}{K+1}$. $\square$

Proof of Theorem 2.1 for $K \geq 5$: Taking θ_* as in Proposition 4.2, $S^\infty \backslash \mathcal{T}$ is a union of almost surely finite connected components:

$$S^\infty \backslash \mathcal{T} = \bigcup_i D_i, \tag{4.11}$$

where the D_i's are nonempty, disjoint and almost surely finite with $D_i = D_{x_i}$ for some x_i.

Fix any i; it suffices to show that D_i is eliminated by the dynamics in finite time. By this we mean that there exists $T_i < \infty$ such that for any $y \in D_i, \sigma_y(t) = +1, \forall t \geq T_i$, and so the droplet D_i fixates to $+1$. We proceed to show this.

For any set $B \subset S^\infty$, let

$$\partial B = \{x \in B \text{ such that there is an edge } e_{xy} \in \mathbb{E}^\infty \text{ with } y \in B^c\}. \qquad (4.12)$$

$\partial(D_i^c) \subset \mathcal{T}$ so $\partial(D_i^c)$ is stable with respect to the dynamics and for any $x \in \partial(D_i^c)$, $\sigma_x(0) = +1$.

Since D_i is finite it contains a longest path, $p = (z, \ldots, w)$. Since p cannot be extended to a longer path, z must have $K + 1$ neighbors in D_i^c. After z's clock first rings, it must be $+1$ and fixated for all later times. This argument can be extended to show D_i is eliminated (i.e., the -1 vertices are all flipped to $+1$) by the dynamics in finite time as follows. Consider the set of vertices in D_i which are not fixated to $+1$ by some time t, and take t to infinity. Suppose this limiting set is nonempty. Since this set is finite, it contains a longest path $\tilde{p} = (\tilde{z}, \ldots, \tilde{w})$. But now $K + 1$ of $\tilde{z}$'s neighbors have spin $+1$ as $t \to \infty$, implying that $\tilde{z}$ had no clock rings in $[T, \infty)$ for some finite T. This event has zero probability of occurring, which contradicts the supposition of a nonempty limit set. $\qquad \square$

The proof of Theorem 2.1 for $K = 3$ and 4 is slightly different than for $K \geq 5$, since for $K = 3$ (respectively, $K = 4$) the $\mathcal{T}_i$'s of Definition 3.1 are not stable with respect to the dynamics: each vertex $v \in \mathcal{T}_i$ has 2 (resp., 3) neighbors of spin $+1$, which is always less than a strict majority. The proof for $K = 3$ or 4 requires a different decomposition of the space S^∞ and definition of stable subsets. With this purpose in mind, we express S^∞ as

$$S^\infty = \bigcup_{i=-\infty}^{\infty} \tilde{S}_i, \qquad (4.13)$$

where $\tilde{S}_i = \mathbb{T}_k \times \{2i, 2i + 1\} = \{(u, j) : u \in \mathbb{T}_K, \text{ and } j = 2i \text{ or } 2i + 1\}$ (see Equation (2.1) for a comparison). We call a vertex $x = (u, 2i)$ or its partner in $\tilde{S}_i$, $\hat{x} = (u, 2i + 1)$, **doubly open** if both $\sigma_x(0) = +1$ and $\sigma_{\hat{x}}(0) = +1$; this occurs with probability θ^2. We proceed by defining a set of fixed vertices in S^∞ in the spirit of Section 2.1.

Definition 4.3. For i fixed, let $\tilde{\mathcal{T}}_i^{+,l}$ be the union of all subgraphs H of $\tilde{S}_i$ that are isomorphic to $\mathbb{T}_l \times \{2i, 2i + 1\}$ such that $\forall x \in H$, x is doubly open.

It is easy to see that $\tilde{\mathcal{T}}_i^{+,K-1}$ is stable for $K = 3$ or 4 with respect to the dynamics on S^∞. Let $\tilde{\mathcal{T}}$ denote the union of $\tilde{\mathcal{T}}_i^{+,K-1}$ across all levels $\tilde{S}_i$, i.e.,

$$\tilde{\mathcal{T}} = \bigcup_{i=-\infty}^{\infty} \tilde{\mathcal{T}}_i, \qquad (4.14)$$

where $\tilde{\mathcal{T}}_i = \tilde{\mathcal{T}}_i^{+,K-1}$.

Proof of Theorem 2.1 for $K = 3$ and 4: We map one independent percolation model, $\sigma_{(u,j)}(0)$ on S^∞ with parameter θ, to another one, $\tilde{\sigma}_{(u,i)}(0)$ on S^∞ with parameter θ^2, by defining $\tilde{\sigma}_{(u,i)}(0) = +1$ (resp., -1) if $(u, 2i)$ is doubly open (resp., is not doubly open). Propositions 3.5 and 4.2 applied to $\tilde{\sigma}$ imply that Proposition 4.2 with $\mathcal{T}$ replaced by $\tilde{\mathcal{T}}$ is valid for the $\tilde{\sigma}(0)$ percolation model. The rest of the proof proceeds as in the case for $K \geq 5$. $\qquad \square$

Proof of Theorem 2.2: The proof proceeds analogously to that of Theorem 2.1, except that the conclusion of Proposition 4.2, that $S^\infty \setminus \mathcal{T}$ almost surely has no infinite components (for θ close to 1), is replaced by an analogous result for $G \setminus \mathcal{T}_G$ with an appropriately defined $\mathcal{T}_G$. We next specify a choice of $\mathcal{T}_G$ for each of our graphs G and leave further details (which are straightforward given the proof of Proposition 4.2) to the reader.

For $G = \mathbb{T}_K$ with *any* $K \geq 3$, we simply take $\mathcal{T}_G = \mathcal{T}^{+,K-1}$ (see Definition 3.4). For $G = S^{\mathrm{semi}}$, $\mathcal{T}_G$ depends on K like it did for $G = S^\infty$ - i.e., for $K \geq 5$, we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\infty} \mathcal{T}_i^{+,K-1}, \tag{4.15}$$

and for $K = 3$ or 4 we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\infty} \tilde{\mathcal{T}}_i^{+,K-1}. \tag{4.16}$$

For $G = S_f^l$ or S_p^l with $K \geq 5$, we take

$$\mathcal{T}_G = \bigcup_{i=0}^{l-1} \mathcal{T}_i^{+,K-1}. \tag{4.17}$$

For $G = S_f^l$ or S_p^l with $K = 3$ or 4, the choice of $\mathcal{T}_G$ depends on whether l is even or odd since in the odd case the layers cannot be evenly paired. If l is even, then we take

$$\mathcal{T}_G = \bigcup_{i=0}^{\frac{l-2}{2}} \tilde{\mathcal{T}}_i^{+,K-1}. \tag{4.18}$$

For l odd (and ≥ 3), we pair off the first $l-3$ layers and then use the final 3 layers to define $\tilde{\tilde{\mathcal{T}}}^{+,K-1}$ in which the use of doubly open sites for $\tilde{\mathcal{T}}^{+,K-1}$ is replaced by **triply open** sites; then we take

$$\mathcal{T}_G = \left(\bigcup_{i=0}^{\frac{l-3}{2}} \tilde{\mathcal{T}}_i^{+,K-1} \right) \cup \tilde{\tilde{\mathcal{T}}}^{+,K-1}. \tag{4.19}$$

$\square$

Appendix A. Galton-Watson Lemmas

The goal of this section is to show that the quantity

$$\lambda^*(\theta) = \min \left\{ (1 - \mu_\theta(\mathrm{Tree}^+[v])^{\epsilon_1}, (1 - \mu_\theta(\mathrm{Tree}^+[v,w])^{\frac{1}{2}\epsilon_2} \right\}, \tag{A.1}$$

which appears at the end of the proof of Proposition 3.5, converges to 0 as $\theta \to 1$. Here v is a leaf of a subtree T of $\mathbb{T}_K$, $\{v,w\}$ is a pair of adjacent vertices of T such that v is a good 2-point (as in Definition 3.3), and ϵ_1, ϵ_2 are fixed constants.

For this purpose we consider independent site percolation on $\mathbb{T}_K$ and let $a_1, a_2, \ldots, a_K$ denote the K neighbors of 0, a distinguished vertex in $\mathbb{T}_K$. We associate to each a_i a tree $A_0[a_i]$ (see Definition 3.2), for $i = 1, \ldots, K$ – see Figure A.5.
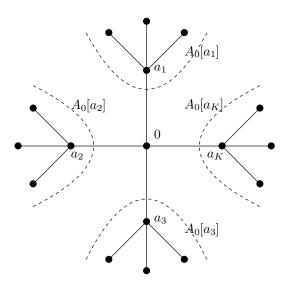
FIGURE A.5. K-ary tree with labeled vertices and branches

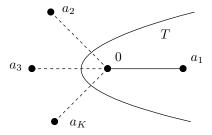Let T be a subtree of $\mathbb{T}_K$ such that T contains a_1 and 0 is a leaf of T – see Figure A.6.

FIGURE A.6. T is a subtree of $\mathbb{T}_K$ and 0 is a leaf of T

Let b be one of the neighbors of a_2 (other than 0) and T' be a subtree of $\mathbb{T}_K$ containing $b, a_2, 0$ and a_1 (but not $a_3, \ldots, a_K$) such that a_2 is a good 2-point of T' – see Figure A.7.
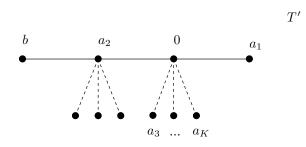
T'

FIGURE A.7. T' is a subtree of $\mathbb{T}_K$ that contains $b, a_2, 0$ and a_1 such that a_2 is a good 2-point of T'

We consider the events $\mathrm{Tree}^+[0]$ with respect to T and $\mathrm{Tree}^+[a_2, 0]$ with respect to T' (see Definition 3.3) and estimate $\mu_\theta(\mathrm{Tree}^+[0])$, $\mu_\theta(\mathrm{Tree}^+[a_2, 0])$ by analyzing a related Galton-Watson process.

Definition A.1. For any vertex $v \in \mathbb{T}_K$, let $C(v)$ denote the $+1$ spin cluster of v, that is, $C(v)$ is the set of vertices u in $\mathbb{T}_K$ such that the path from v to u (including v and u) includes only vertices w, with $\sigma_w = +1$.

Let

$$Z_n = |v \in A_0[a_1] \cap C(a_1) : d(a_1, v) = n|, \tag{A.2}$$

where $d(a_1, v)$ represents the graph distance(i.e., the minimum number of edges) between a_1 and v. $Z_0 = 1$ if and only if $\sigma_{a_1} = +1$, and in general Z_n is the number of vertices in $A_0[a_1]$ at distance n from a_1 that are in a_1's $+1$ spin cluster. Z_n is a Galton-Watson branching process with offspring distribution $\mathrm{Bin}(K - 1, \theta)$.

Let $\mathbb{T}^{\mathrm{root}}[x]$ denote a tree with root x, such that x has coordination number $K - 2$ and all the other vertices have coordination number $K - 1$. The following definition is close to that of $\mathrm{Tree}^+[x]$ (see Definition 3.3 and Figure 3.3), except that here the $(K - 1)$-ary tree in question is rooted.

Definition A.2. Random rooted $(K - 1)$-ary trees of spin $+1$
Consider two vertices $v, v' \in \mathbb{T}_K$ such that v' is a neighbor of v. Let $\mathrm{Part}^+_{v'}[v]$ denote the event that there exists a subgraph H of $\mathbb{T}_K$ isomorphic to $\mathbb{T}^{\mathrm{root}}[v]$ which contains v and is contained in $A_{v'}[v]$, such that for all $u \in H, \sigma_u = +1$.

Consider three vertices x, y, z such that x and z are neighbors of y. Let $\mathrm{Part}^+_{x,z}[y]$ be the event that there exists a subgraph H of $\mathbb{T}_K$ isomorphic to $\mathbb{T}^{\mathrm{root}}[x]$ which contains y and is contained in $A_{x,z}[y]$ (see Figure 3.2), such that for all $u \in H, \sigma_u = +1$.

Define $\tau(\theta)$ as

$$\tau(\theta) = \mu_\theta(\mathrm{Part}^+_0[a_1]). \tag{A.3}$$

$\mathrm{Part}^+_0[a_i]$ and $\mathrm{Part}^+_0[a_j]$ are independent for $i \neq j$ by construction. The event $\mathrm{Tree}^+[0]$ is equivalent to the spin at 0 being $+1$ and the vertices $a_2, \ldots, a_K$ being the roots of $(K - 1)$-ary trees of spin $+1$, so that

$$\mu_\theta(\mathrm{Tree}^+[0]) = \theta \, \tau(\theta)^{K-1}. \tag{A.4}$$

Lemma A.3. $\tau(\theta) \to 1$ *as* $\theta \to 1$.

Proof: The proof is a consequence of Proposition 5.30 from Lyons and Peres (2014) (about occurrence of j-ary subtrees in Galton-Watson processes). $\square$

Define $\tilde{\tau}(\theta)$ as

$$\tilde{\tau}(\theta) = \mu_\theta(\text{Part}^+_{a_1,a_2}[0]). \tag{A.5}$$

The event $\text{Tree}^+[a_2, 0]$ is equivalent to $\{\text{Part}^+_{a_1,a_2}[0] \cap \text{Part}^+_{b,0}[a_2]\}$, so that, by the independence of the events $\text{Part}^+_{a_1,a_2}[0]$ and $\text{Part}^+_{b,0}[a_2]$,

$$\mu_\theta(\text{Tree}^+[a_2, 0]) = \tilde{\tau}(\theta)^2. \tag{A.6}$$

Lemma A.4. $\tilde{\tau}(\theta) \to 1$ *as* $\theta \to 1$.

Proof: This result follows as in the proof of Lemma A.3. $\square$

Equations (A.4) and (A.6) imply that $\mu_\theta(\text{Tree}^+[0])$ and $\mu_\theta(\text{Tree}^+[a_2, 0])$ converge to 1 as $\theta \to 1$, which immediately implies:

Lemma A.5. $\lambda^*(\theta) \to 0$ *as* $\theta \to 1$.

Appendix B. Geometric Lemmas

Let T be a finite tree with N vertices and maximal coordination number $\leq K$. $N_1 \leq N$ of T's vertices are labeled special, such that all of T's leaves are special vertices. We remark that in Section 4, we start with $|\Lambda|$ special vertices in $\mathbb{T}_K$ and then T is the minimal subtree of $\mathbb{T}_K$ that contains all the special vertices.

Lemma B.1. *Let M_1 be the number of leaves in T, M_2 the number of 2-points (vertices with exactly two edges in T), ..., M_K the number of K-points (vertices with exactly K edges in T); $M_1 + \ldots + M_K = N$. Then*

$$M_i \leq M_1 \tag{B.1}$$

for $i = 3, \ldots, K$.

Proof: The proof can be found, for example, as part of Theorem 8.1 in Grimmett (1999). $\square$

Definition B.2. Recall that a *good* 2-point in T is a 2-point both of whose neighbors are 2-points. A *bad* 2-point is a 2-point that is not a good 2-point – see Figure B.8.
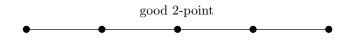
good 2-point

FIGURE B.8. A good 2-point

Lemma B.3. *There exist constants $\epsilon_1, \epsilon_2 \in (0, \infty)$, depending only on K, such that either:*

a) $M_1 \geq \epsilon_1 N_1$, and/or
b) there are at least $\epsilon_2 N_1$ special good 2-points.

Proof: By Lemma B.1

$$\sum_{i=3}^{K} M_i \leq (K-2)M_1, \tag{B.2}$$

and since $\sum_{1}^{K} M_i = N$,

$$(K-1)M_1 + M_2 = M_1 + M_2 + (K-2)M_1 \tag{B.3}$$

$$\geq \sum_{1}^{K} M_i = N. \tag{B.4}$$

Thus either $(K-1)M_1 \geq \frac{N}{K}$ or $M_2 \geq \frac{N(K-1)}{K}$. In the first case, since $N \geq N_1$,

$$M_1 \geq \frac{N}{K(K-1)} \geq \frac{1}{K(K-1)} N_1, \tag{B.5}$$

and letting $\epsilon_1 = \frac{1}{K(K-1)}$ gives a).

In the second case $M_2 \geq \frac{N(K-1)}{K}$, and if a) is not valid with $\epsilon_1 = \frac{1}{K(K-1)}$, then $M_1 \leq \frac{N_1}{K(K-1)}$. To prove b) we need to count the various types of special vertices in T. The set of special vertices is comprised of:

- special good 2-points; let *Good* denote the set of such vertices,
- special bad 2-points; let *Bad* denote the set of such vertices,
- special leaves, special 3-points, ... , special K-points; let *Other* denote the set of such vertices.

Since $|\text{Good}| = N_1 - |\text{Bad}| - |\text{Other}|$, we need to upper bound $|\text{Other}|$ and $|\text{Bad}|$. By Lemma B.1,

$$|\text{Other}| \leq M_1 + M_3 + \ldots + M_K \tag{B.6}$$

$$\leq (K-2)M_1 \tag{B.7}$$

$$\leq \frac{K-2}{K(K-1)} N_1. \tag{B.8}$$

Now $|\text{Bad}| \leq |\{\text{all bad 2-points}\}|$ and it is easy to see that the latter is upper bounded by $M_1 + 3M_3 + \ldots + KM_K$. Thus by Lemma B.1,

$$|\text{Bad}| \leq M_1 + 3M_3 + \ldots + KM_K \tag{B.9}$$

$$\leq M_1(1 + 3 + \ldots + K) \tag{B.10}$$

$$\leq \frac{1}{2} K(K-1)M_1 \tag{B.11}$$

$$\leq \frac{1}{2} N_1, \tag{B.12}$$

since $M_1 \leq \frac{N_1}{K(K-1)}$. Thus

$$|\text{Good}| \quad = \quad N_1 - |\text{Bad}| - |\text{Other}| \tag{B.13}$$

$$\geq \quad N_1 \left(1 - \frac{K-2}{K(K-1)} - \frac{1}{2} \right) \tag{B.14}$$

$$= \quad N_1 \left(\frac{K^2 - 3K + 4}{2K(K-1)} \right). \tag{B.15}$$

We let $\epsilon_2 = \frac{K^2 - 3K + 4}{2K(K-1)} > 0$ and so $|\text{Good}| \geq \epsilon_2 N_1$. $\qquad\square$

Appendix C. Disjointness Lemma

Consider site percolation on $\mathbb{T}_K$ distributed according to the product measure μ_θ with

$$\mu_\theta(\sigma_x = +1) = \theta = 1 - \mu_\theta(\sigma_x = -1), \forall x \in \mathbb{T}_K. \tag{C.1}$$

Let T be a finite subtree of $\mathbb{T}_K$ with $2 \leq N_1 \leq |T|$ of its vertices labeled special, such that all the leaves are special. As in Lemma B.3, in the following lemma ϵ_1 and ϵ_2 are strictly positive, finite and depend only on K. For the events $\text{Tree}^+[v]$ and $\text{Tree}^+[v, w]$, see Definition 3.3.

Lemma C.1. *Disjoint events*
For each such tree T, one or both of the following is valid:

a) *there are at least $\epsilon_1 N_1$ leaves v in T, with the events $\{\text{Tree}^+(v)\}$ mutually independent, and/or*

b) *there are at least $\frac{1}{2}\epsilon_2|\Lambda|$ edges having endpoints v, w in T with v a good special 2-point, and the events $\{\text{Tree}^+(v, w)\}_{v,w}$ mutually independent.*

Proof: Lemma C.1.a follows from Lemma B.3.a, since for each of the $\epsilon_1 N_1$ leaves of T we can define an event $\text{Tree}^+[v]$, and these events depend on the spins of disjoint sets of vertices and are therefore mutually independent.

Otherwise, by Lemma B.3.b there are at least $N_3 = \epsilon_2 N_1$ good special 2-points in T. These are arranged into $p \geq 1$ nonempty maximal chains of adjacent vertices along T. We order the chains and let n_i denote the number of vertices in the ith chain, for $i = 1, \ldots, p$; $n_1, \ldots, n_p \geq 1$ and $n_1 + \ldots + n_p = N_3$. We also order the N_3 good special 2-points, $\{s_1, s_2, \ldots, s_{N_3}\}$, so that they are consecutively ordered in each chain.

Suppose $n_i = 1$ for some i, and the good special 2-point in this chain is s_i^*. Let w_i be one of s_i^*'s neighbors in T and consider the event $\text{Tree}^+[s_i^*, w_i]$. If $n_i = 2$, the ith chain contains two adjacent special points $\{s_i^*(1), s_i^*(2)\}$ and we consider the event $\text{Tree}^+[s_i^*(1), s_i^*(2)]$. Generally for the ith chain, we pair adjacent good special 2-points (other than the last if n_i is odd) so as to consider $\lfloor \frac{n_i+1}{2} \rfloor$ events $\text{Tree}^+[s_i^*(j), s_i^*(j+1)]$, where the last event is $\text{Tree}^+[s_i^*(n_i), w_i]$ if n_i is odd; these events involve disjoint sets of vertices and are therefore independent. Thus in total we can construct

$$\left\lfloor \frac{n_1 + 1}{2} \right\rfloor + \ldots + \left\lfloor \frac{n_p + 1}{2} \right\rfloor \geq \left\lfloor \frac{N_3}{2} \right\rfloor \tag{C.2}$$

mutually independent events. $\qquad\square$

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