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Conditioned limit theorems for products of positive random matrices

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Abstract. Inspired by a recent paper of I. Grama, E. Le Page and M. Peigné (Grama et al., 2014), we consider a sequence $(g_n)_{n\geq 1}$ of i.i.d. random $d\times d$-matrices with non-negative entries and study the fluctuations of the process $(\log|g_n\cdots g_1 x|)_{n\geq 1}$ for any non-zero vector x in $\mathbb{R}^d$ with non-negative coordinates. Our method involves approximating this process by a martingale and studying harmonic functions for its restriction to the upper half line. Under certain conditions, the probability for this process to stay in the upper half real line up to time n decreases as $\frac{c}{\sqrt{n}}$ for some positive constant c.

1. Introduction

Many limit theorems describe the asymptotic behaviour of random walks with i.i.d. increments, for instance the strong law of large numbers, the central limit theorem, the invariant principle... Besides, the fluctuations of these processes are well studied, for example the decay of the probability that they stay inside the half real line up to time n or functional central limit theorems for random walks conditioned to stay positive. A vast literature exists on this subject, see for instance Bolthausen (1976), Iglehart (1974a), Iglehart (1974b), Iglehart (1975), Kaigh (1976) or Shimura (1983), and references therein. The Wiener-Hopf factorization is usually used in this case and so far, it seems to be impossible to adapt in non-abelian context. Recently, much efforts are made to apply the results above for the logarithm of the norm of the product of i.i.d. random matrices since it behaves similarly to a sum of i.i.d. random variables. Many limit theorems arose for the last 60 years,

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initiated by Furstenberg and Kesten (1960), Guivarc'h and Raugi (1985), Le Page (1982)... and recently Benoist and Quint (2016). Let us mention also the works by Hennion (1984) and Hennion and Hervé (2008) for matrices with positive entries. However, the studies on the subject of fluctuation was quite sparse a few years ago. Thanks to the approach of Denisov and Wachtel (2015) for random walks in Euclidean spaces and motivated by branching processes, I. Grama, E. Le Page and M. Peigné recently progressed for invertible matrices (Grama et al., 2014). Here we propose to develop the same strategy for matrices with positive entries by using Hennion and Hervé (2008).

We endow $\mathbb{R}^d$ with the norm $|\cdot|$ defined by $|x| := \sum_{i=1}^{d} |x_i|$ for any column vector $x = (x_i)_{1\leq i\leq d}$. Let $\mathcal{C}$ be the cone of vectors in $\mathbb{R}^d$ with non-negative coordinates

$$\mathcal{C} := \{x \in \mathbb{R}^d : \forall 1 \leq i \leq d, x_i \geq 0\}$$

and $\mathbb{X}$ be the standard simplex defined by

$$\mathbb{X} := \{x \in \mathcal{C}, |x| = 1\}.$$

Let S be the set of $d\times d$ matrices with non-negative entries such that each column contains at least one positive entry; its interior is $\mathring{S} := \{g = (g(i,j))_{1\leq i,j\leq d} / g(i,j) > 0\}$. Endowed with the standard multiplication of matrices, the set S is a semigroup and $\mathring{S}$ is the ideal of S, more precisely, for any $g \in \mathring{S}$ and $h \in S$, it is evident that $gh \in \mathring{S}$.

We consider the following actions:

- the left linear action of S on $\mathcal{C}$ defined by $(g, x) \mapsto gx$ for any $g \in S$ and $x \in \mathcal{C}$,
- the left projective action of S on $\mathbb{X}$ defined by $(g, x) \mapsto g \cdot x := \frac{gx}{|gx|}$ for any $g \in S$ and $x \in \mathbb{X}$.

For any $g = (g(i,j))_{1\leq i,j\leq d} \in S$, without confusion, let

$$v(g) := \min_{1\leq j\leq d}\Big(\sum_{i=1}^{d} g(i,j)\Big) \quad \text{and} \quad |g| := \max_{1\leq j\leq d}\Big(\sum_{i=1}^{d} g(i,j)\Big),$$

then $|\cdot|$ is a norm on S and for any $x \in \mathcal{C}$,

$$0 < v(g)\ |x| \leq |gx| \leq |g|\ |x|. \tag{1.1}$$

We set $N(g) := \max\left(\frac{1}{v(g)}, |g|\right)$; notice that $N(g) \geq 1$ for any $g \in S$.

On some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider a sequence of i.i.d. S-valued matrices $(g_n)_{n\geq 0}$ with the same distribution μ on S. Let $L_0 = Id$ and $L_n := g_n \dots g_1$ for any $n \geq 0$. For any fixed $x \in \mathbb{X}$, we define the $\mathbb{X}$-valued Markov chain $(X_n^x)_{n\geq 0}$ by setting $X_n^x := L_n \cdot x$ for any $n \geq 0$ (or simply X_n if there is no confusion). We denote by P the transition probability of $(X_n)_{n\geq 0}$, defined by: for any $x \in \mathbb{X}$ and any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{C}$,

$$P\varphi(x) := \int_S \varphi(g\cdot x)\mu(dg) = \mathbb{E}[\varphi(L_1 \cdot x)].$$

Hence, for any $n \geq 1$,

$$P^n\varphi(x) = \mathbb{E}[\varphi(L_n \cdot x)].$$

We assume that with positive probability, after finitely many steps, the sequence $(L_n)_{n\geq 1}$ reaches $\mathring{S}$. In mathematical term, it is equivalent to writing as

$$\mathbb{P}\left(\bigcup_{n\geq 1}[L_n \in \mathring{S}]\right) > 0.$$

On the product space $S\times\mathbb{X}$, we define the function ρ by setting for any $(g,x)\in S\times\mathbb{X}$,

$$\rho(g,x) := \log|gx|.$$

Notice that $gx = e^{\rho(g,x)}g\cdot x$; in other terms, the linear action of S on $\mathcal{C}$ corresponds to the couple $(g\cdot x, \rho(g,x))$. This function ρ satisfies the cocycle property $\rho(gh,x) = \rho(g, h\cdot x) + \rho(h,x)$ for any $g,h\in S$ and $x\in\mathbb{X}$ and implies the basic decomposition for any $x\in\mathbb{X}$,

$$\log|L_n x| = \sum_{k=1}^{n}\rho(g_k, X_{k-1}^x).$$

For any $a\in\mathbb{R}$ and $n\geq 1$, let $S_0 := a$ and $S_n = S_n(x,a) := a + \sum_{k=1}^n \rho(g_k, X_{k-1})$. Then the sequence $(X_n, S_n)_{n\geq 0}$ is a Markov chain on $\mathbb{X}\times\mathbb{R}$ with transition probability $\widetilde{P}$ defined by: for any $(x,a)\in\mathbb{X}\times\mathbb{R}$ and any bounded Borel function $\psi:\mathbb{X}\times\mathbb{R}\to\mathbb{C}$,

$$\widetilde{P}\psi(x,a) = \int_S \psi(g\cdot x, a+\rho(g,x))\mu(dg).$$

For any $(x,a)\in\mathbb{X}\times\mathbb{R}$, we denote by $\mathbb{P}_{x,a}$ the probability measure on $(\Omega,\mathcal{F})$ conditioned to the event $[X_0 = x, S_0 = a]$ and by $\mathbb{E}_{x,a}$ the corresponding expectation; for the sake of brevity, by $\mathbb{P}_x$ we denote $\mathbb{P}_{x,a}$ when $S_0 = 0$ and by $\mathbb{E}_x$ the corresponding expectation. Hence for any $n\geq 1$,

$$\widetilde{P}^n\psi(x,a) = \mathbb{E}[\psi(L_n\cdot x, a + \log|L_n x|)] = \mathbb{E}_{x,a}[\psi(X_n, S_n)].$$

Now we consider the restriction $\widetilde{P}_+$ to $\mathbb{X}\times\mathbb{R}^+$ of $\widetilde{P}$ defined by: for any $(x,a)\in\mathbb{X}\times\mathbb{R}^+$ and any bounded function $\psi:\mathbb{X}\times\mathbb{R}\to\mathbb{C}$,

$$\widetilde{P}_+\psi(x,a) = P(\psi\mathbf{1}_{\mathbb{X}\times\mathbb{R}^+_*})(x,a).$$

Let us emphasize that $\widetilde{P}_+$ may not be a Markov kernel on $\mathbb{X}\times\mathbb{R}^+$.

Let $\tau := \min\{n\geq 1 : S_n \leq 0\}$ be the first time the random process $(S_n)_{n\geq 1}$ becomes non-positive; for any $(x,a)\in\mathbb{X}\times\mathbb{R}^+$ and any bounded Borel function $\psi:\mathbb{X}\times\mathbb{R}\to\mathbb{C}$,

$$\widetilde{P}_+\psi(x,a) = \mathbb{E}_{x,a}[\psi(X_1,S_1);\tau>1] = \mathbb{E}[\psi(g_1\cdot x, a+\rho(g_1,x)); a+\rho(g_1,x)>0].$$

A positive $\widetilde{P}_+$-harmonic function V is any function from $\mathbb{X}\times\mathbb{R}^+$ to $\mathbb{R}^+$ satisfying $\widetilde{P}_+V = V$. We extend V by setting $V(x,a) = 0$ for $(x,a)\in\mathbb{X}\times\mathbb{R}^-_*$. In other words, the function V is $\widetilde{P}_+$-harmonic if and only if for any $x\in\mathbb{X}$ and $a\geq 0$,

$$V(x,a) = \mathbb{E}_{x,a}[V(X_1,S_1);\tau>1]. \tag{1.2}$$

From Theorem II.1 in Hennion and Hervé (2008), under conditions P1-P3 introduced below, there exists a unique probability measure ν on $\mathbb{X}$ such that for any bounded Borel function φ from $\mathbb{X}$ to $\mathbb{R}$,

$$(\mu*\nu)(\varphi) = \int_S\int_{\mathbb{X}}\varphi(g\cdot x)\nu(dx)\mu(dg) = \int_{\mathbb{X}}\varphi(x)\nu(dx) = \nu(\varphi).$$

Such a measure is said to be μ-invariant. Moreover, the upper Lyapunov exponent associated with μ is finite and is expressed by

$$\gamma_\mu = \int_S \int_{\mathbb{X}} \rho(g,x)\nu(dx)\mu(dg). \tag{1.3}$$

Now we state the needed hypotheses for later work.

HYPOTHESES

P1 *There exists* $\delta_0 > 0$ *such that* $\displaystyle\int_S N(g)^{\delta_0}\mu(dg) < +\infty$.

P2 *There exists no affine subspaces* A *of* $\mathbb{R}^d$ *such that* $A \cap \mathcal{C}$ *is non-empty and bounded and invariant under the action of all elements of the support of* μ.

P3 *There exists* $n_0 \geq 1$ *such that* $\mu^{*n_0}(\mathring{S}) > 0$.

P4 *The upper Lyapunov exponent* γ_μ *is equal to* 0.

P5 *There exists* $\delta > 0$ *such that* $\mu\{g \in S : \forall x \in \mathbb{X}, \log|gx| \geq \delta\} > 0$.

In this paper, we establish the asymptotic behaviour of $\mathbb{P}_{x,a}(\tau > n)$ by studying the $\widetilde{P}_+$-harmonic function V. More precisely, Proposition 1.1 concerns the existence of a $\widetilde{P}_+$-harmonic function and its properties whereas Theorem 1.2 is about the limit behaviour of the exit time τ.

Proposition 1.1. *Assume hypotheses P1-P5.*

(1) *For any* $x \in \mathbb{X}$ *and* $a \geq 0$, *the sequence* $\Big(\mathbb{E}_{x,a}[S_n; \tau > n]\Big)_{n\geq 0}$ *converges to the function* $V(x,a) := a - \mathbb{E}_{x,a}M_\tau$.

(2) *For any* $x \in \mathbb{X}$ *the function* $V(x,\cdot)$ *is increasing on* $\mathbb{R}^+$.

(3) *There exist* $c > 0$ *and* $A > 0$ *such that for any* $x \in \mathbb{X}$ *and* $a \geq 0$,

$$\frac{1}{c} \vee (a - A) \leq V(x,a) \leq c(1+a).$$

(4) *For any* $x \in \mathbb{X}$, *the function* $V(x,.)$ *satisfies* $\displaystyle\lim_{a\to+\infty} \frac{V(x,a)}{a} = 1$.

(5) *The function* V *is* $\widetilde{P}_+$*-harmonic.*

The function V contains information of the part of the trajectory which stays in $\mathbb{R}^+$ as stated in Theorem 1.2.

Theorem 1.2. *Assume P1-P5. Then for any* $x \in \mathbb{X}$ *and* $a \geq 0$,

$$\mathbb{P}_{x,a}(\tau > n) \sim \frac{2V(x,a)}{\sigma\sqrt{2\pi n}} \text{ as } n \to +\infty.$$

Moreover, there exists a constant c *such that for any* $x \in \mathbb{X}$, $a \geq 0$ *and* $n \geq 1$,

$$\sqrt{n}\mathbb{P}_{x,a}(\tau > n) \leq cV(x,a).$$

As a direct consequence, we prove that the sequence $(\frac{S_n}{\sigma\sqrt{n}})_{n\geq 1}$, conditioned to the event $\tau > n$, converges in distribution to the Rayleigh law as stated below.

Theorem 1.3. *Assume P1-P5. For any* $x \in \mathbb{X}$, $a \geq 0$ *and* $t > 0$,

$$\lim_{n\to+\infty} \mathbb{P}_{x,a}\left(\frac{S_n}{\sqrt{n}} \leq t \mid \tau > n\right) = 1 - \exp\left(-\frac{t^2}{2\sigma^2}\right).$$

In section 2, we approximate the chain $(S_n)_{n\geq 0}$ by a martigale and in section 3, we study the harmonic function V and state the proof of Proposition 1.1. We use the coupling argument to prove Theorem 1.2 and Theorem 1.3 in section 4. At last, in section 5 we check general conditions to apply an invariant principle stated in Theorem 2.1 in Grama et al. (2014).

Throughout this paper, we denote the absolute constants such as $C, c, c_1, c_2, \ldots$ and the constants depending on their indices such as $c_\varepsilon, c_p, \ldots$. Notice that they are not always the same when used in different formulas. The integer part of a real constant a is denoted by $[a]$.

2. Approximation of the chain $(S_n)_{n\geq 0}$

In this section, we discuss the spectral properties of P and then utilise them to approximate the chain $(S_n)_{n\geq 0}$. Throughout this section, we assume that conditions P1-P4 hold true.

2.1. *Spectral properties of the operators P and its Fourier transform.* Following Hennion (1997), we endow $\mathbb{X}$ with a bounded distance d such that g acts on $\mathbb{X}$ as a contraction with respect to d for any $g \in S$. For any $x, y \in \mathbb{X}$, we write:

$$m(x,y) = \min_{1\leq i\leq d}\left\{\frac{x_i}{y_i}|y_i > 0\right\}$$

and it is clear that $0 \leq m(x,y) \leq 1$. For any $x, y \in \mathbb{X}$, let $d(x,y) := \varphi(m(x,y)\, m(y,x))$, where φ is the one-to-one function defined for any $s \in [0,1]$ by $\varphi(s) := \dfrac{1-s}{1+s}$. Setting $c(g) := \sup\{d(g\cdot x, g\cdot y), x, y \in \mathbb{X}\}$ for $g \in S$; the proposition below gives some more properties of d and $c(g)$.

Proposition 2.1. *Hennion (1997) The quantity d is a distance on $\mathbb{X}$ satisfying the following properties:*

1. $\sup\{d(x,y) : x, y \in \mathbb{X}\} = 1$.
2. $|x-y| \leq 2d(x,y)$ *for any* $x, y \in \mathbb{X}$.
3. $c(g) \leq 1$ *for any* $g \in S$, *and* $c(g) < 1$ *if and only if* $g \in \mathring{S}$.
4. $d(g\cdot x, g\cdot y) \leq c(g)\, d(x,y) \leq c(g)$ *for any and* $x, y \in \mathbb{X}$.
5. $c(gh) \leq c(g)\, c(h)$ *for any* $g, h \in S$.

From now on, we consider a sequence $(g_n)_{n\geq 0}$ of i.i.d. S-valued random variables, we set $a_k := \rho(g_k, X_{k-1})$ for $k \geq 1$ and hence $S_n = a + \sum_{k=1}^n a_k$ for $n \geq 1$. In order to study the asymptotic behavior of the process $(S_n)_{n\geq 0}$, we need to consider the "Fourier transform" of the random variables a_k, under $\mathbb{P}_x, x \in \mathbb{X}$, similarly for classical random walks with independent increments on $\mathbb{R}$. Let P_t be the family of "Fourier operators" defined for any $t \in \mathbb{R}$, $x \in \mathbb{X}$ and any bounded Borel function $\varphi : \mathbb{X} \to \mathbb{C}$ by

$$P_t\varphi(x) := \int_S e^{it\rho(g,x)}\varphi(g\cdot x)\mu(dg) = \mathbb{E}_x\left[e^{ita_1}\varphi(X_1)\right] \tag{2.1}$$

and for any $n \geq 1$,

$$P_t^n\varphi(x) = \mathbb{E}[e^{it\log|L_n x|}\varphi(L_n\cdot x)] = \mathbb{E}_x[e^{itS_n}\varphi(X_n)]. \tag{2.2}$$

Moreover, we can imply that

$$\begin{aligned} P^m P_t^n\varphi(x) &= \mathbb{E}\left[e^{it\log|g_{m+n}\cdots g_{m+1}(L_m\cdot x)|}\varphi(L_{m+n}\cdot x)\right] \\ &= \mathbb{E}_x\left[e^{it(a_{m+1}+\cdots+a_{m+n})}\varphi(X_{n+m})\right] \end{aligned} \tag{2.3}$$

and when $\varphi = 1$, we obtain

$$\mathbb{E}_x\left[e^{itS_n}\right] = P_t^n 1(x) \quad \text{and} \quad \mathbb{E}_x\left[e^{it(a_{m+1}+\cdots+a_{m+n})}\right] = P^m P_t^n 1(x).$$

We consider the space $C(\mathbb{X})$ of continuous functions from $\mathbb{X}$ to $\mathbb{C}$ endowed with the norm of uniform convergence $|.|_\infty$. Let L be the subset of Lipschitz functions on $\mathbb{X}$ defined by

$$L := \{\varphi \in C(\mathbb{X}) : |\varphi|_L := |\varphi|_\infty + m(\varphi) < +\infty\},$$

where $m(\varphi) := \sup_{\substack{x,y\in\mathbb{X}\\ x\neq y}} \frac{|\varphi(x)-\varphi(y)|}{d(x,y)}$. The spaces $(C(\mathbb{X}), |\cdot|_\infty)$ and $(L, |\cdot|_L)$ are Banach spaces and the canonical injection from L into $C(\mathbb{X})$ is compact. The norm of a bounded operation A from L to L is denoted by $|A|_{L\to L} := \sup_{\varphi\in L} |A\varphi|_L$. We denote L' the topological dual of L endowed with the norm $|\cdot|_{L'}$ corresponding to $|\cdot|_L$; notice that any probability measure ν on $\mathbb{X}$ belongs to L'.
For further uses, we state here some helpful estimations.

Lemma 2.2. *For $g \in S$, $x, y, z \in \mathbb{X}$ such that $d(x, y) < 1$ and for any $t \in \mathbb{R}$,*

$$|\rho(g, x)| \leq 2 \log N(g), \tag{2.4}$$

and

$$|e^{it\rho(g,y)} - e^{it\rho(g,z)}| \leq \Big(4 \min\{2|t| \log N(g), 1\} + 2C|t|\Big) d(y, z), \tag{2.5}$$

where $C = \sup\{\frac{1}{u} \log \frac{1}{1-u} : 0 < u \leq \frac{1}{2}\} < +\infty$.

Proof. For the first assertion, from (1.1), we can imply that $|\log |gx|| \leq \log N(g)$. For the second assertion, we refer to the proof the Theorem III.2 in Hennion and Hervé (2008).

□

Denote $\varepsilon(t) := \int_S \min\{2|t| \log N(g), 2\}\mu(dg)$. Notice that $\lim_{t\to 0} \varepsilon(t) = 0$.

Proposition 2.3. *Hennion and Hervé (2008) Under hypotheses* **P1, P2, P3** *and* **P4**, *for any $t \in \mathbb{R}$, the operator P_t acts on L and satisfies the following properties:*

(1) *Let $\Pi : L \to L$ be the rank one operator defined by $\Pi(\varphi) = \nu(\varphi)1$ for any function $\varphi \in L$, where ν is the unique P-invariant probability measure on $\mathbb{X}$ and $R := P - \Pi$.*
The operator $R : L \to L$ satisfies
$$\Pi R = R\Pi = 0,$$
and its spectral radius is less than 1; in other words, there exist constants $C > 0$ and $0 < \kappa < 1$ such that $|R^n|_{L\to L} \leq C\kappa^n$ for any $n \geq 1$.

(2) *There exist $\epsilon > 0$ and $0 \leq r_\epsilon < 1$ such that for any $t \in [-\epsilon, \epsilon]$, there exist a complex number λ_t closed to 1 with modulus less than or equal to 1, a rank one operator Π_t and an operator R_t on L with spectral radius less than or equal to r_ϵ such that*
$$P_t = \lambda_t \Pi_t + R_t \quad \text{and} \quad \Pi_t R_t = R_t \Pi_t = 0.$$
Moreover, $C_P := \sup_{\substack{-\epsilon\leq t\leq\epsilon\\ n\geq 0}} |P_t^n|_{L\to L} < +\infty$.

(3) *For any $p \geq 1$,*
$$\sup_{n\geq 0} \sup_{x\in\mathbb{X}} \mathbb{E}_x |\rho(g_{n+1}, X_n)|^p < +\infty. \tag{2.6}$$

Proof. **(a)** We first check that P_t acts on $(L, |\cdot|_L)$ for any $t \in \mathbb{R}$. On one hand, $|P_t\varphi|_\infty \leq |\varphi|_\infty$ for any $\varphi \in L$. On the other hand, by (2.5) for any $x, y \in \mathbb{X}$ such that $x \neq y$,

$$\begin{aligned}
&\frac{|P_t\varphi(x) - P_t\varphi(y)|}{d(x,y)} \\
&\leq \int_S \left(\left| \frac{e^{it\rho(g,x)} - e^{it\rho(g,y)}}{d(x,y)} \right| |\varphi(g\cdot x)| + \left| \frac{\varphi(g\cdot x) - \varphi(g\cdot y)}{d(x,y)} \right| \right) \mu(dg) \\
&\leq |\varphi|_\infty (4\varepsilon(t) + 2C|t|) + \int_S \left(\frac{|\varphi(g\cdot x) - \varphi(g\cdot y)|}{d(g\cdot x, g\cdot y)} \frac{d(g\cdot x, g\cdot y)}{d(x,y)} \right) \mu(dg), \\
&\leq |\varphi|_\infty (4\varepsilon(t) + 2C|t|) + m(\varphi),
\end{aligned}$$

which implies $m(P_t\varphi) \leq |\varphi|_\infty(4\varepsilon(t) + 2C|t|) + m(\varphi) < +\infty$. Therefore $P_t\varphi \in L$.

(b) Let Π be the rank one projection on L defined by $\Pi\varphi = \nu(\varphi)\mathbf{1}$ for any $\varphi \in L$. Let $R := P - \Pi$. By definition, we obtain $P\Pi = \Pi P = \Pi$ and $\Pi^2 = \Pi$ which implies $\Pi R = R\Pi = 0$ and $R^n = P^n - \Pi$ for any $n \geq 1$. Here we only sketch the main steps by taking into account the proof of Theorem III.1 in Hennion and Hervé (2008). Let μ^{*n} be the distribution of the random variable L_n and set

$$c(\mu^{*n}) := \sup \left\{ \int_S \frac{d(g\cdot x, g\cdot y)}{d(x,y)} d\mu^{*n}(g) : x, y \in \mathbb{X}, x \neq y \right\}.$$

Since $c(\cdot) \leq 1$, we have $c(\mu^{*n}) \leq 1$. Furthermore, we can see that $c(\mu^{*(m+n)}) \leq c(\mu^{*m})c(\mu^{*n})$ for any $m, n > 0$. Hence, the sequence $(c(\mu^{*n}))_{n\geq 1}$ is submultiplicative and satisfies $c(\mu^{*n_0}) < 1$ for some $n_0 \geq 1$. Besides, we obtain $m(P^n\varphi) \leq m(\varphi)c(\mu^{*n})$. Moreover, we also obtain $m(\varphi) \leq |\varphi|_L \leq 3m(\varphi)$ for any $\varphi \in Ker\Pi$. Notice that $P^n(\varphi - \Pi\varphi)$ belongs to $Ker\Pi$ for any $\varphi \in L$ and $n \geq 0$. Hence $|P^n(\varphi - \Pi\varphi)|_L \leq 3c(\mu^{*n})|\varphi|_L$ which yields

$$|R^n|_{L\to L} = |P^n - \Pi|_{L\to L} = |P^n(I - \Pi)|_{L\to L} \leq 3c(\mu^{*n}).$$

Therefore, the spectral radius of R is less than or equal to $\kappa := \lim_{n\to+\infty} \Big(c(\mu^{*n})\Big)^{\frac{1}{n}}$ which is strictly less than 1 by hypothesis P3 and Proposition 2.1 (3).

(c) The theory of the perturbation (see Dunford and Schwartz, 1988, Chapter VII, section 6) allows to extend the decomposition $P = \Pi + R$ to the operator P_t when t is closed to 0. Indeed, for $\epsilon > 0$ small enough, there exists $r_\epsilon \in [0, 1[$ such that, for any $t \in [-\epsilon; \epsilon]$, the operator P_t may be decomposed as $P_t = \lambda_t\Pi_t + R_t$, where the spectral radius of R_t is less than or equal to r_ϵ and λ_t is the unique eigenvalue of P_t with modulus greater than r_ϵ; furthermore, the eigenvalue λ_t is simple. In order to control P_t^n, we ask λ_t^n to be bounded. Notice that by Hypothesis P1, the function $t \mapsto P_t$ is analytic near 0. To prove that the sequence $(P_t^n)_t$ is bounded in L, it suffices to check $|\lambda_t| \leq 1$ for any $t \in [-\epsilon, \epsilon]$.
When $\varphi(x) = \mathbf{1}(x)$, equality (2.2) becomes

$$P_t^n\mathbf{1}(x) = \mathbb{E}\left[e^{it\rho(L_n,x)}\right] = \lambda_t^n\Pi_t\mathbf{1}(x) + R_t^n\mathbf{1}(x). \tag{2.7}$$

We have the local expansion of λ_t at 0:

$$\lambda_t = \lambda_0 + t\lambda_0' + \frac{t^2}{2}\lambda_0''[1 + o(1)]. \tag{2.8}$$

Taking the first derivative of (2.7) with respect to t, we may write for any $n \geq 0$,

$$\frac{d}{dt}P_t^n\mathbf{1}(x) = \frac{d}{dt}\Big(\lambda_t^n\Pi_t\mathbf{1}(x) + R_t^n\mathbf{1}(x)\Big) = \mathbb{E}\left[i\rho(L_n,x)e^{it\rho(L_n,x)}\right].$$

Since $\lambda_0 = 1$, $\Pi_0 1(x) = 1$ and $|R^n|_{L\to L} \leq Cr_\epsilon^n$, we can imply that

$$\lambda_0' = \frac{i}{n}\mathbb{E}[\rho(L_n,x)] - \frac{\Pi_0' 1(x)}{n} - \frac{[R_t^n 1(x)]_{t=0}'}{n},$$

which yields $\lambda_0' = i\lim_{n\to+\infty}\frac{1}{n}\mathbb{E}[\rho(L_n,x)] = i\gamma_\mu = 0$. Similarly, taking the second derivative of (2.7) implies $\lambda_0'' = -\lim_{n\to+\infty}\frac{1}{n}\mathbb{E}[\rho(L_n,x)^2]$. Denote $\sigma^2 := \lim_{n\to+\infty}\frac{1}{n}\mathbb{E}_x[S_n^2]$. Applying in our context of matrices with non-negative coefficients the argument developed in Bougerol and Lacroix (1985) Lemma 5.3, we can imply that $\sigma^2 > 0$ and hence $\lambda_0'' = -\sigma^2 < 0$. Therefore, in particular, for t closed to 0, expression (2.8) becomes

$$\lambda_t = 1 - \frac{\sigma^2}{2}t^2[1 + o(1)]$$

which implies $|\lambda_t| \leq 1$ for t small enough.

(d) In particular, inequality (1.1) implies $|\rho(g,x)| \leq \log N(g)$ for any $x \in \mathbb{X}$. Therefore, for any $p \geq 1$, $x \in \mathbb{X}$ and $n \geq 1$, Hypothesis P1 yields

$$\mathbb{E}_x|\rho(g_{n+1},X_n)|^p \leq \frac{p!}{\delta_0^p}\mathbb{E}_x e^{\delta_0|\rho(g_{n+1},X_n)|} \leq \frac{p!}{\delta_0^p}\mathbb{E}N(g_{n+1})^{\delta_0} < +\infty.$$

□

2.2. *Martingale approximation of the chain* $(S_n)_{n\geq 0}$. As announced in the abstract, we approximate the process $(S_n)_{n\geq 0}$ by a martingale $(M_n)_{n\geq 0}$. In order to construct the suitable martingale, we introduce the operator $\overline{P}$ and then find the solution of the Poisson equation as follows. First, it is neccessary to introduce some notation and basic properties. Let $g_0 = I$ and $X_{-1} := X_0$. The sequence $((g_n, X_{n-1}))_{n\geq 0}$ is a Markov chain on $S\times\mathbb{X}$, starting from (Id, x) and with transition operator $\overline{P}$ defined by: for any $(g,x) \in S\times\mathbb{X}$ and any bounded measurable function $\phi : S\times\mathbb{X} \to \mathbb{R}$,

$$\overline{P}\phi(g,x) := \int_{S\times\mathbb{X}}\phi(h,y)\overline{P}((g,x),dhdy) = \int_S \phi(h, g\cdot x)\mu(dh) \tag{2.9}$$

(in other words, the measure $\overline{P}((g,x),dhdy)$ on $S\times\mathbb{X}$ equals $\delta_{g\cdot x}(dy)\mu(dh)$).
Notice that by (2.4), under assumption **P1**, for any $g \in S$ and $x \in \mathbb{X}$, the function $h \mapsto \rho(h, g\cdot x)$ is μ-integrable, so that $\overline{P}\rho(g,x)$ is well defined.

Lemma 2.4. *The function* $\bar{\rho} : x \mapsto \int_S \rho(g,x)\mu(dg)$ *belongs to* L *and for any* $g \in S$, $x \in \mathbb{X}$ *and* $n \geq 1$,

$$\overline{P}^{n+1}\rho(g,x) = P^n\overline{\rho}(g\cdot x). \tag{2.10}$$

Proof. (1) For any $x \in \mathbb{X}$, definition of ρ and (2.4) yield

$$|\overline{\rho}(x)| \leq \int_S |\log|gx||\mu(dg) \leq \int_S 2\log N(g)\mu(dg) \leq \int_S 2N(g)^{\delta_0}\mu(dg) < +\infty.$$

Hence $|\overline{\rho}|_\infty < +\infty$. For any $x, y \in \mathbb{X}$ such that $d(x,y) > \frac{1}{2}$, we can see that

$$|\rho(g,x) - \rho(g,y)| \leq |\rho(g,x) - \rho(g,y)|2d(x,y) \leq 8\log N(g)d(x,y). \tag{2.11}$$

For any $x, y \in \mathbb{X}$ such that $d(x,y) \leq \frac{1}{2}$, applying Lemma III.1 in Hennion and Hervé (2008), we obtain

$$|\rho(g,x) - \rho(g,y)| \leq 2\log\frac{1}{1-d(x,y)} \leq 2Cd(x,y), \tag{2.12}$$

where C is given in Lemma 2.2. For any $x, y \in \mathbb{X}$, by (2.11) and (2.12) we obtain

$$\begin{aligned} |\overline{\rho}(x) - \overline{\rho}(y)| &\leq \int_S |\rho(g,x) - \rho(g,y)|\,\mu(dg) \\ &\leq \int_S [8\log N(g) + 2C]d(x,y)\mu(dg). \end{aligned}$$

Thus $m(\overline{\rho}) = \sup_{x,y\in\mathbb{X}, x\neq y} \dfrac{|\overline{\rho}(x) - \overline{\rho}(y)|}{d(x,y)} < +\infty$.

(2) From (2.9) and definition of ρ, it is obvious that

$$\overline{P}\rho(g,x) = \int_S \rho(h, g\cdot x)\mu(dh) = \overline{\rho}(g\cdot x),$$

which yields

$$\begin{aligned} \overline{P}^2\rho(g,x) &= \overline{P}(\overline{P}\rho)(g,x) = \int_{S\times\mathbb{X}} (\overline{P}\rho)(k,y)\overline{P}((g,x), dkdy) \\ &= \int_{S\times\mathbb{X}} \overline{\rho}(k\cdot y)\overline{P}((g,x), dkdy) \\ &= \int_S \overline{\rho}(k\cdot(g\cdot x))\mu(dk) = P\overline{\rho}(g\cdot x). \end{aligned}$$

By induction, we obtain $\overline{P}^{n+1}\rho(g,x) = P^n\overline{\rho}(g\cdot x)$ for any $n \geq 0$.

□

Formally, the solution $\theta : S\times\mathbb{X} \to \mathbb{R}$ of the equation $\theta - \overline{P}\theta = \rho$ is the function

$$\theta : (g,x) \mapsto \sum_{n=0}^{+\infty} \overline{P}^n\rho(g,x).$$

Notice that we do not have any spectral property for $\overline{P}$ and ρ does not belong to L. However, we still obtain the convergence of this series by taking into account the important relation (2.10), as shown in the following lemma.

Lemma 2.5. *The sum* $\theta = \sum_{n=0}^{+\infty} \overline{P}^n\rho$ *exists and satisfies the Poisson equation* $\rho = \theta - \overline{P}\theta$*. Moreover,*

$$|\overline{P}\theta|_\infty = \sup_{g\in S, x\in\mathbb{X}} |\theta(g,x) - \rho(g,x)| < +\infty; \tag{2.13}$$

and for any $p \geq 1$*, it holds*

$$\sup_{n\geq 0}\sup_{x\in\mathbb{X}} \mathbb{E}_x|\theta(g_{n+1}, X_n)|^p < +\infty. \tag{2.14}$$

Proof. **(1)** Since P acts on $(L, |\cdot|_L)$ and $\overline{\rho} \in L$ from Lemma 2.4, we obtain $P\overline{\rho} \in L$. Thanks to definition of ρ, (1.3) and P4, it follows that

$$\nu(\overline{\rho}) = \int_{\mathbb{X}} \overline{\rho}(x)\nu(dx) = \int_S\int_{\mathbb{X}} \rho(g,x)\nu(dx)\mu(dg) = \gamma_\mu = 0.$$

Proposition 2.3 and the relation (2.10) yield for any $x \in \mathbb{X}$ and $n \geq 0$,

$$\overline{P}^{n+1}\rho(g,x) = P^n\overline{\rho}(g{\cdot}x) = \Pi\overline{\rho}(g{\cdot}x)+R^n\overline{\rho}(g{\cdot}x) = \nu(\overline{\rho})\mathbf{1}(g{\cdot}x)+R^n\overline{\rho}(g{\cdot}x) = R^n\overline{\rho}(g{\cdot}x)$$

and there exist $C > 0$ and $0 < \kappa < 1$ such that for any $x \in \mathbb{X}$ and $n \geq 0$,

$$|R^n\overline{\rho}(x)| \leq |R^n\overline{\rho}|_L \leq |R^n|_{L\to L} \leq C\kappa^n.$$

Hence for any $g \in S$ and $x \in \mathbb{X}$,

$$\left|\sum_{n=1}^{+\infty}\overline{P}^n\rho(g,x)\right| \leq \sum_{n=0}^{+\infty}|P^n\overline{\rho}(g\cdot x)| \leq C\sum_{n=0}^{+\infty}\kappa^n = \frac{C}{1-\kappa} < +\infty.$$

Therefore, the function $\theta = \sum_{n=0}^{+\infty}\overline{P}^n\rho$ exists and obviously satisfies the Poisson equation $\rho = \theta - \overline{P}\theta$. Finally, it is evident that

$$\sup_{g\in S, x\in\mathbb{X}}|\theta(g,x)-\rho(g,x)| = \sup_{g\in S, x\in\mathbb{X}}\left|\sum_{n=1}^{+\infty}\overline{P}^n\rho(g,x)\right| < +\infty.$$

(2) Indeed, from (2.6), (2.13) and Minkowski's inequality, the assertion arrives. □

Now we contruct a martingale to approximate the Markov walk $(S_n)_{n\geq0}$. Hence, from the definition of S_n and the Poisson equation, by adding and removing the term $\overline{P}\theta(g_0, X_{-1})$, we obtain

$$\begin{aligned} S_n &= a + \rho(g_1, X_0) + \ldots + \rho(g_n, X_{n-1}) \\ &= a + \overline{P}\theta(g_0, X_{-1}) - \overline{P}\theta(g_n, X_{n-1}) + \sum_{k=0}^{n-1}\left[\theta(g_{k+1}, X_k) - \overline{P}\theta(g_k, X_{k-1})\right]. \end{aligned}$$

Let $\mathcal{F}_0 := \{\emptyset, \Omega\}$ and $\mathcal{F}_n := \sigma\{g_k : 0 \leq k \leq n\}$ for $n \geq 1$.

Proposition 2.6. *For any $n \geq 0$, $x \in \mathbb{X}$, $a \geq 0$ and $p > 2$, the sequence $(M_n)_{n\geq0}$ defined by*

$$M_0 := S_0 \text{ and } M_n := M_0 + \sum_{k=0}^{n-1}\left[\theta(g_{k+1}, X_k) - \overline{P}\theta(g_k, X_{k-1})\right] \tag{2.15}$$

is a martingale in $L^p(\Omega, \mathbb{P}_{x,a}, (\mathcal{F}_n)_{n\geq0})$ satisfying the properties:

$$\sup_{n\geq0}|S_n - M_n| \leq 2|\overline{P}\theta|_\infty \quad \mathbb{P}_{x,a}\text{-a.s.} \tag{2.16}$$

$$\sup_{n\geq1} n^{-\frac{p}{2}}\sup_{x\in\mathbb{X}}\mathbb{E}_{x,a}|M_n|^p < +\infty. \tag{2.17}$$

From now on, we set $A := 2|\overline{P}\theta|_\infty$.

Proof. By definition (2.15), martingale property arrives.

(1) From the construction of M_n and (2.13), we can see easily that

$$\sup_{n\geq0}|S_n - M_n| = \sup_{n\geq0}\left|\overline{P}\theta(g_0, X_{-1}) - \overline{P}\theta(g_n, X_{n-1})\right| \leq 2\left|\overline{P}\theta\right|_\infty < +\infty \quad \mathbb{P}_{x,a}\text{-a.s.}.$$

(2) Denote $\xi_k := \theta(g_{k+1}, X_k) - \overline{P}\theta(g_k, X_{k-1})$. Thus $M_n = M_0 + \sum_{k=0}^{n-1} \xi_k$. Using Burkholder's inequality, for any $p \geq 1$, there exists some positive constant c_p such that for $0 \leq k < n$,

$$(\mathbb{E}_{x,a}|M_n|^p)^{\frac{1}{p}} \leq c_p \left(\mathbb{E}_{x,a} \left| \sum_{k=0}^{n-1} \xi_k^2 \right|^{\frac{p}{2}} \right)^{\frac{1}{p}}.$$

Now, with $p > 2$, applying Holder's inequality, we obtain

$$\left| \sum_{k=0}^{n-1} \xi_k^2 \right| \leq n^{1-\frac{2}{p}} \left(\sum_{k=0}^{n-1} |\xi_k|^p \right)^{\frac{2}{p}},$$

which implies

$$\mathbb{E}_{x,a} \left| \sum_{k=0}^{n-1} \xi_k^2 \right|^{\frac{p}{2}} \leq n^{\frac{p}{2}-1} \mathbb{E}_{x,a} \sum_{k=0}^{n-1} |\xi_k|^p \leq n^{\frac{p}{2}} \sup_{0 \leq k \leq n-1} \mathbb{E}_{x,a} |\xi_k|^p.$$

Since $(M_n)_n$ is a martingale, by using the convexity property, we can see that for any $k \geq 0$,

$$\left| \overline{P}\theta(g_k, X_{k-1}) \right|^p = \left| \mathbb{E}_{x,a} \Big[|\theta(g_{k+1}, X_k)| | \mathcal{F}_k \Big] \right|^p \leq \mathbb{E}_{x,a} \Big[|\theta(g_{k+1}, X_k)|^p | \mathcal{F}_k \Big],$$

which implies $\mathbb{E}_{x,a} \left| \overline{P}\theta(g_k, X_{k-1}) \right|^p \leq \mathbb{E}_{x,a} \left| \theta(g_{k+1}, X_k) \right|^p$. Therefore, we obtain

$$\begin{aligned}
\Big(\mathbb{E}_{x,a}|M_n|^p \Big)^{\frac{1}{p}} &\leq c_p \left(n^{\frac{p}{2}} \sup_{0 \leq k \leq n-1} \mathbb{E}_{x,a}|\xi_k|^p \right)^{\frac{1}{p}} \leq c_p n^{\frac{1}{2}} \sup_{0 \leq k \leq n-1} \Big(\mathbb{E}_{x,a}|\xi_k|^p \Big)^{\frac{1}{p}} \\
&\leq c_p n^{\frac{1}{2}} \sup_{0 \leq k \leq n-1} \left[\Big(\mathbb{E}_{x,a}|\theta(g_{k+1}, X_k)|^p \Big)^{1/p} + \Big(\mathbb{E}_{x,a}|\overline{P}\theta(g_k, X_{k-1})|^p \Big)^{1/p} \right] \\
&\leq 2c_p n^{\frac{1}{2}} \sup_{0 \leq k \leq n-1} \Big(\mathbb{E}_{x,a}|\theta(g_{k+1}, X_k)|^p \Big)^{\frac{1}{p}}.
\end{aligned}$$

Consequently, we obtain $\mathbb{E}_{x,a}|M_n|^p \leq (2c_p)^p n^{\frac{p}{2}} \sup_{0 \leq k \leq n-1} \mathbb{E}_{x,a}|\theta(g_{k+1}, X_k)|^p$ and the assertion arrives by using (2.14).

□

3. On the $\widetilde{P}_+$-harmonic function V and the proof of Proposition 1.1

In this section we construct explicitly a $\widetilde{P}_+$-harmonic function V and study its properties. We begin with the first time the martingale $(M_n)_{n \geq 0}$ (2.15) visit $]-\infty, 0]$, defined by

$$T = \min\{n \geq 1 : M_n \leq 0\}.$$

The equality $\gamma_\mu = 0$ yields $\liminf_{n \to +\infty} S_n = -\infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{X}$, thus $\liminf_{n \to +\infty} M_n = -\infty$ $\mathbb{P}_x$-a.s., so that $T < +\infty$ $\mathbb{P}_x$-a.s. for any $x \in \mathbb{X}$ and $a \geq 0$.

3.1. *On the properties of T and $(M_n)_n$.* We need to control the first moment of the random variable $|M_{T\wedge n}|$ under $\mathbb{P}_x$; we consider the restriction of this variable to the event $[T \le n]$ in Lemma 3.1 and control the remaining term in Lemma 3.4.

Lemma 3.1. *There exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\varepsilon \in (0, \varepsilon_0), n \ge 1, x \in \mathbb{X}$ and $a \ge n^{\frac{1}{2}-\varepsilon}$,*

$$\mathbb{E}_{x,a}\Big[|M_T|; T \le n\Big] \le c\frac{a}{n^\varepsilon}.$$

Proof. For any $\varepsilon > 0$, consider the event $A_n := \{\max_{0\le k\le n-1} |\xi_k| \le n^{\frac{1}{2}-2\varepsilon}\}$, where $\xi_k = \theta(g_{k+1}, X_k) - \overline{P}\theta(g_k, X_{k-1})$; then

$$\mathbb{E}_{x,a}\Big[|M_T|; T \le n\Big] = \mathbb{E}_{x,a}\Big[|M_T|; T \le n, A_n\Big] + \mathbb{E}_{x,a}\Big[|M_T|; T \le n, A_n^c\Big]. \tag{3.1}$$

On the event $[T \le n] \cap A_n$, we have $|M_T| \le |\xi_{T-1}| \le n^{\frac{1}{2}-2\varepsilon}$. Hence for any $x \in \mathbb{X}$ and $a \ge n^{\frac{1}{2}-\varepsilon}$,

$$\mathbb{E}_{x,a}\Big[|M_T|; T \le n, A_n\Big] \le \mathbb{E}_{x,a}\Big[|\xi_{T-1}|; T \le n, A_n\Big] \le n^{\frac{1}{2}-2\varepsilon} \le \frac{a}{n^\varepsilon}. \tag{3.2}$$

Let $M_n^* := \max_{1\le k\le n} |M_k|$; since $|M_T| \le M_n^*$ on the event $[T \le n]$, it is clear that, for any $x \in \mathbb{X}$ and $a \ge 0$,

$$\begin{aligned}
\mathbb{E}_{x,a}\Big[|M_T|; T \le n, A_n^c\Big] &\le \mathbb{E}_x[M_n^*; A_n^c]\\
&\le \mathbb{E}_{x,a}\Big[M_n^*; M_n^* > n^{\frac{1}{2}+2\varepsilon}, A_n^c\Big] + n^{\frac{1}{2}+2\varepsilon}\mathbb{P}_{x,a}(A_n^c)\\
&\le \int_{n^{\frac{1}{2}+2\varepsilon}}^{+\infty} \mathbb{P}_{x,a}(M_n^* > t)dt + 2n^{\frac{1}{2}+2\varepsilon}\mathbb{P}_{x,a}(A_n^c). \qquad (3.3)
\end{aligned}$$

We bound the probability $\mathbb{P}_{x,a}(A_n^c)$ by using Markov's inequality, martingale definition and (2.14) as follows: for any $p \ge 1$,

$$\begin{aligned}
\mathbb{P}_{x,a}(A_n^c) &\le \sum_{k=0}^{n-1} \mathbb{P}_{x,a}\Big(|\xi_k| > n^{\frac{1}{2}-2\varepsilon}\Big)\\
&\le \frac{1}{n^{(\frac{1}{2}-2\varepsilon)p}} \sum_{k=0}^{n-1} \mathbb{E}_{x,a}|\xi_k|^p\\
&\le \frac{2^p}{n^{(\frac{1}{2}-2\varepsilon)p}} \sum_{k=0}^{n-1} \mathbb{E}_{x,a}|\theta(g_{k+1}, X_k)|^p\\
&= \frac{c_p}{n^{\frac{p}{2}-1-2\varepsilon p}}.
\end{aligned}$$

For any $a \ge n^{\frac{1}{2}-\varepsilon}$, it follows that

$$n^{\frac{1}{2}+2\varepsilon}\mathbb{P}_{x,a}(A_n^c) \le an^{3\varepsilon}\mathbb{P}_{x,a}(A_n^c) \le \frac{c_p a}{n^{\frac{p}{2}-1-2\varepsilon p-3\varepsilon}}. \tag{3.4}$$

Now we control the integral in (3.3). Using Doob's maximal inequality for martingales and (2.17), we receive for any $p \ge 1$,

$$\mathbb{P}_x(M_n^* > t) \le \frac{1}{t^p}\mathbb{E}_x\Big[|M_n|^p\Big] \le c_p\frac{n^{\frac{p}{2}}}{t^p},$$

which implies for any $a \geq n^{\frac{1}{2}-\varepsilon}$,

$$\int_{n^{\frac{1}{2}+2\varepsilon}}^{+\infty} \mathbb{P}_x(M_n^* > t)dt \quad \leq \quad \frac{c_p}{p-1} \frac{n^{\frac{p}{2}}}{n^{(\frac{1}{2}+2\varepsilon)(p-1)}} \leq \frac{c_p}{p-1} \frac{a}{n^{2\varepsilon p-3\varepsilon}}. \tag{3.5}$$

Taking (3.3), (3.4) and (3.5) altogether, we obtain for some c'_p,

$$\mathbb{E}_{x,a}\Big[|M_T|; T \leq n, A_n^c\Big] \leq c'_p \left(\frac{a}{n^{2\varepsilon p-3\varepsilon}} + \frac{a}{n^{\frac{p}{2}-1-2\varepsilon p-3\varepsilon}} \right). \tag{3.6}$$

Finally, from (3.1), (3.2) and (3.6), we obtain for any $a \geq n^{\frac{1}{2}-\varepsilon}$,

$$\mathbb{E}_{x,a}\Big[|M_T|; T \leq n\Big] \leq \frac{a}{n^\varepsilon} + c'_p \frac{a}{n^\varepsilon} \left(\frac{1}{n^{2\varepsilon p-4\varepsilon}} + \frac{1}{n^{\frac{p}{2}-1-2\varepsilon p-4\varepsilon}} \right).$$

Fix $p > 2$. Then there exist $c > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $a \geq n^{\frac{1}{2}-\varepsilon}$,

$$\mathbb{E}_{x,a}\Big[|M_T|, T \leq n\Big] \leq c\frac{a}{n^\varepsilon}$$

which proves the lemma.

□

For fixed $\varepsilon > 0$ and $a \geq 0$, we consider the first time $\nu_{n,\varepsilon}$ when the process $(|M_k|)_{k\geq 1}$ exceeds $2n^{\frac{1}{2}-\varepsilon}$. It is connected to Lemma 4.3 where $\mathbb{P}(\tau_a^{bm} > n)$ is controlled uniformly in a under condition $a \leq \theta_n \sqrt{n}$ with $\lim_{n\to+\infty} \theta_n = 0$ which we take into account here by setting

$$\nu_{n,\varepsilon} := \min\{k \geq 1 : |M_k| \geq 2n^{\frac{1}{2}-\varepsilon}\}.$$

Notice first that for any $\varepsilon > 0, x \in \mathbb{X}$ and $a \geq 0$ the sequence $(\nu_{n,\varepsilon})_{n\geq 1}$ tends to $+\infty$ a.s. on $(\Omega, \mathcal{B}(\Omega), \mathbb{P}_{x,a})$. The following lemma yields to a more precise control of this property.

Lemma 3.2. *For any $\varepsilon \in (0, \frac{1}{2})$, there exists $c_\varepsilon > 0$ such that for any $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$,*

$$\mathbb{P}_{x,a}(\nu_{n,\varepsilon} > n^{1-\varepsilon}) \leq \exp(-c_\varepsilon n^\varepsilon).$$

Proof. Let $m = [B^2 n^{1-2\varepsilon}]$ and $K = [n^\varepsilon/B^2]$ for some positive constant B. By (2.16), for n sufficiently great such that $A \leq n^{\frac{1}{2}-\varepsilon}$, we obtain for any $x \in \mathbb{X}$ and $a \geq 0$,

$$\begin{aligned} \mathbb{P}_{x,a}(\nu_{n,\varepsilon} > n^{1-\varepsilon}) \quad &\leq \quad \mathbb{P}_{x,a}\left(\max_{1\leq k\leq n^{1-\varepsilon}} |M_k| \leq 2n^{\frac{1}{2}-\varepsilon} \right) \\ &\leq \quad \mathbb{P}_{x,a}\left(\max_{1\leq k\leq K} |M_{km}| \leq 2n^{\frac{1}{2}-\varepsilon} \right) \\ &\leq \quad \mathbb{P}_{x,a}\left(\max_{1\leq k\leq K} |S_{km}| \leq 3n^{\frac{1}{2}-\varepsilon} \right). \end{aligned} \tag{3.7}$$

Using Markov property, it follows that, for any $x \in \mathbb{X}$ and $a \geq 0$, from which by iterating K times, we obtain

$$\mathbb{P}_{x,a}\left(\max_{1\leq k\leq K} |S_{km}| \leq 3n^{\frac{1}{2}-\varepsilon} \right) \leq \left(\sup_{b\in\mathbb{R}, x\in\mathbb{X}} \mathbb{P}_{x,b}\left(|S_m| \leq 3n^{\frac{1}{2}-\varepsilon} \right) \right)^K. \tag{3.8}$$

Denote $\mathbb{B}(b; r) = \{c : |b + c| \leq r\}$. Then for any $x \in \mathbb{X}$ and $b \in \mathbb{R}$

$$\mathbb{P}_{x,b}\left(|S_m| \leq 3n^{\frac{1}{2}-\varepsilon} \right) = \mathbb{P}_x\left(\frac{S_m}{\sqrt{m}} \in \mathbb{B}(b/\sqrt{m}; r_n) \right),$$

where $r_n = \frac{3n^{\frac{1}{2}-\varepsilon}}{\sqrt{m}}$. Using the central limit theorem for S_n (Theorem 5.1 property iii) Bougerol and Lacroix (1985)), we obtain for $n \to +\infty$,

$$\sup_{b\in\mathbb{R},x\in\mathbb{X}} \left| \mathbb{P}_x \left(\frac{S_m}{\sqrt{m}} \in \mathbb{B}(b/\sqrt{m}; r_n) \right) - \int_{\mathbb{B}(b/\sqrt{m};r_n)} \phi_{\sigma^2}(u)du \right| \to 0,$$

where $\phi_{\sigma^2}(t) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\sigma^2}\right)$ is the normal density of mean 0 and variance σ^2 on $\mathbb{R}$. Since $r_n \le c_1 B^{-1}$ for some constant $c_1 > 0$, we obtain

$$\sup_{b\in\mathbb{R}} \int_{\mathbb{B}(b/\sqrt{m};r_n)} \phi_{\sigma^2}(u)du \le \int_{-r_n}^{r_n} \phi_{\sigma^2}(u)du \le \frac{2r_n}{\sigma\sqrt{2\pi}} \le \frac{2c_1}{B\sigma\sqrt{2\pi}}.$$

Choosing B and n great enough, for some $q_\varepsilon < 1$, we obtain

$$\sup_{b\in\mathbb{R},x\in\mathbb{X}} \mathbb{P}_{x,b}\left(|S_m| \le 3n^{\frac{1}{2}-\varepsilon}\right) \le \sup_{b\in\mathbb{R}} \int_{\mathbb{B}(b/\sqrt{m};r_n)} \phi_{\sigma^2}(u)du + o(1) \le q_\varepsilon.$$

Implementing this bound in (3.8) and using (3.7), it follows that for some $c_\varepsilon > 0$,

$$\sup_{a>0,x\in\mathbb{X}} \mathbb{P}_{x,a}(\nu_{n,\varepsilon} > n^{1-\varepsilon}) \le q_\varepsilon^K \le q_\varepsilon^{\frac{n^\varepsilon}{B^2}-1} \le e^{-c_\varepsilon n^\varepsilon}.$$

□

Lemma 3.3. *There exists $c > 0$ such that for any $\varepsilon \in (0, \frac{1}{2})$, $x \in \mathbb{X}$, $a \ge 0$ and $n \ge 1$,*

$$\sup_{1\le k\le n} \mathbb{E}_{x,a}[|M_k|; \nu_{n,\varepsilon} > n^{1-\varepsilon}] \le c(1+a)\exp(-c_\varepsilon n^\varepsilon)$$

for some positive constant c_ε which only depends on ε.

Proof. By Cauchy-Schwartz inequality, for any $x \in \mathbb{X}$, $a \ge 0$ and $1 \le k \le n$,

$$\mathbb{E}_{x,a}\left[|M_k|; \nu_{n,\varepsilon} > n^{1-\varepsilon}\right] \le \sqrt{\mathbb{E}_{x,a}|M_k|^2 \mathbb{P}_{x,a}(\nu_{n,\varepsilon} > n^{1-\varepsilon})}.$$

By Minkowsky's inequality, (2.16) and the fact that $\frac{1}{n}\mathbb{E}_x|S_n|^2 \to \sigma^2$ as $n \to +\infty$, it yields

$$\sqrt{\mathbb{E}_{x,a}|M_k|^2} \le a + \sqrt{\mathbb{E}_{x,a}[M_k^2]} \le a + A + \sqrt{\mathbb{E}_{x,a}[S_k^2]} \le c(a + n^{\frac{1}{2}})$$

for some $c > 0$ which does not depend on x. The claim follows by Lemma 3.2.

□

Lemma 3.4. *There exists $c > 0$ and $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$, $a \ge 0$ and $n \ge 1$,*

$$\mathbb{E}_{x,a}[M_n; T > n] \le c(1+a). \tag{3.9}$$

and

$$\lim_{a\to+\infty} \frac{1}{a} \lim_{n\to+\infty} \mathbb{E}_{x,a}[M_n; T > n] = 1. \tag{3.10}$$

Proof. **(1)** On one hand, we claim

$$\mathbb{E}_{x,a}[M_n; T > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}] \quad \le \quad \left(1 + \frac{c'_\varepsilon}{n^\varepsilon}\right) \mathbb{E}_{x,a}\left[M_{[n^{1-\varepsilon}]}; T > [n^{1-\varepsilon}]\right] \tag{3.11}$$

and delay the proof of (3.11) at the end of the first part. On the other hand, by Lemma 3.3, there exists $c > 0$ such that for any $\varepsilon \in (0, \frac{1}{2})$, $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$,

$$\begin{aligned}\mathbb{E}_{x,a}[M_n; T > n, \nu_{n,\varepsilon} > n^{1-\varepsilon}] &\leq \sup_{1\leq k\leq n} \mathbb{E}_{x,a}\left[|M_k|; \nu_{n,\varepsilon} > n^{1-\varepsilon}\right] \\ &\leq c(1+a)\exp(-c_\varepsilon n^\varepsilon). \end{aligned} \tag{3.12}$$

Hence combining (3.11) and (3.12), we obtain for any $x \in \mathbb{X}$ and $a \geq 0$,

$$\mathbb{E}_{x,a}[M_n; T > n] \leq \left(1 + \frac{c'_\varepsilon}{n^\varepsilon}\right) \mathbb{E}_{x,a}\left[M_{[n^{1-\varepsilon}]}; T > [n^{1-\varepsilon}]\right] + c(1+a)\exp(-c_\varepsilon n^\varepsilon). \tag{3.13}$$

Let $k_j := \left[n^{(1-\varepsilon)^j}\right]$ for $j \geq 0$. Notice that $k_0 = n$ and $[k_j^{1-\varepsilon}] \leq k_{j+1}$ for any $j \geq 0$. Since the sequence $((M_n)\mathbf{1}_{[T>n]})_{n\geq 1}$ is a submartingale, by using the bound (3.13), it yields

$$\begin{aligned}\mathbb{E}_{x,a}[M_{k_1}; T > k_1] &\leq \left(1 + \frac{c'_\varepsilon}{{k_1}^\varepsilon}\right) \mathbb{E}_{x,a}\left[M_{[{k_1}^{1-\varepsilon}]}; T > [{k_1}^{1-\varepsilon}]\right] + c(1+a)\exp(-c_\varepsilon k_1^\varepsilon) \\ &\leq \left(1 + \frac{c'_\varepsilon}{{k_1}^\varepsilon}\right) \mathbb{E}_{x,a}[M_{k_2}; T > k_2] + c(1+a)\exp(-c_\varepsilon k_1^\varepsilon).\end{aligned}$$

Let n_0 be a constant and $m = m(n)$ such that $k_m = \left[n^{(1-\varepsilon)^m}\right] \leq n_0$. After m iterations, we obtain

$$\mathbb{E}_{x,a}[M_n; T > n] \leq A_m\Big(\mathbb{E}_{x,a}[M_{k_m}; T > k_m] + c(1+a)B_m\Big), \tag{3.14}$$

where

$$A_m = \prod_{j=1}^m \left(1 + \frac{c'_\varepsilon}{k_{j-1}^\varepsilon}\right) \leq \exp\Big(2^\varepsilon c'_\varepsilon \frac{n_0^{-\varepsilon}}{1 - n_0^{-\varepsilon^2}}\Big), \tag{3.15}$$

and

$$B_m = \sum_{j=1}^m \frac{\exp\left(-c_\varepsilon k_{j-1}^\varepsilon\right)}{\left(1 + \frac{c'_\varepsilon}{k_{j-1}^\varepsilon}\right)\dots\left(1 + \frac{c'_\varepsilon}{k_m^\varepsilon}\right)} \leq c_1 \frac{n_0^{-\varepsilon}}{1 - n_0^{-\varepsilon^2}} \tag{3.16}$$

from Lemma 5.6 in Grama et al. (2014). By choosing n_0 sufficient great, the first assertion of the lemma follows from (3.14), (3.15) and (3.16) taking into account that

$$\mathbb{E}_{x,a}[M_{k_m}; T > k_m] \leq \mathbb{E}_{x,a}[M_{n_0}; T > n_0] \leq \mathbb{E}_{x,a}|M_{n_0}| \leq a + c.$$

Before proving (3.11), we can see that there exist $c > 0$ and $0 < \varepsilon_0 < \frac{1}{2}$ such that for any $\varepsilon \in (0, \varepsilon_0)$, $x \in \mathbb{X}$ and $b \geq n^{\frac{1}{2}-\varepsilon}$,

$$\mathbb{E}_{x,b}[M_n; T > n] \leq \Big(1 + \frac{c}{n^\varepsilon}\Big)\, b. \tag{3.17}$$

Indeed, since $(M_n, \mathcal{F}_n)_{n\geq 1}$ is a $\mathbb{P}_{x,b}$- martingale, we obtain

$$\mathbb{E}_{x,b}[M_n; T \leq n] = \mathbb{E}_{x,b}[M_T; T \leq n]$$

and thus

$$\begin{aligned}\mathbb{E}_{x,b}[M_n; T > n] &= \mathbb{E}_{x,b}[M_n] - \mathbb{E}_{x,a}[M_n; T \leq n] \\ &= b - \mathbb{E}_{x,b}[M_T; T \leq n] \\ &= b + \mathbb{E}_{x,b}[|M_T|; T \leq n].\end{aligned} \tag{3.18}$$

Hence (3.17) arrives by using Lemma 3.1. For (3.11), it is obvious that

$$\mathbb{E}_{x,a}\Big[M_n; T > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\Big] = \sum_{k=1}^{[n^{1-\varepsilon}]} \mathbb{E}_{x,a}\Big[M_n; T > n, \nu_{n,\varepsilon} = k\Big]. \tag{3.19}$$

Denote $U_m(x,a) := \mathbb{E}_{x,a}[M_m; T > m]$. For any $m \ge 1$, by the Markov property applied to $(X_n)_{n\ge 1}$, it follows that

$$\begin{aligned}\mathbb{E}_{x,a}\Big[M_n; T > n, \nu_{n,\varepsilon} = k\Big] &= \int \mathbb{E}_{y,b}[M_{n-k}; T > n-k] \\ &\qquad \mathbb{P}_{x,a}(X_k \in dy, M_k \in db; T > k, \nu_{n,\varepsilon} = k) \\ &= \mathbb{E}_{x,a}\Big[U_{n-k}(X_k, M_k); T > k, \nu_{n,\varepsilon} = k\Big]. \end{aligned} \tag{3.20}$$

From the definition of $\nu_{n,\varepsilon}$, we can see that $[\nu_{n,\varepsilon} = k] \subset \Big[|M_k| \ge n^{\frac{1}{2}-\varepsilon}\Big]$, and by using (3.17), on the event $[T > k, \nu_{n,\varepsilon} = k]$ we have $U_{n-k}(X_k, M_k) \le \Big(1 + \frac{c}{(n-k)^\varepsilon}\Big)M_k$. Therefore (3.20) becomes

$$\mathbb{E}_{x,a}[M_n; T > n, \nu_{n,\varepsilon} = k] \le \left(1 + \frac{c}{(n-k)^\varepsilon}\right)\mathbb{E}_{x,a}[M_k; T > k, \nu_{n,\varepsilon} = k]. \tag{3.21}$$

Combining (3.19) and (3.21), it follows that, for n sufficiently great,

$$\begin{aligned}\mathbb{E}_{x,a}[M_n; T > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}] &\le \sum_{k=1}^{[n^{1-\varepsilon}]}\left(1 + \frac{c}{(n-k)^\varepsilon}\right)\mathbb{E}_{x,a}[M_k; T > k, \nu_{n,\varepsilon} = k] \\ &\le \left(1 + \frac{c'_\varepsilon}{n^\varepsilon}\right)\sum_{k=1}^{[n^{1-\varepsilon}]}\mathbb{E}_{x,a}[M_k; T > k, \nu_{n,\varepsilon} = k],\end{aligned}$$

for some constant $c'_\varepsilon > 0$. Since $(M_n \mathbf{1}_{[T>n]})_{n\ge1}$ is a submartingale, for any $x \in \mathbb{X}$, $a \ge 0$ and $1 \le k \le [n^{1-\varepsilon}]$,

$$\mathbb{E}_{x,a}[M_k; T > k, \nu_{n,\varepsilon} = k] \le \mathbb{E}_{x,a}\Big[M_{[n^{1-\varepsilon}]}; T > [n^{1-\varepsilon}], \nu_{n,\varepsilon} = k\Big].$$

This implies

$$\begin{aligned}\mathbb{E}_{x,a}[M_n; T > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}] &\le \left(1 + \frac{c'_\varepsilon}{n^\varepsilon}\right)\sum_{k=1}^{[n^{1-\varepsilon}]}\mathbb{E}_{x,a}\left[M_{[n^{1-\varepsilon}]}; T > [n^{1-\varepsilon}], \nu_{n,\varepsilon} = k\right] \\ &\le \left(1 + \frac{c'_\varepsilon}{n^\varepsilon}\right)\mathbb{E}_{x,a}\left[M_{[n^{1-\varepsilon}]}; T > [n^{1-\varepsilon}]\right].\end{aligned}$$

(2) Let $\delta > 0$. From (3.15) and (3.16), by choosing n_0 sufficiently great, we obtain $A_m \le 1 + \delta$ and $B_m \le \delta$. Together with (3.14), since $(M_n \mathbf{1}_{[T>n]})_{n\ge1}$ is a submartingale, we obtain for $k_m \le n_0$,

$$\mathbb{E}_{x,a}[M_n; T > n] \le (1+\delta)\Big(\mathbb{E}_{x,a}[M_{n_0}; T > n_0] + c(1+a)\delta\Big).$$

Moreover, the sequence $\mathbb{E}_{x,a}[M_n; T > n]$ is increasing, thus it converges $\mathbb{P}_{x,a}$-a.s. and

$$\lim_{n\to+\infty}\mathbb{E}_{x,a}[M_n; T > n] \le (1+\delta)\Big(\mathbb{E}_{x,a}[M_{n_0}; T > n_0] + c(1+a)\delta\Big).$$

By using (3.18), we obtain

$$a \leq \lim_{n\to+\infty} \mathbb{E}_{x,a}[M_n; T > n] \leq (1+\delta)\Big(a + \mathbb{E}_x|M_{n_0}| + c(1+a)\delta\Big).$$

Hence the assertion follows since $\delta > 0$ is arbitrary.

□

3.2. *On the stopping time τ.* We now state some useful properties of τ and S_τ.

Lemma 3.5. *There exists $c > 0$ such that for any $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$,*

$$\mathbb{E}_{x,a}[S_n, \tau > n] \leq c(1+a).$$

Proof. (2.16) yields $\mathbb{P}_x(\tau_a \leq T_{a+A}) = 1$ and $A + M_n \geq S_n > 0$ on the event $[\tau > n]$. By (3.9), it follows that

$$\begin{aligned}
\mathbb{E}_{x,a}[S_n; \tau > n] &\leq \mathbb{E}_{x,a}[A + M_n; \tau > n] \\
&\leq \mathbb{E}_{x,a+A}[M_n; T > n] \\
&\leq c_1(1 + a + A) \leq c_2(1+a).
\end{aligned}$$

□

Proposition 3.6. *There exists $c > 0$ such that for any $x \in \mathbb{X}$ and $a \geq 0$,*

$$\mathbb{E}_{x,a}|S_\tau| \leq c(1+a) < +\infty$$

and

$$\mathbb{E}_{x,a}|M_\tau| \leq c(1+a) < +\infty. \tag{3.22}$$

Proof. By (2.16), since $(M_n)_n$ is a martingale, we can see that

$$\begin{aligned}
-\mathbb{E}_{x,a}[S_\tau; \tau \leq n] &\leq -\mathbb{E}_{x,a}[M_\tau; \tau \leq n] + A \\
&= \mathbb{E}_{x,a}[M_n; \tau > n] - \mathbb{E}_{x,a}[M_n] + A \\
&\leq \mathbb{E}_{x,a}[S_n; \tau > n] + 2A.
\end{aligned}$$

Hence by Lemma 3.5, for any $x \in \mathbb{X}$ and $a \geq 0$,

$$\begin{aligned}
\mathbb{E}_{x,a}\left[|S_\tau|; \tau \leq n\right] &\leq \mathbb{E}_{x,a}\left|S_{\tau\wedge n}\right| \\
&= \mathbb{E}_{x,a}\left[S_n; \tau > n\right] - \mathbb{E}_{x,a}\left[S_\tau; \tau \leq n\right] \\
&\leq 2\mathbb{E}_{x,a}[S_n; \tau > n] + 2A \\
&\leq c(1+a) + 2A.
\end{aligned}$$

By Lebesgue's Dominated Convergence Theorem, it yields

$$\mathbb{E}_{x,a}|S_\tau| = \lim_{n\to+\infty} \mathbb{E}_{x,a}\left[|S_\tau|; \tau \leq n\right] \leq c(1+a) + 2A < +\infty.$$

By (2.16), the second assertion arrives.

□

3.3. *Proof of Proposition 1.1.*
Denote $\tau_a := \min\{n \geq 1 : S_n \leq -a\}$ and $T_a := \min\{n \geq 1 : M_n \leq -a\}$ for any $a \geq 0$. Then $\mathbb{E}_{x,a} M_\tau = a + \mathbb{E}_x M_{\tau_a}$ and $\mathbb{P}_{x,a}(\tau > n) = \mathbb{P}_x(\tau_a > n)$.

(1) By (3.22) and Lebesgue's Dominated Convergence Theorem, for any $x \in \mathbb{X}$ and $a \geq 0$,

$$\lim_{n \to +\infty} \mathbb{E}_{x,a}[M_\tau; \tau \leq n] = \mathbb{E}_{x,a} M_\tau = a - V(x,a),$$

where $V(x,a)$ is the quantity defined by: for $x \in \mathbb{X}$ and $a \in \mathbb{R}$,

$$V(x,a) := \begin{cases} -\mathbb{E}_x M_{\tau_a} & \text{if} \quad a \geq 0, \\ 0 & \text{if} \quad a < 0. \end{cases}$$

Since $(M_n, \mathcal{F}_n)_{n \geq 1}$ is a $\mathbb{P}_{x,a}$-martingale,

$$\mathbb{E}_{x,a}[M_n; \tau > n] \quad = \quad \mathbb{E}_{x,a} M_n - \mathbb{E}_{x,a}[M_n; \tau \leq n] = a - \mathbb{E}_{x,a}[M_\tau; \tau \leq n], \tag{3.23}$$

which implies

$$\lim_{n \to +\infty} \mathbb{E}_{x,a}[M_n; \tau > n] = V(x,a).$$

Since $|S_n - M_n| \leq A$ $\mathbb{P}_x$-a.s. and $\lim\limits_{n \to +\infty} \mathbb{P}_{x,a}(\tau > n) = 0$, it follows that

$$\lim_{n \to +\infty} \mathbb{E}_{x,a}[S_n; \tau > n] = \lim_{n \to +\infty} \mathbb{E}_{x,a}[M_n; \tau > n] = V(x,a).$$

(2) The assertion arrives by taking into account that $0 \leq a \leq a'$ implies $\tau_a \leq \tau_{a'}$ and

$$\mathbb{E}_x[a + S_n; \tau_a > n] \leq \mathbb{E}_x[a' + S_n; \tau_{a'} > n].$$

(3) Lemma 3.5 and assertion 1 imply that $V(x,a) \leq c(1+a)$ for any $x \in \mathbb{X}$ and $a \geq 0$. Besides, (3.23) and (2.16) yield

$$\mathbb{E}_{x,a}[M_n; \tau > n] \geq a - \mathbb{E}_{x,a}[S_\tau; \tau \leq n] - A \geq a - A,$$

which implies

$$V(x,a) \geq a - A. \tag{3.24}$$

Now we prove $V(x,a) \geq 0$. Assertion 2 implies $V(x,0) \leq V(x,a)$ for any $x \in \mathbb{X}$ and $a \geq 0$. From P5, let $E_\delta := \{g \in S : \forall x \in \mathbb{X}, \log |gx| \geq \delta\}$ and choose a positive constant k such that $k\delta > 2A$. Hence, for any $g_1, \ldots, g_k \in E_\delta$ and any $x \in \mathbb{X}$, we obtain $\log |g_k \ldots g_1 x| \geq k\delta > 2A$. It yields

$$\begin{aligned} V(x,0) &= \lim_{n \to +\infty} \mathbb{E}_x[S_n; \tau > n] \\ &\geq \liminf_{n \to +\infty} \int_{E_\delta} \cdots \int_{E_\delta} \mathbb{E}_{g_k \ldots g_1 \cdot x, \log |g_k \ldots g_1 x|}[S_{n-k}; \tau > n-k] \mu(dg_1) \ldots \mu(dg_k) \\ &\geq \liminf_{n \to +\infty} \int_{E_\delta} \cdots \int_{E_\delta} V(g_k \ldots g_1 \cdot x, 2A) \mu(dg_1) \ldots \mu(dg_k) \\ &\geq A\Big(\mu(E_\delta)\Big)^k > 0, \end{aligned}$$

where the last inequality comes from (3.24) by applying to $a = 2A$.

(4) Equation (3.24) yields $\lim\limits_{a \to +\infty} \dfrac{V(x,a)}{a} \geq 1$. By (2.16), it yields that $\mathbb{P}_x(\tau_a < T_{A+a}) = 1$, which implies

$$\begin{aligned} \mathbb{E}_{x,a}[S_n; \tau > n] &\leq \mathbb{E}_{x,a}[A + M_n; \tau > n] \\ &\leq \mathbb{E}_{x,a}[A + M_n; T_A > n] = \mathbb{E}_{x,a+A}[M_n; T > n]. \end{aligned}$$

From (3.10), we obtain $\lim_{n\to+\infty} \dfrac{V(x,a)}{a} \leq 1$.

(5) For any $x \in \mathbb{X}$, $a \geq 0$ and $n \geq 1$, we set $V_n(x,a) := \mathbb{E}_{x,a}[S_n; \tau > n]$. By assertion 1, we can see $\lim_{n\to+\infty} V_n(x,a) = V(x,a)$. By Markov property, we obtain

$$\begin{aligned} V_{n+1}(x,a) &= \mathbb{E}_{x,a}\left[\mathbb{E}\left[S_1 + \sum_{k=1}^{n} \rho(g_{k+1}, X_k); S_1 > 0, \dots, S_{n+1} > 0 | \mathcal{F}_1\right]\right] \\ &= \mathbb{E}_{x,a}\left[V_n(X_1, S_1); \tau > 1\right]. \end{aligned}$$

By Lemma 3.5, we obtain $\sup_{x\in\mathbb{X}, a\geq 0} V_n(x,a) \leq c(1+a)$ which implies $\mathbb{P}$-a.s.

$$V_n(X_1, S_1)\mathbf{1}_{[\tau>1]} \leq c(1+S_1)\mathbf{1}_{[\tau>1]}.$$

Lebesgue's Dominated Convergence Theorem and (1.2) yield

$$\begin{aligned} V(x,a) = \lim_{n\to+\infty} V_{n+1}(x,a) &= \lim_{n\to+\infty} \mathbb{E}_{x,a}[V_n(X_1,S_1); \tau > 1] \\ &= \mathbb{E}_{x,a}[V(X_1,S_1); \tau > 1] \\ &= \widetilde{P}_+ V(x,a). \end{aligned}$$

□

4. Coupling argument and proof of Theorems 1.2 and 1.3

First, we check that the weak invariance principle with rate stated in Grama et al. (2014) (Theorem 2.1) may be applied to the sequence $(\rho(g_k, X_{k-1}))_{k\geq 0}$.The hypotheses C1, C2 and C3 of this theorem are given in terms of Fourier transform of the partial sums of S_n; combining the expressions (2.1), (2.2), (2.3) and the properties of the Fourier operators $(P_t)_t$, we verify in the next section that these conditions are satisfied in our context. This leads to the following simpler but sufficient statement.

Theorem 4.1. *Assume P1-P4. There exist*

- $\varepsilon_0 > 0$, *and* $c_0 > 0$,
- *a probability space* $(\widetilde{\Omega}, B(\widetilde{\Omega}))$,
- *a family* $(\widetilde{\mathbb{P}}_x)_{x\in\mathbb{X}}$ *of probability measures on* $(\widetilde{\Omega}, B(\widetilde{\Omega}))$,
- *a sequence* $(\tilde{a}_k)_k$ *of real-valued random variables on* $(\widetilde{\Omega}, B(\widetilde{\Omega}))$ *such that* $\mathcal{L}\Big((\tilde{a}_k)_k/\widetilde{\mathbb{P}}_x\Big) = \mathcal{L}\Big((a_k)_k/\mathbb{P}_x\Big)$ *for any* $x \in \mathbb{X}$,
- *and a sequence* $(\widetilde{W}_i)_{i\geq 1}$ *of independent standard normal random variables on* $(\widetilde{\Omega}, \mathcal{B}(\widetilde{\Omega}))$

such that for any $x \in \mathbb{X}$,

$$\widetilde{\mathbb{P}_x}\left(\sup_{1\leq k\leq n}\left|\sum_{i=1}^{k}(\tilde{a}_i - \sigma\widetilde{W}_i)\right| > n^{\frac{1}{2}-\varepsilon_0}\right) \leq c_0 n^{-\varepsilon_0}. \tag{4.1}$$

Notice that the fact (4.1) holds true for ε_0 implies (4.1) holds true for ε, whenever $\varepsilon \leq \varepsilon_0$. In order to simplify the notations, we identify $(\widetilde{\Omega}, \mathcal{B}(\tilde{\Omega}))$ and $(\Omega, \mathcal{B}(\Omega))$ and consider that the process $(\log|L_n x|)_{n\geq 0}$ satisfies the following property: there exists

$\varepsilon_0 > 0$ and $c_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0]$ and $x \in \mathbb{X}$,

$$\mathbb{P}\left(\sup_{0\le t\le 1} |\log|L_{[nt]}x| - \sigma B_{nt}| > n^{\frac{1}{2}-\varepsilon}\right)$$
$$= \mathbb{P}_x\left(\sup_{0\le t\le 1} |S_{[nt]} - \sigma B_{nt}| > n^{\frac{1}{2}-\varepsilon}\right) \le c_0 n^{-\varepsilon}, \tag{4.2}$$

where $(B_t)_{t\ge 0}$ is a standard Brownian motion on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$ and $\sigma > 0$ is used in the proof of Proposition 2.3, part c). For any $a \ge 0$, let τ_a^{bm} be the first time the process $(a + \sigma B_t)_{t\ge 0}$ becomes non-positive:

$$\tau_a^{bm} = \inf\{t \ge 0 : a + \sigma B_t \le 0\}.$$

The following lemma is due to Lévy (1937, Theorem 42.I, pp. 194-195).

Lemma 4.2. (1) *For any $a \ge 0$ and $n \ge 1$,*

$$\mathbb{P}(\tau_a^{bm} > n) = \mathbb{P}\left(\sigma \inf_{0\le u\le n} B_u > -a\right) = \frac{2}{\sigma\sqrt{2\pi n}} \int_0^a \exp\left(-\frac{s^2}{2n\sigma^2}\right) ds.$$

(2) *For any a, b such that $0 \le a < b < +\infty$ and $n \ge 1$,*

$$\mathbb{P}(\tau_a^{bm} > n, a + \sigma B_n \in [a, b])$$
$$= \frac{1}{\sigma\sqrt{2\pi n}} \int_a^b \left[\exp\left(-\frac{(s-a)^2}{2n\sigma^2}\right) - \exp\left(-\frac{(s+a)^2}{2n\sigma^2}\right)\right] ds.$$

From Lemma 4.2, we can obtain the next result.

Lemma 4.3. (1) *There exists a positive constant c such that for any $a \ge 0$ and $n \ge 1$,*

$$\mathbb{P}(\tau_a^{bm} > n) \le c\frac{a}{\sigma\sqrt{n}}. \tag{4.3}$$

(2) *For any sequence of real numbers $(\alpha_n)_n$ such that $\alpha_n \to 0$ as $n \to +\infty$, there exists a positive constant c such that for any $a \in [0, \alpha_n\sqrt{n}]$,*

$$\left|\mathbb{P}(\tau_a^{bm} > n) - \frac{2a}{\sigma\sqrt{2\pi n}}\right| \le c\frac{\alpha_n}{\sqrt{n}} a. \tag{4.4}$$

We use the coupling result described in Theorem 4.1 above to transfer the properties of the exit time τ_a^{bm} to the exit time τ_a for great a.

4.1. *Proof of Theorem 1.2.*
(1) Let $\varepsilon \in (0, \min\{\varepsilon_0; \frac{1}{2}\})$ and $(\theta_n)_{n\ge 1}$ be a sequence of positive numbers such that $\theta_n \to 0$ and $\theta_n n^{\varepsilon/4} \to +\infty$ as $n \to +\infty$. For any $x \in \mathbb{X}$ and $a \ge 0$, we have the decomposition

$$P_n(x, a) := \mathbb{P}_{x,a}(\tau > n) = \mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} > n^{1-\varepsilon}) + \mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}). \tag{4.5}$$

It is obvious that from Lemma 3.2, we obtain

$$\sup_{x\in\mathbb{X}, a\ge 0} \mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} > n^{1-\varepsilon}) \le \sup_{x\in\mathbb{X}, a\ge 0} \mathbb{P}_{x,a}(\nu_{n,\varepsilon} > n^{1-\varepsilon}) \le e^{-c_\varepsilon n^\varepsilon}. \tag{4.6}$$

For the second term, by Markov's property,

$$\mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}) = \mathbb{E}_{x,a}\left[P_{n-\nu_n}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}); \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right] \tag{4.7}$$
$$= I_n(x, a) + J_n(x, a),$$

where

$$I_n(x,a) := \mathbb{E}_{x,a}\left[P_{n-\nu_n}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}); S_{\nu_{n,\varepsilon}} \le \theta_n n^{\frac{1}{2}}, \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right],$$

$$\text{and } J_n(x,a) := \mathbb{E}_{x,a}\left[P_{n-\nu_n}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}); S_{\nu_{n,\varepsilon}} > \theta_n n^{\frac{1}{2}}, \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right].$$

Now we control the quantity $P_{n-\nu_n}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}})$ by using the following lemma. The proofs of the lemmas stated in this subsection are postponed to the next subsection.

Lemma 4.4. (1) *There exists $c > 0$ such that for any n sufficiently great, $x \in \mathbb{X}$ and $a \in [n^{\frac{1}{2}-\varepsilon}, \theta_n n^{\frac{1}{2}}]$,*

$$\left|\mathbb{P}_{x,a}(\tau > n) - \frac{2a}{\sigma\sqrt{2\pi n}}\right| \le c\frac{a\theta_n}{\sqrt{n}}. \tag{4.8}$$

(2) *There exists $c > 0$ such that for any $x \in \mathbb{X}$, $a \ge n^{\frac{1}{2}-\varepsilon}$ and $n \ge 1$,*

$$\mathbb{P}_{x,a}(\tau > n) \le c\frac{a}{\sqrt{n}}. \tag{4.9}$$

Notice that for any $x \in \mathbb{X}, a \ge 0$ and $0 \le k \le n^{1-\varepsilon}$,

$$P_n(x,a) \le P_{n-k}(x,a) \le P_{n-[n^{1-\varepsilon}]}(x,a). \tag{4.10}$$

By definition of $\nu_{n,\varepsilon}$ and (2.16), as long as $A \le n^{\frac{1}{2}-\varepsilon}$, we have $\mathbb{P}_{x,a}$-a.s.

$$S_{\nu_{n,\varepsilon}} \ge M_{\nu_{n,\varepsilon}} - A \ge 2n^{\frac{1}{2}-\varepsilon} - A \ge n^{\frac{1}{2}-\varepsilon}. \tag{4.11}$$

Using (4.8) and (4.10), (4.11) with θ_n replaced by $\theta_n\left(\frac{n}{n-n^{1-\varepsilon}}\right)^{\frac{1}{2}}$, for n sufficiently great, on the event $\left[S_{\nu_{n,\varepsilon}} \le \theta_n n^{\frac{1}{2}}, \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right]$, we obtain $\mathbb{P}_{x,a}$-a.s.

$$P_{n-\nu_{n,\varepsilon}}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}) = \frac{2(1+o(1))S_{\nu_{n,\varepsilon}}}{\sigma\sqrt{2\pi n}}.$$

Let

$$I_n'(x,a) := \mathbb{E}_{x,a}\left[S_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right], \tag{4.12}$$

$$J_n'(x,a) := \mathbb{E}_{x,a}\left[S_{\nu_{n,\varepsilon}}; S_{\nu_{n,\varepsilon}} > \theta_n n^{\frac{1}{2}}, \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right]. \tag{4.13}$$

Hence

$$\begin{aligned} I_n(x,a) &= \frac{2(1+o(1))}{\sigma\sqrt{2\pi n}}\mathbb{E}_{x,a}\left[S_{\nu_{n,\varepsilon}}; S_{\nu_{n,\varepsilon}} \le \theta_n n^{\frac{1}{2}}, \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right] \\ &= \frac{2(1+o(1))}{\sigma\sqrt{2\pi n}}\left[I_n'(x,a) - J_n'(x,a)\right], \\ J_n(x,a) &= \frac{c(1+o(1))}{\sqrt{n}}J_n'(x,a). \end{aligned}$$

Therefore (4.5) becomes

$$\left|\mathbb{P}_{x,a}(\tau > n) - \frac{2(1+o(1))}{\sigma\sqrt{2\pi n}}I_n'(x,a)\right| \le C\left(n^{-\frac{1}{2}}J_n'(x,a)\right) + C'\left(e^{-c_\varepsilon n^\varepsilon}\right).$$

The first assertion of Theorem 1.2 immediately follows by noticing that the term J_n' is negligible and $\mathbb{P}_{x,a}(\tau > n)$ is dominated by the term I_n' as shown in the lemma below.

Lemma 4.5.

$$\lim_{n\to+\infty} I'_n(x,a) = V(x,a) \quad \textit{and} \quad \lim_{n\to+\infty} n^{2\varepsilon} J'_n = 0,$$

where I'_n and J'_n are defined in (4.12) and (4.13).

(2) By using Proposition 1.1 (3), it suffices to prove $\sqrt{n}\mathbb{P}_{x,a}(\tau > n) \le c(1+a)$ for n great enough. For n sufficiently great, using (4.9) and (4.11), we obtain $\mathbb{P}_{x,a}$-a.s.

$$P_{n-[n^{1-\varepsilon}]}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}) \le c\frac{S_{\nu_{n,\varepsilon}}}{\sqrt{n}}.$$

Combined with (4.7), it yields

$$\mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}) \le \frac{c}{\sqrt{n}} I'_n. \tag{4.14}$$

Since $\tau_a < T_{a+A}$ $\mathbb{P}$-a.s. and (3.9), it follows that

$$I'_n(x,a) \le \mathbb{E}_{x,a+A}[M_{\nu_{n,\varepsilon}}; T > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}] \le c(1+a+A).$$

Hence (4.14) becomes

$$\mathbb{P}_{x,a}(\tau > n, \nu_{n,\varepsilon} \le n^{1-\varepsilon}) \le \frac{c}{\sqrt{n}}(1+a+A). \tag{4.15}$$

Combining (4.5), (4.6) and (4.15), we obtain for n great enough,

$$\mathbb{P}_{x,a}(\tau > n) \le e^{-c_\varepsilon n^\varepsilon} + \frac{c}{\sqrt{n}}(1+a+A) \le c'(1+a).$$

□

4.2. *Proof of Theorem 1.3.*
Let us decompose $\mathbb{P}_{x,a}(S_n \le t\sqrt{n} | \tau > n)$ as follows:

$$\frac{\mathbb{P}_{x,a}(S_n \le t\sqrt{n}, \tau > n)}{\mathbb{P}_{x,a}(\tau > n)} = D_{n,1} + D_{n,2} + D_{n,3}, \tag{4.16}$$

where

$$\begin{aligned} D_{n,1} &:= \frac{\mathbb{P}_{x,a}(S_n \le t\sqrt{n}, \tau > n, \nu_{n,\varepsilon} > n^{1-\varepsilon})}{\mathbb{P}_{x,a}(\tau > n)}, \\ D_{n,2} &:= \frac{\mathbb{P}_{x,a}(S_n \le t\sqrt{n}, \tau > n, S_n > \theta_n\sqrt{n}, \nu_{n,\varepsilon} \le n^{1-\varepsilon})}{\mathbb{P}_{x,a}(\tau > n)}, \\ D_{n,3} &:= \frac{\mathbb{P}_{x,a}(S_n \le t\sqrt{n}, \tau > n, S_n \le \theta_n\sqrt{n}, \nu_{n,\varepsilon} \le n^{1-\varepsilon})}{\mathbb{P}_{x,a}(\tau > n)}. \end{aligned}$$

Lemma 3.2 and Theorem 1.2 imply

$$\lim_{n\to+\infty} D_{n,1} = 0. \tag{4.17}$$

Theorem 1.2 and Proposition 1.1 (3) imply

$$\begin{aligned}
D_{n,2} &\leq \frac{\mathbb{P}_{x,a}(\tau > n, S_n > \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon})}{\mathbb{P}_{x,a}(\tau > n)} \\
&= \frac{1}{\mathbb{P}_{x,a}(\tau > n)}\mathbb{E}_{x,a}\Big[P_{n-\nu_{n,\varepsilon}}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}); \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} > \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big] \\
&\leq c\frac{\mathbb{E}_{x,a}\Big[1 + S_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} > \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big]}{\mathbb{P}_{x,a}(\tau > n)\sigma\sqrt{n - n^{1-\varepsilon}}} \\
&\leq c'\frac{\mathbb{E}_{x,a}\Big[S_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} > \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big] + \mathbb{P}_{x,a}(\tau > \nu_{n,\varepsilon})}{V(x,a)\sqrt{1 - n^{-\varepsilon}}}.
\end{aligned}$$

Since $\mathbb{P}_{x,a}(\tau < +\infty) = 1$ and $\mathbb{P}_{x,a}(\nu_{n,\varepsilon} < +\infty) = 0$, Lemma 4.5 yields

$$\lim_{n\to+\infty} D_{n,2} = 0. \tag{4.18}$$

Now we control $D_{n,3}$. Let $H_m(x,a) := \mathbb{P}_{x,a}(S_m \leq t\sqrt{n}, \tau > m)$. We claim the following lemma and postpone its proof at the end of this section.

Lemma 4.6. *Let $\varepsilon \in (0, \varepsilon_0), t > 0$ and $(\theta_n)_{n\geq 1}$ be a sequence such that $\theta_n \to 0$ and $\theta_n n^{\varepsilon/4} \to +\infty$ as $n \to +\infty$. Then for any $x \in \mathbb{X}$, $n^{1/2-\varepsilon} \leq a \leq \theta_n\sqrt{n}$ and $1 \leq k \leq n^{1-\varepsilon}$,*

$$\mathbb{P}_{x,a}\Big(S_{n-k} \leq t\sqrt{n}, \tau > n - k\Big) = \frac{2a}{\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du(1 + o(1)).$$

It is noticeable that on the event $[\tau > k, S_k \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} = k]$, the random variable $H_{n-k}(X_k, S_k)$ satisfies the hypotheses of Lemma 4.6. Hence

$$\begin{aligned}
&\mathbb{P}_{x,a}(S_n \leq t\sqrt{n}, \tau > n, S_n \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}) \\
&= \mathbb{E}_{x,a}\Big[H_{n-\nu_{n,\varepsilon}}(X_{\nu_{n,\varepsilon}}, S_{\nu_{n,\varepsilon}}); \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big] \\
&= \sum_{k=1}^{[n^{1-\varepsilon}]}\mathbb{E}_{x,a}\Big[H_{n-k}(X_k, S_k); \tau > k, S_k \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} = k\Big] \\
&= \frac{2(1 + o(1))}{\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\left(\frac{-u^2}{2\sigma^2}\right)du\,\mathbb{E}_{x,a}\Big[S_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big].
\end{aligned}$$

Lemma 4.5 yield as $n \to +\infty$,

$$\mathbb{E}_{x,a}\Big[S_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, S_{\nu_{n,\varepsilon}} \leq \theta_n\sqrt{n}, \nu_{n,\varepsilon} \leq n^{1-\varepsilon}\Big] = V(x,a)(1 + o(1)).$$

Therefore, Theorem 1.2 yields

$$\begin{aligned}
D_{n,3} &= \frac{2V(x,a)(1 + o(1))}{\mathbb{P}_{x,a}(\tau > n)\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\left(\frac{-u^2}{2\sigma^2}\right)du \\
&= \frac{1 + o(1)}{\sigma^2}\int_0^t u\exp\left(\frac{-u^2}{2\sigma^2}\right)du. \qquad (4.19)
\end{aligned}$$

The assertion of the theorem arrives by combining (4.16), (4.17), (4.18) and (4.19). □

4.3. *Proof of Lemma 4.4.*
(1) Fix $\varepsilon > 0$ and let

$$A_{n,\varepsilon} := \left[\sup_{0\le t\le 1} |S_{[nt]} - \sigma B_{nt}| \le n^{\frac{1}{2}-2\varepsilon} \right].$$

For any $x \in \mathbb{X}$, (4.2) implies $\mathbb{P}_x(A^c_{n,\varepsilon}) \le c_0 n^{-2\varepsilon}$. Denote $a^\pm := a \pm n^{\frac{1}{2}-2\varepsilon}$ and notice that for $a \in [n^{\frac{1}{2}-\varepsilon}, \theta_n\sqrt{n}]$,

$$0 \le a^\pm \le 2\theta_n\sqrt{n}. \tag{4.20}$$

Using (4.4) and (4.20), for any $x \in \mathbb{X}$ and $a \in [n^{\frac{1}{2}-\varepsilon}, \theta_n\sqrt{n}]$, we obtain

$$-\frac{ca^\pm\theta_n}{\sqrt{n}} \pm \frac{2n^{-2\varepsilon}}{\sigma\sqrt{2\pi}} \le \mathbb{P}_x(\tau^{bm}_{a^\pm} > n) - \frac{2a}{\sigma\sqrt{2\pi n}} \le \frac{ca^\pm\theta_n}{\sqrt{n}} \pm \frac{2n^{-2\varepsilon}}{\sigma\sqrt{2\pi}}. \tag{4.21}$$

For any $a \ge n^{\frac{1}{2}-\varepsilon}$, we have $\left[\tau^{bm}_{a^-} > n\right] \cap A^c_{n,\varepsilon} \subset [\tau_a > n] \cap A^c_{n,\varepsilon} \subset \left[\tau^{bm}_{a^+} > n\right] \cap A^c_{n,\varepsilon}$, which yields

$$\mathbb{P}_x(\tau^{bm}_{a^-} > n) - \mathbb{P}_x(A^c_{n,\varepsilon}) \le \mathbb{P}_x(\tau_a > n) \le \mathbb{P}_x(\tau^{bm}_{a^+} > n) + \mathbb{P}_x(A^c_{n,\varepsilon})$$

for any $x \in \mathbb{X}$. It follows that

$$\begin{cases} \mathbb{P}_x(\tau_a > n) - \mathbb{P}_x(\tau^{bm}_{a^+} > n) \le c_0 n^{-2\varepsilon}, \\ \mathbb{P}_x(\tau^{bm}_{a^-} > n) - \mathbb{P}_x(\tau_a > n) \le c_0 n^{-2\varepsilon}. \end{cases} \tag{4.22}$$

The fact that $\theta_n n^{\varepsilon/4} \to +\infty$ yields for n great enough

$$\theta_n \frac{a}{\sqrt{n}} \ge \frac{n^{\frac{1}{2}-\varepsilon}}{n^\varepsilon\sqrt{n}} = n^{-2\varepsilon}. \tag{4.23}$$

From (4.21), (4.22) and (4.23), it follows that for any $a \in [n^{\frac{1}{2}-\varepsilon}, \theta_n\sqrt{n}]$,

$$\left| \mathbb{P}_x(\tau_a > n) - \frac{2a}{\sigma\sqrt{2\pi n}} \right| \le c(1+\theta_n)n^{-2\varepsilon} + c_1\frac{\theta_n a}{\sqrt{n}} \le c_2\frac{\theta_n a}{\sqrt{n}}.$$

(2) For n great enough, condition $a \ge n^{\frac{1}{2}-\varepsilon}$ implies $a^+ \le 2a$. From (4.3) and (4.22) , since $n^{-2\varepsilon} \le \frac{a}{\sqrt{n}}$, for any $x \in \mathbb{X}$,

$$\mathbb{P}_x(\tau_a > n) \le c\frac{a}{\sigma\sqrt{n}} + c_0 n^{-2\varepsilon} \le c_1\frac{a}{\sqrt{n}}.$$

□

4.4. *Proof of Lemma 4.5.*
(1) We prove that $\lim_{n\to+\infty} \mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right] = V(x,a)$. Then, the assertion arrives by using (2.16) and taking into account that $\mathbb{P}_x(\tau_a < +\infty) = 1$ and $\mathbb{P}_x(\lim_{n\to+\infty} \nu_{n,\varepsilon} = +\infty) = 1$. For $x \in \mathbb{X}$ and $a \ge 0$, we obtain

$$\begin{aligned} \mathbb{E}_{x,a}&\left[M_{\nu_{n,\varepsilon}}; \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right] = \mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]}; \tau > \nu_{n,\varepsilon}, \nu_{n,\varepsilon} \le n^{1-\varepsilon}\right] \\ &= \mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]}; \tau > \nu_{n,\varepsilon} \wedge [n^{1-\varepsilon}]\right] \\ &\qquad - \mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]}; \tau > \nu_{n,\varepsilon} \wedge [n^{1-\varepsilon}], \nu_{n,\varepsilon} > n^{1-\varepsilon}\right]. \end{aligned}$$

By using Lemma 3.3,

$$\mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]}; \tau > \nu_{n,\varepsilon} \wedge [n^{1-\varepsilon}], \nu_{n,\varepsilon} > n^{1-\varepsilon}\right] \le c(1+a)e^{-c_\varepsilon n^\varepsilon}.$$

Using the facts that $(M_n)_{n\geq 0}$ is a martingale and $\mathbb{P}_x\left(\lim_{n\to+\infty}\nu_{n,\varepsilon}=+\infty\right)=1$, we obtain

$$\begin{aligned}
&\lim_{n\to+\infty}\mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}};\tau>\nu_{n,\varepsilon},\nu_{n,\varepsilon}\leq n^{1-\varepsilon}\right]\\
&=\lim_{n\to+\infty}\mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]};\tau>\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]\right]\\
&=a-\lim_{n\to+\infty}\mathbb{E}_{x,a}\left[M_{\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]};\tau\leq\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]\right]\\
&=a-\lim_{n\to+\infty}\mathbb{E}_{x,a}\left[M_{\tau};\tau\leq\nu_{n,\varepsilon}\wedge[n^{1-\varepsilon}]\right]\\
&=a-\mathbb{E}_{x,a}[M_\tau]=V(x,a).
\end{aligned}$$

(2) Let $b=a+A$. Remind that $M_n^*=\max_{1\leq k\leq n}|M_k|$. We obtain

$$\begin{aligned}
&\mathbb{E}_{x,a}\left[S_{\nu_{n,\varepsilon}};S_{\nu_{n,\varepsilon}}>\theta_n n^{\frac{1}{2}},\tau>\nu_{n,\varepsilon},\nu_{n,\varepsilon}\leq n^{1-\varepsilon}\right]\\
&\leq\mathbb{E}_{x,b}\left[M_{\nu_{n,\varepsilon}};M_{\nu_{n,\varepsilon}}>\theta_n n^{\frac{1}{2}},\nu_{n,\varepsilon}\leq n^{1-\varepsilon}\right]\\
&\leq\mathbb{E}_{x,b}\left[M^*_{[n^{1-\varepsilon}]};M^*_{[n^{1-\varepsilon}]}>\theta_n n^{\frac{1}{2}}\right].
\end{aligned}$$

Since $\theta_n n^{\varepsilon/4}\to+\infty$ as $n\to+\infty$, it suffices to prove that for any $\delta>0$, $x\in\mathbb{X}$ and $b\in\mathbb{R}$,

$$\lim_{n\to+\infty}n^{2\varepsilon}\mathbb{E}_x\left[b+M_n^*;M_n^*>n^{\frac{1}{2}+\delta}\right]=0.$$

Obviously, by (3.5),

$$\begin{aligned}
\mathbb{E}_x\left[b+M_n^*;M_n^*>n^{\frac{1}{2}+\delta}\right]&\leq b\mathbb{P}_x\left(M_n^*>n^{\frac{1}{2}+\delta}\right)+\mathbb{E}_x\left[M_n^*;M_n^*>n^{\frac{1}{2}+\delta}\right]\\
&=\left(b+n^{\frac{1}{2}+\delta}\right)\mathbb{P}_x\left(M_n^*>n^{\frac{1}{2}+\delta}\right)+\int_{n^{\frac{1}{2}+\delta}}^{+\infty}\mathbb{P}_x(M_n^*>t)dt\\
&\leq c\left(b+n^{\frac{1}{2}+\delta}\right)n^{-p\delta}+cn^{-p\delta+\frac{1}{2}+\delta}.
\end{aligned}$$

Since p can be taken arbitrarily great, it follows that $\lim_{n\to+\infty}n^{2\varepsilon}J_n'=0$.

□

4.5. *Proof of Lemma 4.6.*
Recall that $a^{\pm}=a\pm n^{1/2-2\varepsilon}$ and denote $t^{\pm}=t\pm 2n^{-2\varepsilon}$. For any $1\leq k\leq n^{1-\varepsilon}$,

$$\{\tau_{a^-}^{bm}\}\cap A_{n,\varepsilon}\subset\{\tau_a>n-k\}\cap A_{n,\varepsilon}\subset\{\tau_{a^+}^{bm}\}\cap A_{n,\varepsilon}$$

and

$$\begin{aligned}
\{a^-+\sigma B_{n-k}\leq t^-\sqrt{n}\}\cap A_{n,\varepsilon}&\subset\{a+S_{n-k}\leq t\sqrt{n}\}\cap A_{n,\varepsilon}\\
&\subset\{a^++\sigma B_{n-k}\leq t^+\sqrt{n}\}\cap A_{n,\varepsilon},
\end{aligned}$$

which imply

$$\begin{aligned}
&\mathbb{P}_x(\tau_{a^-}^{bm}>n-k,a^-+\sigma B_{n-k}\leq t^-\sqrt{n})-\mathbb{P}_x(A_{n,\varepsilon}^c)\\
&\leq\mathbb{P}_x(\tau_a>n-k,a+S_{n-k}\leq t\sqrt{n})\leq \qquad (4.24)\\
&\mathbb{P}_x(\tau_{a^+}^{bm}>n-k,a^++\sigma B_{n-k}\leq t^+\sqrt{n})+\mathbb{P}_x(A_{n,\varepsilon}^c).
\end{aligned}$$

Moreover, by Lemma 4.2, we obtain

$$\begin{aligned}&\mathbb{P}_x\Big(\tau_{a^+}^{bm} > n-k, a^+ + \sigma B_{n-k} \le t^+\sqrt{n}\Big)\\&= \frac{2a}{\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du(1+o(1))\end{aligned} \tag{4.25}$$

and similarly,

$$\begin{aligned}&\mathbb{P}_x\Big(\tau_{a^-}^{bm} > n-k, a^- + \sigma B_{n-k} \le t^-\sqrt{n}\Big)\\&= \frac{2a}{\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du(1+o(1)).\end{aligned} \tag{4.26}$$

Therefore, from (4.24), (4.25), (4.26) and $\mathbb{P}_x(A_{n,\varepsilon}^c) \le cn^{-2\varepsilon}$, it follows that

$$\mathbb{P}_x\Big(\tau_a > n-k, a + S_{n-k} \le t\sqrt{n}\Big) = \frac{2a}{\sigma^3\sqrt{2\pi n}}\int_0^t u\exp\Big(-\frac{u^2}{2\sigma^2}\Big)du(1+o(1)).1$$

□

5. On conditions C1-C3 of Theorem 2.1 in Grama et al. (2014)

Let $k_{gap}, M_1, M_2 \in \mathbb{N}$ and $j_0 < \ldots < j_{M_1+M_2}$ be natural numbers. Denote $a_{k+J_m} = \sum_{l\in J_m} a_{k+l}$, where $J_m = [j_{m-1}, j_m), m = 1, \ldots, M_1+M_2$ and $k \ge 0$. Consider the vectors $\bar{a}_1 = (a_{J_1}, \ldots, a_{J_{M_1}})$ and $\bar{a}_2 = (a_{k_{gap}+J_{M_1+1}}, \ldots, a_{k_{gap}+J_{M_1+M_2}})$. Denote by $\phi_x(s,t) = \mathbb{E}e^{is\bar{a}_1+it\bar{a}_2}$, $\phi_{x,1}(s) = \mathbb{E}e^{is\bar{a}_1}$ and $\phi_{x,2}(s) = \mathbb{E}e^{it\bar{a}_2}$ the characteristic functions of $(\bar{a}_1, \bar{a}_2)$, $\bar{a}_1$ and $\bar{a}_2$, respectively. For the sake of brevity, we denote $\phi_1(s) = \phi_{x,1}(s), \phi_2(t) = \phi_{x,2}(t)$ and $\phi(s,t) = \phi_x(s,t)$.

We first check that conditions C1-C3 hold and then prove the needed lemmas.

5.1. *Statement and proofs of conditions C1-C3.* **C1**: There exist positive constants $\varepsilon_0 \le 1, \lambda_0, \lambda_1, \lambda_2$ such that for any $k_{gap} \in \mathbb{R}, M_1, M_2 \in \mathbb{Z}_+$, any sequence $j_0 < \ldots < j_{M_1+M_2}$ and any $s \in \mathbb{R}^{M_1}, t \in \mathbb{R}^{M_2}$ satisfying $|(s,t)|_\infty \le \varepsilon_0$,

$$|\phi(s,t) - \phi_1(s)\phi_2(t)| \le \lambda_0\exp(-\lambda_1 k_{gap})\left(1 + \max_{m=1,\ldots,M_1+M_2} card(J_m)\right)^{\lambda_2(M_1+M_2)}.$$

C2: There exists a positive constant δ such that $\sup_{n\ge0}|a_n|_{L^{2+2\delta}} < +\infty$.
C3: There exist a positive constant C and a positive number σ such that for any $\gamma > 0$, any $x \in \mathbb{X}$ and any $n \ge 1$,

$$\sup_{m\ge0}\left|n^{-1}Var_{\mathbb{P}_x}\left(\sum_{i=m}^{m+n-1} a_i\right) - \sigma^2\right| \le Cn^{-1+\gamma}.$$

Proposition 5.1. *Condition 1 is satisfied under hypotheses P1-P5.*

Proof. First, we prove the following lemma.

Lemma 5.2. *There exist two positive constants C and κ such that $0 < \kappa < 1$ and*

$$|\phi(s,t) - \phi_1(s)\phi_2(t)| \le CC_P^{M_1+M_2}\kappa^{k_{gap}},$$

where C_P is defined in Proposition 2.3.

Proof. In fact, the characteristic functions of the ramdom variables $\bar{a}_1, \bar{a}_2$ and $(\bar{a}_1, \bar{a}_2)$ can be written in terms of operator respectively as follows:

$$\begin{aligned} \phi_1(s) &= \mathbb{E}_x[e^{is\bar{a}_1}] = P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}\mathbf{1}(x), \\ \phi_2(t) &= \mathbb{E}_x[e^{it\bar{a}_2}] = P^{k_{gap}+j_{M_1}-1}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}(x), \\ \phi(s,t) &= \mathbb{E}_x[e^{is\bar{a}_1+it\bar{a}_2}] = P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}P^{k_{gap}}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}(x). \end{aligned} \tag{5.1}$$

Now we decompose $\phi(s,t)$ into the sum of $\phi_\Pi(s,t)$ and $\phi_R(s,t)$ by using the spectral decomposition $P = \Pi + R$ in Proposition 2.3, where

$$\begin{aligned} \phi_\Pi(s,t) &= P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}\Pi P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}(x), \\ \phi_R(s,t) &= P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}R^{k_{gap}}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}(x). \end{aligned}$$

Since $\Pi(\varphi) = \nu(\varphi)\mathbf{1}$ for any $\varphi \in L$ and P_t acts on L, we obtain

$$\phi_\Pi(s,t) = P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}\mathbf{1}(x)\nu\left(P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right).$$

Then setting $\psi_2(t) = \nu(P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1})$ yields

$$\begin{aligned} \phi(s,t) &= \phi_1(s)\psi_2(t) + \phi_R(s,t) \\ &= \phi_1(s)\phi_2(t) + \phi_1(s)[\psi_2(t) - \phi_2(t)] + \phi_R(s,t), \end{aligned}$$

which implies

$$|\phi(s,t) - \phi_1(s)\phi_2(t)| \le |\phi_1(s)||\psi_2(t) - \phi_2(t)| + |\phi_R(s,t)|. \tag{5.2}$$

On the one hand, we can see that $|\phi_1(s)| = \left|\left(P^{j_0-1}P_{s_1}^{|J_1|}\dots P_{s_{M_1}}^{|J_{M_1}|}\mathbf{1}\right)(x)\right| \le C_P^{1+M_1}$ and $|\phi_R(s,t)| \le C_P^{1+M_1+M_2}C_R\kappa^{k_{gap}}$. On the other hand, since ν is P-invariant measure and $(\nu - \delta_x)(\mathbf{1}) = 0$, by using again the expression $P = \Pi + R$, we obtain

$$\begin{aligned} |\psi_2(t) - \phi_2(t)| &= \left|(\nu - \delta_x)\left(P^{k_{gap}+j_{M_1}-1}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &\le \left|(\nu - \delta_x)\left(\Pi P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &\qquad + \left|(\nu - \delta_x)\left(R^{k_{gap}+j_{M_1}-1}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &= \left|(\nu - \delta_x)(\mathbf{1})\nu\left(P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &\qquad + \left|(\nu - \delta_x)\left(R^{k_{gap}+j_{M_1}-1}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &= \left|(\nu - \delta_x)\left(R^{k_{gap}+j_{M_1}-1}P_{t_1}^{|J_{M_1+1}|}\dots P_{t_{M_2}}^{|J_{M_1+M_2}|}\mathbf{1}\right)\right| \\ &\le CC_P^{M_2}\kappa^{k_{gap}+j_{M_1}-1}. \end{aligned} \tag{5.3}$$

Therefore, (5.2) follows.

□

Second, let $\lambda_2 = \max\{1, \log_2 C_P\}$. Since $\max_{m=1,\dots,M_1+M_2} card(J_m) \ge 1$, we obtain

$$C_P^{M_1+M_2} \le 2^{\lambda_2(M_1+M_2)} \le \left(1 + \max_{m=1,\dots,M_1+M_2} card(J_m)\right)^{\lambda_2(M_1+M_2)},$$

which implies that

$$|\phi(s,t)-\phi_1(s)\phi_2(t)| \le C\kappa^{k_{gap}}\left(1+\max_{m=1,\dots,M_1+M_2} card(J_m)\right)^{\lambda_2(M_1+M_2)}.$$

Finally, let $\lambda_0 = C$ and $\lambda_1 = -\log\kappa$. Then the assertion arrives.

□

Proposition 5.3. *Condition 2 is satisfied under hypotheses P1-P5.*

Proof. Condition P1 implies that there exists $\delta_0 > 0$ such that $\mathbb{E}[N(g)^{\delta_0}] < +\infty$ and since $\mathbb{E}[N(g)^{\delta_0}] = \mathbb{E}[\exp(\delta_0 \log N(g))] = \sum_{k=0}^{+\infty} \frac{\delta_0^k}{k!}\mathbb{E}[(\log N(g))^k]$, we obtain $\mathbb{E}|a_n|^k \le \mathbb{E}[(\log N(g))^k] < +\infty$ for any $n \ge 0$ and any $k \ge 0$.

□

Proposition 5.4. *Condition 3 is satisfied under hypotheses P1-P5. More precisely, there exists a positive constant σ such that for any $x \in \mathbb{X}$ and any $n \ge 1$,*

$$\sup_{m\ge 0}\left| Var_{\mathbb{P}_x}\left(\sum_{k=m}^{m+n-1} a_k\right) - n\sigma^2\right| < +\infty. \tag{5.4}$$

Proof. For any integer $m, n \ge 0$, we denote $S_{m,n} = \sum_{k=m}^{m+n-1} a_k$, $V_x(X) = Var_{\mathbb{P}_x}(X) = \mathbb{E}_x(X^2) - (\mathbb{E}_x X)^2$ and $Cov_x(X,Y) = Cov_{\mathbb{P}_x}(X,Y)$. Then

$$V_x(S_{m,n}) \quad = \quad \sum_{k=m}^{m+n-1} V_x(a_k) + 2\sum_{k=m}^{m+n-1}\sum_{l=1}^{m+n-k-1} Cov_x(a_k, a_{k+l}) \tag{5.5}$$

and (5.4) becomes $\sup_{m\ge 0}|V_x(S_{m,n}) - n\sigma^2| < +\infty$. We claim two lemmas and postpone their proofs until the end of this section.

Lemma 5.5. *There exist $C > 0$ and $0 < \kappa < 1$ such that for any $x \in \mathbb{X}$, any $k \ge 0$ and any $l \ge 0$,*

$$|Cov_x(a_k, a_{k+l})| \le C\kappa^l. \tag{5.6}$$

Lemma 5.6. *There exist $C > 0$, $0 < \kappa < 1$ and a sequence $(s_n)_{n\ge 0}$ of real numbers such that for any $x \in \mathbb{X}$, any $k \ge 0$ and any $l \ge 0$,*

$$|Cov_x(a_k, a_{k+l}) - s_l| \le C\kappa^k, \tag{5.7}$$

$$|s_l| \le C\kappa^l. \tag{5.8}$$

For the first term of the right side of (5.5), by combining Lemma 5.5 and Lemma 5.6, we obtain

$$|Cov_x(a_k, a_{k+l}) - s_l| \le C\kappa^{\max\{k,l\}}. \tag{5.9}$$

Inequality (5.7) implies $|V_x(a_k) - s_0| \le C\kappa^k$, which yields for any integer $m, n \ge 0$,

$$\left|\sum_{k=m}^{m+n-1} V_x(a_k) - ns_0\right| \le \sum_{k=m}^{m+n-1} |V_x(a_k) - s_0| \le C\sum_{k=m}^{m+n-1}\kappa^k \le \frac{C}{1-\kappa} < +\infty. \tag{5.10}$$

For the second term of the right side of (5.5), we can see that

$$\begin{aligned}
&\left|\sum_{k=m}^{m+n-1}\sum_{l=1}^{m+n-k-1} Cov_x(a_k,a_{k+l}) - \sum_{k=m}^{m+n-1}\sum_{l=1}^{+\infty} s_l\right| \\
\leq \ & \sum_{k=m}^{m+n-1}\sum_{l=1}^{m+n-k-1} |Cov_x(a_k,a_{k+l}) - s_l| + \sum_{k=m}^{m+n-1}\sum_{l=m+n-k}^{+\infty} |s_l| \\
= \ & \Sigma_1(x,m,n) + \Sigma_2(x,m,n). \qquad (5.11)
\end{aligned}$$

On the one hand, by (5.7) and (5.9), we can see that for any $x \in \mathbb{X}$, any $m \geq 0$ and any $n \geq 1$,

$$\begin{aligned}
\Sigma_1(x,m,n) \ &\leq \ \sum_{k=0}^{+\infty}\sum_{l=1}^{k} C\kappa^k + \sum_{k=0}^{+\infty}\sum_{l=k+1}^{+\infty} C\kappa^l \\
&\leq \ \sum_{k=0}^{+\infty} Ck\kappa^k + \sum_{k=0}^{+\infty} C\frac{\kappa^{k+1}}{1-\kappa} < +\infty. \qquad (5.12)
\end{aligned}$$

Similarly, on the other hand, by (5.8) we obtain for any $x \in \mathbb{X}$, any $m \geq 0$ and any $n \geq 1$,

$$\Sigma_2(x,m,n) \leq \sum_{k=0}^{n-1}\sum_{l=n-k}^{+\infty} C\kappa^l \leq \frac{C}{(1-\kappa)^2} < +\infty. \qquad (5.13)$$

Combining (5.5),(5.10),(5.11),(5.12) and (5.13), we obtain

$$\sup_{m\geq 0}\left| V_x(S_{m,n}) - n\sum_{l=0}^{+\infty} s_l \right| < +\infty. \qquad (5.14)$$

In fact, by using Lemma 2.1 in Le Page et al. (2017+), Theorem 5 in Hennion (1997) implies that the sequence $(\frac{S_n}{\sqrt{n}})_{n\geq 1}$ converges weakly to a normal law with variance σ^2. Meanwhile, under hypothesis P2, Corollary 3 in Hennion (1997) implies that the sequence $(|R_n|)_{n\geq 1}$ is not tight and thus $\sigma^2 > 0$, see Hennion (1997) for the definition and basic properties. Therefore, we can see that $Var_x S_n \sim n\sigma^2$ with $\sigma^2 > 0$, which yields $\sum_{l=0}^{+\infty} s_l = \sigma^2$.

□

5.2. *Proof of Lemma 5.5.*
Let $g(x) = \begin{cases} x & \text{if } |x| \leq 1, \\ 0 & \text{if } |x| > 2. \end{cases}$ such that g is C^∞ on $\mathbb{R}$ and $|g(x)| \leq |x|$ for any $x \in \mathbb{R}$. Then $g \in L^1(\mathbb{R}) \cap C_c^1(\mathbb{R})$. Therefore, the Fourier transform of g is $\hat{g}$ defined as follows:

$$\hat{g}(t) := \int_{\mathbb{R}} e^{-itx} g(x)dx,$$

and the Inverse Fourier Theorem yields

$$g(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{itx}\hat{g}(t)dt.$$

Let $g_T(x) := Tg(\frac{x}{T})$ for any $T > 0$. Then $|\hat{g}_T|_1 = T|\hat{g}|_1 < +\infty$. Let $h_T(x,y) = g_T(x)g_T(y)$. Then $\hat{h}_T(x,y) = \hat{g}_T(x)\hat{g}_T(y)$. Let V and V' be two i.i.d. random variables with mean 0, independent of a_l for any $l \geq 0$ whose characteristic functions have the support included in the interval $[-\varepsilon_0, \varepsilon_0]$ for ε_0 defined in C1. Assume that $\mathbb{E}_{|}V|^n < +\infty$ for any $n > 0$. Let $Z_k = a_k + V$ and $Z'_{k+l} = a_{k+l} + V'$ and denote by $\widetilde{\phi}_1(s), \widetilde{\phi}_2(t)$ and $\widetilde{\phi}(s,t)$ the characteristic functions of Z_k, Z'_{k+l} and (Z_k, Z'_{k+l}), respectively.

We use the same notations introduced at the beginning of this section by setting $\phi_1(s) = \mathbb{E}_x[e^{isa_k}], \phi_2(t) = \mathbb{E}_x[e^{ita_{k+l}}]$ and $\phi(s,t) = \mathbb{E}_x[e^{isa_k + ita_{k+l}}]$. We also denote φ the characteristic function of V, that yields

$$\begin{aligned}
\widetilde{\phi}_1(s) &= \mathbb{E}_{[}e^{isZ_k}] = \mathbb{E}_{[}e^{isa_k}]\mathbb{E}_{[}e^{isV}] = \phi_1(s)\varphi(s), \\
\widetilde{\phi}_2(t) &= \mathbb{E}_{[}e^{itZ'_{k+l}}] = \mathbb{E}_{[}e^{ita_{k+l}}]\mathbb{E}_{[}e^{itV'}] = \phi_2(t)\varphi(t), \\
\widetilde{\phi}(s,t) &= \mathbb{E}_{[}e^{isZ_k + itZ'_{k+l}}] = \mathbb{E}_{[}e^{isa_k + ita_{k+l}}]\mathbb{E}_{[}e^{isV}]\mathbb{E}_{[}e^{itV'}] = \phi(s,t)\varphi(s)\varphi(t).
\end{aligned} \tag{5.15}$$

Then we can see that $\widetilde{\phi}_1$ and $\widetilde{\phi}_2$ have the support in $[-\varepsilon_0, \varepsilon_0]$. We perturb a_k and a_{k+l} by adding the random variables V and V' with mean 0 and the support of their characteristic functions are on $[-\varepsilon_0, \varepsilon_0]$. We explicit the quantity $Cov_x(a_k, a_{k+l})$:

$$Cov_x(a_k, a_{k+l}) = \mathbb{E}_x[a_k, a_{k+l}] - \mathbb{E}_x a_k \mathbb{E}_x a_{k+l}. \tag{5.16}$$

On the one hand, we can see that

$$\begin{aligned}
\mathbb{E}_x[a_k a_{k+l}] = \mathbb{E}_x[Z_k Z'_{k+l}] &= \mathbb{E}_x[h_T(Z_k; Z'_{k+l})] + \mathbb{E}_x[Z_k Z'_{k+l}] - \mathbb{E}_x[h_T(Z_k; Z'_{k+l})] \\
&= \frac{1}{(2\pi)^2}\mathbb{E}_x \int\int \hat{h}_T(s,t) e^{isZ_k + itZ'_{k+l}} ds dt + R_0 \\
&= \frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)\mathbb{E}_x\left[e^{isZ_k + itZ'_{k+l}}\right] ds dt + R_0 \\
&= \frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)\widetilde{\phi}(s,t) ds dt + R_0,
\end{aligned} \tag{5.17}$$

where $R_0 = \mathbb{E}_x[Z_k Z'_{k+l}] - \mathbb{E}_x[h_T(Z_k; Z'_{k+l})]$. On the other hand, we obtain

$$\begin{aligned}
\mathbb{E}_x a_k = \mathbb{E}_x Z_k &= \mathbb{E}_x g_T(Z_k) + \mathbb{E}_x Z_k - \mathbb{E}_x g_T(Z_k) \\
&= \frac{1}{2\pi}\int \hat{g}_T(s)\widetilde{\phi}_1(s) ds + R_1,
\end{aligned} \tag{5.18}$$

where $R_1 = \mathbb{E}_x Z_k - \mathbb{E}_x g_T(Z_k)$ and

$$\begin{aligned}
\mathbb{E}_x a_{k+l} = \mathbb{E}_x Z'_{k+l} &= \mathbb{E}_x g_T(Z'_{k+l}) + \mathbb{E}_x Z'_{k+l} - \mathbb{E}_x g_T(Z'_{k+l}) \\
&= \frac{1}{2\pi}\int \hat{g}_T(t)\widetilde{\phi}_2(t) dt + R_2,
\end{aligned} \tag{5.19}$$

where $R_2 = \mathbb{E}_x Z'_{k+l} - \mathbb{E}_x g_T(Z'_{k+l})$. From (5.16), (5.17), (5.18) and (5.19), since $\hat{h}_T(s,t) = \hat{g}_T(s)\hat{g}_T(t)$, we obtain

$$\begin{aligned}
Cov_x(a_k, a_{k+l}) &= \frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)\widetilde{\phi}(s,t) ds dt + R_0 \\
&\quad - \left(\frac{1}{2\pi}\int \hat{g}_T(s)\widetilde{\phi}_1(s) ds + R_1\right)\left(\frac{1}{2\pi}\int \hat{g}_T(t)\widetilde{\phi}_2(t) dt + R_2\right) \\
&= \frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)\left[\widetilde{\phi}(s,t) - \widetilde{\phi}_1(s)\widetilde{\phi}_2(t)\right] ds dt + R
\end{aligned} \tag{5.20}$$

where $R = R_0 - R_1 R_2 - R_1 \frac{1}{2\pi}\int \hat{g}_T(t)\widetilde{\phi}_2(t)dt - R_2 \frac{1}{2\pi}\int \hat{g}_T(s)\widetilde{\phi}_1(s)ds$. Since $\hat{g}_T \in L_1(\mathbb{R})$ and applying Lemma 5.2 for $j_0 = k, j_1 = k+1, j_2 = k+2, k_{gap} = l, M_1 = M_2 = 1$, we obtain

$$\begin{aligned}
&|Cov_x(a_k, a_{k+l})| \\
&\le \frac{1}{(2\pi)^2}\int\int \left|\hat{h}_T(s,t)\right| \left|\widetilde{\phi}(s,t) - \widetilde{\phi}_1(s)\widetilde{\phi}_2(t)\right| dsdt + |R| \\
&\le \frac{1}{(2\pi)^2}\int\int \left|\hat{h}_T(s,t)\right| |\phi(s,t)\varphi(s)\varphi(t) - \phi_1(s)\phi_2(t)\varphi(s)\varphi(t)| \, dsdt + |R| \\
&\le \sup_{|s|,|t|\le\varepsilon_0} |\phi(s,t) - \phi_1(s)\phi_2(t)| \left(\int |\hat{g}_T(s)| \, ds\right)^2 + |R| \\
&\le CT^2\kappa^l + |R|. \qquad (5.21)
\end{aligned}$$

It remains to bound of $|R|$. On the one hand, we can see that

- $|R_1| = |\mathbb{E}_x[Z_k - g_T(Z_k)]| = \mathbb{E}_x\left|[Z_k - g_T(Z_k)]\mathbf{1}_{[|Z_k|>T]}\right| \le 2T^{-1}\mathbb{E}_x|Z_k|^2$,
- $\left|\frac{1}{2\pi}\int \hat{g}_T(s)\widetilde{\phi}_1(s)ds\right| = |\mathbb{E}_x g_T(Z_k)| \le \mathbb{E}_x|Z_k| \le \mathbb{E}_x|a_k| + \mathbb{E}_x|V| \le C$,
- $|R_2| = \left|\mathbb{E}_x[Z'_{k+l} - g_T(Z'_{k+l})]\right| \le 2T^{-1}\mathbb{E}_x|Z'_{k+l}|^2$,
- $\left|\frac{1}{2\pi}\int \hat{g}_T(t)\widetilde{\phi}_2(t)dt\right| = \left|\mathbb{E}_x g_T(Z'_{k+l})\right| \le \mathbb{E}_x|a_{l+k}| + \mathbb{E}_x|V'| \le C$.

On the other hand, similarly for $|R_0|$, we obtain

$$\begin{aligned}
&|R_0| = \\
&= \mathbb{E}_x\left[\left|Z_k Z'_{k+l} - h_T(Z_k, Z'_{k+l})\right| \left(\mathbf{1}_{[|Z_k|>T]} + \mathbf{1}_{[|Z_k|\le T]}\right)\left(\mathbf{1}_{[|Z'_{k+l}|>T]} + \mathbf{1}_{[|Z'_{k+l}|\le T]}\right)\right] \\
&\le \mathbb{E}_x\left[\left|Z_k Z'_{k+l} - h_T(Z_k, Z'_{k+l})\right| \left(\mathbf{1}_{[|Z_k|>T]} + \mathbf{1}_{[|Z'_{k+l}|>T]}\right)\right] \\
&\le 2\mathbb{E}_x\left|Z_k Z'_{k+l}\mathbf{1}_{[|Z_k|>T]}\right| + 2\mathbb{E}_x\left|Z_k Z'_{k+l}\mathbf{1}_{[|Z'_{k+l}|>T]}\right|.
\end{aligned}$$

For any positive δ, let $q_\delta = \frac{\delta+1}{\delta}$, by Holder's inequality, we obtain

$$\mathbb{E}_x\left|Z_k Z'_{k+l}\mathbf{1}_{[|Z_k|>T]}\right| \le \left(\mathbb{E}_x|Z_k|^{2+2\delta}\right)^{\frac{1}{2+2\delta}}\left(\mathbb{E}_x|Z'_{k+l}|^{2+2\delta}\right)^{\frac{1}{2+2\delta}}\mathbb{P}_x(|Z_k|>T)^{\frac{1}{q_\delta}}.$$

By Minkowski's inequality,

$$\left(\mathbb{E}_x|Z_k|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} \le \left(\mathbb{E}_x|a_k|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} + \left(\mathbb{E}_x|V|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} < C,$$

$$\left(\mathbb{E}_x|Z'_{k+l}|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} \le \left(\mathbb{E}_x|a_{l+k}|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} + \left(\mathbb{E}_x|V'|^{2+2\delta}\right)^{\frac{1}{2+2\delta}} < C.$$

By Markov's inequality,

$$\mathbb{P}_x(|Z_k|>T) \le \frac{1}{T^{q_\delta}}\mathbb{E}_x|Z_k|^{q_\delta} \le \frac{C}{T^{q_\delta}},$$

$$\mathbb{P}_x(|Z'_{k+l}|>T) \le \frac{1}{T^{q_\delta}}\mathbb{E}_x|Z'_{k+l}|^{q_\delta} \le \frac{C}{T^{q_\delta}}.$$

Hence $|R_0| \le CT^{-1}$ for $T>1$ and thus $|R| \le CT^{-1}$.

Thus, (5.21) becomes $|Cov_x(a_k, a_{k+l})| \le CT^2\kappa^l + CT^{-1}$. By choosing $T = \kappa^{-\alpha}$ with $\alpha > 0$, we obtain

$$|Cov_x(a_k, a_{k+l})| \le C\kappa^{l-2\alpha} + C\kappa^{\alpha} \le C' \max\{\kappa^{l-2\alpha}, \kappa^{\alpha}\}.$$

Now we choose $\alpha > 0$ such that $l - 2\alpha > 0$, for example, let $\alpha = \frac{l}{4}$, we obtain

$$|Cov_x(a_k, a_{k+l})| \le C\kappa^{\frac{l}{4}}.$$

5.3. *Proof of Lemma 5.6.*
Inequality (5.8) follows by setting $k = l$ in (5.6) and (5.7). It suffices to prove (5.7). Recall the definition in (5.15) and let

$$\begin{aligned} \psi(s) &= \nu(P_s\mathbf{1})\varphi(s), \\ \psi(s,t;l) &= \nu(P_sP^{l-1}P_t\mathbf{1})\varphi(s)\varphi(t), \\ \widetilde{\psi}(s,t;l) &= \psi(s,t;l) - \psi(s)\psi(t), \\ \widetilde{\phi}_0(s,t) &= \widetilde{\phi}(s,t) - \widetilde{\phi}_1(s)\widetilde{\phi}_2(t), \\ s_{l,T} &= \frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)\widetilde{\psi}(s,t;l)dsdt. \end{aligned} \tag{5.22}$$

Then (5.20) implies

$$|Cov_x(a_k, a_{k+l}) - s_{l,T}| \le \left|\frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)[\widetilde{\phi}_0(s,t) - \widetilde{\psi}(s,t;l)]dsdt\right| + |R|.$$

We claim that

$$\left|\frac{1}{(2\pi)^2}\int\int \hat{h}_T(s,t)[\widetilde{\phi}_0(s,t) - \widetilde{\psi}(s,t;l)]dsdt\right| \le C\kappa^{k-1}T^2, \tag{5.23}$$

which implies

$$|Cov_x(a_k, a_{k+l}) - s_{l,T}| \quad \le \quad C\kappa^{k-1}T^2 + CT^{-1}, \tag{5.24}$$

which yields for any $k, m \ge 1$,

$$|Cov_x(a_k, a_{k+l}) - Cov_x(a_m, a_{m+l})| \le C\kappa^{\min\{k-1,m-1\}}T^2 + CT^{-1}. \tag{5.25}$$

By choosing $T = \kappa^{-\frac{1}{4}\min\{k-1,m-1\}}$, we obtain

$$|Cov_x(a_k, a_{k+l}) - Cov_x(a_m, a_{m+l})| \le C\kappa^{\min\{\frac{k-1}{4},\frac{m-1}{4}\}}. \tag{5.26}$$

Hence we can say that $(Cov_x(a_k, a_{k+l}))_l$ is a Cauchy sequence, thus it converges to some limit, denoted by $s_l(x)$. When $k \to +\infty$, (5.24) becomes

$$|s_l(x) - s_{l,T}| \le CT^{-1}.$$

Now let $T = T(\ell) = \kappa^{-\ell}$, we obtain $|s_l(x) - s_{l,T(\ell)}| \le C\kappa^{\ell}$. Let $\ell \to +\infty$, we can see that $s_{l,T(\ell)} \to s_l(x)$. Since $s_{l,T(\ell)}$ does not depend on x, so is $s_l(x)$, i.e. $s_l(x) = s_l$. Now let $m \to +\infty$ in (5.26), we obtain

$$|Cov_x(a_k, a_{k+l}) - s_l| \le C\kappa^{\frac{k-1}{4}}.$$

Now we prove the claim (5.23). By definitions in (5.15) and (5.22), we obtain

$$\left|\widetilde{\phi}_0(s,t) - \widetilde{\psi}(s,t;l)\right| \le \left|\widetilde{\phi}(s,t) - \psi(s,t;l)\right| + \left|\widetilde{\phi}_1(s)\widetilde{\phi}_2(t) - \psi(s)\psi(t)\right|. \tag{5.27}$$

On the one hand, we can see that

$$\begin{aligned}
&\left|\widetilde{\phi}(s,t)-\psi(s,t;k)\right| \\
=\ &\left|P^{k-1}P_sP^{l-1}P_t\mathbf{1}(x)\varphi(s)\varphi(t)-\nu(P_sP^{l-1}P_t\mathbf{1})\varphi(s)\varphi(t)\right| \\
=\ &\left|\Pi P_sP^{l-1}P_t\mathbf{1}(x)+R^{k-1}P_s\Pi P_t\mathbf{1}(x)+R^{k-1}P_sR^{l-1}P_t\mathbf{1}(x)-\nu(P_sP^{l-1}P_t\mathbf{1})\right| \\
=\ &\left|R^{k-1}P_s\mathbf{1}(x)\nu(P_t\mathbf{1})+R^{k-1}P_sR^{l-1}P_t\mathbf{1}(x)\right|\leq C\kappa^{k-1}. \qquad (5.28)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\left|\widetilde{\phi}_1(s)\widetilde{\phi}_2(t)-\psi(s)\psi(t)\right| &= \left|[\widetilde{\phi}_1(s)-\psi(s)]\widetilde{\phi}_2(t)+\psi(s)[\widetilde{\phi}_2(t)-\psi(t)]\right| \\
&\leq \left|\widetilde{\phi}_1(s)-\psi(s)\right|+\left|\widetilde{\phi}_2(t)-\psi(t)\right| \\
&\leq \left|\phi_1(s)\varphi(s)-\psi(s)\right|+\left|\phi_2(t)\varphi(t)-\psi(t)\right|,
\end{aligned}$$

where as long as $k \geq 2$,

$$\begin{aligned}
\left|\phi_1(s)\varphi(s)-\psi(s)\right| &= \left|\left[\Pi P_s\mathbf{1}(x)+R^{k-1}P_s\mathbf{1}(x)\right]\mathbb{E}_x[e^{isV}]-\nu(P_s\mathbf{1})\mathbb{E}_x[e^{isV}]\right| \\
&\leq \left|[\Pi P_s\mathbf{1}(x)-\nu(P_s\mathbf{1})]+R^{k-1}P_s\mathbf{1}(x)\right| \\
&= \left|R^{k-1}P_s\mathbf{1}(x)\right|\leq C\kappa^{k-1}.
\end{aligned}$$

Similarly, we obtain

$$\left|\widetilde{\phi}_1(s)\widetilde{\phi}_2(t)-\psi(s)\psi(t)\right|\leq C\kappa^{k-1}. \qquad (5.29)$$

Therefore, (5.27), (5.28) and (5.29) imply $\left|\widetilde{\phi}_0(s,t)-\widetilde{\psi}(s,t;l)\right| \leq C\kappa^{k-1}$ which yields the assertion of the claim.

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