ALEA, Lat. Am. J. Probab. Math. Stat. **15**, 353–374 (2018)
DOI: 10.30757/ALEA.v15-15

Large time unimodality for classical and free Brownian motions with initial distributions

Takahiro Hasebe and Yuki Ueda

Department of Mathematics, Hokkaido University,
Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
E-mail address: thasebe@math.sci.hokudai.ac.jp
URL: https://www.math.sci.hokudai.ac.jp/~thasebe/

Department of Mathematics, Hokkaido University,
Kita 10, Nishi 8, Kita-Ku, Sapporo, Hokkaido, 060-0810, Japan
E-mail address: yuuki1114@math.sci.hokudai.ac.jp

Abstract. We prove that classical and free Brownian motions with initial distributions are unimodal for sufficiently large time, under some assumption on the initial distributions. The assumption is almost optimal in some sense. Similar results are shown for a symmetric stable process with index 1 and a positive stable process with index 1/2. We also prove that free Brownian motion with initial symmetric unimodal distribution is unimodal, and discuss strong unimodality for free convolution.

1. Introduction

A Borel measure μ on $\mathbb{R}$ is *unimodal* if there exist $a \in \mathbb{R}$ and a function $f\colon \mathbb{R} \to [0,\infty)$ which is non-decreasing on $(-\infty, a)$ and non-increasing on (a,∞), such that

$$\mu(dx) = \mu(\{a\})\delta_a + f(x)\,dx. \tag{1.1}$$

The most outstanding result on unimodality is Yamazato's theorem (Yamazato, 1978) saying that all classical selfdecomposable distributions are unimodal. After this result, in Hasebe and Thorbjørnsen (2016), Hasebe and Thorbjørnsen proved the free analog of Yamazato's result: all freely selfdecomposable distributions are unimodal. The unimodality has several other similarity points between the classical and free probability theories Hasebe and Sakuma (2017). However, it is not true that the unimodality shows complete similarity between the classical and free probability theories. For example, classical compound Poisson processes are likely to be

Received by the editors October 27th, 2017; accepted March 23th, 2018.
2010 *Mathematics Subject Classification.* Primary 46L54; Secondary 60E07; 60G52, 60J65.
Key words and phrases. unimodal, strongly unimodal, Brownian motion, Cauchy process, positive stable process.
Research supported by JSPS Grant-in-Aid for Young Scientists (B) 15K17549 (for TH) and JSPS and MAEDI Japan–France Integrated Action Program (SAKURA) (for both authors).

non-unimodal in large time (Wolfe, 1978), while free Lévy processes with compact support become unimodal in large time (Hasebe and Sakuma, 2017). In this paper, we mainly focus on the unimodality of classical and free Brownian motions with initial distributions and consider whether the classical and free versions share similarity points or not. In free probability theory, the semicircle distribution is the free analog of the normal distribution. In random matrix theory, it appears as the limit of eigenvalue distributions of Wigner matrices as the size of the random matrices goes to infinity. Free Brownian motion is defined as a process with free independent increments, started at 0 and distributed as the centered semicircle distribution $S(0,t)$ at time $t>0$. Furthermore, we can provide free Brownian motion with initial distribution μ which is a process with free independent increments distributed as μ at $t=0$ and as $\mu \boxplus S(0,t)$ at time $t>0$. The definition of $\boxplus$, called additive free convolution, is provided in Section 2.1. The additive free convolution is the distribution of the sum of two random variables which are freely independent (for the definition of free independence see Nica and Speicher, 2006). In Biane (1997), Biane gave the density function formula of free Brownian motion with initial distribution (see Section 2.2). In our studies, we first consider the symmetric Bernoulli distribution $\mu := \frac{1}{2}\delta_{+1} + \frac{1}{2}\delta_{-1}$ as an initial distribution and compute the density function of $\mu \boxplus S(0,t)$. Then we see that the probability distribution $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4$ (and it is not unimodal for $0<t<4$). This computation leads to a natural problem:

Problem 1.1. For which class of probability measures μ on $\mathbb{R}$ does the free Brownian motion with initial distribution μ become unimodal for sufficiently large time?

We then answer to this problem as follows (formulated as Theorem 3.2 and Proposition 3.4 in Section 3.1):

Theorem 1.2. (1) *Let μ be a compactly supported probability measure on $\mathbb{R}$ and $D_\mu := \sup\{|x-y| : x,y \in \mathrm{supp}(\mu)\}$. Then $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4D_\mu^2$.*

(2) *Let $f\colon \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then there exists a probability measure μ on $\mathbb{R}$ such that $\mu \boxplus S(0,t)$ is not unimodal for any $t>0$ and*

$$\int_{\mathbb{R}} f(x)\, d\mu(x) < \infty. \tag{1.2}$$

Note that such a measure μ is not compactly supported by (1).

The function f can grow very fast such as e^{x^2}, and so, a tail decay of the initial distribution does not imply large time unimodality.

The corresponding classical problem is natural, that is, for which class of initial distributions on $\mathbb{R}$ does Brownian motion become unimodal for sufficiently large time $t>0$? We prove in this direction the following results (formulated as Theorem 4.3 and Proposition 4.2 in Section 4, respectively):

Theorem 1.3. (1) *Let μ be a probability measure on $\mathbb{R}$ such that*

$$\alpha := \int_{\mathbb{R}} e^{\varepsilon x^2}\, d\mu(x) < \infty \tag{1.3}$$

for some $\varepsilon > 0$. Then the distribution $\mu * N(0,t)$ is unimodal for all $t \geq \frac{36\log(2\alpha)}{\varepsilon}$.

(2) *There exists a probability measure μ on $\mathbb{R}$ satisfying that*

$$\int_{\mathbb{R}} e^{A|x|^p}\, d\mu(x) < \infty \quad \text{for all } A > 0 \text{ and } 0 < p < 2 \tag{1.4}$$

such that $\mu * N(0,t)$ is not unimodal for any $t > 0$.

Thus, in the classical case, the tail decay (1.3) is sufficient and almost necessary to guarantee the large time unimodality. From these results, we conclude that the classical and free versions of the problems of unimodality for Brownian motions with initial distributions share common features as both become unimodal in large time under each assumption.

Further related results in this paper are as follows. In Section 3.2 we prove that $\mu \boxplus S(0,t)$ is always unimodal whenever μ is symmetric around 0 and unimodal (see Theorem 3.6). In Section 3.3, we define freely strongly unimodal probability measures as a natural free analogue of strongly unimodal probability measures. We conclude that the semicircle distributions are not freely strongly unimodal (Lemma 3.9). More generally, we have the following result (restated as Theorem 3.10 later):

Theorem 1.4. *Let λ be a probability measure with finite variance, not being a delta measure. Then λ is not freely strongly unimodal.*

On the other hand, there are many strongly unimodal distributions with finite variance in classical probability including the normal distributions and exponential distributions. Thus the notion of strong unimodality breaks the similarity between the classical and free probability theories.

In Section 5, we focus on other processes with classical/free independent increments with initial distributions. We consider a symmetric stable process with index 1 and a positive stable process with index 1/2. Then we prove large time unimodality similar to the case of Brownian motion but under different tail decay assumptions.

2. Preliminaries

2.1. *Definition of free convolution.* Let μ be a probability measure on $\mathbb{R}$. The Cauchy transform

$$G_\mu(z) := \int_{\mathbb{R}} \frac{1}{z-x}\, d\mu(x) \tag{2.1}$$

is analytic on the complex upper half plane $\mathbb{C}^+$. We define the truncated cones

$$\Gamma^{\pm}_{\alpha,\beta} := \{z \in \mathbb{C}^{\pm} : |\mathrm{Re}(z)| < \alpha|\mathrm{Im}(z)|,\ \mathrm{Im}(z) > \beta\}, \qquad \alpha > 0, \beta \in \mathbb{R}. \tag{2.2}$$

In Bercovici and Voiculescu (1993, Proposition 5.4), it was proved that for any $\gamma < 0$ there exist $\alpha, \beta > 0$ and $\delta < 0$ such that G_μ is univalent on $\Gamma^+_{\alpha,\beta}$ and $\Gamma^-_{\gamma,\delta} \subset G_\mu(\Gamma^+_{\alpha,\beta})$. Therefore the right inverse function G_μ^{-1} exists on $\Gamma^-_{\gamma,\delta}$. We define the *R-transform of μ* by

$$R_\mu(z) := G_\mu^{-1}(z) - \frac{1}{z}, \quad z \in \Gamma^-_{\gamma,\delta}. \tag{2.3}$$

Then for any probability measures μ and ν on $\mathbb{R}$ there exists a unique probability measure λ on $\mathbb{R}$ such that

$$R_\lambda(z) = R_\mu(z) + R_\nu(z), \tag{2.4}$$

for all z in the intersection of the domains of the three transforms. We denote $\lambda := \mu \boxplus \nu$ and call it the *(additive) free convolution* of μ and ν.

2.2. *Free convolution with semicircle distributions.* Almost all the materials and methods in this section are following (Biane, 1997). The *semicircle distribution* $S(0,t)$ of mean 0 and variance $t > 0$ is the probability measure with density

$$\frac{1}{2\pi t}\sqrt{4t - x^2}, \tag{2.5}$$

supported on the interval $[-2\sqrt{t}, 2\sqrt{t}]$. We then compute its Cauchy transform and its R-transform (see, for example, Nica and Speicher, 2006):

$$G_{S(0,t)}(z) = \frac{z - \sqrt{z^2 - 4t}}{2t}, \quad z \in \mathbb{C}^+, \tag{2.6}$$

where the branch of the square root on $\mathbb{C} \setminus \mathbb{R}^+$ is such that $\sqrt{-1} = \mathrm{i}$. Moreover

$$R_{S(0,t)}(z) = tz, \quad z \in \mathbb{C}^-. \tag{2.7}$$

Let μ be a probability measure on $\mathbb{R}$ and let $\mu_t = \mu \boxplus S(0,t)$. Then we define the following set:

$$U_t := \left\{ u \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2}\, d\mu(x) > \frac{1}{t} \right\}, \tag{2.8}$$

and the following function from $\mathbb{R}$ to $[0,\infty)$ by setting

$$v_t(u) := \inf \left\{ v \geq 0 \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2 + v^2}\, d\mu(x) \leq \frac{1}{t} \right\}. \tag{2.9}$$

Then v_t is continuous on $\mathbb{R}$ and one has $U_t = \{x \in \mathbb{R} \mid v_t(u) > 0\}$. Moreover, for every $u \in U_t$, $v_t(u)$ is the unique solution $v > 0$ of the equation

$$\int_{\mathbb{R}} \frac{1}{(x-u)^2 + v^2}\, d\mu(x) = \frac{1}{t}. \tag{2.10}$$

We define

$$\Omega_{t,\mu} := \{x + \mathrm{i}y \in \mathbb{C} \mid y > v_t(x)\}. \tag{2.11}$$

Then the map $H_t(z) := z + R_{S(0,t)}(G_\mu(z))$, defined on $\mathbb{C}^+$, is a homeomorphism from $\overline{\Omega_{t,\mu}}$ to $\mathbb{C}^+ \cup \mathbb{R}$ and it is conformal from $\Omega_{t,\mu}$ onto $\mathbb{C}^+$. Hence there exists an inverse function $F_t : \mathbb{C}^+ \cup \mathbb{R} \to \overline{\Omega_{t,\mu}}$. By Biane (1997, Lemma 1, Proposition 1), the domain $\Omega_{t,\mu}$ is equal to the connected component of the set $H_t^{-1}(\mathbb{C}^+)$ which contains $\mathrm{i}y$ for large $y > 0$ and for all $z \in \mathbb{C}^+$ one has

$$\begin{aligned} G_{\mu_t}(z) &= G_{\mu_t}(H_t(F_t(z))) \\ &= G_{\mu_t}\Big(F_t(z) + R_{S(0,t)}\big(G_\mu(F_t(z))\big)\Big) \\ &= G_\mu(F_t(z)). \end{aligned} \tag{2.12}$$

Moreover G_{μ_t} has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$. Then μ_t is Lebesgue absolutely continuous by Stieltjes inversion and the density function $p_t \colon \mathbb{R} \to [0,\infty)$ of μ_t is given by

$$p_t(\psi_t(x)) = -\frac{1}{\pi} \lim_{y \to +0} \mathrm{Im}(G_{\mu_t}(\psi_t(x) + \mathrm{i}y)) = \frac{v_t(x)}{\pi t}, \quad x \in \mathbb{R}, \tag{2.13}$$

where

$$\psi_t(x) = H_t(x + \mathrm{i}v_t(x)) = x + t \int_{\mathbb{R}} \frac{(x-u)}{(x-u)^2 + v_t(x)^2}\, d\mu(u). \tag{2.14}$$

It can be shown that ψ_t is a homeomorphism of $\mathbb{R}$. Furthermore the topological support of p_t is given by $\psi_t(\overline{U_t})$, and p_t extends to a continuous function on $\mathbb{R}$ and is real analytic in $\{x \in \mathbb{R} : p_t(x) > 0\}$.

Example 2.1. Let $\mu := \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_{+1}$ be the symmetric Bernoulli distribution. Recall that the topological support of the density function p_t is equal to $\psi_t(\overline{U_t})$. If $0 < t \leq 1$, we have

$$U_t = \left\{ u \in \mathbb{R} \;\middle|\; \sqrt{\frac{2 + t - \sqrt{t^2 + 8t}}{2}} < |u| < \sqrt{\frac{2 + t + \sqrt{t^2 + 8t}}{2}} \right\}, \tag{2.15}$$

and if $t > 1$, we have

$$U_t = \left\{ u \in \mathbb{R} \;\middle|\; -\sqrt{\frac{2 + t + \sqrt{t^2 + 8t}}{2}} < u < \sqrt{\frac{2 + t + \sqrt{t^2 + 8t}}{2}} \right\}. \tag{2.16}$$

If $0 < t \leq 1$ then the number of connected components of U_t is two, and therefore μ_t is not unimodal. If $t > 1$ then U_t is connected. By the density formula (2.13), the unimodality of μ_t is equivalent to the unimodality of the function $v_t(u)$, the latter being expressed in the form

$$v_t(u) = \sqrt{\frac{-(2u^2 + 2 - t) + \sqrt{t^2 + 16u^2}}{2}}. \tag{2.17}$$

Calculating its first derivative then shows that μ_t is unimodal if and only if $t \geq 4$; see Figures 2.1–2.6.

According to Example 2.1, we have a question about what class of initial distributions implies the large time unimodality of free Brownian motion. We will give a result for this in Section 3.1.

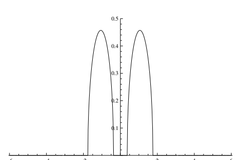

FIGURE 2.1. $p_{0.25}(x)$

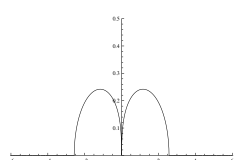

FIGURE 2.2. $p_1(x)$

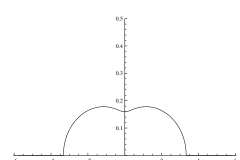

FIGURE 2.3. $p_2(x)$

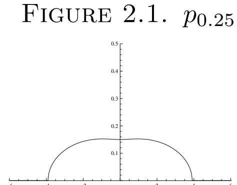

FIGURE 2.4. $p_3(x)$

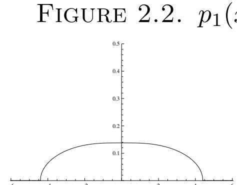

FIGURE 2.5. $p_4(x)$

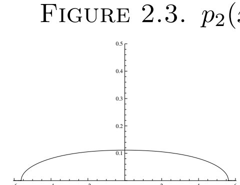

FIGURE 2.6. $p_7(x)$

2.3. *A basic lemma for unimodality.* A unimodal distribution is allowed to have a plateau or a discontinuous point in its density such as the uniform distribution or the exponential distribution. If we exclude such cases, then unimodality can be characterized in terms of levels of density.

Lemma 2.2. *Let μ be a probability measure on $\mathbb{R}$ that is Lebesgue absolutely continuous. Suppose that its density $p(x)$ extends to a continuous function on $\mathbb{R}$ and is real analytic in $\{x \in \mathbb{R} : p(x) > 0\}$. Then μ is unimodal if and only if, for any $a > 0$, the equation $p(x) = a$ has at most two solutions x.*

The proof is just to use the intermediate value theorem and the fact that a real analytic function never has plateaux. Note that the idea of the above lemma was first introduced by Haagerup and Thorbjørnsen (2014) to prove that a "free gamma distribution" is unimodal.

3. Free Brownian motion with initial distribution

3.1. *Large time unimodality for free Brownian motion.* The semicircular distribution is the free analogue of the normal distribution. Therefore, a process with free independent increments is called *(standard) free Brownian motion with initial distribution* μ if its distribution at time t is given by $\mu \boxplus S(0,t)$. In this section, we firstly prove that free Brownian motion with compactly supported initial distribution is unimodal for sufficiently large time, by using Biane's density formula (see Section 2.2).

Lemma 3.1. *Suppose that $t > 0$. The following statements are equivalent.*

(1) $\mu \boxplus S(0,t)$ is unimodal.
(2) For any $R > 0$ the equation

$$\int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x) = \frac{1}{t} \tag{3.1}$$

has at most two solutions $u \in \mathbb{R}$.

Proof: We adopt the notations $\mu_t = \mu \boxplus S(0,t), v_t, p_t$ and ψ_t in Section 2.2. This proof is similar to Hasebe and Thorbjørnsen (2016, Proposition 3.8). Recall that μ_t is absolutely continuous with respect to Lebesgue measure and the density function p_t is continuous on $\mathbb{R}$ since its Cauchy transform G_{μ_t} has a continuous extension to $\mathbb{C}^+ \cup \mathbb{R}$, and the density function is real analytic in $\psi_t(U_t)$ and continuous on $\mathbb{R}$ by Section 2.2.

Suppose (2) first. Since ψ_t is a homeomorphism of $\mathbb{R}$, by Lemma 2.2 it suffices to show that for any $a > 0$ the equation

$$a = p_t(\psi_t(u)) = \frac{v_t(u)}{\pi t}, \quad u \in \mathbb{R} \tag{3.2}$$

has at most two solutions $u \in \mathbb{R}$. Since $v_t(u)$ is a unique solution of the equation (2.10) if it is positive, then we have

$$\left\{ u \in \mathbb{R} \;\middle|\; a = \frac{v_t(u)}{\pi t} \right\} = \left\{ u \in \mathbb{R} \;\middle|\; \int_{\mathbb{R}} \frac{1}{(x-u)^2 + (\pi a t)^2}\, d\mu(x) = \frac{1}{t} \right\}. \tag{3.3}$$

By the assumption (2), the equation $a = \frac{v_t(u)}{\pi t}$ has at most two solutions in $\mathbb{R}$.

Conversely, suppose that (2) does not hold. Then for some $R > 0$ there are three distinct solutions $u_1, u_2, u_3 \in \mathbb{R}$ to (3.1). By (3.3) this shows that $v_t(u_i) = R$ and hence $p_t(\psi_t(u_i)) = \frac{R}{\pi t}$ for $i = 1, 2, 3$. Again by Lemma 2.2, μ_t is not unimodal. □

Now we are ready to prove Theorem 1.2 (1).

Theorem 3.2. *Let μ be a compactly supported probability measure, and let D_μ be the diameter of the support: $D_\mu = \sup\{|x-y| : x, y \in \mathrm{supp}(\mu)\}$. Then $\mu \boxplus S(0,t)$ is unimodal for $t \geq 4D_\mu^2$.*

Proof: Applying a shift we may assume that μ is supported on $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$. For fixed $R > 0$, we define by

$$\xi_R(u) := \int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x), \quad u \in \mathbb{R}. \tag{3.4}$$

Then we have

$$\begin{aligned} \xi_R'(u) &= \int_{-\frac{D_\mu}{2}}^{\frac{D_\mu}{2}} \frac{2(x-u)}{\{(x-u)^2 + R^2\}^2}\, d\mu(x), \\ \xi_R''(u) &= \int_{-\frac{D_\mu}{2}}^{\frac{D_\mu}{2}} \frac{6(x-u)^2 - 2R^2}{\{(x-u)^2 + R^2\}^3}\, d\mu(x). \end{aligned} \tag{3.5}$$

If $u < -\frac{D_\mu}{2}$, then $x - u > -\frac{D_\mu}{2} - (-\frac{D_\mu}{2}) = 0$ for all $x \in \mathrm{supp}(\mu)$, so that $\xi_R'(u) > 0$. If $u > \frac{D_\mu}{2}$, then $x - u < \frac{D_\mu}{2} - \frac{D_\mu}{2} = 0$ for all $x \in \mathrm{supp}(\mu)$, so that $\xi_R'(u) < 0$. We take $t \geq 4D_\mu^2$ and consider the form of $\xi_R(u)$ on $u \in (-\frac{D_\mu}{2}, \frac{D_\mu}{2})$.
If $0 < R < \sqrt{3}D_\mu$, then $(x-u)^2 + R^2 < (\frac{D_\mu}{2} + \frac{D_\mu}{2})^2 + (\sqrt{3}D_\mu)^2 = 4D_\mu^2 \leq t$, so that

$$\xi_R(u) = \int_{\mathbb{R}} \frac{1}{(x-u)^2 + R^2}\, d\mu(x) > \frac{1}{t}, \quad u \in \left(-\frac{D_\mu}{2}, \frac{D_\mu}{2}\right). \tag{3.6}$$

Hence the equation $\xi_R(u) = \frac{1}{t}$ has at most two solutions $u \in \mathbb{R}$ if $t \geq 4D_\mu^2$. If $R \geq \sqrt{3}D_\mu$, then the function $\xi_R(u)$ satisfies the following three conditions:

- $\xi_R'(u) > 0$ for all $u < -\frac{D_\mu}{2}$ and $\xi_R'(u) < 0$ for all $u > \frac{D_\mu}{2}$,
- ξ_R' is continuous on $\mathbb{R}$,
- $\xi_R''(u) < 0$ for all $u \in \left(-\frac{D_\mu}{2}, \frac{D_\mu}{2}\right)$.

By the intermediate value theorem, ξ_R' has a unique zero in $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$ and hence ξ_R takes a unique local maximum in the interval $[-\frac{D_\mu}{2}, \frac{D_\mu}{2}]$ (and therefore it is a unique global maximum on $\mathbb{R}$). Therefore the equation $\xi_R(u) = \frac{1}{t}$ has at most two solutions if $t \geq 4D_\mu^2$. Hence we have that the free convolution $\mu \boxplus S(0,t)$ is unimodal if $t \geq 4D_\mu^2$ by Lemma 3.1. □

Problem 3.3. What is the optimal universal constant $C > 0$ such that $\mu \boxplus S(0,t)$ is unimodal for all $t \geq CD_\mu^2$ and all probability measures μ with compact support?

We have already shown that $C \leq 4$. Recall from Example 2.1 that if μ is the symmetric Bernoulli distribution on $\{-1, 1\}$, then $\mu \boxplus S(0,t)$ is unimodal if and only if $t \geq 4$. Since the diameter D_μ of the support of μ is 2, we conclude that $2 \leq C \leq 4$.

The next question is whether there exists a probability measure μ such that $\mu \boxplus S(0,t)$ is not unimodal for any $t > 0$ or at least for sufficiently large $t > 0$.

Such a distribution must have an unbounded support if it exists. Such an example can be constructed with an idea similar to Huang (2015, Proposition 4.13) (see also Hasebe and Sakuma, 2017, Example 5.3).

Proposition 3.4. *Let $f\colon \mathbb{R} \to [0,\infty)$ be a Borel measurable function. Then there exists a probability measure μ such that $\mu \boxplus S(0,t)$ is not unimodal at any $t>0$ and*

$$\int_{\mathbb{R}} f(x)\, d\mu(x) < \infty. \tag{3.7}$$

Proof: We follow the notations in Section 2.2. Let $\{w_n\}_n$ and $\{a_n\}_n$ be sequences in $\mathbb{R}$ satisfying

- $w_n > 0, \sum_{n=1}^{\infty} w_n = 1$,
- $a_{n+1} > a_n, \quad n \geq 1$,
- $\lim_{n\to\infty} (a_{n+1} - a_n) = \infty$.

Consider the probability measure $\mu := \sum_{n=1}^{\infty} w_n \delta_{a_n}$ on $\mathbb{R}$ and define the function

$$X_\mu(u) := \int_{\mathbb{R}} \frac{1}{(u-x)^2}\, d\mu(x) = \sum_{n=1}^{\infty} \frac{w_n}{(u-a_n)^2}. \tag{3.8}$$

Recall that U_t is the set of $u \in \mathbb{R}$ such that $X_\mu(u) > 1/t$. Set $b_k := \frac{a_{k+1}+a_k}{2}$ for all $k \in \mathbb{N}$. Then we have

$$|b_k - a_n| \geq \frac{a_{k+1}-a_k}{2}, \quad k,n \in \mathbb{N} \tag{3.9}$$

and so

$$X_\mu(b_k) \leq \left(\frac{2}{a_{k+1}-a_k}\right)^2 \sum_{n=1}^{\infty} w_n = \left(\frac{2}{a_{k+1}-a_k}\right)^2 \to 0$$

as $k \to \infty$. This means that for each $t > 0$ there exists an integer $K(t) > 0$ such that $X_\mu(b_k) < \frac{1}{t}$ for all $k \geq K(t)$. This implies that, for $k \geq K(t)$, the closure of U_t does not contain b_k. Note that the set U_t is an open set. Therefore, there exists a sequence $\{\varepsilon_k(t)\}_{k=K(t)+1}^{\infty}$ of positive numbers such that

$$U_t \cap (b_{K(t)}, \infty) = \bigcup_{k=K(t)+1}^{\infty} (a_k - \varepsilon_k(t), a_k + \varepsilon_k(t)), \tag{3.10}$$

where the closures of the intervals are disjoint. Thus the set $\mathrm{supp}(\mu_t) = \psi_t(\overline{U_t})$ consists of infinitely many connected components for all $t > 0$, and hence μ_t is not unimodal for any $t > 0$.

In the above construction, the weights w_n are only required to be positive. Therefore, we may take

$$w_n = \frac{c}{n^2 \max\{f(a_n), 1\}}, \tag{3.11}$$

where $c > 0$ is a normalizing constant. Then the integrability condition (3.7) is satisfied. □

Proposition 3.4 shows that tail decay estimates of the initial distribution do not guarantee the large time unimodality. In Section 4 we shall see that the situation is different for classical Brownian motion.

3.2. *Free Brownian motion with symmetric initial distribution.* This section proves a unimodality result of different flavor. It is well known that if μ and ν are symmetric unimodal distributions, then $\mu * \nu$ is also (symmetric) unimodal (see Sato, 2013, Exercise 29.22). The free analogue of this statement is not known.

Conjecture 3.5. *Let μ and ν be symmetric unimodal distributions. Then $\mu \boxplus \nu$ is unimodal.*

We can give a positive answer in the special case when one distribution is a semicircle distribution.

Theorem 3.6. *Let μ be a symmetric unimodal distribution on $\mathbb{R}$. Then $\mu \boxplus S(0,t)$ is unimodal for any $t > 0$.*

Remark 3.7. There is a unimodal probability measure μ such that $\mu \boxplus S(0,1)$ is not unimodal (see Lemma 3.9). Hence we cannot remove the assumption of symmetry of μ.

Proof of Theorem 3.6. By Lemma 3.1 it suffices to show that for any $R > 0$ the equation

$$\int_{\mathbb{R}} \frac{1}{(u-x)^2 + R^2}\, d\mu(x) = \frac{1}{t} \tag{3.12}$$

has at most two solutions $u \in \mathbb{R}$. Up to a constant multiplication, the LHS is the density of $\mu * C_R$, which is unimodal since μ and C_R are symmetric unimodal. Since the density of $\mu * C_R$ is real analytic, Lemma 2.2 implies that the equation (3.12) has at most two solutions. This shows that $\mu \boxplus S(0,t)$ is unimodal. □

3.3. *Freely strong unimodality.* In classical probability, a probability measure is said to be *strongly unimodal* if $\mu * \nu$ is unimodal for all unimodal distributions ν on $\mathbb{R}$. A distribution is strongly unimodal if and only if the distribution is Lebesgue absolutely continuous, supported on an interval and a version of its density is log-concave (see Sato, 2013, Theorem 52.3). From this characterization, the normal distributions are strongly unimodal. We discuss the free version of strong unimodality.

Definition 3.8. A probability measure μ on $\mathbb{R}$ is said to be *freely strongly unimodal* if $\mu \boxplus \nu$ is unimodal for all unimodal distributions ν on $\mathbb{R}$.

Lemma 3.9. *The semicircle distributions are not freely strongly unimodal.*

Proof: The Cauchy distribution is not strongly unimodal since its density is not log-concave. Hence, there exists a unimodal probability measure μ such that $\mu * C_1$ is not unimodal. Since the density of $\mu * C_1$ is real analytic on $\mathbb{R}$, by Lemma 2.2 there exists $t > 0$ such that the equation

$$\pi \frac{d(\mu * C_1)}{dx}(x) = \int_{\mathbb{R}} \frac{1}{(x-y)^2 + 1}\, d\mu(y) = \frac{1}{t} \tag{3.13}$$

has at least three distinct solutions $x_1, x_2, x_3 \in \mathbb{R}$. Lemma 3.1 then implies that $S(0,t) \boxplus \mu$ is not unimodal. By changing the scaling, we conclude that no semicircle distributions are freely strongly unimodal. □

Theorem 3.10. *Let λ be a probability measure with finite variance, not being a delta measure. Then λ is not freely strongly unimodal.*

Proof: We may assume that λ has mean 0. Suppose that λ is freely strongly unimodal. A simple induction argument shows that $\lambda^{\boxplus n}$ is freely strongly unimodal for all $n \in \mathbb{Z}_+$. We derive a contradiction below. By Lemma 3.9 we can take a unimodal probability measure μ such that $\mu \boxplus S(0,1)$ is not unimodal. Our hypothesis shows that the measure $\mathrm{D}_{\sqrt{nv}}(\mu) \boxplus \lambda^{\boxplus n}$ is unimodal, where v is the variance of λ and $\mathrm{D}_c(\rho)$ is the push-forward of a measure ρ by the map $x \mapsto cx$ for $c \in \mathbb{R}$. By the free central limit theorem (Maassen, 1992, Theorem 5.2), the unimodal distributions $\mathrm{D}_{1/\sqrt{nv}}(\mathrm{D}_{\sqrt{nv}}(\mu) \boxplus \lambda^{\boxplus n}) = \mu \boxplus \mathrm{D}_{1/\sqrt{nv}}(\lambda^{\boxplus n})$ weakly converge to $\mu \boxplus S(0,1)$ as $n \to \infty$. Since the set of unimodal distributions is weakly closed (see Sato, 2013, Exercise 29.20), we conclude that $\mu \boxplus S(0,1)$ is unimodal, a contradiction. □

Problem 3.11. Does there exist a freely strongly unimodal probability measure that is not a delta measure?

4. Classical Brownian motion with initial distribution

This section discusses large time unimodality for *classical standard Brownian motion with an initial distribution* μ, which is a process with independent increments distributed as $\mu * N(0,t)$ at time $t \geq 0$. Before going to the general case, one example will be helpful in understanding unimodality for $\mu * N(0,t)$.

Example 4.1. Elementary calculus shows that

$$\frac{1}{2}(\delta_{-1} + \delta_1) * N(0,t) \tag{4.1}$$

is unimodal if and only if $t \geq 1$; see Figures 4.7-4.12.

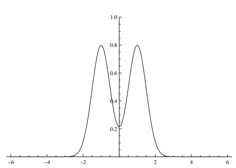

FIGURE 4.7. $t = 0.25$

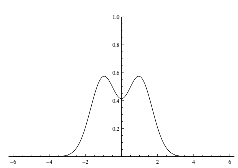

FIGURE 4.8. $t = 0.5$

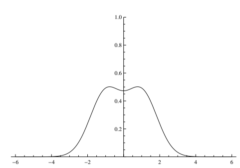

FIGURE 4.9. $t = 0.75$

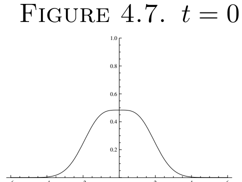

FIGURE 4.10. $t = 1$

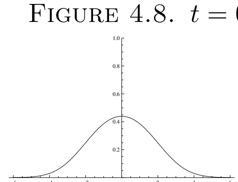

FIGURE 4.11. $t = 2$

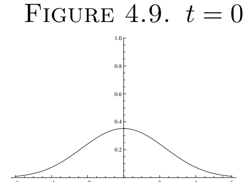

FIGURE 4.12. $t = 4$

This simple example suggests that Brownian motion becomes unimodal for sufficiently large time, possibly under some condition on the initial distribution. In some sense, we give an almost optimal condition on the initial distribution for the large time unimodality to hold. We start by providing an example of initial distribution with which the Brownian motion never becomes unimodal.

Proposition 4.2. *There exists a probability measure μ on $\mathbb{R}$ such that $\mu * N(0,t)$ is not unimodal at any $t > 0$, and*

$$\int_{\mathbb{R}} e^{A|x|^p} d\mu(x) < \infty \tag{4.2}$$

for all $A > 0$ and $0 < p < 2$.

Proof: We consider sequences $\{w_n\}_{n\in\mathbb{N}} \subset (0,\infty)$ and $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\sum_{n=1}^{\infty} w_n = 1, \tag{4.3}$$

$$b_k := \inf_{n\in\mathbb{Z}_+\setminus\{k\}} |a_k - a_n - 1| \to \infty \text{ as } k \to \infty. \tag{4.4}$$

Moreover we assume that for all $t > 0$ there exists $k_0 = k_0(t) \in \mathbb{N}$ such that

$$k \geq k_0(t) \Rightarrow w_k e^{-\frac{1}{2t}} > b_k e^{-\frac{b_k^2}{2t}}. \tag{4.5}$$

Define a probability measure μ by setting $\mu := \sum_{n=1}^{\infty} w_n \delta_{a_n}$ and the following function:

$$f_t(x) := \sqrt{2\pi t} \cdot \frac{d(\mu * N(0,t))}{dx}(x) = \sum_{n=1}^{\infty} w_n e^{-\frac{(x-a_n)^2}{2t}}, \quad x \in \mathbb{R}. \tag{4.6}$$

Then we have

$$f_t'(x) = \sum_{n=1}^{\infty} w_n \Big(-\frac{x-a_n}{t}\Big) e^{-\frac{(x-a_n)^2}{2t}}, \quad x \in \mathbb{R}. \tag{4.7}$$

If needed we may replace $k_0(t)$ by a larger integer so that $b_k > \sqrt{t}$ holds for all $k \geq k_0(t)$. For $k \geq k_0(t)$ we have

$$\begin{aligned} f_t'(a_k - 1) &= \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \Big(\frac{a_k - a_n - 1}{t}\Big) e^{-\frac{(a_k-a_n-1)^2}{2t}} \\ &\geq \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \frac{|a_k - a_n - 1|}{t} e^{-\frac{(a_k-a_n-1)^2}{2t}} \\ &\geq \frac{w_k}{t} e^{-\frac{1}{2t}} - \sum_{n\in\mathbb{N}, n\neq k} w_n \frac{b_k}{t} e^{-\frac{b_k^2}{2t}} \\ &\geq \frac{1}{t}\Big(w_k e^{-\frac{1}{2t}} - b_k e^{-\frac{b_k^2}{2t}}\Big) > 0, \end{aligned} \tag{4.8}$$

where we used the fact that $x \mapsto |x| e^{-\frac{x^2}{2t}}$ takes the global maximum at $x = \pm\sqrt{t}$ on the third inequality and the assumption (4.5) on the last line. This implies that $\mu * N(0,t)$ is not unimodal for any $t > 0$.

Next we take specific sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfying the conditions (4.3)–(4.5). Set $a_k = a^k$, $k \in \mathbb{N}$ where $a \geq 2$. Note that there exists some constant $c > 0$ such that $b_k \geq c a^k$ for all $k \in \mathbb{N}$. Hence we have that $b_k \to \infty$ as $k \to \infty$. Moreover we set

$$w_k := M e^{-\frac{b_k^2}{k}}, \tag{4.9}$$

where $M > 0$ is a normalized constant, that is, $M = (\sum_{k=1}^{\infty} e^{-\frac{b_k^2}{k}})^{-1}$ (note that the series converges). Since $b_k \to \infty$ as $k \to \infty$, we have

$$\frac{b_k e^{-\frac{b_k^2}{2t}}}{w_k} = \frac{1}{M} b_k e^{\frac{b_k^2}{k} - \frac{b_k^2}{2t}} \to 0, \quad \text{as } k \to \infty. \tag{4.10}$$

For all $t > 0$ there exists $k_0 = k_0(t) \in \mathbb{N}$ such that $k \geq k_0$ implies that $e^{-\frac{1}{2t}} > b_k e^{-\frac{b_k^2}{2t}} / w_k$. Therefore the sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfy the condition (4.5). Finally we show that $\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$ has the property (4.2). For every $A > 0$ and $0 < p < 2$, using the inequality $b_k \geq c a^k$ shows that

$$\begin{aligned} \int_{\mathbb{R}} e^{A|x|^p}\, d\mu(x) &= \sum_{k=1}^{\infty} w_k e^{A|a_k|^p} = M \sum_{k=1}^{\infty} e^{-k^{-1} b_k^2} e^{A a^{kp}} \\ &\leq M \sum_{k=1}^{\infty} e^{-A a^{pk}(c^2 A^{-1} k^{-1} a^{(2-p)k} - 1)} < \infty. \end{aligned} \tag{4.11}$$

Thus the proof is complete. □

Note that for any sequences $\{w_n\}_n$ and $\{a_n\}_n$ satisfying the conditions (4.3)–(4.5) the distribution $\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}$ has the following property:

$$\int_{\mathbb{R}} e^{\varepsilon x^2}\, d\mu(x) = \infty \text{ for all } \varepsilon > 0. \tag{4.12}$$

Then we have a natural question whether the large time unimodality of $\mu * N(0,t)$ holds or not if the initial distribution μ does not satisfy the condition (4.12). We solve this question as follows.

Theorem 4.3. *Let μ be a probability measure on $\mathbb{R}$ such that*

$$\alpha := \int_{\mathbb{R}} e^{\varepsilon x^2}\, d\mu(x) < \infty. \tag{4.13}$$

for some $\varepsilon > 0$. Then $\mu * N(0,t)$ is unimodal for $t \geq \frac{36 \log(2\alpha)}{\varepsilon}$.

Remark 4.4. The proof becomes much easier if we assume that μ has a compact support.

Proof of Theorem 4.3: For $x > 0$ we have

$$\mu(|y| > x) = \int_{|y|>x} 1\, d\mu(y) \leq \int_{|y|>x} \frac{e^{\varepsilon y^2}}{e^{\varepsilon x^2}}\, d\mu(y) \leq \alpha e^{-\varepsilon x^2}. \tag{4.14}$$

Let

$$f_t(x) := \sqrt{2\pi t} \cdot \frac{d(\mu * N(0,t))}{dx}(x) = \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y). \tag{4.15}$$

Then we have

$$\begin{aligned} f_t'(x) &= \int_{\mathbb{R}} \left(\frac{y-x}{t}\right) e^{-\frac{(y-x)^2}{2t}}\, d\mu(y) \\ &= \int_x^{\infty} \frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}}\, d\mu(y) - \int_{-\infty}^{x} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y). \end{aligned} \tag{4.16}$$

For $x > 0$, we have

$$\int_x^{\infty} \frac{y-x}{t} e^{-\frac{(y-x)^2}{2t}}\, d\mu(y) \leq \frac{\sqrt{t}}{t} e^{-\frac{1}{2}} \mu((x,\infty)) \leq \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2} - \varepsilon x^2}. \tag{4.17}$$

For $x \geq \frac{\sqrt{t}}{2}$, we have

$$\int_{-\infty}^{x} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \geq \int_{-3x}^{x-\frac{\sqrt{t}}{3}} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \geq \frac{1}{t} \min\left\{ \frac{\sqrt{t}}{3} e^{-\frac{1}{18}}, 4xe^{-\frac{8x^2}{t}} \right\} \mu\left(\left[-3x, x - \frac{\sqrt{t}}{3} \right] \right). \tag{4.18}$$

Now the function $g(x) := 4xe^{-\frac{8x^2}{t}}$ has a local maximum at $x = \frac{\sqrt{t}}{4}$, and hence for all $x \geq \frac{\sqrt{t}}{2}$,

$$g(x) \leq 2e^{-2}\sqrt{t} < \frac{1}{3} e^{-\frac{1}{18}} \sqrt{t}, \tag{4.19}$$

where $2e^{-2} \approx 0.2706$ and $\frac{1}{3}e^{-\frac{1}{18}} \approx 0.3153$. Hence we have

$$\begin{aligned} \int_{-\infty}^{x} \frac{x-y}{t} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) &\geq \frac{4x}{t} e^{-\frac{8x^2}{t}} \mu\left(\left[-3x, x - \frac{\sqrt{t}}{3} \right] \right) \\ &\geq \frac{4x}{t} e^{-\frac{8x^2}{t}} \mu\left(\left[-\frac{\sqrt{t}}{6}, \frac{\sqrt{t}}{6} \right] \right) \\ &\geq \frac{2}{\sqrt{t}} e^{-\frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}), \end{aligned} \tag{4.20}$$

where the last inequality holds thanks to (4.14) and $x \geq \frac{\sqrt{t}}{2}$. Therefore if $x \geq \frac{\sqrt{t}}{2}$ then

$$\begin{aligned} f_t'(x) &\leq \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2} - \varepsilon x^2} - \frac{2}{\sqrt{t}} e^{-\frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \\ &= \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2} - \varepsilon x^2} \left\{ 1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\varepsilon x^2 - \frac{8x^2}{t}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \right\}. \end{aligned} \tag{4.21}$$

If $t \geq \frac{16}{\varepsilon}$ then $e^{\varepsilon x^2 - \frac{8x^2}{t}} \geq e^{\frac{1}{2}\varepsilon x^2} \geq e^{\frac{\varepsilon t}{8}}$. Hence if $t \geq \frac{16}{\varepsilon}$ then

$$f_t'(x) \leq \frac{\alpha}{\sqrt{t}} e^{-\frac{1}{2} - \varepsilon x^2} \left\{ 1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\frac{\varepsilon t}{8}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \right\}. \tag{4.22}$$

If $t \geq \frac{36 \log(2\alpha)}{\varepsilon}$ then $1 - \alpha e^{-\frac{\varepsilon t}{36}} \geq \frac{1}{2}$ and we have

$$1 - \frac{2e^{\frac{1}{2}}}{\alpha} e^{\frac{\varepsilon t}{8}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \leq 1 - e^{\frac{1}{2}} \cdot 2^{\frac{9}{2}} \cdot \alpha^{\frac{7}{2}} < 0. \tag{4.23}$$

By taking $t \geq \max\{\frac{16}{\varepsilon}, \frac{36\log(2\alpha)}{\varepsilon}\} = \frac{36\log(2\alpha)}{\varepsilon}$, we have

$$f_t'(x) < 0 \quad \text{if } x \geq \frac{\sqrt{t}}{2}. \tag{4.24}$$

Similarly, we have

$$f_t'(x) > 0 \quad \text{if } x \leq -\frac{\sqrt{t}}{2}. \tag{4.25}$$

Next, we will show that $f_t''(x) < 0$ for all $x \in \mathbb{R}$ with $|x| < \frac{\sqrt{t}}{2}$. We then calculate the following

$$\begin{aligned} f_t''(x) &= \int_{\mathbb{R}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &= \int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &\quad - \int_{|x-y|\le\sqrt{t}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y). \end{aligned} \tag{4.26}$$

Since the function $h(u) := \frac{u^2-t}{t^2} e^{-\frac{u^2}{2t}}$ has a local maximum at $u = \pm\sqrt{3t}$, we have

$$\int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \le \frac{2}{t} e^{-\frac{3}{2}} \mu([x-\sqrt{t}, x+\sqrt{t}]^c). \tag{4.27}$$

For all $x \in \mathbb{R}$ with $|x| < \frac{\sqrt{t}}{2}$, we have $[x-\sqrt{t}, x+\sqrt{t}]^c \subset \left[-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\right]^c$, and therefore

$$\int_{|x-y|>\sqrt{t}} \frac{(x-y)^2 - t}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \le \frac{2}{t} e^{-\frac{3}{2}} \mu\Big(\Big[-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\Big]^c\Big) \le \frac{2}{t} e^{-\frac{3}{2}} \alpha e^{-\frac{\varepsilon t}{4}}. \tag{4.28}$$

Since the function $-h(u)$ is decreasing on $[0, \sqrt{3t}]$, we have

$$\begin{aligned} \int_{|x-y|\le\sqrt{t}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) &\ge \int_{|x-y|\le\frac{2\sqrt{t}}{3}} \frac{t-(x-y)^2}{t^2} e^{-\frac{(x-y)^2}{2t}}\, d\mu(y) \\ &\ge \frac{t-\frac{4}{9}t}{t^2} e^{-\frac{\frac{4}{9}t}{2t}} \mu\Big(\Big[x - \frac{2\sqrt{t}}{3}, x + \frac{2\sqrt{t}}{3}\Big]\Big) \\ &= \frac{5}{9t} e^{-\frac{2}{9}} \mu\Big(\Big[x - \frac{2\sqrt{t}}{3}, x + \frac{2\sqrt{t}}{3}\Big]\Big) \\ &\ge \frac{5}{9t} e^{-\frac{2}{9}} \mu\Big(\Big[-\frac{\sqrt{t}}{6}, \frac{\sqrt{t}}{6}\Big]\Big) \\ &\ge \frac{5}{9t} e^{-\frac{2}{9}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}). \end{aligned} \tag{4.29}$$

Therefore

$$\begin{aligned} f_t''(x) &\le \frac{2}{t} e^{-\frac{3}{2}} \alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9t} e^{-\frac{2}{9}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \\ &= \frac{1}{t} \Big\{ 2e^{-\frac{3}{2}} \alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9} e^{-\frac{2}{9}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \Big\}. \end{aligned} \tag{4.30}$$

If $t \ge \frac{36 \log(2\alpha)}{\varepsilon}$, then

$$2e^{-\frac{3}{2}} \alpha e^{-\frac{\varepsilon t}{4}} - \frac{5}{9} e^{-\frac{2}{9}} (1 - \alpha e^{-\frac{\varepsilon t}{36}}) \le \frac{e^{-\frac{3}{2}}}{(2\alpha)^8} - \frac{5}{18} e^{-\frac{2}{9}} < 0, \tag{4.31}$$

and therefore $f_t''(x) < 0$ if $|x| < \frac{\sqrt{t}}{2}$.

To summarize, if $t \geq \frac{36\log(2\alpha)}{\varepsilon}$, then we have the following properties:

$$\begin{aligned} x \leq -\frac{\sqrt{t}}{2} &\Rightarrow f_t'(x) > 0, \\ x \geq \frac{\sqrt{t}}{2} &\Rightarrow f_t'(x) < 0, \\ |x| < \frac{\sqrt{t}}{2} &\Rightarrow f_t''(x) < 0. \end{aligned} \tag{4.32}$$

Hence $f_t(x)$ has a unique local maximum in $\left(-\frac{\sqrt{t}}{2}, \frac{\sqrt{t}}{2}\right)$ when $t \geq \frac{36\log(2\alpha)}{\varepsilon}$, which is a unique global maximum on $\mathbb{R}$ as well. Therefore $\mu * N(0,t)$ is unimodal for $t \geq \frac{36\log(2\alpha)}{\varepsilon}$. □

Remark 4.5. If μ is unimodal then $\mu * N(0,t)$ is unimodal for all $t > 0$. This is a consequence of the strong unimodality of the normal distribution $N(0,t)$ (see Section 3.3), in contrast with the failure of freely strong unimodality of the semicircle distribution (see Lemma 3.9).

We close this section by placing a problem for future research.

Problem 4.6. Estimate the position of the mode of classical Brownian motion with initial distributions satisfying the assumption (4.13). Our proof shows that for $t \geq \frac{36}{\varepsilon}\log(2\alpha)$, the mode is located in the interval $[-\sqrt{t}/2, \sqrt{t}/2]$. How about free Brownian motion?

5. Large time unimodality for stable processes with index 1 and index 1/2 with initial distributions

We investigate large time unimodality for stable processes with index 1 (following Cauchy distributions) and index 1/2 (following Lévy distributions) with initial distributions.

5.1. *Cauchy process with initial distribution.* Let $\{C_t\}_{t\geq 0}$ be the symmetric Cauchy distribution

$$C_t(dx) := \frac{t}{\pi(x^2+t^2)} \cdot 1_{\mathbb{R}}(x)\,dx, \quad x \in \mathbb{R}, \qquad C_0 = \delta_0, \tag{5.1}$$

which forms both classical and free convolution semigroups. A *Cauchy process with initial distribution* μ is a process with independent increments that follows the law $\mu * C_t$ at time $t \geq 0$. It is known that the Cauchy distribution satisfies the identity

$$\mu \boxplus C_t = \mu * C_t \tag{5.2}$$

for any μ and $t \geq 0$, and so the distributions $\mu * C_t$ can also be realized as the laws at time $t \geq 0$ of a process with free independent increments with initial distribution μ. The authors do not know a written proof of (5.2) in the literature, so give a proof below. The Cauchy transform of the Cauchy distribution is given by $G_{C_t}(z) = 1/(z+\mathrm{i}t)$ on $\mathbb{C}^+$ (e.g. by the residue theorem), and hence $R_{C_t}(z) = -\mathrm{i}tz$. Then $R_{\mu\boxplus C_t}(z) = R_\mu(z) - \mathrm{i}tz$, and after some computation we can check that $G_{\mu\boxplus C_t}(z) = G_\mu(z+\mathrm{i}t)$. The Stieltjes inversion formula implies that for $t > 0$ the free convolution $\mu \boxplus C_t$ has the density

$$-\frac{1}{\pi}\mathrm{Im}(G_\mu(x+\mathrm{i}t)) = \int_{\mathbb{R}} \frac{t}{\pi((x-y)^2+t^2)}\,d\mu(y), \tag{5.3}$$

which is exactly the density of $\mu * C_t$.

As in the cases of free and classical BMs, taking μ to be the symmetric Bernoulli $\frac{1}{2}(\delta_{-1}+\delta_1)$ is helpful. By calculus we can show that $\mu * C_t$ is unimodal if and only if $t \geq \sqrt{3}$; see Figures 5.13-5.18. Thus it is again natural to expect that a Cauchy process becomes unimodal for sufficiently large time, under some condition on the initial distribution. We start from the following counterexample.

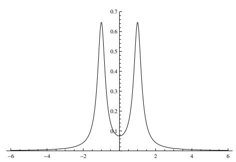

FIGURE 5.13. $t = 0.25$

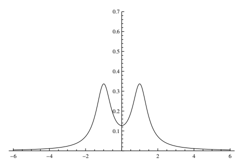

FIGURE 5.14. $t = 0.5$

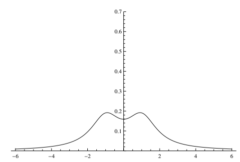

FIGURE 5.15. $t = 1$

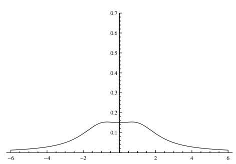

FIGURE 5.16. $t = \sqrt{2}$

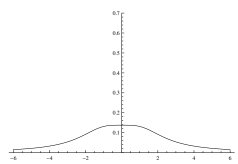

FIGURE 5.17. $t = \sqrt{3}$

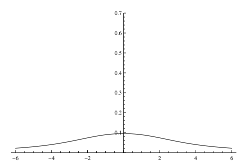

FIGURE 5.18. $t = 3$

Proposition 5.1. *There exists a probability measure μ on $\mathbb{R}$ such that $\mu * C_t$ is not unimodal for any $t > 0$, and*

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) < \infty, \quad 0 < p < 3. \tag{5.4}$$

Proof: Let $\{w_n\}_{n\geq 1}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} w_n = 1$ and $\{a_n\}_{n\geq 1}$ be a sequence of real numbers. Consider the probability measure

$$\mu = \sum_{n=1}^{\infty} w_n \delta_{a_n}. \tag{5.5}$$

Suppose that the sequence

$$b_k = \inf_{n \in \mathbb{Z}_+ \setminus \{k\}} |a_k - a_n - 1|, \qquad k \in \mathbb{Z}_+ \tag{5.6}$$

satisfies the condition

$$\lim_{k\to\infty} w_k b_k^3 = \infty. \tag{5.7}$$

When $\{a_n\}$ is increasing, this condition means that the distance between a_n and a_{n+1} grows sufficiently fast. Let

$$f_t(x) := \frac{\pi}{t} \frac{d(\mu * C_t)}{dx}(x) = \sum_{n=1}^{\infty} \frac{w_n}{(x-a_n)^2 + t^2}. \tag{5.8}$$

Then we obtain

$$f_t'(x) = \sum_{n=1}^{\infty} \frac{-2w_n(x-a_n)}{[(x-a_n)^2+t^2]^2}, \tag{5.9}$$

and so for sufficiently large k such that $b_k > 0$ and for each $t > 0$

$$\begin{aligned} f_t'(a_k-1) &= \frac{2w_k}{(1+t^2)^2} + \sum_{n\geq 1, n\neq k} \frac{-2w_n(a_k-a_n-1)}{[(a_k-a_n-1)^2+t^2]^2} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \sum_{n\geq 1, n\neq k} \frac{2w_n}{|a_k-a_n-1|^3} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \sum_{n\geq 1, n\neq k} \frac{2w_n}{b_k^3} \\ &\geq \frac{2w_k}{(1+t^2)^2} - \frac{2}{b_k^3} = 2w_k\left(\frac{1}{(1+t^2)^2} - \frac{1}{w_k b_k^3}\right). \end{aligned} \tag{5.10}$$

The condition (5.7) shows that $f_t'(a_k-1)$ is positive for sufficiently large $k \in \mathbb{Z}_+$. This shows that $\mu * C_t$ is not unimodal for any $t > 0$.

If we take the particular sequences $a_n = a^n$ and $w_n = cn^r a^{-3n}$, where $a \geq 2, r > 0$ and $c > 0$ is a normalizing constant, then the sequence b_n satisfies $b_n \geq Ca^n$ for some constant $C > 0$ independent of n. Then the conditions (5.7) and (5.4) hold true. □

In the above construction, for any positive weights $\{w_n\}_n$ and any sequence $\{a_n\}_n$ that satisfies (5.7), the third moment of μ is always infinite,

$$\int_{\mathbb{R}} |x|^3 \, d\mu(x) = \infty, \tag{5.11}$$

due to the inequality $|a_k| \geq b_k - |a_1| - 1$ and the condition (5.7). The next question is then whether there exists a probability measure μ with a finite third moment such that $\mu * C_t$ is not unimodal for any $t > 0$, or at least for sufficiently large $t > 0$. The complete answer is given below.

Theorem 5.2. *Let μ be a probability measure on $\mathbb{R}$ which has a finite absolute third moment*

$$\beta := \int_{\mathbb{R}} |x|^3 \, d\mu(x) < \infty.$$

Then $\mu * C_t$ is unimodal for $t \geq 20\beta^{\frac{1}{3}}$.

Proof: The Markov inequality implies the tail estimate

$$\mu([-x,x]^c) \leq \frac{\beta}{x^3}, \qquad x > 0. \tag{5.12}$$

It suffices to prove that the function

$$f_t(x) := \frac{\pi}{t}\frac{d(\mu * C_t)}{dx}(x) = \int_{\mathbb{R}} \frac{1}{(x-y)^2+t^2} \, d\mu(y) \tag{5.13}$$

has a unique local maximum for large $t > 0$. Suppose first that $x > 0$. The derivative f_t' splits into the positive and negative parts

$$f_t'(x) = \int_{y>x} \frac{2(y-x)}{[(x-y)^2+t^2]^2} \, d\mu(y) - \int_{y\leq x} \frac{2(x-y)}{[(x-y)^2+t^2]^2} \, d\mu(y). \tag{5.14}$$

By calculus the function $u \mapsto \frac{2u}{(u^2+t^2)^2}$ takes a global maximum at the unique point $u = t/\sqrt{3}$. Using the tail estimate (5.12) yields the estimate of the positive part

$$\int_{y>x} \frac{2(y-x)}{[(x-y)^2+t^2]^2}\, d\mu(y) \le \int_{y>x} \frac{2\frac{t}{\sqrt{3}}}{(\frac{t^2}{3}+t^2)^2}\, d\mu(y) \le \frac{\beta}{t^3x^3}. \tag{5.15}$$

On the other hand the negative part can be estimated as

$$\int_{y\le x} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y) \ge \int_{-x<y<x/2} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y). \tag{5.16}$$

Elementary calculus shows that for $-x < y < x/2$,

$$\frac{2(x-y)}{[(x-y)^2+t^2]^2} \ge \min\left\{\frac{4x}{(4x^2+t^2)^2}, \frac{x}{(x^2/4+t^2)^2}\right\}, \tag{5.17}$$

and if we further restrict to the case $x \ge t/4$, then

$$\begin{aligned} &\min\left\{\frac{4x}{(4x^2+t^2)^2}, \frac{x}{(x^2/4+t^2)^2}\right\} \\ &\quad\ge \min\left\{\frac{4x}{(4x^2+16x^2)^2}, \frac{x}{(x^2/4+16x^2)^2}\right\} \ge \frac{10^{-3}}{x^3}. \end{aligned} \tag{5.18}$$

Thus we obtain

$$\begin{aligned} \int_{y\le x} \frac{2(x-y)}{[(x-y)^2+t^2]^2}\, d\mu(y) &\ge \frac{10^{-3}}{x^3}\mu((-x/2,x/2)) \ge \frac{10^{-3}}{x^3}\left(1-\frac{\beta}{(x/2)^3}\right) \\ &\ge \frac{10^{-3}}{x^3}\left(1-\frac{8^3\beta}{t^3}\right), \qquad x \ge t/4. \end{aligned} \tag{5.19}$$

Comparing (5.15) and (5.19), taking $t \ge 20\beta^{1/3}$ guarantees that the positive part is smaller than the negative part, and hence

$$f_t'(x) < 0, \qquad x \ge \frac{t}{4}. \tag{5.20}$$

Similarly, if $t \ge 20\beta^{1/3}$ then

$$f_t'(x) > 0, \qquad x \le -\frac{t}{4}. \tag{5.21}$$

In order to show that f_t' has a unique zero, it suffices to show that $f_t''(x) < 0$ for $|x| \le t/4$. Now we have

$$\begin{aligned} f_t''(x) = &\int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2-t^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) \\ &- \int_{|y-x|\le t/\sqrt{3}} \frac{2[t^2-3(y-x)^2]}{[(x-y)^2+t^2]^3}\, d\mu(y). \end{aligned} \tag{5.22}$$

The function $u \mapsto 2(3u^2-t^2)/(u^2+t^2)^3$ attains a global maximum at $u = \pm t$ and a global minimum at $u = 0$. Therefore, the positive part can be estimated as follows:

$$\begin{aligned} \int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2-t^2]}{[(x-y)^2+t^2]^3}\, d\mu(y) &\le \int_{|y-x|>t/\sqrt{3}} \frac{2(3t^2-t^2)}{(t^2+t^2)^3}\, d\mu(y) \\ &= \frac{1}{2t^4}\mu\left(\left[x-\frac{t}{\sqrt{3}}, x+\frac{t}{\sqrt{3}}\right]^c\right). \end{aligned} \tag{5.23}$$

For all x such that $|x| \le t/4$, we have the inclusion

$$\left[x - \frac{t}{\sqrt{3}}, x + \frac{t}{\sqrt{3}}\right]^c \subseteq \left[-\frac{t}{5}, \frac{t}{5}\right]^c, \tag{5.24}$$

and hence we obtain

$$\int_{|y-x|>t/\sqrt{3}} \frac{2[3(y-x)^2 - t^2]}{[(x-y)^2 + t^2]^3}\, d\mu(y) \le \frac{1}{2t^4}\mu\left(\left[-\frac{t}{5}, \frac{t}{5}\right]^c\right) \le \frac{5^3\beta}{2t^7}. \tag{5.25}$$

On the other hand, the negative part has the estimate

$$\int_{|y-x|\le t/\sqrt{3}} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2 + t^2]^3}\, d\mu(y) \ge \int_{|y-x|\le t/2} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2 + t^2]^3}\, d\mu(y). \tag{5.26}$$

By calculus, the function $u \mapsto (t^2 - 3u^2)/(u^2 + t^2)^3$ is decreasing on $[0, t]$, and so

$$\begin{aligned} &\int_{|y-x|\le t/2} \frac{2[t^2 - 3(y-x)^2]}{[(x-y)^2 + t^2]^3}\, d\mu(y) \\ &\ge \int_{|y-x|\le t/2} \frac{2(t^2 - 3\frac{t^2}{4})}{(\frac{t^2}{4} + t^2)^3}\, d\mu(y) = \frac{32}{125t^4}\mu\left(\left[x - \frac{t}{2}, x + \frac{t}{2}\right]\right) \\ &\ge \frac{32}{125t^4}\mu\left(\left[-\frac{t}{4}, \frac{t}{4}\right]\right) \ge \frac{1}{4t^4}\left(1 - \frac{4^3\beta}{t^3}\right) \end{aligned} \tag{5.27}$$

for all $|x| \le t/4$. The positive part (5.25) is smaller than the negative part (5.27) if we take t in such a way that $t \ge 10\beta^{1/3}$. Thus $f_t''(x) < 0$ for all $|x| \le t/4$ and $t \ge 10\beta^{1/3}$. □

5.2. *Positive stable process with index* 1/2 *with initial distribution.* A positive stable process with index 1/2 has the distribution

$$L_t(dx) := \frac{t}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{t^2}{2x}}}{x^{3/2}} \cdot 1_{(0,\infty)}(x)\, dx, \quad x \in \mathbb{R}, \tag{5.28}$$

at time $t \ge 0$ which is called the *Lévy distribution.* We restrict to the case where the initial distribution is compactly supported.

Theorem 5.3. *If μ is a compactly supported on $\mathbb{R}$ with diameter D_μ then $\mu * L_t$ is unimodal for all $t \ge (90/4)^{1/4} D_\mu^{1/2}$.*

Proof: By performing a translation we may assume that μ is supported on $[0, \gamma]$, where $\gamma = D_\mu$. We set the following function:

$$g_{t,y}(x) := \frac{e^{-\frac{t^2}{2(x-y)}}}{(x-y)^{3/2}} 1_{(0,\infty)}(x-y), \qquad x, y \in \mathbb{R}. \tag{5.29}$$

Note that this function is C^∞ with respect to x. Consider the following function

$$f_t(x) := \frac{\sqrt{2\pi}}{t} \cdot \frac{d(\mu * L_t)}{dx}(x) = \int_0^\gamma g_{t,y}(x)\, d\mu(y), \qquad x \in \mathbb{R}, \tag{5.30}$$

which is supported on $(0, \infty)$ and has the derivative

$$\frac{d}{dx} f_t(x) = \int_0^\gamma \frac{t^2 - 3(x-y)}{2(x-y)^{7/2}} e^{-\frac{t^2}{2(x-y)}} 1_{(0,\infty)}(x-y)\, d\mu(y). \tag{5.31}$$

If $0 < x < t^2/3$ then $t^2 - 3(x-y) > t^2 - 3 \cdot t^2/3 = 0$ for all $0 \le y \le \gamma$, and therefore $f_t'(x) > 0$. Moreover, if $t^2/3 + \gamma < x$ then we have that $t^2 - 3(x-y) <$

$t^2 - 3(t^2/3 + \gamma) + 3\gamma = 0$ for all $0 \le y \le \gamma$, and therefore $f_t'(x) < 0$. For $t^2/3 < x < t^2/3 + \gamma$, the second derivative of f_t is given by

$$f_t''(x) = \int_0^\gamma \frac{15(x-y)^2 - 10t^2(x-y) + t^4}{4(x-y)^{11/2}} e^{-\frac{t^2}{2(x-y)}} 1_{(0,\infty)}(x-y)\, d\mu(y). \tag{5.32}$$

Note that $t^2/3 - \gamma < x - y < t^2/3 + \gamma$. Since for $X \in \mathbb{R}$

$$15X^2 - 10t^2X + t^4 < 0 \quad \text{iff} \quad \frac{t^2}{3} - \frac{2t^2}{3\sqrt{10}} < X < \frac{t^2}{3} + \frac{2t^2}{3\sqrt{10}}, \tag{5.33}$$

if $\frac{2t^2}{3\sqrt{10}} \ge \gamma$ then $f_t''(x) < 0$.

To summarize, we have obtained that if $t^2 \ge \frac{3\sqrt{10}}{2}\gamma$ then

- $f_t'(x) > 0, \qquad x < t^2/3,$
- $f_t'(x) < 0, \qquad x > t^2/3 + \gamma,$
- $f_t''(x) < 0, \qquad t^2/3 < x < t^2/3 + \gamma.$

Hence f_t has a unique local maximum in $[t^2/3, t^2/3 + \gamma]$, which is a unique global maximum on $\mathbb{R}$ as well. Therefore $\mu * L_t$ is unimodal for all $t^2 \ge \frac{3\sqrt{10}}{2}\gamma$. □

We give a counterexample for large time unimodality for positive stable processes of index 1/2 when the initial distribution is not compactly supported.

Proposition 5.4. *There exists a probability measure μ on $\mathbb{R}$ such that $\mu * L_t$ is not unimodal for any $t > 0$, and*

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) < \infty, \qquad 0 < p < \frac{5}{2}. \tag{5.34}$$

Proof: Let $\{w_n\}_{n\ge1}$ be a sequence of positive numbers such that $\sum_{n=1}^\infty w_n = 1$ and $\{a_n\}_{n\ge1}$ be a sequence of real numbers such that the sequence

$$b_k = \inf_{n\in\mathbb{N}\setminus\{k\}} |a_k - a_n|, \qquad k \in \mathbb{N} \tag{5.35}$$

satisfies

$$\lim_{k\to\infty} w_k b_k^{5/2} = \infty. \tag{5.36}$$

Consider the probability measure

$$\mu = \sum_{n=1}^\infty w_n \delta_{a_n}. \tag{5.37}$$

Let

$$f_t(x) := \frac{\sqrt{2\pi}}{t} \frac{d(\mu * L_t)}{dx}(x) = \sum_{n\ge1, a_n < x} w_n \frac{e^{-\frac{t^2}{2(x-a_n)}}}{(x-a_n)^{3/2}}. \tag{5.38}$$

Then we obtain

$$f_t'(x) = -\sum_{n\ge1, a_n<x} w_n h(x - a_n), \tag{5.39}$$

where $h(x) = \frac{3x - t^2}{2x^{7/2}} e^{-\frac{t^2}{2x}}$, and for each $k \in \mathbb{N}$ and each $t > 0$

$$f_t'(a_k + t^2/6) = w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \sum_{\substack{n: n\ne k \\ a_n < a_k + t^2/6}} w_n h(a_k - a_n + t^2/6). \tag{5.40}$$

Since $b_k \to \infty$ as $k \to \infty$, there exists a positive integer $k(t)$ such that $b_k + t^2/6 > \frac{5+\sqrt{10}}{15}t^2$ for all $k \geq k(t)$. For $k \geq k(t)$ and $n \neq k$ such that $a_n < a_k + t^2/6$, we can show that $a_k - a_n \geq b_k$; otherwise, by the definition of b_k it must hold that $a_k - a_n \leq -b_k < -\frac{5+2\sqrt{10}}{30}t^2$, which contradicts $a_n < a_k + t^2/6$. Since the map h is positive and strictly decreasing on $(\frac{5+\sqrt{10}}{15}t^2, \infty)$, for $k \geq k(t)$ we have

$$f_t'(a_k + t^2/6) \geq w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \sum_{\substack{n:n\neq k \\ a_n < a_k + t^2/6}} w_n h(b_k + t^2/6) \tag{5.41}$$

$$\geq w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - h(b_k + t^2/6) \tag{5.42}$$

$$= w_k \cdot \frac{54\sqrt{6}}{t^5} e^{-3} - \frac{3b_k - t^2/2}{2(b_k + t^2/6)^{7/2}} e^{-\frac{t^2}{2(b_k + t^2/6)}}. \tag{5.43}$$

The condition (5.36) shows that $f_t'(a_k + t^2/6)$ is positive for sufficiently large $k \in \mathbb{N}$. This shows that $\mu * L_t$ is not unimodal for any $t > 0$.

If we take the particular sequences $a_k = a^k$ and $w_k = cka^{-\frac{5}{2}k}$ where $a > 2$ and $c > 0$ is a normalizing constant, then the sequence b_k satisfies $b_k \geq Ca^k$ for some constant $C > 0$. Then the condition (5.36) holds true and

$$\int_{\mathbb{R}} |x|^p \, d\mu(x) = \sum_{k\geq 1} w_k |a_k|^p = c \sum_{k \geq 1} k a^{(p-5/2)k}. \tag{5.44}$$

Hence the above integral is finite if and only if $0 < p < 5/2$. □

In the above construction, for any positive weights $\{w_n\}_n$ and any sequence $\{a_n\}_n$ that satisfies (5.36), the 5/2-th moment of μ is always infinite, that is,

$$\int_{\mathbb{R}} |x|^{5/2} \, d\mu(x) = \infty. \tag{5.45}$$

We conjecture that if the integral in (5.45) is finite then $\mu * L_t$ is unimodal in large time. More generally, considering results on Cauchy processes in Section 5.1, it is natural to expect the following.

Conjecture 5.5. *Suppose that S_t (resp. T_t) is the law at time $t \geq 0$ of a classical (resp. free) strictly stable process of index $\alpha \in (0, 2)$. If μ is a probability measure such that*

$$\int_{\mathbb{R}} |x|^{2+\alpha} \, d\mu(x) < \infty, \tag{5.46}$$

then $\mu * S_t$ (resp. $\mu \boxplus T_t$) is unimodal for sufficiently large $t > 0$.

Acknowledgements

The authors express sincere thanks to the referee who gave useful comments to improve the manuscript.

References

H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.* **42** (3), 733–773 (1993). MR1254116.

P. Biane. On the free convolution with a semi-circular distribution. *Indiana Univ. Math. J.* **46** (3), 705–718 (1997). MR1488333.

U. Haagerup and S. Thorbjørnsen. On the free gamma distributions. *Indiana Univ. Math. J.* **63** (4), 1159–1194 (2014). MR3263926.

T. Hasebe and N. Sakuma. Unimodality for free Lévy processes. *Ann. Inst. Henri Poincaré Probab. Stat.* **53** (2), 916–936 (2017). MR3634280.

T. Hasebe and S. Thorbjørnsen. Unimodality of the freely selfdecomposable probability laws. *J. Theoret. Probab.* **29** (3), 922–940 (2016). MR3540484.

H.-W. Huang. Supports of measures in a free additive convolution semigroup. *Int. Math. Res. Not. IMRN* (12), 4269–4292 (2015). MR3356753.

H. Maassen. Addition of freely independent random variables. *J. Funct. Anal.* **106** (2), 409–438 (1992). MR1165862.

A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-85852-6; 0-521-85852-6. MR2266879.

K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2013). ISBN 978-1-107-65649-9. MR3185174.

S. J. Wolfe. On the unimodality of infinitely divisible distribution functions. *Z. Wahrsch. Verw. Gebiete* **45** (4), 329–335 (1978). MR511778.

M. Yamazato. Unimodality of infinitely divisible distribution functions of class L. *Ann. Probab.* **6** (4), 523–531 (1978). MR0482941.