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Sample path properties of reflected Gaussian processes

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Abstract. We consider a stationary queueing process Q_X fed by a centered Gaussian process X with stationary increments and variance function satisfying classical regularity conditions. A criterion when, for a given function f, $\mathbb{P}(Q_X(t) > f(t) \text{ i.o.})$ equals 0 or 1 is provided. Furthermore, an Erdös–Révész type law of the iterated logarithm is proven for the last passage time $\xi(t) = \sup\{s : 0 \le s \le t, Q_X(s) \ge f(s)\}$. Both of these findings extend previously known results that were only available for the case when X is a fractional Brownian motion.

1. Introduction and Main Results

Let $X = \{X(t) : t \ge 0\}$ be a centered Gaussian process with stationary increments and almost surely continuous sample paths. Given $c > 0$, consider a *reflected* (at 0) Gaussian process $Q_X = \{Q_X(t) : t \ge 0\}$ given by the following formula

$$Q_X(t) = X(t) - ct + \max\left(Q_X(0), -\inf_{s\in[0,t]}(X(s) - cs)\right). \tag{1.1}$$

It is well known in queueing and risk theory, e.g., Reich (1958), that the unique stationary solution of (1.1) has the following representation

$$Q_X(t) = \sup_{-\infty<s\le t} (X(t) - X(s) - c(t-s)).$$

Due to numerous application, Q_X has been studied in the literature under different levels of generality, e.g., Norros (1994); Hüsler and Piterbarg (1999); Dębicki (2002); Hüsler and Piterbarg (2004); Dieker (2005); Hashorva et al. (2013); Liu et al. (2015).

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Let f be any positive nondecreasing function on $\mathbb{R}$. Kolmogorov's zero-one law implies that the process Q_X crosses the function f infinitely many times with probability 0 or 1. Assume that $\mathbb{P}(Q_X(t) > f(t) \text{ i.o.}) = 1$ and define $\xi_f = \{\xi_f(t) : t \geq 0\}$ as the last crossing time before time t, that is,

$$\xi_f(t) = \sup\{s : 0 \leq s \leq t, Q_X(s) \geq f(s)\}.$$

By the assumption on f it follows that

$$\lim_{t\to\infty} \xi_f(t) = \infty \quad \text{and} \quad \limsup_{t\to\infty}(\xi_f(t) - t) = 0 \quad \text{a.s.}$$

The purpose of this paper is to provide a tractable criterion to verify the zero-one law as well as to give the asymptotic lower bound on $\xi_f(t) - t$. Erdös and Révész (1990) investigated the lower bound in the case when Q_X is substituted by Brownian motion W and $f(t) = \sqrt{2t\log_2 t}$ with $\log_2 t = \log\log t$. Subsequently similar results are known as Erdös–Révész type law of the iterated logarithm.

In the reminder of the paper we impose the following assumptions on variance function σ^2 of X:

AI: $\lim_{t\to\infty} \sigma^2(t)/t^{2\alpha_\infty} = A_\infty$, for some $A_\infty > 0$, $\alpha_\infty \in (0,1)$. Further, σ^2 is positive and twice continuously differentiable on $(0,\infty)$ with its first derivative $\dot{\sigma}^2$ and second derivative $\ddot{\sigma}^2$ being ultimately monotone at ∞.

AII: $\lim_{t\to 0^+} \sigma^2(t)/t^{2\alpha_0} = A_0$, for some $A_0 > 0$, $\alpha_0 \in (0,1]$.

Assumptions **AI-AII** allow us to cover models that play important role in Gaussian storage models, including both aggregations of *fractional Brownian motions* and *integrated stationary Gaussian processes*; see, e.g., Norros (1994); Hüsler and Piterbarg (1999); Dieker (2005); Dębicki (2002). In further analysis we tacitly assume that the variance function σ^2 of X satisfies both **AI** and **AII**. Our first contribution is the following criterion; see, e.g., Watanabe (1970); Qualls and Watanabe (1971) for similar results in the classical setting of *non-reflected* stationary Gaussian process.

Theorem 1.1. *For all positive and nondecreasing functions f on some interval $[T,\infty)$, $T > 0$,*

$$\mathbb{P}(Q_X(t) > f(t) \quad \text{i.o.}) = 0 \quad \textit{or} \quad 1,$$

according as the integral

$$\int_T^\infty \frac{\psi(f(u))}{f(u)}\,\mathrm{d}u \quad \textit{is finite or infinite,}$$

where

$$\psi(u) := \mathbb{P}\left(\sup_{t\in[0,u]} Q_X(t) > u\right).$$

With $\overleftarrow{m}$ being the generalized inverse of

$$m(u) = \inf_{t\geq 0} \frac{u(1+ct)}{\sigma(ut)},$$

define function f_p by

$$f_p(t) = \overleftarrow{m}\left(\sqrt{2\left(\log t + \left(\frac{\gamma - 1}{2(1-\alpha_\infty)} - p\right)\log_2 t\right)}\right), \qquad \gamma = \begin{cases} \frac{2(1-\alpha_\infty)}{\alpha_\infty} & \alpha_\infty \geq 1/2 \\ \frac{2(1+\alpha_0-2\alpha_\infty)}{\alpha_0} & \alpha_\infty < 1/2 \end{cases}, \tag{1.2}$$

and a positive constant $\mathscr{C}$ as

$$\mathscr{C} = \frac{1}{2}(\mathcal{H}_{\eta_{\alpha_\infty}})^2\sqrt{\frac{A}{B}}\zeta_{\alpha_\infty}\left(\frac{\sqrt{2A_\infty}}{A}\right)^{\frac{\gamma-1}{1-\alpha_\infty}},$$

where the remaining constants are defined in Equation 3.43. Since the exact asymptotics of $\psi(u)$, as u grows large, were found in (Dębicki and Liu, 2016), c.f., Theorem 3.1, it follows that

$$\frac{\psi(f_p(u))}{f_p(u)} = \mathscr{C}(u\log^{1-p} u)^{-1}(1+o(u)), \text{ as } u \to \infty. \tag{1.3}$$

Hence, by Theorem 1.1, $\mathbb{P}\left(Q_{B_H}(t) > f_p(t) \text{ i.o.}\right) = 1$ provided that $p \geq 0$, which leads to the following conclusion after deriving the exact asymptotics of f_p.

Corollary 1.2.

$$\limsup_{t\to\infty} \frac{Q_X(t)}{(\log t)^{\frac{1}{2(1-\alpha_\infty)}}} = \left(\frac{2A_\infty}{A^2}\right)^{\frac{1}{2(1-\alpha_\infty)}} \quad a.s.$$

Our second contribution is as follows.

Theorem 1.3. *If $p > 1$, then*

$$\liminf_{t\to\infty} \frac{\xi_{f_p}(t) - t}{h_p(t)} = -1 \quad \text{a.s.}$$

If $p \in (0, 1]$, then

$$\liminf_{t\to\infty} \frac{\log\left(\xi_{f_p}(t)/t\right)}{h_p(t)/t} = -1 \quad \text{a.s.},$$

where

$$h_p(t) = p\frac{f_p(t)}{\psi(f_p(t))}\log_2 t.$$

Theorem 1.3 shows that for t big enough, there exists an s in $[t - h_p(t), t]$ such that $Q_X(s) \geq f_p(s)$ and that the length of the interval $h_p(t)$ is smallest possible. Theorem 1.1 and Theorem 1.3 generalize the main results of Dębicki and Kosiński (2017), which considered the special case when $X \equiv B_H$ is a fractional Brownian motion with any Hurst parameter $H \in (0, 1)$; see also (Shao, 1992; Dębicki and Kosiński, 2018) for similar results for non-reflected Gaussian processes and Gaussian order statistics. The organization of the rest of paper is as follows. The notation and examples of Gaussian processes X that fall under our framework are displayed in Section 2 followed by properties of the storage process Q_X in Section 3. Section 4 gives two useful tools and some auxiliary lemmas for the proof of the main results which are presented in Section 5.

2. Notation and Special Cases

We write $f(u) \sim g(u)$ if $\lim_{u\to\infty} f(u)/g(u) = 1$. By $\overleftarrow{\sigma}$ we denote the generalized inverse function to σ, Ψ denotes the tail distribution function of the standard Normal random variable. Function f is ultimately monotone if there exists a constant $M > 0$ such that f is monotone over (M, ∞). For a centered continuous Gaussian process with stationary increments $V = \{V(t) : t \in \mathbb{R}\}$, such that $V(0) = 0$,

$$\mathbb{C}\mathrm{ov}(V(t), V(s)) = \frac{\sigma_V^2(t) + \sigma_V^2(s) - \sigma_V^2(t-s)}{2}, \tag{2.1}$$

we introduce *the generalized Pickands' constant* on a compact set $E \subset \mathbb{R}^d$ as

$$\mathcal{H}_V(E) = \mathbb{E}\exp\left(\sup_{t\in E}\left(\sqrt{2}V(t) - \sigma_V^2(t)\right)\right).$$

Let

$$\mathcal{H}_V = \lim_{S\to\infty} \frac{\mathcal{H}_V([0,S])}{S}.$$

We refer to (Dębicki and Kosiński, 2014) for the finiteness of $\mathcal{H}_V(E)$ and to (Dębicki et al., 2017; Dębicki and Hashorva, 2017) for the fact that $\mathcal{H}_V \in (0, \infty)$. Furthermore, see (Dębicki, 2002; Dieker, 2005) for the analysis of other properties of Pickands'-type constants.

Special cases. *Fractional Brownian motion*. Let $B_H = \{B_H(t) : t \geq 0\}$ denote fBm with Hurst index $H \in (0, 1]$ which is a centered Gaussian processes with continuous sample paths and covariance function satisfying

$$\mathbb{C}\mathrm{ov}\left(B_H(t), B_H(s)\right) = \frac{|s|^{2H} + |t|^{2H} - |t-s|^{2H}}{2}, \quad s, t \geq 0.$$

Direct calculations show that

$$\sigma^2(t) = |t|^{2H}, \quad m(u) = Au^{1-H},$$

$$A = \left(\frac{H}{c(1-H)}\right)^{-H} \frac{1}{1-H}, \quad B = \left(\frac{H}{c(1-H)}\right)^{-H-2} H,$$

$$\overleftarrow{m}(u) = A^{-\frac{1}{1-H}} u^{\frac{1}{1-H}}, \quad f_p(u) = \left(\frac{2}{A^2}\left(\log u + \left(\frac{2-3H}{2H(1-H)} - p\right)\log_2 u\right)\right)^{\frac{1}{2(1-H)}},$$

$$h_p(u) = p\mathscr{C}^{-1} u \log^{1-p} u \log_2 u, \quad \mathscr{C} = \frac{1}{2}(\mathcal{H}_{B_H})^2 \sqrt{\frac{A}{B}} \left(\frac{\sqrt{2}(\tau^*)^{2H}}{1 + c\tau^*}\right)\left(\frac{\sqrt{2}}{A}\right)^{\frac{2-3H}{H(1-H)}},$$

with $\tau^* = \frac{H}{c(1-H)}$. This coincides with (Dębicki and Kosiński, 2017, Theorem 1 and 2).

Short-range dependent Gaussian integrated processes. Let $X(t) = \int_0^t Y(s)\,\mathrm{d}s$ where Y is a centered stationary Gaussian process with unit variance and correlation function $r(t) = \mathbb{C}\mathrm{ov}(Y(s+t), Y(s)), s \geq 0, t \geq 0$. We say that X possesses short-range dependence property if:

- **S1**: r is a continuous function on $[0, \infty)$ such that, $\lim_{t\to\infty} tr(t) = 0$;
- **S2**: r is decreasing over $[0, \infty)$ and $\int_0^\infty r(t)\,\mathrm{d}t = \frac{1}{G}$ for some $0 < G < \infty$;
- **S3**: $\int_0^\infty s^2|r(s)|\,\mathrm{d}s < \infty$.

The above assumptions go line by line the same as the assumptions in Dębicki (2002) except a little modification. **S1-S3** cover wide range of stationary Gaussian processes such as the process with correlation function

$$r(t) = e^{-|t|^{\alpha}}, \quad \alpha \in (0,2].$$

In particular if $r(t) = e^{-|t|}$, X is the so-called Ornstein-Uhlenbeck process. Apparently, if **S1-S3** are satisfied, then

$$\sigma^2(t) = 2\int_0^t \int_0^s r(v)\,\mathrm{d}v\,\mathrm{d}s$$

satisfies **AI-AII**. Note that

$$\sigma^2(t) \sim t^2, \quad \text{as } t \to 0, \quad \sigma^2(t) \sim \frac{2}{G}t, \text{ as } t \to \infty.$$

Dębicki (2002, Proposition 6.1) shows that

$$m^2(u) = 2Gu + 2G^2G_1 + o(1), \text{ as } u \to \infty,$$

with $G_1 = \int_0^\infty tr(t)\,\mathrm{d}t$. This indicates that $m(u)$ can be replaced by $\widehat{m}(u) = \sqrt{2Gu + 2G^2G_1}$ in Theorem 3.1 and Theorem 1.3. Under this replacement, we have that

$$\overleftarrow{\widehat{m}(u)} = \frac{u^2}{2G} - GG_1, \quad f_p(t) = \frac{1}{G}\left(\log t + (1-p)\log_2 t\right) - GG_1$$

and

$$h_p(u) = p\mathscr{C}^{-1}u\log^{1-p}u\log_2 u, \quad \mathscr{C} = \frac{2(\mathcal{H}_{\eta_{1/2}})^2\left(\overleftarrow{\sigma}\left(\frac{\sqrt{2}}{cG}\right)\right)^{-2}}{A^{3/2}\sqrt{B}G},$$

with $\eta_{1/2} = \frac{cG}{\sqrt{2}}X(\overleftarrow{\sigma}(\frac{\sqrt{2}}{cG})t)$, $A = 2c^{1/2}$, and $B = \frac{1}{2}c^{5/2}$.

3. Properties of the storage process

Before we present our auxiliary results, we need to introduce some notation and state some properties of the supremum of the process Q_X as derived in (Piterbarg, 2001; Hüsler and Piterbarg, 2004). We begin with the relation

$$\mathbb{P}\left(\sup_{t\in[0,T]} Q_X(t) > u\right) = \mathbb{P}\left(\sup_{\substack{s\in[0,T/u]\\ \tau\geq 0}} Z_u(s,\tau) > m(u)\right), \quad \text{for any} \quad T>0, \tag{3.1}$$

where

$$Z_u(s,\tau) = \frac{X(u(\tau+s)) - X(us)}{u(1+c\tau)}m(u).$$

Note that $Z_u(s,\tau)$ is a Gaussian field, stationary in s, but not in τ. The variance $\sigma_u^2(\tau)$ of $Z_u(s,\tau)$ equals $\frac{\sigma^2(u\tau)}{(u(1+c\tau))^2}m^2(u)$ and $\sigma_u(\tau)$ has a single maximum point at $\tau(u)$ for u sufficiently large with $\lim_{u\to\infty}\tau(u) = \tau^*$, where

$$\tau^* = \frac{\alpha_\infty}{c(1-\alpha_\infty)}. \tag{3.2}$$

Taylor's formula shows that, for each $u > 0$ sufficiently large,

$$\sigma_u(\tau) = \sigma_u(\tau(u)) + \dot{\sigma}_u(\tau(u))(\tau - \tau(u)) + \frac{1}{2}\ddot{\sigma}_u(\xi)(\tau - \tau(u))^2$$

with $\xi \in (\tau, \tau(u))$. Noting that $\sigma_u(\tau(u)) = 1$, for u sufficiently large, $\dot{\sigma}_u(\tau(u)) = 0$ and for $\lim_{u\to\infty} \delta_u = 0$

$$\lim_{u\to\infty} \sup_{|\tau-\tau(u)|<\delta_u} \left| \frac{1}{2}\ddot{\sigma}_u(\xi) - \frac{B}{2A} \right| = 0,$$

we have

$$\lim_{u\to\infty} \sup_{\tau\neq\tau(u), |\tau-\tau(u)|<\delta_u} \left| \frac{1-\sigma_u(\tau)}{\frac{B}{2A}(\tau-\tau(u))^2} - 1 \right| = 0 \tag{3.3}$$

with $\lim_{u\to\infty} \delta_u = 0$, where

$$A = \left(\frac{\alpha_\infty}{c(1-\alpha_\infty)}\right)^{-\alpha_\infty} \frac{1}{1-\alpha_\infty}, \quad B = \left(\frac{\alpha_\infty}{c(1-\alpha_\infty)}\right)^{-\alpha_\infty-2} \alpha_\infty. \tag{3.4}$$

Let $r_{u,u'}(s,\tau,s',\tau')$ be the correlation function of $Z_u(s,\tau)$ and $Z_{u'}(s',\tau')$. Then

$$\begin{aligned} r_{u,u'}(s,\tau,s',\tau') = \Big(2\sigma(u\tau)\sigma(u'\tau')\Big)^{-1} \cdot \Big(&-\sigma^2(|us-u's'+u\tau-u'\tau'|) \\ &+\sigma^2(|us-u's'+u\tau|)+\sigma^2(|us-u's'-u'\tau'|)-\sigma^2(|us-u's'|)\Big). \end{aligned}$$

Denote by

$$r_u(s,\tau,s',\tau') = r_{u,u}(s,\tau,s',\tau').$$

Then Lemma 5.4 in Dębicki and Liu (2016) gives that, with $\delta_u > 0$ and $\lim_{u\to\infty} \delta_u = 0$,

$$\lim_{u\to\infty} \sup_{(s,\tau)\neq(s',\tau'), |\tau-\tau(u)|, |\tau'-\tau(u)|, |s-s'|\le\delta_u} \left| \frac{1-r_u(s,\tau,s',\tau')}{\frac{\sigma^2(u|s-s'+\tau-\tau'|)+\sigma^2(u|s-s'|)}{2\sigma^2(u\tau^*)}} - 1 \right| = 0. \tag{3.5}$$

Now assume that

$$\frac{u\tau+u'\tau'}{|us-u's'|} < \frac{1}{2}, \tag{3.6}$$

and without loss of generality, $us > u's'$. Then Taylor's formula gives that

$$r_{u,u'}(s,\tau,s',\tau') = \frac{-\ddot{\sigma^2}(|us-u's'+v_1-v_2|)u\tau u'\tau'}{2\sigma(u\tau)\sigma(u'\tau')},$$

with $v_1 \in (0, u\tau), v_2 \in (0, u'\tau')$. Noting that by (3.6)

$$|us-u's'+v_1-v_2| \ge u\tau + u'\tau,$$

in light of (Bingham et al., 1987, Theorem 1.7.2) and by **AI-AII** we have

$$\frac{|us-u's'+v_1-v_2|^2\ddot{\sigma^2}(|us-u's'+v_1-v_2|)}{\sigma^2(|us-u's'+v_1-v_2|)} \to 2\alpha_\infty(2\alpha_\infty-1), \text{ as } u\tau, u'\tau' \to \infty.$$

Hence

$$r_{u,u'}(s,\tau,s',\tau') \sim -\alpha_\infty(2\alpha_\infty-1)\left|\frac{\sqrt{u\tau u'\tau'}}{us-u's'+v_1-v_2}\right|^{2\lambda}, \text{ as } u\tau, u'\tau' \to \infty, \tag{3.7}$$

where $\lambda = 1-\alpha_\infty > 0$. This implies that for any $0<\epsilon<\frac{1}{2}$ if

$$\frac{u\tau + u'\tau'}{|us-u's'|} < \epsilon,$$

then, for $u\tau$ and $u\tau'$ both sufficiently large,

$$|r_{u,u'}(s,\tau,s',\tau')| \le (1-2\epsilon)^{2(\alpha_\infty-1)} \left|\frac{\sqrt{u\tau u'\tau'}}{us-u's'}\right|^{2\lambda}. \tag{3.8}$$

Next we focus on the case when $u \sim u'$, $|s-s'|\le M$ and $|\tau-\tau_0|, |\tau'-\tau^*| \le \delta(u,u')$ with τ^* defined in (3.2) and $\lim_{u,u'\to\infty}\delta(u,u')=0$. In light of **AI** and **AII**, noting that σ^2 is bounded over any compact interval, using uniform convergence theorem in Bingham et al. (1987) we have that, for $u\sim u'$,

$$\lim_{u,u'\to\infty} \sup_{|s-s'|\le M, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} \Bigg| \frac{\sigma^2(|us-u's'+u\tau|)+\sigma^2(|us-u's'-u'\tau'|)}{\sigma^2(u)}$$
$$-|s-s'+\tau^*|^{2\alpha_\infty} - |s-s'-\tau^*|^{2\alpha_\infty}\Bigg| = 0,$$

$$\lim_{u,u'\to\infty} \sup_{|s-s'|\le M, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} \Bigg| \frac{\sigma^2(|us-u's'+u\tau-u'\tau'|)+\sigma^2(|us-u's'|)}{\sigma^2(u)}$$
$$-2|s-s'|^{2\alpha_\infty}\Bigg| = 0,$$

$$\lim_{u,u'\to\infty} \sup_{|s-s'|\le M, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} \left| \frac{\sigma(u\tau)\sigma(u\tau')}{\sigma^2(u)} - |\tau^*|^{2\alpha_\infty}\right| = 0.$$

Hence for $u\sim u'$

$$\lim_{u,u'\to\infty} \sup_{|s-s'|\le M, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} |r_{u,u'}(s,\tau,s',\tau') - g(s-s')| = 0, \tag{3.9}$$

with

$$g(t) = \frac{|t+\tau^*|^{2\alpha_\infty} + |t-\tau^*|^{2\alpha_\infty} - 2|t|^{2\alpha_\infty}}{2(\tau^*)^{2\alpha_\infty}}.$$

Note that $g(0)=1$ and for any $0<\delta<1$, there exists $0<c_\delta<1/2$ such that

$$\inf_{|t|<c_\delta} g(t) > \delta, \quad \sup_{|t|>\delta} g(t) < 1-c_\delta. \tag{3.10}$$

The proof of (3.10) is postponed to Appendix. Following (3.9) and (3.10) , we have that with $u\sim u'$, for u sufficiently large,

$$\inf_{|s-s'|<c_\delta, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} r_{u,u'}(s,\tau,s',\tau') > \delta/2, \tag{3.11}$$

$$\sup_{|s-s'|>\delta, |\tau-\tau^*|, |\tau'-\tau^*|\le\delta(u,u')} r_{u,u'}(s,\tau,s',\tau') < 1-c_\delta/2 < 1. \tag{3.12}$$

3.1. *Asymptotics.* Let $\tau^*(u) = (\log m(u))/m(u)$ and $J(u) = \{\tau : |\tau-\tau(u)| \le \tau^*(u)\}$. Due to the following lemma, while analyzing tail asymptotics of the supremum of Z_u, we can restrict the considered domain of (s,τ) to a strip $J(u)$.

Lemma 3.1 (Dębicki and Liu (2016), Lemma 5.6 and Theorem 3.3)**.** *There exists a positive constant C such that for any $v,T>0$,*

$$\mathbb{P}\left(\sup_{(s,\tau)\in[0,T]\times(J(u))^c} Z_u(s,\tau) > m(u)\right) \le CT\frac{u^\gamma}{m(u)}\Psi(m(u))\exp\left(-\frac{b}{4}\log^2(m(u))\right), \tag{3.13}$$

where $b = B/(2A)$. Furthermore, for any $T > 0$ such that, there exist $c \in (0, \frac{1}{2})$ and $H' \in (-\gamma/2, 0)$, such that $u^{H'} < T < \exp(cm^2(u))$ for u sufficiently large,

$$\mathbb{P}\left(\sup_{(s,\tau)\in[0,T]\times J(u)} Z_u(s,\tau) > m(u)\right) = (\mathcal{H}_{\eta_{\alpha_\infty}})^2\sqrt{\frac{2A\pi}{B}}\zeta_{\alpha_\infty} T\frac{u^\gamma}{m(u)}\Psi(m(u))(1+o(1)), \tag{3.14}$$

where

$$\eta_{\alpha_\infty}(t) = \begin{cases} B_{\alpha_\infty}(t) & \alpha_\infty > 1/2 \\ \frac{1+c\tau^*}{\sqrt{2}A_\infty\tau^*}X(\overleftarrow{\sigma}(\frac{\sqrt{2}A_\infty\tau^*}{1+c\tau^*})t) & \alpha_\infty = 1/2 \\ B_{\alpha_0}(t) & \alpha_\infty < 1/2 \end{cases},$$

$$\zeta_{\alpha_\infty} = \begin{cases} \left(\frac{\sqrt{2}A_\infty(\tau^*)^{2\alpha_\infty}}{1+c\tau^*}\right)^{-2/\alpha_\infty} & \alpha_\infty > 1/2 \\ \left(\overleftarrow{\sigma}\left(\frac{\sqrt{2}A_\infty\tau^*}{1+c\tau^*}\right)\right)^{-2} & \alpha_\infty = 1/2 \\ \left(\frac{\sqrt{2}A_\infty(\tau^*)^{2\alpha_\infty}}{\sqrt{A_0}(1+c\tau^*)}\right)^{-2/\alpha_0} & \alpha_\infty < 1/2 \end{cases},$$

with γ defined in (1.2) and τ^* given by (3.2).

3.2. *Discretization.* For a fixed $T, \theta > 0$ and some $u > 0$, let us define a discretization of the set $[0,T]\times J(u)$ as follows

$$s_l = lq(u),\ 0 \le l \le L,\ L = [T/q(u)],\ q(u) = \theta\frac{\Delta(u)}{u},\ \Delta(u) = \overleftarrow{\sigma}\left(\frac{\sqrt{2}\sigma^2(u\tau^*)}{u(1+c\tau^*)}\right)$$

$$\tau_n = \tau(u) + nq(u),\ 0 \le |n| \le N,\ N = [\tau^*(u)/q(u)],\ E_{l,n}(u) = [s_l, s_{l+1}]\times[\tau_n, \tau_{n+1}].$$

Along the similar lines as in (Hüsler and Piterbarg, 2004, Lemma 6) we get the following lemma.

Lemma 3.2. *There exist positive constants $K_1, K_2, u_0 > 0$, such that, for any $\theta = \theta(u) > 0$ with $\lim_{u\to\infty}\theta(u) = 0$, $u \ge u_0$ and $\eta \in (0, \min(\alpha_0, \alpha_\infty))$*

$$\mathbb{P}\left(\max_{\substack{0\le l\le L\\ 0\le|n|\le N}} Z_u(s_l,\tau_n) \le m(u) - \frac{\theta^\eta}{m(u)}, \sup_{\substack{s\in[0,T]\\ \tau\in J(u)}} Z_u(s,\tau) > m(u)\right)$$
$$\le K_1\frac{u^\gamma}{m(u)}\Psi(m(u))e^{-\frac{\theta^{-2H}}{K_2}}$$

with $H \in (0, \min(\alpha_0, \alpha_\infty) - \eta)$.

Proof: Conditioning on $Z_u(s_l,\tau_n) = m(u) - \frac{\theta^\eta}{m(u)}$, we have for u sufficiently large

$$\mathbb{P}\left(Z_u(s_l,\tau_n) \le m(u) - \frac{\theta^\eta}{m(u)}, \sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau) > m(u)\right)$$
$$= \int_{\theta^\eta}^\infty \frac{1}{\sqrt{2\pi}m(u)\sigma_u(\tau_n)}e^{-\frac{(m(u)-y/m(u))^2}{2\sigma_u^2(\tau_n)}}$$
$$\times\mathbb{P}\left(\sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau) > m(u)\middle| Z_u(s_l,\tau_n) = m(u) - \frac{y}{m(u)}\right)\mathrm{d}y$$

$$
\leq \frac{K}{\sqrt{2\pi}m(u)} e^{-\frac{m^2(u)}{2\sigma_u^2(\tau_n)}} \int_{\theta^\eta}^{\infty} e^{2y} \times \mathbb{P}\left(\sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau) - m(u) > 0 \Big| Z_u(s_l,\tau_n) = m(u) - \frac{y}{m(u)} \right) \mathrm{d}y.
$$

Moreover,

$$
Z_u(s,\tau) - m(u) \Big| Z_u(s_l,\tau_n) = m(u) - \frac{y}{m(u)} \stackrel{d}{=} Y_u(s,\tau) + h(u,y),
$$

holds for $(s,\tau) \in E_{l,n}(u)$, where

$$
Y_u(s,\tau) = Z_u(s,\tau) - r_u(s,\tau,s_l,\tau_n)\frac{\sigma_u(\tau)}{\sigma_u(\tau_n)} Z_u(s_l,\tau_n),
$$

$$
h(u,y) = r_u(s,\tau,s_l,\tau_n)\frac{\sigma_u(\tau)}{\sigma_u(\tau_n)} \left(m(u) - \frac{y}{m(u)} \right) - m(u).
$$

Taylor's formula gives that

$$
\begin{aligned}
m(u)h(u,y) &= -m^2(u)(1 - r_u(s,\tau,s_l,\tau_n)) - m^2(u) r_u(s,\tau,s_l,\tau_n) \left(1 - \frac{\sigma_u(\tau)}{\sigma_u(\tau_n)} \right) \\
&\quad - r_u(s,\tau,s_l,\tau_n)\frac{\sigma_u(\tau)}{\sigma_u(\tau_n)} y \\
&\leq -m^2(u) r_u(s,\tau,s_l,\tau_n) \frac{\dot{\sigma}_u(\tau)(\tau_n - \tau) + (1/2)\ddot{\sigma}_u(v)(\tau - \tau_n)^2}{\sigma_u(\tau_n)} \\
&\quad - r_u(s,\tau,s_l,\tau_n)\frac{\sigma_u(\tau)}{\sigma_u(\tau_n)} y,
\end{aligned}
$$

with $v \in (\tau_n, \tau)$. Using the fact that $\dot{\sigma}_u(\tau(u)) = 0$ and $\sup_{\tau \in J(u)} |\ddot{\sigma}_u(\tau)| \leq \frac{2B}{A}$ for u sufficiently large, by Taylor's formula, we have

$$
\begin{aligned}
m^2(u)|\dot{\sigma}_u(\tau)(\tau_n - \tau)| &= m^2(u)|(\dot{\sigma}_u(\tau) - \dot{\sigma}_u(\tau(u)))(\tau_n - \tau)| \\
&= m^2(u)|\tau_n - \tau||\ddot{\sigma}(v_1)(\tau(u) - \tau)| \\
&\leq \frac{2B}{A} m^2(u) q(u) \tau^*(u) \\
&= \theta \frac{2B}{A} \frac{m(u)\Delta(u)}{u} \log m(u),
\end{aligned}
$$

with $v_1 \in (\tau, \tau(u))$. Note that by **AI-AII**

$$
\frac{m(u)\Delta(u)}{u} \log m(u) \sim \mathbb{Q} u^{v_2} \log u, \quad \text{with} \quad v_2 = \begin{cases} 2 - \alpha_\infty - 1/\alpha_\infty & \alpha_\infty \geq 1/2 \\ \frac{2\alpha_\infty - 1}{\alpha_0} - \alpha_\infty & \alpha_\infty < 1/2 \end{cases}.
$$

Since $v_2 < 0$ for all $\alpha_\infty \in (0,1]$, then

$$
m^2(u)|\dot{\sigma}_u(\tau)(\tau_n - \tau)| = o(\theta), \quad u \to \infty.
$$

$$
m^2(u)|\ddot{\sigma}_u(\theta)(\tau - \tau_n)^2| \leq K \left(\frac{m(u)\Delta(u)}{u} \right)^2 \theta^2 = o(\theta^2).
$$

Due to the fact that $y \geq \theta^\eta$ with $0 < \eta < 1$, we have

$$
h(u,y) \leq -y(1 + o(1)).
$$

Consequently, for u sufficiently large

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau)-m(u)>0\,\middle|\,Z_u(s_l,\tau_n)=m(u)-\frac{y}{m(u)}\right)\\
&\quad\le \mathbb{P}\left(\sup_{(s,\tau)\in[0,1]^2} m(u)\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}>\frac{y}{\sup_{\tau\in[0,1]}\sigma_u(\tau_n+q\tau)}(1+o(1))\right)\\
&\quad\le \mathbb{P}\left(\sup_{(s,\tau)\in[0,1]^2} m(u)\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}>\frac{y}{2}\right)
\end{aligned}$$

By (3.5) for u large enough

$$\begin{aligned}
&m^2(u)\,\mathbb{V}\mathrm{ar}\left(\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}-\frac{Y_u(s_l+qs',\tau_n+q\tau')}{\sigma_u(\tau_n+q\tau')}\right)\\
&\le 8m^2(u)(1-r_u(s_l+qs,\tau_n+q\tau,s_l+qs',\tau_n+q\tau'))\\
&\le 16m^2(u)\frac{\sigma^2(uq(u)|s-s'|)+\sigma^2(uq(u)|s+\tau-s'-\tau'|)}{2\sigma^2(u\tau^*)}\\
&\le K\frac{\sigma^2(\Delta(u)\theta|s-s'|)+\sigma^2(\Delta(u)\theta|s+\tau-s'-\tau'|)}{\sigma^2(\Delta(u))}\\
&\le K\left(\frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}\theta^{2\eta'}|s-s'|^{2\eta'}\right.\\
&\qquad\qquad\left.+\frac{h(\Delta(u)\theta|s+\tau-s'-\tau'|)}{h(\Delta(u))}\theta^{2\eta'}|s+\tau-s'-\tau'|^{2\eta'}\right)\\
&\le K\left(\frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}+\frac{h(\Delta(u)\theta|s+\tau-s'-\tau'|)}{h(\Delta(u))}\right)\theta^{2\eta'}(|s-s'|^{2\eta'}+|\tau-\tau'|^{2\eta'}),
\end{aligned}$$

with $s,s',\tau,\tau'\in[0,1]$,

where $h(t)=\frac{\sigma^2(t)}{t^{2\eta'}}$ and $\eta'\in(\eta,\min(\alpha_0,\alpha_\infty))$. Then it follows from **AI** and **AII** that $h(t)>0, t>0$ is a regularly varying function at both 0 and ∞ with indices $2(\alpha_0-\eta')>0$ and $2(\alpha_\infty-\eta')>0$ respectively; see Bingham et al. (1987) for the definition and properties of regularly varying functions. Next we focus on the boundedness of $\sup_{s,s'\in[0,1]}\frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}$. If $\lim_{u\to\infty}\Delta(u)=\infty$, noting that h is bounded over any compact interval, then uniform convergence theorem in Bingham et al. (1987) gives that

$$\lim_{u\to\infty}\sup_{s,s'\in[0,1]}\left|\frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}-(\theta|s-s'|)^{2(\alpha_\infty-\eta')}\right|=0,$$

implying that there exists $K_1>0$ such that for u large enough

$$\sup_{s,s'\in[0,1]}\frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}<K_1.$$

For the case $\lim_{u\to\infty}\Delta(u)=0$, uniform convergence theorem in Bingham et al. (1987) can similarly show that the above argument holds. For $\lim_{u\to\infty}\Delta(u)\in(0,\infty)$, it is obvious that

$$\lim_{u\to\infty}\sup_{s,s'\in[0,1]}h(\Delta(u)\theta|s-s'|)=0,\qquad \lim_{u\to\infty}h(\Delta(u))\in(0,\infty).$$

Thus the boundedness of $\sup_{s,s'\in[0,1]} \frac{h(\Delta(u)\theta|s-s'|)}{h(\Delta(u))}$ also holds. The boundedness of $\sup_{s,s',\tau,\tau'\in[0,1]} \frac{h(\Delta(u)\theta|s+\tau-s'-\tau'|)}{h(\Delta(u))}$ can be given similarly. Thus we have that

$$\begin{aligned}
m^2(u)\,\mathbb{V}\mathrm{ar}&\left(\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}-\frac{Y_u(s_l+qs',\tau_n+q\tau')}{\sigma_u(\tau_n+q\tau')}\right)\\
&\le K\theta^{2\eta'}(|s-s'|^{2\eta'}+|\tau-\tau'|^{2\eta'}),\quad s,s',\tau,\tau'\in[0,1],
\end{aligned}$$

with $\eta'\in(\eta,\min(\alpha_0,\alpha_\infty))$. Similarly

$$\sup_{s,\tau\in[0,1]} m^2(u)\,\mathbb{V}\mathrm{ar}\left(\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}\right)\le K\theta^{2\eta'}.$$

Hence in light of Piterbarg inequality (Piterbarg, 1996, Theorem 8.1 or Dębicki and Liu, 2016, Lemma 5.1), we have for u sufficiently large

$$\begin{aligned}
&\mathbb{P}\left(\sup_{(s,\tau)\in[0,1]^2} m(u)\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}>\frac{y}{2}\right)\\
&\quad\mathbb{P}\left(\sup_{(s,\tau)\in[0,1]^2}\theta^{-\eta'}m(u)\frac{Y_u(s_l+qs,\tau_n+q\tau)}{\sigma_u(\tau_n+q\tau)}>\frac{y}{2}\theta^{-\eta'}\right)\\
&\quad\le K_1(y\theta^{-\eta'})^{2/\eta'-1}e^{-\frac{(y\theta^{-\eta'})^2}{K}}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\mathbb{P}&\left(Z_u(s_l,\tau_n)\le m(u)-\frac{\theta^\eta}{m(u)},\sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau)>m(u)\right)\\
&\le\frac{K_1}{\sqrt{2\pi}m(u)}e^{-\frac{m^2(u)}{2\sigma_u^2(\tau_n)}}\int_{\theta^\eta}^\infty e^{2y}(y\theta^{-\eta'})^{2/\eta'-1}e^{-\frac{(y\theta^{-\eta'})^2}{K}}dy\\
&\le\frac{K_1}{\sqrt{2\pi}m(u)}e^{-\frac{m^2(u)}{2\sigma_u^2(\tau_n)}}\theta^{\eta'}\int_{\theta^{\eta-\eta'}}^\infty e^{2y\theta^{\eta'}}y^{2/\eta'-1}e^{-\frac{y^2}{K}}dy\\
&\le\frac{K_1}{\sqrt{2\pi}m(u)}e^{-\frac{m^2(u)}{2\sigma_u^2(\tau_n)}}e^{-\frac{\theta^{2(\eta-\eta')}}{K_2}}.
\end{aligned}$$

Using the above inequality and (3.3), we have that

$$\begin{aligned}
\mathbb{P}&\left(\max_{\substack{0\le l\le L\\0\le|n|\le N}} Z_u(s_l,\tau_n)\le m(u)-\frac{\theta^\eta}{m(u)},\sup_{(s,\tau)\in[0,T]\times J(u)} Z_u(s,\tau)>m(u)\right)\\
&\le\sum_{0\le l\le L,|n|\le N}\mathbb{P}\left(Z_u(s_l,\tau_n)\le m(u)-\frac{\theta^\eta}{m(u)},\sup_{(s,\tau)\in E_{l,n}(u)} Z_u(s,\tau)>m(u)\right)\\
&\le\sum_{0\le l\le L,|n|\le N}\frac{K_1}{\sqrt{2\pi}m(u)}e^{-\frac{m^2(u)}{2\sigma_u^2(\tau_n)}}e^{-\frac{\theta^{2(\eta-\eta')}}{K_2}}\\
&\le L\frac{K_1}{\sqrt{2\pi}m(u)}e^{-\frac{\theta^{2(\eta-\eta')}}{K_2}}\sum_{|n|\le N}e^{-\frac{m^2(u)(1+B(nq)^2/(4A))}{2}}\\
&\le K_1\left(\frac{u}{m(u)\Delta(u)}\right)^2e^{-\frac{m^2(u)}{2}}\theta^{-2}e^{-\frac{\theta^{2(\eta-\eta')}}{K_2}}
\end{aligned}$$

$$\le K_1 \frac{u^\gamma}{m(u)} \Psi(m(u)) e^{-\frac{\theta^{2(\eta-\eta')}}{K_2}}.$$

This completes the proof. □

Finally, by following the same arguments as in Dębicki and Liu (2016, Theorems 3.3) with the supremum functional substituted by its discrete counterpart, the maximum, we state the following result. Note that the asymptotic result below is a discrete version of (3.14) in Theorem 3.1.

Lemma 3.3. *For any $T, \theta > 0$, as $u \to \infty$,*

$$\mathbb{P}\left(\max_{\substack{0 \le l \le L \\ 0 \le |n| \le N}} Z_u(s_l, \tau_n) > m(u) \right) = (\mathcal{H}^\theta_{\eta_{\alpha_\infty}})^2 \sqrt{\frac{2A\pi}{B}} \zeta_{\alpha_\infty} T \frac{u^\gamma}{m(u)} \Psi(m(u))(1 + o(1)),$$

where $\mathcal{H}^\theta_{\eta_{\alpha_\infty}} = \lim_{S\to\infty} S^{-1} \mathbb{E} \exp\left(\sup_{t \in \theta\mathbb{Z} \cap [0,S]} \left(\sqrt{2}\eta_{\alpha_\infty}(t) - \mathbb{V}\mathrm{ar}(\eta_{\alpha_\infty}(t)) \right) \right)$.

By the monotone convergence theorem, it follows that $\mathcal{H}^\theta_{\eta_{\alpha_\infty}} \to \mathcal{H}_{\eta_{\alpha_\infty}}$ as $\theta \to 0$, since $H_{\eta_{\alpha_\infty}}$ is a positive, finite constant and η_{α_∞} has almost surely continuous sample paths. Consequently, when the discretization parameter θ decreases to zero so that the number of discretization points grows to infinity, we recover (3.14).

4. Auxiliary Lemmas

We begin with some auxiliary lemmas that are later needed in the proofs. The first lemma is (Leadbetter et al., 1983, Theorem 4.2.1).

Lemma 4.1 (Berman's inequality)**.** *Suppose $\xi_1, \dots, \xi_n$ are standard normal variables with covariance matrix $\Lambda^1 = (\Lambda^1_{i,j})$ and $\eta_1, \dots, \eta_n$ similarly with covariance matrix $\Lambda^0 = (\Lambda^0_{i,j})$. Let $\rho_{i,j} = \max(|\Lambda^1_{i,j}|, |\Lambda^0_{i,j}|)$ and let $u_1, \dots, u_n$ be real numbers. Then,*

$$\mathbb{P}\left(\bigcap_{j=1}^n \{\xi_j \le u_j\} \right) - \mathbb{P}\left(\bigcap_{j=1}^n \{\eta_j \le u_j\} \right) \le \frac{1}{2\pi} \sum_{1 \le i < j \le n} \left(\Lambda^1_{i,j} - \Lambda^0_{i,j} \right)^+ (1 - \rho^2_{i,j})^{-\frac{1}{2}} \exp\left(-\frac{u_i^2 + u_j^2}{2(1 + \rho_{i,j})} \right).$$

The following lemma is a general form of the Borel-Cantelli lemma; cf. (Spitzer, 1964).

Lemma 4.2 (Borel-Cantelli lemma)**.** *Consider a sequence of event $\{E_k\}_{k=0}^\infty$. If*

$$\sum_{k=0}^\infty \mathbb{P}(E_k) < \infty,$$

then $\mathbb{P}(E_n \ \text{i.o.}) = 0$. Whereas, if

$$\sum_{k=0}^\infty \mathbb{P}(E_k) = \infty \quad \text{and} \quad \liminf_{n\to\infty} \frac{\sum_{1 \le k \ne t \le n} \mathbb{P}(E_k E_t)}{\left(\sum_{k=1}^n \mathbb{P}(E_k) \right)^2} \le 1,$$

then $\mathbb{P}(E_n \ \text{i.o.}) = 1$.

Lemma 4.3. *For any $\varepsilon \in (0,1)$, there exist positive constants K and ρ depending only on $\varepsilon, \alpha_0, \alpha_\infty$ and p such that*

$$
\begin{aligned}
&\mathbb{P}\left(\sup_{S<t\le T} \frac{Q_X(t)}{f_p(t)} \le 1\right) \\
&\le \exp\left(-(1-\varepsilon)\int_{S+f_p(S)}^{T} \frac{1}{f_p(u)} \mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right) \mathrm{d}u\right) + KS^{-\rho},
\end{aligned}
$$

for any $T - f_p(S) \ge S \ge K$, with $f_p(T)/f_p(S) \le \mathcal{C}$ and $\mathcal{C}$ being some universal positive constant.

Proof: Let $\varepsilon \in (0,1)$ be some positive constant. For the remainder of the proof let K and ρ be two positive constants depending only on $\varepsilon, \alpha_0, \alpha_\infty$ and p that may differ from line to line. For any $k \ge 0$ put $s_0 = S$, $y_0 = f_p(s_0)$, $t_0 = s_0 + y_0$, $x_0 = f_p(t_0)$ and

$$
\begin{aligned}
&s_k = t_{k-1} + \varepsilon x_{k-1}, \quad y_k = f_p(s_k), \quad t_k = s_k + y_k, \quad x_k = f_p(t_k),\\
&I_k = (s_k, t_k], \quad \tilde{I}_k = \frac{I_k}{x_k} = (\tilde{s}_k, \tilde{t}_k], \quad |\tilde{I}_k| = \frac{y_k}{x_k}.
\end{aligned}
$$

From this construction, it is easy to see that the intervals I_k are disjoint. Furthermore, $\delta(I_k, I_{k+1}) = \varepsilon x_k$, and $1-\varepsilon \le y_k/x_k \le 1$, for any $k \ge 0$ and sufficiently large S. Note that, for any $k \ge 0$, $|I_k| \ge f_p(S)$, therefore if $T(S,\varepsilon)$ is the smallest number of intervals $\{I_k\}$ needed to cover $[S,T]$, then $T(S,\varepsilon) \le [(T-S)/(f_p(S)(1+\varepsilon))]$. Moreover, since $f_p(T)/f_p(S)$ is bounded by the constant $\mathcal{C} > 0$ not depending on S and ε, it follows that, $x_k/x_t \le \mathcal{C}$ for any $0 \le t < k \le T(S,\varepsilon)$.

Now let us introduce a discretization of the set $\tilde{I}_k \times J(x_k)$ as in Subsection 3.2. That is, for some $\theta = \frac{\Delta(S)}{S}$, define grid points

$$
\begin{aligned}
&s_{k,l} = \tilde{s}_k + l q_k, \quad 0 \le l \le L_k, \quad L_k = [(1-\varepsilon)/q_k], \quad q_k = \theta \frac{\Delta(x_k)}{x_k},\\
&\tau_{k,n} = \tau(x_k) + n q_k, \quad 0 \le |n| \le N_k, \quad N_k = [\tau^*(x_k)/q_k].
\end{aligned}
$$

Since f_p is an increasing function, it easily follows that,

$$
\begin{aligned}
\mathbb{P}\left(\sup_{S<t\le T} \frac{Q_X(t)}{f_p(t)} \le 1\right) &\le \mathbb{P}\left(\bigcap_{k=0}^{T(S,\varepsilon)} \left\{\sup_{t\in I_k} Q_X(t) \le x_k\right\}\right)\\
&\le \mathbb{P}\left(\bigcap_{k=0}^{T(S,\varepsilon)} \left\{\sup_{\substack{s\in I_k/x_k\\ \tau\in J(x_k)}} Z_{x_k}(s,\tau) \le m(x_k)\right\}\right)\\
&\le \mathbb{P}\left(\bigcap_{k=0}^{T(S,\varepsilon)} \left\{\max_{\substack{0\le l\le L_k\\ 0\le|n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) \le m(x_k)\right\}\right)\\
&\le \prod_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\max_{\substack{0\le l\le L_k\\ 0\le|n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) \le m(x_k)\right) + \sum_{0\le t<k\le T(S,\varepsilon)} C_{k,t} =: P_1 + P_2,
\end{aligned}
$$

where the last inequality follows from Berman's inequality with

$$C_{k,t} = \sum_{\substack{0\le l\le L_k\\ 0\le p\le L_t}} \sum_{\substack{|n|\le N_k\\ |m|\le N_t}} \frac{|r_{x_k,x_t}(s_{k,l},\tau_{k,n},s_{t,p},\tau_{t,m})|}{\sqrt{1-r^2_{x_k,x_t}(s_{k,l},\tau_{k,n},s_{t,p},\tau_{t,m})}} \times \exp\left(-\frac{\frac{1}{2}(m^2(x_k)+m^2(x_t))}{1+|r_{x_k,x_t}(s_{k,l},\tau_{k,n},s_{t,p},\tau_{t,m})|}\right).$$

Estimation of P_1:

Since for any u the process Z_u is stationary in the first variable, from Theorem 3.3 we have that, as $S\to\infty$ (noting that $\theta = \frac{\Delta(S)}{S} \to 0$)

$$\mathbb{P}\left(\max_{\substack{0\le l\le L_k\\ 0\le |n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) > m(x_k)\right) \sim \mathbb{P}\left(\sup_{(s,\tau)\in \tilde{I}_k\times J(x_k)} Z_{x_k}(s,\tau) > m(x_k)\right)$$

uniformly with respect to $0\le k\le T(S,\varepsilon)$. Hence for any $\varepsilon\in(0,1)$, sufficiently large S and small θ,

$$\begin{aligned}
P_1 &\le \exp\left(-\sum_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\max_{\substack{0\le l\le L_k\\ 0\le |n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) > m(x_k)\right)\right)\\
&\le \exp\left(-(1-\frac{\varepsilon}{8})\sum_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\sup_{(s,\tau)\in \tilde{I}_k\times J(x_k)} Z_{x_k}(s,\tau) > m(x_k)\right)\right)
\end{aligned}$$

Then, by (3.1) combined with Theorem 3.1,

$$\begin{aligned}
P_1 &\le \exp\left(-(1-\frac{\varepsilon}{4})\sum_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\sup_{\substack{s\in\tilde{I}_k\\ \tau\ge 0}} Z_{x_k}(s,\tau) > m(x_k)\right)\right)\\
&= \exp\left(-(1-\frac{\varepsilon}{4})\sum_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\sup_{t\in[0,\frac{y_k}{x_k}f_p(t_k)]} Q_X(t) > f_p(t_k)\right)\right)\\
&\le \exp\left(-(1-\frac{\varepsilon}{2})\sum_{k=0}^{T(S,\varepsilon)} \mathbb{P}\left(\sup_{t\in[0,f_p(t_k)]} Q_X(t) > f_p(t_k)\right)\frac{f_p(s_k)}{f_p(t_k)}\right)\\
&\le \exp\left(-(1-\varepsilon)\int_{S+f_p(S)}^{T} \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right) \mathrm{d}u\right).
\end{aligned}$$

Estimation of P_2:

For any $0\le t<k\le T(S,\varepsilon)$, $0\le l\le L_k$, $0\le p\le L_t$, we have

$$\begin{aligned}
x_k s_{k,l} - x_t s_{t,p} &= (s_k + x_k l q_k) - (s_t + x_t p q_t)\\
&= \sum_{i=t}^{k-1}(y_i+\varepsilon x_i) + x_k l q_k - x_t p q_t \ge \sum_{i=t}^{k-1}(y_i+\varepsilon x_i) - y_t\\
&\ge \sum_{i=t+1}^{k-1} y_i.
\end{aligned}$$

Recall that $\lambda = 1-\alpha_\infty$. Hence we can find $s_0 > 2$ such that for S sufficiently large, $k-t \geq 2s_0$, $0 \leq l \leq L_k$, $0 \leq p \leq L_t$, $|n| \leq N_k$ and $|m| \leq N_t$

$$\frac{x_k \tau_{k,n} + x_t \tau_{t,m}}{|x_k s_{k,l} - x_t s_{t,p}|} \leq \frac{x_k(\tau_{k,n} + \tau_{t,m})}{\sum_{i=t+1}^{k-1} y_i} < 1/3,$$

which applied to (3.8) indicates that for $k-t \geq s_0$ and S sufficiently large,

$$\begin{aligned} r^*_{k,t} := & \sup_{\substack{0\leq l\leq L_k, 0\leq p\leq L_t \\ |n|\leq N_k, |m|\leq N_t}} |r_{x_k,x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})| \\ & \leq 3^{2(1-\alpha_\infty)} \sup_{\substack{0\leq l\leq L_k, 0\leq p\leq L_t \\ |n|\leq N_k, |m|\leq N_t}} \left| \frac{\sqrt{x_k \tau_{k,n} x_t \tau_{t,m}}}{\sum_{i=t+1}^{k-1} y_i} \right|^{2\lambda} \\ & \leq 3^{2(1-\alpha_\infty)} \sup_{\substack{0\leq l\leq L_k, 0\leq p\leq L_t \\ |n|\leq N_k, |m|\leq N_t}} \left| \frac{x_k \tau_{k,n}}{\sum_{i=t+1}^{k-1} y_i} \right|^{\lambda} \left| \frac{x_t \tau_{t,m}}{\sum_{i=t+1}^{k-1} y_i} \right|^{\lambda} \\ & \leq K \left| \frac{x_t}{\sum_{i=t+1}^{k-1} y_i} \right|^{\lambda} \leq K(k-t)^{-\lambda} \leq \frac{\lambda}{4}. \end{aligned} \tag{4.1}$$

For $1 \leq k-t \leq s_0$, it follows that $x_k \sim x_t$, $\tau_{k,l} \to \tau^*$ and $\tau_{t,p} \to \tau^*$ as $S \to \infty$, and $s_{k,l} - s_{t,p} \geq \epsilon x_t / x_k > \epsilon/2$ for S sufficiently large. Therefore, by (3.12) there exists a positive constant $\zeta \in (0,1)$ depending only on ε such that for S sufficiently large

$$\sup_{1\leq k-t\leq s_0} r^*_{k,t} = \sup_{1\leq k-t\leq s_0} \sup_{\substack{0\leq l\leq L_k, 0\leq p\leq L_t \\ |n|\leq N_k, |m|\leq N_t}} |r_{x_k,x_t}(s_{k,l}, \tau_{k,n}, s_{t,p}, \tau_{t,m})| \leq \zeta < 1. \tag{4.2}$$

Finally, note that; c.f., (1.2),

$$N_k \leq L_k \leq \frac{2(1-\varepsilon)x_k}{\theta \Delta(x_k)} \leq K x_k^{2\gamma} \leq K (\log t_k)^{\frac{\gamma}{(1-\alpha_\infty)}},$$

$$\exp\left(-\frac{m^2(x_k)}{2}\right) = \frac{(\log t_k)^{p-\frac{\gamma-1}{2(1-\alpha_\infty)}}}{t_k},$$

so that

$$\begin{aligned} P_2 & \leq \frac{4}{\sqrt{1-\zeta^2}} \sum_{0\leq t<k\leq T(S,\varepsilon)} L_k L_t N_k N_t r^*_{k,t} \exp\left(-\frac{m^2(x_k)+m^2(x_t)}{2(1+r^*_{k,t})}\right) \\ & \leq K \left(\sum_{\substack{0<k-t\leq s_0 \\ 0\leq t<k\leq T(S,\varepsilon)}} + \sum_{\substack{k-t>s_0 \\ 0\leq t<k\leq T(S,\varepsilon)}} \right) (\cdot) \\ & \leq K \left(\sum_{k=0}^{\infty} x_k^{8\gamma} \exp\left(-\frac{m^2(x_k)}{1+\zeta}\right) \right. \\ & \qquad \left. + \sum_{\substack{k-t>s_0 \\ 0\leq t<k\leq T(S,\varepsilon)}} (x_k x_t)^{4\gamma} (k-t)^{-\lambda} \exp\left(-\frac{m^2(x_k)+m^2(x_t)}{2(1+\frac{\lambda}{4})}\right) \right) \end{aligned}$$

$$\begin{aligned}
&\le K\left(\sum_{k=0}^{\infty} t_k^{-\frac{2}{1+\sqrt{\zeta}}} + \sum_{\substack{k-t>s_0\\0\le t<k\le T(S,\varepsilon)}} t_k^{-\frac{1}{1+\frac{\lambda}{2}}} t_t^{-\frac{1}{1+\frac{\lambda}{2}}}(k-t)^{-\lambda}\right)\\
&\le K\left(\sum_{k=[S]}^{\infty} k^{-\frac{2}{1+\sqrt{\zeta}}} + \sum_{[S]\le t<k\le\infty} k^{-\frac{1}{1+\frac{\lambda}{2}}} t^{-\frac{1}{1+\frac{\lambda}{2}}}(k-t)^{-\lambda}\right)\\
&\le KS^{-\rho},
\end{aligned}$$

where the last inequality follows from basic algebra. □

Let $S > 0$ be any fixed number, $a_0 = S$, $y_0 = f_p(a_0)$ and $b_0 = a_0 + y_0$. For $i > 0$, define

$$a_i = b_{i-1}, \quad y_i = f_p(a_i), \quad b_i = a_i + y_i, \quad M_i = (a_i, b_i], \quad \tilde{M}_i = \frac{M_i}{y_i} = (\tilde{a}_i, \tilde{b}_i]. \tag{4.3}$$

From this construction, it is easy to see that the intervals M_i are disjoint, $\cup_{j=0}^{i} M_j = (S, b_i]$, and $|\tilde{M}_i| = 1$. Now let us introduce a discretization of the set $\tilde{M}_i \times J(y_i)$ as in Subsection 3.2. That is, for $\theta = \frac{\Delta(S)}{S}$, define grid points

$$\begin{aligned}
s_{i,l} &= \tilde{a}_i + lq_i, \quad 0\le l\le L_i, \quad L_i = [1/q_i], \quad q_i = \theta\frac{\Delta(y_i)}{y_i}, \qquad (4.4)\\
\tau_{i,n} &= \tau(y_i) + nq_i, \quad 0\le |n|\le N_i, \quad N_i = [\tau^*(y_i)/q_i].
\end{aligned}$$

With the above notation, we have the following lemma.

Lemma 4.4. *For any $\varepsilon \in (0,1)$, there exists positive constants K and ρ depending only on $\varepsilon, \alpha_0, \alpha_\infty$ and p such that, with $\theta_i = (m(y_i))^{-4/\hat{\alpha}}$, where $\hat{\alpha} = \min(\alpha_0, \alpha_\infty)$,*

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{i=0}^{[(T-S)/f_p(S)]}\left\{\max_{\substack{0\le l\le L_i\\0\le|n|\le N_i}} Z_{y_i}(s_{i,l},\tau_{i,n}) \le m(y_i) - \frac{\theta_i^{\frac{\hat{\alpha}}{2}}}{m(y_i)}\right\}\right)\\
&\qquad \ge \frac{1}{4}\exp\left(-(1+\varepsilon)\int_S^T \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right)\mathrm{d}u\right) - KS^{-\rho},
\end{aligned}$$

for any $T - f_p(S) \ge S \ge K$, with $f_p(T)/f_p(S) \le \mathcal{C}$ and $\mathcal{C}$ being some universal positive constant.

Proof: Put

$$\hat{m}(y_i) = m(y_i) - \frac{\theta_i^{\frac{\hat{\alpha}}{2}}}{m(y_i)}, \quad I = [(T-S)/f_p(S)].$$

Similarly as in the proof of Theorem 4.3 we find that Berman's inequality implies

$$\begin{aligned}
&\mathbb{P}\left(\bigcap_{i=0}^{I}\left\{\max_{\substack{0\le l\le L_i\\0\le|n|\le N_i}} Z_{y_i}(s_{i,l},\tau_{i,n}) \le m(y_i) - \frac{\theta_i^{\frac{\hat{\alpha}}{2}}}{m(y_i)}\right\}\right)\\
&\ge \prod_{i=0}^{I}\mathbb{P}\left(\max_{\substack{0\le l\le L_i\\0\le|n|\le N_i}} Z_{y_i}(s_{i,l},\tau_{i,n}) \le m(y_i) - \frac{\theta_i^{\frac{\hat{\alpha}}{2}}}{m(y_i)}\right) - \sum_{0\le i<j\le I} D_{i,j} =: P_1' + P_2',
\end{aligned}$$

where

$$D_{i,j} = \frac{1}{2\pi} \sum_{\substack{0\le l\le L_i\\ 0\le p\le L_j}} \sum_{\substack{|n|\le N_i\\ |m|\le N_j}} \frac{(\tilde r_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m}))^+}{\sqrt{1-\tilde r^2_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m})}} \times \exp\left(-\frac{\frac12(\hat m^2(y_i)+\hat m^2(y_j))}{1+|\tilde r_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m})|}\right),$$

with

$$\tilde r_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m}) = -r_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m}).$$

Estimation of P_1':

By Theorem 3.1 the correction term $\theta_i^{\hat\alpha/2}/m(y_i)$ does not change the order of asymptotics of the tail of Z_{y_i}. Furthermore, the tail asymptotics of the supremum on the strip $(s,\tau)\in \tilde M_i \times J(y_i)$ are of the same order if $\tau\ge 0$. Hence, for every $\varepsilon>0$,

$$\begin{aligned}
P_1' &\ge \frac14 \exp\left(-\sum_{i=0}^{I} \mathbb{P}\left(\max_{\substack{0\le l\le L_i\\ 0\le|n|\le N_i}} Z_{y_i}(s_{i,l},\tau_{i,n}) > \hat m(y_i)\right)\right)\\
&\ge \frac14 \exp\left(-\sum_{i=0}^{I} \mathbb{P}\left(\sup_{\substack{s\in\tilde M_i\\ \tau\in J(y_i)}} Z_{y_i}(s,\tau) > m(y_i) - \frac{\theta_i^{\frac{\hat\alpha}{2}}}{m(y_i)}\right)\right)\\
&\ge \frac14 \exp\left(-(1+\varepsilon)\sum_{i=0}^{I} \mathbb{P}\left(\sup_{\substack{s\in\tilde M_i\\ \tau\ge 0}} Z_{y_i}(s,\tau) > m(y_i)\right)\right)\\
&= \frac14 \exp\left(-(1+\varepsilon)\sum_{i=0}^{I} \mathbb{P}\left(\sup_{t\in[0,f_p(a_i)]} Q_X(t) > f_p(a_i)\right)\right)\\
&\ge \frac14 \exp\left(-(1+\varepsilon)\int_S^T \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right) \mathrm{d}u\right),
\end{aligned}$$

provided that S is sufficiently large along the same lines as the estimation of P_1 in Theorem 4.3.

Estimation of P_2':

Clearly, for $j\ge i+2$, and any $0\le l\le L_i$, $0\le p\le L_j$; c.f. (4.3),

$$y_j s_{j,p} - y_i s_{i,l} = a_j + y_j p q_j - (a_i + y_i l q_i) \ge \sum_{k=i+1}^{j-1} y_k.$$

Hence there exists $s_0\ge 2$ such that for $j-i\ge s_0$, $0\le l\le L_i$, $0\le p\le L_j$, $|n|\le N_i$, $|m|\le N_j$ and S sufficiently large

$$\frac{y_j\tau_{j,m}+y_i\tau_{i,n}}{|y_j s_{j,p} - y_i s_{i,l}|} \le \frac{y_j(\tau_{j,m}+\tau_{i,n})}{\sum_{k=i+1}^{j-1} y_k} \le \frac13.$$

Analogously as the derivation of (4.1), by (3.8) for $j-i \geq s_0$ and S sufficiently large

$$r^*_{i,j} := \sup_{\substack{0\leq l\leq L_i, 0\leq p\leq L_j\\ |n|\leq N_i, |m|\leq N_j}} |\tilde{r}_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m})| \leq K(k-t)^{-\lambda} \leq \frac{\lambda}{4},$$

where $\lambda = 1-\alpha_\infty$. For $1 \leq j-i \leq s_0$, it follows that $y_i \sim y_j$, $\tau_{i,n} \to \tau^*$ and $\tau_{j,m} \to \tau^*$ as $S \to \infty$, and $s_{i,l} - s_{j,p} \geq y_{i+1}/y_j > \frac{1}{2}$ for $2 \leq j-i \leq s_0$ and S sufficiently large. Therefore, by (3.12) there exists a positive constant $\zeta_1 \in (0,1)$ depending only on ε such that for S sufficiently large

$$\sup_{2\leq j-i\leq s_0} r^*_{i,j} = \sup_{2\leq j-i\leq s_0} \sup_{\substack{0\leq l\leq L_i, 0\leq p\leq L_j\\ |n|\leq N_i, |m|\leq N_j}} |r_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m})| \leq \zeta_1. \tag{4.5}$$

Moreover, by (3.11) there exist positive constants $\delta \in (0,1)$ and $c_\delta \in (0,\frac{1}{2})$, $M < 1$, such that, for sufficiently large S,

$$\inf_{|y_i-y_j|<c_\delta, |\tau-\tau^*|, |\tau'-\tau^*|\leq M} r_{y_i,y_j}(s,\tau,s',\tau') > \frac{\delta}{2}.$$

Hence for sufficiently large S and $0 \leq l \leq L_i$, $0 \leq p \leq L_j$, $|n| \leq N_i$, $|m| \leq N_j$

$$(\tilde{r}_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m}))^+ = 0, \quad \text{if} \quad j = i+1, \quad |s_{i,l}-s_{j,p}| \leq c_\delta. \tag{4.6}$$

By (3.12) there exits $\zeta_2 \in (0,1)$ such that for S sufficiently large and $0 \leq l \leq L_i$, $0 \leq p \leq L_j$, $|n| \leq N_i$, $|m| \leq N_j$

$$|\tilde{r}_{y_i,y_j}(s_{i,l},\tau_{i,n},s_{j,p},\tau_{j,m})| \leq \zeta_2, \quad \text{if} \quad j = i+1, \quad |s_{i,l}-s_{j,p}| \geq c_\delta. \tag{4.7}$$

Let $\zeta = \max(\zeta_1,\zeta_2)$. Therefore, by (4.2)–(4.7) we obtain

$$\begin{aligned} P'_2 \leq & \sum_{\substack{0\leq i\leq I-1\\ 1\leq j-i\leq s_0}} \sum_{\substack{0\leq l\leq L_i\\ 0\leq p\leq L_j}} \sum_{\substack{|n|\leq N_i\\ |m|\leq N_j}} \frac{1}{\sqrt{1-\zeta}} \exp\left(-\frac{\frac{1}{2}(\hat{m}^2(y_i)+\hat{m}^2(y_j))}{1+\zeta}\right) \\ & + \sum_{\substack{0\leq i\leq I-2\\ i+s_0\leq j\leq I}} \sum_{\substack{0\leq l\leq L_i\\ 0\leq p\leq L_j}} \sum_{\substack{|n|\leq N_i\\ |m|\leq N_j}} \frac{r^*_{i,j}}{\sqrt{1-r^*_{i,j}}} \exp\left(-\frac{\frac{1}{2}(\hat{m}^2(y_i)+\hat{m}^2(y_j))}{1+r^*_{i,j}}\right). \end{aligned}$$

Completely similar to the estimation of P_2 in the proof of Theorem 4.3, we can arrive that there exist positive constants K and ρ such that, for sufficiently large S,

$$P'_2 \leq KS^{-\rho}.$$

□

The next lemma is a straightforward modification of (Watanabe, 1970, Lemma 3.1 and Lemma 4.1), see also (Qualls and Watanabe, 1971, Lemma 1.4).

Lemma 4.5. *It is enough to proof Theorem 1.1 for any nondecreasing function f such that,*

$$\overleftarrow{m}\left(\sqrt{\log t}\right) \leq f(t) \leq \overleftarrow{m}\left(\sqrt{3\log t}\right), \tag{4.8}$$

for all $t \geq T$, and T large enough.

5. Proof of the main results

Proof of Theorem 1.1: Note that the case $\mathscr{I}_f < \infty$ is straightforward and does not need any additional knowledge on the process Q_X apart from the property of stationarity. Indeed, consider the sequence of intervals M_i as in Theorem 4.4. Then, for any $\varepsilon > 0$ and sufficiently large T,

$$\sum_{k=[T]+1}^{\infty} \mathbb{P}\left(\sup_{t\in M_k} Q_X(t) > f(a_k)\right) = \sum_{k=[T]}^{\infty} \mathbb{P}\left(\sup_{t\in[0,f(b_k)]} Q_X(t) > f(b_k)\right) \le \mathscr{I}_f < \infty,$$

and the Borel-Cantelli lemma completes this part of the proof since f is an increasing function.

Now let f be an increasing function such that $\mathscr{I}_f \equiv \infty$. With the same notation as in Theorem 4.3 with f instead of f_p, we find that, for any $S, \varepsilon, \theta > 0$,

$$\begin{aligned}\mathbb{P}\left(Q_X(s) > f(s) \text{ i.o.}\right) &\ge \mathbb{P}\left(\left\{\sup_{t\in I_k} Q_X(t) > f(t_k)\right\} \quad \text{i.o.}\right)\\ &\ge \mathbb{P}\left(\left\{\max_{\substack{0\le l\le L_k\\ 0\le |n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) > m(x_k)\right\} \text{ i.o.}\right).\end{aligned}$$

Let

$$E_k = \left\{\max_{\substack{0\le l\le L_k\\ 0\le |n|\le N_k}} Z_{x_k}(s_{k,l},\tau_{k,n}) \le m(x_k)\right\}.$$

For sufficiently large S and sufficiently small θ; c.f., estimation of P_1, we get

$$\sum_{k=0}^{\infty} \mathbb{P}\left(E_k^c\right) \ge (1-\varepsilon)\int_{S+f(S)}^{\infty} \frac{1}{f(u)}\mathbb{P}\left(\sup_{t\in[0,f(u)]} Q_X(t) > f(u)\right) du = \infty. \tag{5.1}$$

Note that

$$1 - \mathbb{P}\left(E_i^c \quad \text{i.o.}\right) = \lim_{m\to\infty}\prod_{k=m}^{\infty}\mathbb{P}\left(E_k\right) + \lim_{m\to\infty}\left(\mathbb{P}\left(\bigcap_{k=m}^{\infty} E_k\right) - \prod_{k=m}^{\infty}\mathbb{P}\left(E_k\right)\right).$$

The first limit is zero as a consequence of (5.1), and the second limit will be zero because of the asymptotic independence of the events E_k. Indeed, there exist positive constants K and ρ, depending only on $\alpha_0, \alpha_\infty, \varepsilon, \lambda$, such that for any $n > m$,

$$A_{m,n} = \left|\mathbb{P}\left(\bigcap_{k=m}^{n} E_k\right) - \prod_{k=m}^{n}\mathbb{P}\left(E_k\right)\right| \le K(S+m)^{-\rho},$$

by the same calculations as in the estimate of P_2 in Theorem 4.3 after realizing that, by Theorem 4.5, we might restrict ourselves to the case when (4.8) holds. Therefore $\mathbb{P}\left(E_i^c \text{ i.o.}\right) = 1$, which finishes the proof. □

Proof of Theorem 1.3:

Let $\xi_p \equiv \xi_{f_p}$ for short.

Step 1. Let $p > 1$, then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t\to\infty} \frac{\xi_p(t) - t}{h_p(t)} \ge -(1+2\varepsilon) \quad \text{a.s.}$$

Proof: Let $\{T_k : k \geq 1\}$ be a sequence such that $T_k \to \infty$, as $k \to \infty$. Put $S_k = T_k - (1+2\varepsilon)h_p(T_k)$. Since $h_p(t) = O(t\log^{1-p} t \log_2 t)$, then, for $p > 1$, $S_k \sim T_k$, as $k \to \infty$, and from Theorem 4.3 it follows that

$$\begin{aligned}\mathbb{P}\Big(\frac{\xi_p(T_k) - T_k}{h_p(T_k)} \leq -(1+2\varepsilon)^2\Big) &= \mathbb{P}\left(\xi_p(T_k) \leq S_k\right) = \mathbb{P}\left(\sup_{S_k < t \leq T_k} \frac{Q_X(t)}{f_p(t)} < 1\right) \\ &\leq \exp\left(-(1-\varepsilon)\int_{S_k + f_p(S_k)}^{T_k} \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right) \mathrm{d}u\right) + 2KT_k^{-\rho}.\end{aligned}$$

Moreover, as $k \to \infty$,

$$\begin{aligned}&\int_{S_k + f_p(S_k)}^{T_k} \frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]} Q_X(t) > f_p(u)\right) \mathrm{d}u \\ &\sim (1+2\varepsilon)h_p(T_k)\frac{1}{f_p(T_k)}\mathbb{P}\left(\sup_{t\in[0,f_p(T_k)]} Q_X(t) > f_p(T_k)\right) = (1+2\varepsilon)p\log_2 T_k.\end{aligned} \tag{5.2}$$

Now take $T_k = \exp(k^{1/p})$. Then,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\xi_p(T_k) \leq S_k\right) \leq 2K \sum_{k=1}^{\infty} k^{-(1+\varepsilon/2)} < \infty.$$

Hence by the Borel-Cantelli lemma,

$$\liminf_{k\to\infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \geq -(1+2\varepsilon) \quad \text{a.s.}. \tag{5.3}$$

Since $\xi_p(t)$ is a non-decreasing random function of t, for every $T_k \leq t \leq T_{k+1}$, we have

$$\frac{\xi_p(t) - t}{h_p(t)} \geq \frac{\xi_p(T_k) - T_k}{h_p(T_k)} - \frac{T_{k+1} - T_k}{h_p(T_k)}.$$

For $p > 1$ elementary calculus implies

$$\lim_{k\to\infty} \frac{T_{k+1} - T_k}{h_p(T_k)} = 0,$$

so that

$$\liminf_{t\to\infty} \frac{\xi_p(t) - t}{h_p(t)} \geq \liminf_{k\to\infty} \frac{\xi_p(T_k) - T_k}{h_p(T_k)} \quad \text{a.s.},$$

which finishes the proof of this step. □

Step 2. Let $p > 1$, then, for every $\varepsilon \in (0,1)$,

$$\liminf_{t\to\infty} \frac{\xi_p(t) - t}{h_p(t)} \leq -(1-\varepsilon) \quad \text{a.s.}$$

Proof: As in the proof of the lower bound, put

$$T_k = \exp(k^{(1+\varepsilon^2)/p}), \quad S_k = T_k - (1-\varepsilon)h_p(T_k), \quad k \geq 1.$$

Let

$$B_k = \{\xi_p(T_k) \leq S_k\} = \left\{\sup_{S_k < t \leq T_k} \frac{Q_X(t)}{f_p(t)} < 1\right\}.$$

It suffices to show $\mathbb{P}(B_n \text{ i.o.}) = 1$, that is

$$\lim_{m\to\infty} \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) = 1. \tag{5.4}$$

Let

$$a_0^k = S_k, \quad y_0^k = f_p(a_0^k), \quad b_0^k = a_0^k + y_0^k,$$

$$a_i^k = b_{i-1}^k, \quad y_i^k = f_p(a_i^k), \quad b_i^k = a_i^k + y_i^k, \quad M_i^k = (a_i^k, b_i^k], \quad \tilde{M}_i^k = \frac{M_i^k}{y_i^k} = (\tilde{a}_i^k, \tilde{b}_i^k].$$

Define J_k to be the biggest number such that $b_{J_k-1}^k \le T_k$ and $b_{J_k}^k > T_k$. Note that $J_k \le [(T_k - S_k)/f_p(S_k)]$.

Since f_p is an increasing function,

$$B_k \supset \bigcap_{i=0}^{J_k} \left\{ \sup_{t\in M_i^k} \frac{Q_X(t)}{f_p(t)} < 1 \right\} \supset \bigcap_{i=0}^{J_k} \left\{ \sup_{t\in M_i^k} Q_X(t) < y_i^k \right\}$$
$$= \bigcap_{i=0}^{J_k} \left\{ \sup_{\substack{s\in \tilde{M}_i^k \\ \tau \ge 0}} Z_{y_i^k}(s,\tau) < m(y_i^k) \right\}.$$

Analogously to (4.4), define a discretization of the set $\tilde{M}_i^k \times J(y_i^k)$ as follows

$$s_{i,l}^k = \tilde{a}_i^k + l q_i^k,\ 0 \le l \le L_i^k,\ L_i^k = [1/q_i^k],\ q_i^k = \theta_i^k \frac{\Delta(y_i^k)}{y_i^k},\ \theta_i^k = \left(m(y_i^k)\right)^{-4/\hat{\alpha}},$$
$$\tau_{i,n}^k = \tau(y_i^k) + n q_i^k,\ 0 \le |n| \le N_i^k,\ N_i^k = [\tau^*(y_i^k)/q_i^k].$$

Recall that $\hat{\alpha} = \min(\alpha_0, \alpha_\infty)$ and let

$$A_k = \bigcap_{i=0}^{J_k} \left\{ \max_{\substack{0\le l\le L_i^k \\ 0\le |n|\le N_i^k}} Z_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \le m(y_i^k) - \frac{(\theta_i^k)^{\frac{\hat{\alpha}}{2}}}{m(y_i^k)} \right\}.$$

Observe that

$$\mathbb{P}\left(\bigcup_{k=m}^{\infty} A_k\right) \le \mathbb{P}\left(\bigcup_{k=m}^{\infty} B_k\right) + \sum_{k=m}^{\infty} \mathbb{P}\left(A_k \cap B_k^c\right).$$

Furthermore,

$$\sum_{k=m}^{\infty} \mathbb{P}\left(A_k \cap B_k^c\right)$$
$$\le \sum_{k=m}^{\infty}\sum_{i=0}^{J_k} \mathbb{P}\left(\max_{\substack{0\le l\le L_i^k \\ 0\le |n|\le N_i^k}} Z_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \le m(y_i^k) - \frac{(\theta_i^k)^{\frac{\hat{\alpha}}{2}}}{m(y_i^k)},\ \sup_{\substack{s\in \tilde{M}_i^k \\ \tau \ge 0}} Z_{y_i^k}(s,\tau) \ge m(y_i^k) \right)$$
$$\le \sum_{k=m}^{\infty}\sum_{i=0}^{J_k} \mathbb{P}\left(\max_{\substack{0\le l\le L_i^k \\ 0\le |n|\le N_i^k}} Z_{y_i^k}(s_{i,l}^k, \tau_{i,n}^k) \le m(y_i^k) - \frac{(\theta_i^k)^{\frac{\hat{\alpha}}{2}}}{m(y_i^k)},\ \sup_{\substack{s\in \tilde{M}_i^k \\ \tau \in J(y_i^k)}} Z_{y_i^k}(s,\tau) \ge m(y_i^k) \right)$$

$$+\sum_{k=m}^{\infty}\sum_{i=0}^{J_k}\mathbb{P}\left(\sup_{\substack{s\in\tilde{M}_i^k\\ \tau\notin J(y_i^k)}} Z_{y_i^k}(s,\tau)\geq m(y_i^k)\right). \tag{5.5}$$

By Theorem 3.1 and Theorem 3.2, for sufficiently large m and some $K_1, K_2 > 0$, the first sum is bounded from above by

$$\begin{aligned}
&\sum_{k=m}^{\infty}\sum_{i=0}^{J_k}K_1\frac{(y_i^k)^\gamma}{m(y_i^k)}\Psi(m(y_i^k))e^{-(m(y_i^k))^3/K_2}\\
&\qquad\leq \sum_{k=m}^{\infty}\sum_{i=0}^{J_k}K_1\mathbb{P}\left(\sup_{(s,\tau)\in[0,1]\times\mathbb{R}_+} Z_{y_i^k}(s,\tau)>m(y_i^k)\right)e^{-(m(y_i^k))^3/K_2}\\
&\qquad\leq \sum_{k=m}^{\infty}\sum_{i=0}^{J_k}K_1\mathbb{P}\left(\sup_{t\in[0,f_p(a_i^k)]} Q_X(t)>f_p(a_i^k)\right)e^{-(\log a_i^k)^{3/2}/K_2}\\
&\qquad\leq K\int_m^\infty \frac{\psi(f_p(x))}{f_p(x)}e^{-\log^{3/2}(x)/K_2}\,\mathrm{d}x<\infty.
\end{aligned}$$

Note that by (3.13), for sufficiently large m, the term in (5.5) is bounded from above by

$$\begin{aligned}
&K\sum_{k=m}^{\infty}\sum_{i=0}^{J_k}\frac{(y_i^k)^\gamma}{m(y_i^k)}\Psi(m(y_i^k))\exp\left(-\frac{b}{4}\log^2 m(y_i^k)\right)\\
&\leq K\int_m^\infty\frac{\psi(f_p(x))}{f_p(x)}e^{-\frac{b}{4}(\frac{1}{2}\log_2 x))^2}\,\mathrm{d}x\\
&<\infty.
\end{aligned}$$

Therefore

$$\lim_{m\to\infty}\sum_{k=m}^{\infty}\mathbb{P}\left(A_k\cap B_k^c\right)=0$$

and

$$\lim_{m\to\infty}\mathbb{P}\left(\bigcup_{k=m}^{\infty}B_k\right)\geq\lim_{m\to\infty}\mathbb{P}\left(\bigcup_{k=m}^{\infty}A_k\right).$$

To finish the proof of (5.4), we only need to show that

$$\mathbb{P}\left(A_n \text{ i.o.}\right)=1. \tag{5.6}$$

Similarly to (5.2), we have

$$\int_{S_k}^{T_k}\frac{1}{f_p(u)}\mathbb{P}\left(\sup_{t\in[0,f_p(u)]}Q_X(t)>f_p(u)\right)\mathrm{d}u\sim(1-\varepsilon)p\log_2 T_k.$$

Now from Theorem 4.4 it follows that

$$\mathbb{P}\left(A_k\right)\geq\frac{1}{4}\exp\left(-(1-\varepsilon^2)p\log_2 T_k\right)-KS_k^{-\rho}\geq\frac{1}{8}k^{-(1-\varepsilon^4)},$$

for every k sufficiently large. Hence,

$$\sum_{k=1}^{\infty}\mathbb{P}\left(A_k\right)=\infty. \tag{5.7}$$

Applying Berman's inequality, we get for $t < k$

$$\mathbb{P}\left(A_k A_t\right) \le \mathbb{P}\left(A_k\right)\mathbb{P}\left(A_t\right) + Q_{k,t}, \tag{5.8}$$

where,

$$Q_{k,t} = \sum_{\substack{0\le i\le J_k\\ 0\le j\le J_t}} \sum_{\substack{0\le l\le L_i^k\\ 0\le p\le L_j^t}} \sum_{\substack{|n|\le N_i^k\\ |m|\le N_j^t}} \frac{|r_{y_i^k,y_j^t}(s_{i,l}^k,\tau_{i,n}^k,s_{j,p}^t,\tau_{j,m}^t)|}{\sqrt{1-r^2_{y_i^k,y_j^t}(s_{i,l}^k,\tau_{i,n}^k,s_{j,p}^t,\tau_{j,m}^t)}}$$
$$\times \exp\left(-\frac{(m(y_i^k)-(m(y_i^k))^{-3})^2+(m(y_j^t)-(m(y_j^t))^{-3})^2}{2(1+|r_{y_i^k,y_j^t}(s_{i,l}^k,\tau_{i,n}^k,s_{j,p}^t,\tau_{j,m}^t)|)}\right).$$

For any $0\le i\le J_k$, $0\le j\le J_t$, $0\le l\le L_i^k$, $0\le p\le L_j^t$, and $t<k$,

$$y_i^k s_{i,l}^k - y_j^t s_{j,p}^t = a_i^k + y_i^k l q_i^k - \left(a_j^t + y_j^t p q_j^t\right) \ge S_k - T_t \ge S_k - T_{k-1} \ge \frac{1}{2}(T_k - T_{k-1}),$$

where the last inequality holds for k large enough since it is easy to see that

$$\frac{S_{k+1}-T_k}{T_{k+1}-T_k} \sim 1, \qquad \text{as } k\to\infty.$$

Thus, sufficiently large k and every $0\le t<k$, and a generic constant $K>0$, similarly to (4.1) we have,

$$\sup_{\substack{0\le i\le J_k\\ 0\le j\le J_t\\ 0\le l\le L_i^k, 0\le p\le L_j^t\\ |n|\le N_i^k, |m|\le N_j^t}} |r_{y_i^k,y_j^t}(s_{i,l}^k,\tau_n^k,s_{j,p}^t,\tau_{j,m}^t)| \le K(T_k-T_{k-1})^{-\lambda/2} \le \frac{\min(1,\lambda)}{32}.$$

Therefore, for some generic constant K not depending on k and t which may vary between lines, for every $t<k$ sufficiently large,

$$\begin{aligned}
Q_{k,t} &\le K \sum_{\substack{0\le i\le J_k\\ 0\le j\le J_t}} L_i^k L_j^t N_i^k N_j^t (T_k-T_{k-1})^{-\lambda/2} \exp\left(-\frac{(m(y_i^k))^2+(m(y_j^t))^2}{2(1+\frac{\lambda}{16})}\right)\\
&\le K(T_k-T_{k-1})^{-\lambda/2}(L_{J_k}^k L_{J_t}^t)^2\\
&\qquad \times \sum_{\substack{0\le i\le J_k\\ 0\le j\le J_t}} \left(a_i^k \log^{\frac{\gamma-1}{2(1-\alpha_\infty)}-p} a_i^k\right)^{-\frac{1}{1+\frac{\lambda}{16}}} \left(a_j^t \log^{\frac{\gamma-1}{2(1-\alpha_\infty)}-p} a_j^t\right)^{-\frac{1}{1+\frac{\lambda}{16}}}\\
&\le K(T_k-T_{k-1})^{-\lambda/2}\left(\log T_k\right)^{\upsilon}\left(T_k\right)^{\frac{\frac{\lambda}{8}}{1+\frac{\lambda}{8}}}\left(T_t\right)^{\frac{\frac{\lambda}{8}}{1+\frac{\lambda}{8}}}\\
&\le K T_k^{-\lambda/8} \le K\exp(-\lambda k^{(1+\varepsilon^2)/p}/8),
\end{aligned}$$

with $\upsilon>0$ a fixed constant. Hence we have,

$$\sum_{0\le t<k<\infty} Q_{k,t} < \infty. \tag{5.9}$$

Now (5.6) follows from (5.7)-(5.9) and the general form of the Borel-Cantelli lemma. $\square$

Step 3. If $p \in (0,1]$, then, for every $\varepsilon \in (0, \frac{1}{4})$,

$$\liminf_{t\to\infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \geq -(1+2\varepsilon) \quad \text{a.s.} \tag{5.10}$$

and

$$\liminf_{t\to\infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} \leq -(1-\varepsilon) \quad \text{a.s.,} \tag{5.11}$$

Proof: Put

$$T_k = \exp(k), \quad S_k = T_k \exp\left(-(1+2\varepsilon)h_p(T_k)/T_k\right).$$

Proceeding the same as in the proof of (5.3), one can obtain that

$$\liminf_{k\to\infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \geq -(1+2\varepsilon) \quad \text{a.s.}$$

On the other hand it is clear that

$$\liminf_{t\to\infty} \frac{\log(\xi_p(t)/t)}{h_p(t)/t} = \liminf_{k\to\infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \quad \text{a.s.}$$

since

$$\liminf_{k\to\infty} \frac{\log(T_k/T_{k+1})}{h_p(T_k)/T_k} = 0.$$

This proves (5.10).

Let

$$T_k = \exp\left(k^{1+\varepsilon^2}\right), \quad S_k = T_k \exp\left(-(1-\varepsilon)h_p(T_k)/T_k\right).$$

Noting that

$$\frac{S_{k+1} - T_k}{S_{k+1}} \sim 1 \quad \text{as } k \to \infty,$$

along the same line as in the proof of (5.4), we also have

$$\liminf_{k\to\infty} \frac{\log(\xi_p(T_k)/T_k)}{h_p(T_k)/T_k} \leq -(1-\varepsilon) \quad \text{a.s.,}$$

which proves (5.11). □

6. Appendix

Proof of (3.10). Let $g_1(t) = g(\tau^* t)$. Then it suffices to prove the claim in (3.10) for

$$g_1(t) = \frac{|1+t|^{2\alpha_\infty} + |1-t|^{2\alpha_\infty} - 2|t|^{2\alpha_\infty}}{2}.$$

Note that $g_1(t) = g_1(-t), t \geq 0$, it is sufficient to prove the argument for $t \geq 0$. We distinguish three scenarios: $0 < \alpha_\infty < 1/2$, $\alpha_\infty = 1/2$ and $1/2 < \alpha_\infty < 1$.
We first focus on $\alpha_\infty = 1/2$. If $\alpha_\infty = 1/2$, then

$$g_1(t) = \begin{cases} 1-t & 0 \leq t \leq 1 \\ 0 & t \geq 1, \end{cases}$$

which implies that (3.10) holds for $g_1(t)$.
Next we consider $0 < \alpha_\infty < 1/2$. For $0 < t \leq 1$, the first derivative of g_1

$$\dot{g}_1(t) = \alpha_\infty \left((1+t)^{2\alpha_\infty - 1} - (1-t)^{2\alpha_\infty - 1} - 2t^{2\alpha_\infty - 1}\right) < 0.$$

Moreover, for $t > 1$, by the convexity of $t^{2\alpha_\infty - 1}$

$$\dot{g}_1(t) = \alpha_\infty \left((1+t)^{2\alpha_\infty - 1} + (t-1)^{2\alpha_\infty - 1} - 2t^{2\alpha_\infty - 1}\right) > 0.$$

Additionally, direct calculation shows that $\lim_{t\to\infty} g_1(t) = 0$. This means that for $0 < \alpha_\infty < 1/2$, $g_1(t)$ is strictly decreasing over $(0,1)$ and increasing over $(1,\infty)$ with $g_1(0) = 1$, $g_1(1) < 0$ and $\lim_{t\to\infty} g_1(t) = 0$. This implies that for any $0 < \delta < 1$,

$$\sup_{t>\delta} g_1(t) < 1.$$

Thus (3.10) holds for g_1 with $0 < \alpha_\infty < 1/2$.

Finally, we focus on $1/2 < \alpha_\infty < 1$. For $0 < t < 1$, using the fact that $s^{2\alpha_\infty-2}$ is strictly decreasing over $(0,\infty)$, we have

$$\begin{aligned}\dot{g}_1(t) &= \alpha_\infty \left((1+t)^{2\alpha_\infty-1} - (1-t)^{2\alpha_\infty-1} - 2t^{2\alpha_\infty-1}\right)\\ &\le \alpha_\infty \left((1+t)^{2\alpha_\infty-1} - (1-t)^{2\alpha_\infty-1} - (2t)^{2\alpha_\infty-1}\right)\\ &= \alpha_\infty(2\alpha_\infty-1)\left(\int_{1-t}^{1+t} s^{2\alpha_\infty-2}ds - \int_0^{2t} s^{2\alpha_\infty-2}ds\right) < 0.\end{aligned}$$

For $t > 1$, by the convexity of $t^{2\alpha_\infty-1}$,

$$\dot{g}_1(t) = \alpha_\infty \left((1+t)^{2\alpha_\infty-1} + (t-1)^{2\alpha_\infty-1} - 2t^{2\alpha_\infty-1}\right) < 0.$$

Additionally, direct calculation shows that $\lim_{t\to\infty} g_1(t) = 0$. Thus we have that $g_1(t)$ is strictly decreasing over $(0,\infty)$ with $g_1(0) = 1$ and $\lim_{t\to\infty} g_1(t) = 0$. Clearly, for any $0 < \delta < 1$,

$$\sup_{t>\delta} g_1(t) < 1,$$

implying that (3.10) holds for $1/2 < \alpha_\infty < 1$. This completes the proofs.

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