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First Passage Time Densities through Hölder curves

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Abstract. We prove that for a standard Brownian motion, there exists a first-passage-time density function through a Hölder curve with exponent greater than 1/2. With a property of local time of a standard Brownian motion and adopting the theories of partial differential equations in Cannon (1984) and the strategies in Fasano (2005) and Carinci et al. (2016), we find a sufficient condition for existence of the density function. We also show that this density function is proportional to the space derivative of the Green function of the heat equation with Dirichlet boundary condition at the moving boundary.

1. Introduction

In this paper we will study the probability of the hitting time to a moving boundary. To state the main result, we need some notations on a Brownian motion. Let us call $P_{r,s}, r \in \mathbb{R}, s \geq 0$, the law on $C([s,\infty))$ of the Brownian motion B_t, $t \geq s$, which starts from r at time s, i.e. $B_s = r$. For each $t > s$ the law of B_t is absolutely continuous with respect to the Lebesgue measure and has a density $G_{s,t}(r,\cdot)$ which is the Gaussian $G(\cdot,t;r,s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left(-\frac{(\cdot - r)^2}{2(t-s)}\right)$. We denote by $E_{r,s}$ the expectation under $P_{r,s}$. For a given curve $X = \{t \to X_t\}$, $s \geq 0$ and $r < X_s$, we define

$$\tau^X_{r,s} = \inf\{t \geq s : B_t \geq X_t\}, \ \textit{and} = \infty \text{ if the set is empty,} \tag{1.1}$$

where $B_s = r$ and denote by $F^X_{r,s}(dt)$ the distribution of $\tau^X_{r,s}$ induced by $P_{r,s}$. For s=0, we use abbreviated forms P_r, E_r, τ^X_r, $F^X_r(dx)$ instead of $P_{r,0}$, $E_{r,0}$, $\tau^X_{r,0}$, $F^X_{r,0}(dx)$ respectively whenever it is needed. In addition, for $t > 0$, let us call $d\mu_{r_0}(\cdot,t)$ the positive measure on $(-\infty, X_t)$ such that

$$\int_{(-\infty,X_t)} d\mu_{r_0}(x,t) f(x) = E_{r_0}[f(B_t); \tau^X_{r_0} > t] \tag{1.2}$$

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for all $f \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} f \Subset (-\infty, X_t)$. The main result in the paper is:

Theorem 1.1. *If X is Hölder continuous on any finite interval in $[0,\infty)$ with exponent $\gamma \in (1/2, 1]$ and $r_0 < X_0$, then*

(1) $d\mu_{r_0}(x,t) = G^X_{0,t}(r_0, x)dx$ *where for all* $x < X_t$,

$$G^X_{0,t}(r_0,x) = G_{0,t}(r_0,x) - \int_{[0,t)} F^X_{r_0}(ds) G_{s,t}(X_s, x). \tag{1.3}$$

(2) $F^X_{r_0}(ds)$ *has a density function* p *on* $[0,\infty)$, *namely* $F^X_{r_0}(ds) = p(s)ds$.

(3) $p(t) = -\dfrac{1}{2}\dfrac{\partial}{\partial x} G^X_{0,t}(r_0,x)\Big|_{x=X_t^-}$ *for all* $t > 0$.

(4) $G^X_{0,t}(r_0,x)$ *solves*

$$v_t = \frac{1}{2} v_{xx}, \quad -\infty < x < X_t, \ t > 0, \tag{1.4}$$

$$\lim_{(x,t)\to(X_s,s)} v(x,t) = 0, \ s > 0, \tag{1.5}$$

$$\lim_{(x,t)\to(y,0)} v(x,t) = \delta_{r_0}(y), \ -\infty < y < X_0. \tag{1.6}$$

Remarks

Item 4 of Theorem 1.1 states that for any $r_0 < X_0$ the function $G^X_{0,\cdot}(r_0,\cdot)$ given by (1.3) is the Green function of the heat equation with Dirichlet boundary conditions at the moving boundary X. In Alili and Patie (2014), when a moving boundary is infinitely differentiable, it is showed that the space derivative of the Green function of the heat equation at the boundary is proportional to the hitting time density function. Likewise by items 2 and 3 of Theorem 1.1, the space derivative of $G^X_{0,\cdot}(r_0,\cdot)$ at the moving boundary is proportional to the hitting time density function p.

For the case when $X_t = a + bt$, it is well known(see for instance Karatzas and Shreve, 1991) that for $r < a$, τ^X_r has a probability density function given by $f(t) = \dfrac{a-r}{\sqrt{2\pi t^3}} \exp\left(-\dfrac{(a+bt-r)^2}{2t}\right)\mathbf{1}_{t>0}$. In addition, there is another result in Ricciardi et al. (1984) when $X_t = a + bt^{\frac{1}{p}}$ for $p \geq 2$, $r < a$, then τ^X_r has a probability density function. In Borovkov and Downes (2010), it is showed that if the boundary behaves as a Lipschitz curve in a local time, then the first passage time density can be expressed explicitly.

In Peskir and Shiryaev (2006), it is proved that for any continuous curve X_t and $r < X_0$, there is a distribution F^X_r of τ^X_r which satisfies the following integral equation(called the Master Equation):

$$\Psi\left(\frac{z-r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{z-X_s}{\sqrt{t-s}}\right) F^X_r(ds), \tag{1.7}$$

where $z \geq X_t$, $t > 0$ and $\Psi(z) = \displaystyle\int_z^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)dx$. This can be proved intuitively as follows; the left hand side of (1.7) is the probability that a Brownian motion starts at r at time 0 and reaches z greater or equal to X_t at time t. Then it should hit the boundary at least once which implies the right hand side of of (1.7). Moreover, it is showed that if X_t is C^1, then there exists a continuous density function f of F_r^X. There is an extension of this result, in Ricciardi et al. (1984), to curves X which are differentiable with $\left|\frac{dX_t}{dt}\right| \le Ct^{-\alpha}$ for some constant $C > 0$ and $\alpha < 1/2$.

By (1.7), for any continuous curve X_t and $r < X_0$, we have

$$\Psi\left(\frac{X_t - r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{X_t - X_s}{\sqrt{t-s}}\right) F_r^X(ds). \tag{1.8}$$

which can be regarded as an integral equation for $F_r^X(ds)$. In Taillefumier and Magnasco (2014), the equation (1.8) is studied when X is Hölder continuous with exponent greater than 1/2. It is proved that there exists a unique continuous function q such that

$$\Psi\left(\frac{X_t - r}{\sqrt{t}}\right) = \int_0^t \Psi\left(\frac{X_t - X_s}{\sqrt{t-s}}\right) q(s)ds. \tag{1.9}$$

To conclude that $F_r^X(ds) = q(s)ds$, one still needs that $F_r^X(ds)$ is absolutely continuous with respect to Lebesgue measure. The analysis in Taillefumier and Magnasco (2014) as well as the proof of Theorem 1.1 uses extensively the work by Cannon Cannon (1984) on the heat equation with the moving boundary.

2. Preliminaries: a weaker form of Theorem 1.1

We define D as

$$D := \{(x,t) : -\infty < x < X_t,\ t > 0\}$$

and

$$u(x,t) := \int_{-\infty}^0 h(\xi) G_{0,t}^X(\xi, x) d\xi \tag{2.1}$$

for given $h \in C_c^\infty((-\infty, X_0); \mathbb{R}_+)$ and all $(x,t) \in D$.

If X is Hölder continuous with exponent $\gamma \in [1/2, 1]$, then u solves the heat equation in D with the initial condition h and Dirichlet boundary condition.(See Theorem 2.1 below) However uniqueness fails for the heat equation with the initial function h and Dirichlet boundary conditions (see Remark 2.2 and 2.3 below).

To have the uniqueness, first of all, we restrict time variable of D in a finite interval. Thus we fix $T > 0$ and define the parabolic cylinder D_T which is a subset of D as

$$D_T := \{(x,t) : -\infty < x < X_t,\ 0 < t \le T\}.$$

Consider the following initial-boundary value problem

$$\begin{cases} v \in C(\overline{D_T}) \cap C^{2,1}(D_T), \\ v_t = \dfrac{1}{2} v_{xx}, \quad (x,t) \in D_T, \\ v(X_t, t) = 0,\ 0 < t \le T, \\ v(x,0) = h(x),\ -\infty < x < X_0, \\ \lim\limits_{x \to -\infty} \sup\limits_{0<t<T} |v(x,t)| = 0. \end{cases} \tag{2.2}$$

We prove the following weaker form of Theorem 1.1.

Theorem 2.1. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in [1/2, 1]$ and let $X_0 = 0$.*

(1) *The function u defined in (2.1) is the unique solution of (2.2).*

(2) *If $\gamma \in (1/2, 1]$, then u has the left hand derivative at the boundary $u_x(X_t^-, t)$ which is continuous on $(0, \infty)$. Moreover, for all $t > 0$, $p_h(t) := -\frac{1}{2}u_x(X_t^-, t)$ satisfies*

$$p_h(t) = -\int_{-\infty}^{0} h(\xi)G_x(X_t, t; \xi, 0)d\xi + \int_0^t G_x(X_t, t; X_\tau, \tau)p_h(\tau)d\tau. \tag{2.3}$$

Remark 2.2. The uniqueness for (2.2) is not guaranteed if we do not assume $v \in C(\overline{D_T})$. When $X_t = 0$ for all t and h is identically 0, if $v(x,t)$ is given by $\frac{1}{\sqrt{2\pi t}}\left\{-\frac{x}{t}\right\}\exp\left(-\frac{x^2}{2t}\right)$, then this satisfies the heat equation with the initial data 0 and is also 0 on the boundary, but this is not continuous at $(0, 0)$.

Remark 2.3. We need the condition $\lim_{x\to-\infty}\sup_{0<t<T}|v(x,t)| = 0$ to have uniqueness. Indeed, for $X_t = 0$ for all $t \geq 0$, the function

$$v(x,t) = \sum_{n=0}^{\infty} f^{(n)}\left(\frac{t}{2}\right)\frac{x^{2n+1}}{(2n+1)!},$$

where

$$f(t) = \begin{cases} \exp\left(-\frac{1}{t^2}\right), & t > 0, \\ 0, & t \leq 0. \end{cases}$$

satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Furthermore, $v \in C(\overline{D_T}) \cap C^{2,1}(D_T)$.

Remark 2.4. It can be shown $\lim_{t\to 0} p_h(t) = 0$ by (2.3) and Lemma 3.3 below. Thus $u_x(X_t^-, t)$ is continuous on $[0, \infty)$.

Following Cannon (1984), we introduce $C^0_{(\nu)}((0,T])$, $0 < \nu \leq 1$, as the subspace of $C((0,T])$ that consists of those functions φ such that

$$\|\varphi\|_T^{(\nu)} = \sup_{0<t\leq T} t^{1-\nu}|\varphi(t)| < \infty.$$

Then $C^0_{(\nu)}((0,T])$ is a Banach space under the norm $\|\cdot\|_T^{(\nu)}$.

We also introduce the following lemmas from Cannon (1984) which play an essential role in our analysis.

Lemma 2.5 (*jump relation*)**.** *For $\varphi \in C^0_{(\nu)}((0,T])$, we have*

$$\lim_{x\to X_t^{\pm}} \frac{\partial w_\varphi}{\partial x}(x,t) = \mp\varphi(t) + \int_0^t G_x(X_t, t; X_\tau, \tau)\varphi(\tau)d\tau, \tag{2.4}$$

where $w_\varphi(x,t) = \int_0^t G(x, t; X_\tau, \tau)\varphi(\tau)d\tau$.

For two continuous curves s_1, s_2 such that $s_1(t) < s_2(t), t \in [0,T]$, let us set $E_T := \{(x,t) : s_1(t) < x < s_2(t),\ 0 < t \leq T\}$ and $B_T := \{(s_i(t), t) : 0 \leq t \leq T,\ i \in \{1,2\}\} \cup \{(x, 0) : s_1(0) < x < s_2(0)\}$.

Lemma 2.6 (*The Weak Maximum(Minimum) Principle*)**.** *For a solution u of $u_t = \frac{1}{2}u_{xx}$ in E_T, which is continuous in $E_T \cup B_T$,*

$$\max_{E_T\cup B_T} u = \max_{B_T} u. \quad \left(\min_{E_T\cup B_T} u = \min_{B_T} u.\right) \tag{2.5}$$

Before going to the proof of Theorem 2.1, we need the following proposition.

Proposition 2.7. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in [1/2, 1]$. If the starting point of the Brownian motion is close to X, the first hitting time converges to 0. Precisely, $\lim_{\xi\to X_0} P_\xi\left[\tau_\xi^X > s\right] = 0$ for all $s > 0$.*

Proof: Without loss of generality, we may reduce this problem as the case for Brownian motion starting at 0 and $X_0 = \epsilon > 0$ and let ϵ go to 0. For $s > 0$, let $m := \sup_{0\le t_1<t_2\le s} \frac{|X_{t_2} - X_{t_1}|}{|t_2 - t_1|^\gamma}$. Since $\limsup_{t\downarrow 0} \frac{B_t}{\sqrt{t}} = \infty$ a.e., for $M > m$, we have a sequence $t_k \downarrow 0$ such that $M\sqrt{t_k} \le B_{t_k}$ a.e. and $M\sqrt{t_k} - mt_k^\gamma \downarrow 0$ for all k. Thus, for $0 < t \le s$, we deduce that a.e.

$$\begin{aligned}\sup_{0\le l\le t}\{B_l - X_l\} \ge \sup_{0\le l\le t}\{B_l - (\epsilon + ml^\gamma)\} &\ge \sup_{t_k\le t}\{B_{t_k} - (\epsilon + mt_k^\gamma)\} \\ &\ge \sup_{t_k\le t}\left\{M\sqrt{t_k} - (\epsilon + mt_k^\gamma)\right\}.\end{aligned}$$

Therefore,

$$P_0\left(\sup_{0\le l\le t}\{B_l - X_l\} < 0\right) \le P_0\left(\sup_{t_k\le t}\left\{M\sqrt{t_k} - mt_k^\gamma\right\} < \epsilon\right).$$

For each sufficently small $\epsilon > 0$, there is a greatest $k(\epsilon)$ such that $M\sqrt{t_{k(\epsilon)}} - mt_{k(\epsilon)}^\gamma \ge \epsilon$. Thus we obtain that $\tau_0^X \le t_{k(\epsilon)}$ a.e. for all sufficiently small $\epsilon > 0$. Since $k(\epsilon)$ is an increasing function as ϵ decreases and $\lim_{\epsilon\to 0} k(\epsilon) = \infty$, the proposition follows. □

Proof of Theorem 2.1: Let us define $X'_\tau := X_{t-\tau}$ for all $0 \le \tau \le t$. Using the invariance of the law of the Brownian motion under time reversal, we have

$$u(x,t) = \int_{-\infty}^{0} h(r')G_{0,t}^X(r',x)dr' = E_x[h(B_t); \tau_x^{X'} > t]. \tag{2.6}$$

Using this equality, we also have

$$|u(x,t)| = \left|E_x[h(B_t); \tau_x^{X'} > t]\right| \le \|h\|_\infty P_x[\tau_x^{X'} > t]. \tag{2.7}$$

For $s > 0$, let us choose $0 < s^* < s$. Then for all (x,t) sufficiently close to (X_s, s), we obtain

$$P_x[\tau_x^{X'} > t] \le P_x[\tau_x^{X'} > s^*] \tag{2.8}$$

which vanishes when $(x,t) \to (X_s, s)$ by Proposition 2.7 so that $\lim_{(x,t)\to(X_s,s)} u(x,t) = 0$. In addition, we have

$$|u(x,t)| = \left|E_x[h(B_t); \tau_x^{X'} > t]\right| \le E_x[|h(B_t)|] = \int_{-\infty}^{0} |h(\xi)|G_{0,t}(x,\xi)d\xi \tag{2.9}$$

which also vanishes when $(x,t)\to(0,0)$, since the support of h is strictly away from 0. To prove that u satisfies the initial data h, we write $y_t = \min_{s\in[0,t]} X'_s$. For $x<0$ and any positive $\lambda>0$,

$$P_x\left[\tau_x^{X'}\le t\right]\le P_x\left[\max_{s\in[0,t]} B_s\ge y_t\right]=P_x\left[\max_{s\in[0,t]}\exp(\lambda B_s)\ge\exp(\lambda y_t)\right]. \quad (2.10)$$

Since the exponential of Brownian motion is a positive submartingale, we can apply Doob's inequality, then

$$P_x\left[\max_{s\in[0,t]}\exp(\lambda B_s)\ge\exp(\lambda y_t)\right]\le\frac{E_x[\exp(\lambda B_t)]}{\exp(\lambda y_t)}=\exp\left(\frac{1}{2}\lambda^2 t-\lambda(y_t-x)\right). \quad (2.11)$$

From (2.10) and (2.11), we get

$$\lim_{(x,t)\to(y,0)} P_x\left[\tau_x^{X'}\le t\right]\le\exp(-\lambda(X_0-y))=\exp(\lambda y) \quad (2.12)$$

so that the left hand side vanishes since $\lambda>0$ is arbitrary and we are assuming $y<X_0=0$. Thus we obtain that

$$\lim_{(x,t)\to(y,0)}\int_{-\infty}^{0} h(\xi)G_{0,t}^{X}(\xi,x)d\xi=\lim_{(x,t)\to(y,0)} E_x[h(B_t)]=h(y). \quad (2.13)$$

By the properties of the Gaussian kernel, we deduce that u solves (2.2). We now prove uniqueness. If v_1, v_2 satisfy all the above conditions, then $v_1-v_2\in C(\overline{D_T})\cap C^{2,1}(D_T)$ satisfies the heat equation with the initial data 0 and is also 0 on the boundary. Moreover, $\lim_{x\to-\infty}\sup_{0<t<T}|v_1(x,t)-v_2(x,t)|=0$ so that by the weak maximum(minimum) principle, v_1-v_2 is all 0 in D_T. Therefore, $v_1=v_2$ so that item 1 is proved.

To prove item 2, assuming that $\gamma\in(1/2,1]$, we define, $(x,t)\in D_T$,

$$v(x,t):=\int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi+\int_0^t G(x,t;X_\tau,\tau)\varphi(\tau)d\tau, \quad (2.14)$$

where $\varphi\in C^0_{(\gamma)}((0,T])$ satisfies

$$0=\int_{-\infty}^{0} h(\xi)G(X_t,t;\xi,0)d\xi+\int_0^t G(X_t,t;X_\tau,\tau)\varphi(\tau)d\tau. \quad (2.15)$$

If we apply the Abel inverse operator A defined by

$$(AF)(t)=\frac{1}{\pi}\frac{d}{dt}\int_0^t\frac{F(\eta)}{(t-\eta)^{\frac{1}{2}}}d\eta$$

on both sides of (2.15), then from Chapter 14 in Cannon (1984) we have a equivalent Volterra integral equation of the second kind:

$$\varphi(t)=\psi(t)+\int_0^t H(t,\tau)\varphi(\tau)d\tau, \quad (2.16)$$

where $\psi\in C^0_{(\gamma)}((0,T])$ and $|H(t,\tau)|\le C(t-\tau)^{2\gamma-2}$. The existence of $\varphi\in C^0_{(\gamma)}((0,T])$ which satisfies (2.16) can be proved similarly to the proof of Proposition 3.2 that we will show later. Then v is well-defined and solves (2.2) so that

$v = u$ since u is the unique solution of (2.2). By the jump relation, we have

$$u_x(X_t^-, t) = \int_{-\infty}^0 h(\xi) G_x(X_t, t; \xi, 0) d\xi + \varphi(t) + \int_0^t G_x(X_t, t; X_\tau, \tau)\varphi(\tau) d\tau \quad (2.17)$$

so that $u_x(X_t^-, t) \in C^0_{(\gamma)}((0,T])$. Since T is arbitrary, we deduce $u_x(X_t^-, t)$ is continuous on $(0, \infty)$.

To show (2.3), let us fix $(x,t) \in D_T$ and let us define $D^{(t)}_{\epsilon,r} := \{(\xi,\tau) : r < \xi < X_\tau - \epsilon,\ \epsilon < \tau < t - \epsilon\}$ for each $\epsilon > 0$ and $r \in \mathbb{R}$. By the Green's identity, we have

$$\frac{1}{2}(u_\xi G - u G_\xi)_\xi - (uG)_\tau = 0 \implies \oint_{\partial D^{(t)}_{\epsilon,r}} \frac{1}{2}(u_\xi G - u G_\xi) d\tau + (uG) d\xi = 0. \quad (2.18)$$

It can be also shown that $\lim_{x \to -\infty} \sup_{0<t<T} |u_x(x,t)| = 0$ by the properties of the Gaussian kernel. Hence we obtain another representation of u by letting $\epsilon \to 0$, $r \to -\infty$,

$$u(x,t) = \int_{-\infty}^0 h(\xi) G(x,t;\xi,0) d\xi + \frac{1}{2}\int_0^t G(x,t;X_\tau,\tau) u_x(X_\tau^-, \tau) d\tau. \quad (2.19)$$

Differentiating both sides of (2.19) with respect to x and applying the jump relation, we get

$$\frac{1}{2}u_x(X_t^-, t) = \int_{-\infty}^0 h(\xi) G_x(X_t, t; \xi, 0) d\xi + \frac{1}{2}\int_0^t G_x(X_t, t; X_\tau, \tau) u_x(X_\tau^-, \tau) d\tau. \quad (2.20)$$

which implies (2.3). □

3. Proof of Theorem 1.1

To prove item 1, let X be a continuous curve defined on $[0,\infty)$ and let $r_0 < X_0$. Using the strong Markov property, we have, for $f \in C_c^\infty(\mathbb{R})$ with supp $f \Subset (-\infty, X_t)$ and $0 \le s \le t$,

$$E_{r_0}[f(B_t) | \tau^X_{r_0} = s] = E_{X_s,s}[f(B_t)]. \quad (3.1)$$

Thus we get

$$E_{r_0}[f(B_t); \tau^X_{r_0} \le t] = \int_{[0,t]} F^X_{r_0}(ds) E_{X_s,s}[f(B_t)]. \quad (3.2)$$

It follows that item 1 holds with $G^X_{0,t}(r_0, x)$ given by (1.3).

To prove item 2, from now on, X is a Hölder continuous curve defined on $[0,\infty)$ with exponent $\gamma \in (1/2, 1]$ and $X_0 = 0$. Comparing the definition (2.1) of u and (2.19), using (1.3), we see the following equality:

$$\int_{[0,t)} G_{\tau,t}(X_\tau, x) \int_{-\infty}^0 h(\xi) F^X_\xi(d\tau) d\xi = -\frac{1}{2}\int_0^t G(x,t;X_\tau,\tau) u_x(X_\tau^-, \tau) d\tau. \quad (3.3)$$

Set $F^X_h(d\tau) := \displaystyle\int_{-\infty}^0 h(\xi) F^X_\xi(d\tau) d\xi$.

Proposition 3.1. $F^X_h(d\tau) = -\frac{1}{2} u_x(X_\tau^-, \tau) d\tau$.

For the proof of Proposition 3.1, we introduce the mass lost $\Delta_I^X(u)$, $I = [t_1, t_2] \subset [0, T]$, $t_1 \le t_2$, is defined by

$$\Delta_I^X(u) = \int_{-\infty}^{X_{t_1}} u(r, t_1)dr - \int_{-\infty}^{X_{t_2}} u(r, t_2)dr. \tag{3.4}$$

If we see the right hand side of (2.19), we can extend u to $\bar{u}$ defined in $\{(x,t) : x \in \mathbb{R},\ 0 < t \le T\}$ as

$$\bar{u}(x,t) = \int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi + \frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau. \tag{3.5}$$

Then this satisfies the heat equation with $\lim_{(x,t)\to(y,0)} \bar{u}(x,t) = 0$ for all $y \ge 0$ and also satisfies $\bar{u}(X_t, t) = 0$ for all $0 < t \le T$. Moreover, by the properties of Gaussian kernel, we have

$$\lim_{x\to\infty} \sup_{0<t<T} |\bar{u}(x,t)| = 0. \tag{3.6}$$

It follows that $\bar{u}(x,t) = 0$ in $\{(x,t) : x \ge X_t, 0 < t \le T\}$ by the weak maximum(minimum) principle. Thus we assume that u is defined $\{(x,t) : x \in \mathbb{R},\ 0 \le t \le T\}$ such that it is 0 in $\{(x,t) : x \ge X_t, 0 \le t \le T\}$.

Heuristically

$$\Delta_I^X(u) = -\int\int_{t_1}^{t_2} u_t(x,t)dtdx = -\int\int_{t_1}^{t_2} \frac{1}{2}u_{xx}(x,t)dxdt = -\frac{1}{2}\int_{t_1}^{t_2} u_x(X_t^-,t)dt.$$

Since we do not control u_{xx} at the moving boundary, we cannot make this argument rigorously. Thus we use a different approach.

Proof of Proposition 3.1: It suffices to show

$$-\frac{1}{2}\int_I u_x(X_t^-,t)dt = \Delta_I^X(u) = F_h^X(I).$$

If we integrate both sides of (2.19), then

$$\begin{aligned}\int_{-\infty}^{\infty} u(x,t)dx = &\int_{-\infty}^{\infty}\int_{-\infty}^{0} h(\xi)G(x,t;\xi,0)d\xi dx \\ &+ \int_{-\infty}^{\infty}\frac{1}{2}\int_0^t G(x,t;X_\tau,\tau)u_x(X_\tau^-,\tau)d\tau dx.\end{aligned}$$

Applying Fubini's theorem, we get

$$\int_{-\infty}^{X_t} u(x,t)dx = \int_{-\infty}^{\infty} u(x,t)dx = \int_{-\infty}^{0} h(\xi)d\xi + \frac{1}{2}\int_0^t u_x(X_\tau^-,\tau)d\tau. \tag{3.7}$$

Thus we get the first equality of the proposition,

$$\Delta_{[t_1,t_2]}^X(u) = -\frac{1}{2}\int_{t_1}^{t_2} u_x(X_\tau^-,\tau)d\tau. \tag{3.8}$$

From (1.2) and item 1 of Theorem 1.1, we get

$$P_\xi[\tau_\xi^X > t] = \int_{-\infty}^{X_t} G_{0,t}^X(\xi,x)dx. \tag{3.9}$$

For $0 = t_1 < t_2$, using Fubini's theorem again, we get

$$\begin{aligned}\Delta_I^X(u) &= \int_{-\infty}^{0} h(\xi)d\xi - \int_{-\infty}^{X_{t_2}}\int_{-\infty}^{0} h(\xi)G_{0,t_2}^X(\xi,x)d\xi dx \\ &= \int_{-\infty}^{0} h(\xi)d\xi - \int_{-\infty}^{0} h(\xi)P_\xi[\tau_\xi^X > t_2]d\xi = \int_{-\infty}^{0} h(\xi)P_\xi[0 \le \tau_\xi^X \le t_2]d\xi.\end{aligned}$$

For $0 < t_1 < t_2$, similarly,

$$\begin{aligned}\Delta_I^X(u) &= \int_{-\infty}^{X_{t_1}}\int_{-\infty}^{0} h(\xi)G_{0,t_1}^X(\xi,x)d\xi dx - \int_{-\infty}^{X_{t_2}}\int_{-\infty}^{0} h(\xi)G_{0,t_2}^X(\xi,x)d\xi dx \\ &= \int_{-\infty}^{0} h(\xi)P_\xi[t_1 < \tau_\xi^X \le t_2]d\xi.\end{aligned}$$

Then for $I_\epsilon = [t_1 - \epsilon, t_1]$, we get

$$\begin{aligned}\lim_{\epsilon\to 0}\Delta_{I_\epsilon}^X(u) &= \lim_{\epsilon\to 0} -\frac{1}{2}\int_{I_\epsilon} u_x(X_t^-,t)dt = 0 = \lim_{\epsilon\to 0}\int_{-\infty}^{0} h(\xi)P_\xi[t_1 - \epsilon < \tau_\xi^X \le t_1]d\xi \\ &= \int_{-\infty}^{0} h(\xi)P_\xi[\tau_\xi^X = t_1]d\xi.\end{aligned}$$

Finally we conclude that

$$\Delta_I^X(u) = \int_{-\infty}^{0} h(\xi)P_\xi[t_1 \le \tau_\xi^X \le t_2]d\xi = F_h^X(I). \tag{3.10}$$

□

By approximating the initial delta measure of Theorem 1.1, we prove the proposition below.

Proposition 3.2. *Let X be a Hölder continuous curve on any finite interval in $[0,\infty)$ with exponent $\gamma \in (1/2, 1]$ and let $X_0 = 0$. We fix $r_0 < X_0 = 0$ and choose a sequence $\{h_n\} \subset C_c^\infty(\mathbb{R};\mathbb{R}_+)$ with supp $h_n = [r_0 - \frac{1}{n}, r_0 + \frac{1}{n}] \Subset (-\infty, 0)$ and $\|h_n\|_1 = 1$. For each h_n, there exists a corresponding solution u_n with $-\frac{1}{2}\frac{\partial u_n}{\partial x}\Big|_{(X_t^-,t)} =: p_n(t)$ in the sense of Theorem 2.1. Then we have the following statements:*

(1) *There is a unique $p \in C([0,\infty))$ with $p(0) = 0$ such that*

$$p(t) = -G_x(X_t, t; r_0, 0) + \int_0^t G_x(X_t, t; X_\tau, \tau)p(\tau)d\tau \text{ for all } t > 0. \tag{3.11}$$

(2) *p_n converges to p in $C^0_{(\eta)}((0,T])$ for all $0 < \eta < 1/2$.*

We use the following lemma extensively to prove Proposition 3.2:

Lemma 3.3. $\displaystyle\int_0^t \tau^{\alpha_1}(t-\tau)^{\alpha_2}d\tau = \frac{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(2+\alpha_1+\alpha_2)}t^{1+\alpha_1+\alpha_2}$ *for $\alpha_1, \alpha_2 > -1$, where Γ is the gamma function.*

Proof of Proposition 3.2: Let $T_s > 0$. Since $|G_x(X_t,t;X_\tau,\tau)| = \left|\frac{1}{\sqrt{2\pi(t-\tau)}}\left\{-\frac{X_t-X_\tau}{t-\tau}\right\}\exp\left(-\frac{(X_t-X_\tau)^2}{2(t-\tau)}\right)\right| \le \frac{C_0}{(t-\tau)^{\frac{3}{2}-\gamma}}$, we deduce that for $q_1,q_2 \in C([0,T_s])$, $0<t\le T_s$,

$$\left|\int_0^t G_x(X_t,t;X_\tau,\tau)(q_1(\tau)-q_2(\tau))d\tau\right| \le \int_0^t \frac{C_0\|q_1-q_2\|_{T_s}}{(t-\tau)^{\frac{3}{2}-\gamma}}d\tau$$
$$= C_1 t^{\gamma-\frac{1}{2}}\|q_1-q_2\|_{T_s} \le C_1 T_s^{\gamma-\frac{1}{2}}\|q_1-q_2\|_{T_s}.$$

We define $F : C([0,T_s]) \to C([0,T_s])$ as, for $q \in C([0,T_s])$,

$$(Fq)(t) = -G_x(X_t,t;r_0,0) + \int_0^t G_x(X_t,t;X_\tau,\tau)q(\tau)d\tau \ \textit{for}\ t>0, \tag{3.12}$$

and $(Fq)(0)=0$. Since $\lim_{t\to 0} G_x(X_t,t;r_0,0)=0$, it is well-defined. If we choose T_s such that $C_1T_s^{\gamma-\frac{1}{2}}<1$, then F is a contraction mapping so that F has a unique fixed point since $C([0,T_s])$ is a Banach space. Let's call this p_{T_s}.

Now we assume that we have p_{T_s} for some $T_s>0$. For $T^\star > T_s$, we define $H : C([T_s,T^\star]) \to C([T_s,T^\star])$ as, for $q\in C([T_s,T^\star])$,

$$(Hq)(t) = -\,G_x(X_t,t;r_0,0) + \int_0^{T_s} G_x(X_t,t;X_\tau,\tau)p_{T_s}(\tau)d\tau$$
$$+ \int_{T_s}^t G_x(X_t,t;X_\tau,\tau)q(\tau)d\tau.$$

Then for $q_1,q_2\in C([T_s,T^\star])$, we have

$$\|Hq_1-Hq_2\|_\infty \le C_2(t-T_s)^{\gamma-\frac{1}{2}}\|q_1-q_2\|_\infty \le C_2(T^\star-T_s)^{\gamma-\frac{1}{2}}\|q_1-q_2\|_\infty. \tag{3.13}$$

Similarly, if we choose $T^\star$ such that $C_2(T^\star-T_s)^{\gamma-\frac{1}{2}}<1$, then H is a contraction mapping so that H has a unique fixed point since $C([T_s,T^\star])$ is a Banach space.

Therefore, if we have p defined $[0,T_s]$, p_{T_s}, then we can extend this to time T_s+C_3 where C_3 is an independent constant. Thus if we repeat this step inductively, we have p defined on $[0,\infty)$ which satisfies (3.11).

We now prove that p_n converges to p in $C^0_{(\eta)}((0,T_s])$ for all sufficiently small $T_s>0$. By (2.3),

$$p_n(t) = -\int_{-\infty}^0 h_n(\xi)G_x(X_t,t;\xi,0)d\xi + \int_0^t G_x(X_t,t;X_\tau,\tau)p_n(\tau)d\tau. \tag{3.14}$$

For $0<t\le T_s$, taking the difference between (3.11) and (3.14), we get

$$\begin{aligned} t^{1-\eta}|p_n(t)-p(t)| \ &\le\ t^{1-\eta}\left|\int_{-\infty}^0 h_n(\xi)[G_x(X_t,t;\xi,0)-G_x(X_t,t;r_0,0)]d\xi\right| \\ &+\ t^{1-\eta}\left|\int_0^t G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right|. \end{aligned}$$

For the second term of the right hand side, we have

$$t^{1-\eta}\left|\int_0^t G_x(X_t,t;X_\tau,\tau)(p_n(\tau)-p(\tau))d\tau\right| \le C_0t^{1-\eta}\int_0^t \frac{\|p_n-p\|^{(\eta)}_{T_s}}{(t-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau$$
$$= C_1t^{\gamma-\frac{1}{2}}\|p_n-p\|^{(\eta)}_{T_s} \le C_1T_s^{\gamma-\frac{1}{2}}\|p_n-p\|^{(\eta)}_{T_s}.$$

Let us choose $T_s > 0$ such that $C_1 T_s^{\gamma-\frac{1}{2}} < 1$. Then

$$(1 - C_1 T_s^{\gamma-\frac{1}{2}})\|p_n - p\|_{T_s}^{(\eta)} \le t^{1-\eta}\left|\int_{-\infty}^{0} h_n(\xi)[G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)]d\xi\right|.$$
$$\le \int_{-\infty}^{0} h_n(\xi) \sup_{\substack{0<t\le T_s \\ \xi\in[r_0-\frac{1}{n}, r_0+\frac{1}{n}]}} t^{1-\eta}\left|G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)\right| d\xi.$$

For all sufficiently large n, $0 < t \le \frac{1}{n}$ and $r_0 - \dfrac{1}{n} \le \xi \le r_0 + \dfrac{1}{n}$, there exists C_2 such that

$$|G_x(X_t,t;\xi,0)| = \left|\frac{1}{\sqrt{2\pi}}\left\{-\frac{X_t-\xi}{t^{\frac{3}{2}}}\right\}\exp\left(-\frac{(X_t-\xi)^2}{2t}\right)\right| \le C_2,$$

Thus

$$\sup_{\substack{0<t\le \frac{1}{n} \\ \xi\in[r_0-\frac{1}{n}, r_0+\frac{1}{n}]}} t^{1-\eta}\left|G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)\right| \le \sup_{0<t\le\frac{1}{n}} 2t^{1-\eta}C_2 \le \frac{2C_2}{n^{1-\eta}}.$$

For all sufficiently large n, $\frac{1}{n} < t \le T_s$ and $r_0 - \dfrac{1}{n} \le \xi \le r_0 + \dfrac{1}{n}$, since $|G_{xx}(\cdot,t;\cdot,0)| \le \dfrac{C_3}{t^{\frac{3}{2}}}$, we have

$$|G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)| \le \frac{C_3|\xi - r_0|}{t^{\frac{3}{2}}} \le \frac{C_3}{nt^{\frac{3}{2}}},$$

Thus

$$\sup_{\substack{\frac{1}{n}<t\le T_s \\ \xi\in[r_0-\frac{1}{n}, r_0+\frac{1}{n}]}} t^{1-\eta}\left|G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)\right| \le \sup_{\frac{1}{n}<t\le T_s} \frac{C_3}{nt^{\frac{1}{2}+\eta}} \le \frac{C_3}{n^{\frac{1}{2}-\eta}}.$$

Therefore, we conclude that p_n converges to p in $C^0_{(\eta)}((0,T_s])$ for all sufficiently small $T_s > 0$.

To extend from T_s to $T^\star$, assuming that p_n converges to p in $C^0_{(\eta)}((0,T_s])$ for some $T_s > 0$ and writing $\|p_n - p\|_{[T_s,T^\star]} = \sup_{\tau\in[T_s,T^\star]} |p_n(\tau) - p(\tau)|$, for $T_s \le t \le T^\star$, we deduce that

$$|p_n(t) - p(t)| \le \left|\int_{-\infty}^{0} h_n(\xi)[G_x(X_t,t;\xi,0) - G_x(X_t,t;r_0,0)]d\xi\right|$$
$$+ \left|\int_0^{T_s} G_x(X_t,t;X_\tau,\tau)(p_n(\tau) - p(\tau))d\tau\right| + \left|\int_{T_s}^{t} G_x(X_t,t;X_\tau,\tau)(p_n(\tau) - p(\tau))d\tau\right|$$
$$\le C_3\int_{-\infty}^{0} h_n(\xi)\frac{|\xi - r_0|}{t^{\frac{3}{2}}}d\xi + C_0\int_0^{T_s}\frac{\|p_n - p\|_{T_s}^{(\eta)}}{(t-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau + C_0\int_{T_s}^{t}\frac{\|p_n - p\|_{[T_s,T^\star]}}{(t-\tau)^{\frac{3}{2}-\gamma}}d\tau$$
$$\le \frac{C_3}{nt^{\frac{3}{2}}} + C_0\int_0^{T_s}\frac{\|p_n - p\|_{T_s}^{(\eta)}}{(T_s-\tau)^{\frac{3}{2}-\gamma}\tau^{1-\eta}}d\tau + C_4\|p_n - p\|_{[T_s,T^\star]}(t - T_s)^{\gamma-\frac{1}{2}}$$
$$\le \frac{C_3}{nT_s^{\frac{3}{2}}} + C_1 T_s^{\gamma-\frac{3}{2}+\eta}\|p_n - p\|_{T_s}^{(\eta)} + C_4\|p_n - p\|_{[T_s,T^\star]}(T^\star - T_s)^{\gamma-\frac{1}{2}}.$$

Let us choose $T^\star > T_s$ such that $C_4(T^\star - T_s)^{\gamma-\frac{1}{2}} < 1$, then we have

$$(1 - C_4(T^\star - T_s)^{\gamma-\frac{1}{2}})\|p_n - p\|_{[T_s,T^\star]} \le \frac{C_3}{nT_s^{\frac{3}{2}}} + C_1 T_s^{\gamma-\frac{3}{2}+\eta}\|p_n - p\|_{T_s}^{(\eta)}.$$

The right term vanishes when n goes to ∞ so that p_n converges to p in $C^0_{(\eta)}((0, T_s + C_5])$ for some independent constant $C_5 > 0$. By repeating this argument inductively, it follows that p_n converges to p in $C^0_{(\eta)}((0, T])$. □

Now we can prove p from Proposition 3.2 is the density function of $F^X_{r_0}$. Hence we have

$$\lim_{n\to\infty} \int_{-\infty}^{0} h_n(\xi) P_\xi(\tau^X_\xi \in I) d\xi = \lim_{n\to\infty} \int_I p_n(t)dt = \int_I p(t)dt.$$

For $I = [0, t] \subset [0, T]$, $P_\xi(\tau^X_\xi \in I)$ is an increasing function of ξ, so if we choose h_n so that $\lim_{n\to\infty} \int_{r_0}^{r_0+\frac{1}{n}} h_n(\xi)d\xi = 1$, we get

$$\lim_{\xi\to r_0^+} P_\xi(\tau^X_\xi \in I) \le \lim_{n\to\infty} \int_{r_0}^{r_0+\frac{1}{n}} h_n(\xi) P_\xi(\tau^X_\xi \in I) d\xi = \lim_{n\to\infty} \int_{-\infty}^{0} h_n(\xi) P_\xi(\tau^X_\xi \in I) d\xi.$$

Similarly, if we choose h_n so that $\lim_{n\to\infty} \int_{r_0-\frac{1}{n}}^{r_0} h_n(\xi)d\xi = 1$, we get

$$\lim_{n\to\infty} \int_{-\infty}^{0} h_n(\xi) P_\xi(\tau^X_\xi \in I) d\xi = \lim_{n\to\infty} \int_{r_0-\frac{1}{n}}^{r_0} h_n(\xi) P_\xi(\tau^X_\xi \in I) d\xi \le \lim_{\xi\to r_0^-} P_\xi(\tau^X_\xi \in I).$$

Therefore, we obtain that

$$F^X_{r_0}([0, t]) = P_{r_0}(\tau^X_{r_0} \le t) = \int_0^t p(\tau)d\tau, \tag{3.15}$$

which implies that p is the density function of $F^X_{r_0}$, thus item 2 is proved.

By Theorem 2.1 and the properties of the Gaussian kernel, $G^X_{0,t}(r_0, x)$ solves (1.4), (1.5) and (1.6). Hence G^X is the Green function of the heat equation with Dirichlet boundary condition which implies item 4. Furthermore, $G^X_{0,t}(r_0, x)$ can be written as

$$G^X_{0,t}(r_0, x) = G_{0,t}(r_0, x) - \int_0^t G_{\tau,t}(X_\tau, x)p(\tau)d\tau. \tag{3.16}$$

Differentiating with respect to x, applying the jump relation and (3.11), we have

$$\begin{aligned} \frac{\partial}{\partial x} G^X_{0,t}(r_0, x)\Big|_{x=X_t^-} &= \frac{\partial}{\partial x} G_{0,t}(r_0, X_t) - p(t) - \int_0^t \frac{\partial}{\partial x} G_{\tau,t}(X_\tau, X_t)p(\tau)d\tau \\ &= -2p(t). \end{aligned} \tag{3.17}$$

Thus $p(t) = -\frac{1}{2}\frac{\partial}{\partial x} G^X_{0,t}(r_0, x)\Big|_{x=X_t^-}$, so it proves item 3 of Theorem 1.1.

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