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A Conformally Invariant Growth Process of SLE Excursions

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Abstract. We construct an aggregation process of chordal SLE_κ excursions in the unit disk, starting from the boundary, growing towards all inner points simultaneously, invariant under all conformal self-maps of the disk. We prove that this conformal growth process of excursions, abbreviated as CGE_κ, exists iff $\kappa \in [0, 4)$, and that it does not create additional fractalness: the Hausdorff dimension of the closure of all the SLE_κ arcs attached is $1 + \kappa/8$ almost surely. We determine the dimension of points that are approached by CGE_κ at an atypical rate.

1. Introduction

Planar aggregation processes based on harmonic measure, usually called Laplacian growth models, have been extensively studied in the physics and mathematics literature, key examples being Diffusion Limited Aggregation and a family of models using iterated conformal maps, the Hastings-Levitov models; see Witten and Sander (1983); Halsey (2000); Hastings and Levitov (1998); Carleson and Makarov (2001); Rohde and Zinsmeister (2005); Norris and Turner (2012); Johansson Viklund et al. (2015). The most interesting versions, which produce fractal growth according to

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simulations, are notoriously hard to study: the discrete ones do not have meaningful scaling limits, the continuum models do not have enough symmetries that would make their analysis easier.

This motivated Itai Benjamini to suggest a model where both the building blocks and the aggregation measure are fully conformally invariant. Firstly, there is a sigma-finite infinite measure on pairs of points on the boundary of the complex unit disk $\mathbb{U}$, unique up to a global constant factor, which is invariant under all conformal self-maps of the disk, the Möbius transformations: it has density

$$H_{\mathbb{U}}(z,w) = c|z-w|^{-2},$$

called the **boundary Poisson kernel**. Secondly, once we have a pair of points $z, w \in \partial\mathbb{U}$, we can take a **chordal SLE$_\kappa$ arc** γ in $\mathbb{U}$ between z and w, with $\kappa \in [0,8)$. (For background on the Schramm-Loewner Evolution, see Schramm, 2000; Werner, 2004; Lawler, 2005.) Then, we can take a point $z \in \mathbb{U}$ and a conformal uniformization map from the component of $\mathbb{U} \setminus \gamma$ that contains z back to $\mathbb{U}$, normalized at z, and try and iterate this procedure. However, since our measure on $\partial\mathbb{U} \times \partial\mathbb{U}$ is not finite, we cannot take iid random pairs (z_i, w_i) one after the other. Instead, we need to take a Poisson point process on $\partial\mathbb{U} \times \partial\mathbb{U} \times [0,\infty)$ with intensity measure $H_{\mathbb{U}}(z,w)\,dz\,dw\,dt$, take all the arrivals $\{(z_i,w_i) : i \in I_r(t)\}$ with time index in $[0,t)$ and arc-length larger than a small positive cutoff $r > 0$, and do the above iterative procedure for these finitely many pairs of points. (See Figure 1.1 for an illustration.) Then, we let $r \to 0$, and hope that the process, using the increasing set $I_r(t)$ of arrivals, will converge to a process $(D_t^z, t \geq 0)$, the connected component of z at time t. Moreover, using conformal invariance, we can try to define the process targeted at all points $z \in \mathbb{U}$ simultaneously: as long as $D_t^z = D_t^w$, the processes targeted towards z and w coincide, and after the disconnection time they continue independently. Our first result says that this envisioned procedure actually works, but only for $\kappa \in [0,4)$:

Theorem 1.1. *For $\kappa \in [0,4)$, there exists a growth process*

$$(D_t, t \geq 0) = (D_t^z, t \geq 0, z \in \mathbb{U})$$

of SLE$_\kappa$ excursions, targeted at all points, with the property that $(D_t, t \geq 0)$ and $(\varphi(D_t), t \geq 0)$ have the same law (with no time-change) for all Möbius transformations φ of $\mathbb{U}$. We will abbreviate this Conformal Growth of Excursions by CGE$_\kappa$.

Maybe disappointingly, CGE$_\kappa$ does not produce considerable extra fractalness, beyond what is already inherent in the SLE$_\kappa$ arcs, which have dimension $1 + \kappa/8$ Beffara (2008):

Theorem 1.2. *Fix $\kappa \in [0,4)$. Suppose that $(D_t^0, t \geq 0)$ is CGE$_\kappa$ targeted at the origin. Define Γ to be the closure of the union $\cup_{t\geq 0}\partial D_t^0$. Then,* $\dim(\Gamma) = 1 + \kappa/8$ *almost surely.*

Consider now the conformal radius of D_t^z seen from z, denoted by $\mathrm{CR}(z; D_t^z)$. We can derive the asymptotic decay of the conformal radius. For $\lambda \in \mathbb{R}$, define the Laplace exponent

$$\Lambda_\kappa(\lambda) = \log \mathbb{E}\left[\mathrm{CR}(z; D_1^z)^{-\lambda}\right]. \tag{1.1}$$

As we will see, $\Lambda_\kappa(\lambda)$ is finite when $\lambda < 1 - \kappa/8$, and we have almost surely that

$$\lim_{t\to\infty} \frac{-\log \mathrm{CR}(z; D_t^z)}{t} = \Lambda'_\kappa(0) \in (0,\infty). \tag{1.2}$$

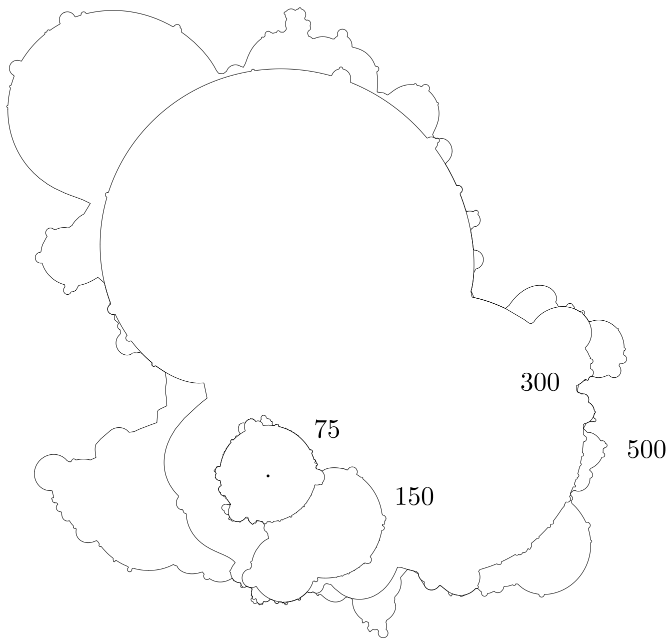


FIGURE 1.1. Four stages (75, 150, 300, 500 arcs) of $\mathrm{CGE}_{\kappa=0}$, growing towards ∞ (that is, the process targeted originally at 0, marked with a black dot, but inverted through the unit circle for better visibility), with some positive cutoff for the sizes of the SLE_0 arcs, which are just semicircles.

From (1.2), we know that typically the conformal radius $\mathrm{CR}(z; D_t^z)$ decays like $\exp(-t\Lambda_\kappa'(0))$. We are also interested in those points z where $\mathrm{CR}(z; D_t^z)$ decays in an abnormal way. Define, for $\alpha \geq 0$, the random set

$$\Theta(\alpha) = \left\{ z \in \mathbb{U} : \lim_{t\to\infty} \frac{-\log \mathrm{CR}(z; D_t^z)}{t} = \alpha \right\}. \tag{1.3}$$

Clearly, when $\alpha \neq \Lambda_\kappa'(0)$, the points in the set $\Theta(\alpha)$ have an abnormal decaying rate of $\mathrm{CR}(z; D_t^z)$. The Hausdorff dimension of $\Theta(\alpha)$ can be estimated through *Fenchel-Legendre transform* of Λ_κ. The Fenchel-Legendre transform of Λ_κ is defined by, for $\alpha \in \mathbb{R}$,

$$\Lambda_\kappa^*(\alpha) = \sup_{\lambda\in\mathbb{R}} (\lambda\alpha - \Lambda_\kappa(\lambda)).$$

Theorem 1.3. *Define*

$$\alpha_{min} = \sup\{\alpha > 0 : 2\alpha - \Lambda_\kappa^*(\alpha) \leq 0\}. \tag{1.4}$$

We have almost surely,

$$\begin{cases} \dim(\Theta(\alpha)) \leq 2 - \Lambda_\kappa^*(\alpha)/\alpha, & \alpha \geq \alpha_{min}; \\ \Theta(\alpha) = \emptyset, & \alpha < \alpha_{min}. \end{cases}$$

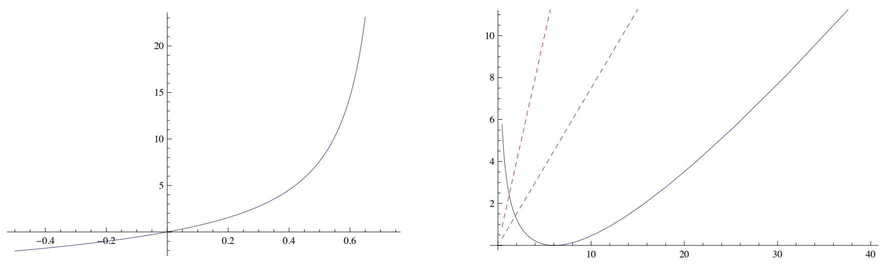


FIGURE 1.2. With $\kappa = 2$, numerical approximations of $\Lambda_\kappa(\lambda)$, blowing up at $\lambda = 1 - \kappa/8$, and its Fenchel-Legendre transform $\Lambda_\kappa^*(\lambda)$, with an asymptotic slope $1 - \kappa/8$.

The CGE_κ process $(D_t, t \geq 0)$ targeted at all points naturally yields a fragmentation process of the unit disk, raising interesting questions. First of all, for any $z, w \in \mathbb{U}$, we can define $T(z,w)$ to be the first time t such that z, w are not in the same connected component of D_t. We call $T(z,w)$ the disconnection time, for which we have the following estimate:

Theorem 1.4. *Fix $\kappa \in [0,4)$ and let $\Lambda_\kappa(\lambda)$ be the Laplace exponent defined in (1.1). Let $z, w \in \mathbb{U}$ be distinct and $T(z,w)$ be the disconnection time of CGE_κ targeted at all points. Then there exists a constant $C \in (0,\infty)$ (only depending on κ) such that*

$$|\mathbb{E}[T(z,w)] - G_{\mathbb{U}}(z,w)/\Lambda_\kappa'(0)| \leq C,$$

where $G_{\mathbb{U}}$ is Green's function of the unit disc.

Finally, let us discuss the most obvious question: in what discrete models can one find a structure that has our CGE_κ as a scaling limit? The full conformal invariance of the process targeted at all points suggests that probably one should look for structures that can be defined not only as growth processes, but also as static objects, similar to the Conformal Loop Ensembles CLE_κ (Sheffield and Werner, 2012); note however that CLE_κ exists for a different subset of κ values: for $\kappa \in (8/3, 4]$ if the loops are simple, and for $\kappa \in (8/3, 8)$ in general. A good candidate for a suitable discrete model is **Wilson's algorithm** (Wilson, 1996), which generates a Uniform Spanning Tree (UST) from iteratively adding Loop-Erased Random Walk trajectories, which converge to SLE_2 arcs (Schramm, 2000; Lawler et al., 2004). Furthermore, Temperley's bijection gives a coupling between the domino height function on $\mathbb{Z}^2$ and the UST (see Kenyon, 2000), and the winding field of the branches in the UST converges to Gaussian Free Field (GFF) (Miller and Sheffield, 2017; Berestycki et al., 2016). This implies that CGE_2 may emerge naturally in the GFF. For general $\kappa \in (2,4)$, a similar construction was conjectured for interacting dimers Giuliani et al. (2017). This gives another candidate related to CGE_κ in the limit.

Overview of the paper. In Section 2, we define the boundary Poisson kernel and the SLE_κ and $\mathrm{SLE}_\kappa(\rho)$ processes, and prove an overshoot estimate for subordinators.

In Section 3, we prove Theorem 1.1: we construct the growth process for $\kappa < 4$, prove that it is conformally invariant, and show that it does not exist for $\kappa \geq 4$. The proofs are based on known estimates on the probability that chordal SLE_κ comes close to a point on the boundary or inside the domain, and on conformal martingales related to these questions, describing the Laplace transform of the capacity of an SLE_κ arc. We also prove Theorem 1.3 on the dimension of points with abnormal decay, using Large Deviations theory.

In Section 4, we give estimates on the disconnection time, proving Theorem 1.4 in particular. Here a key ingredient is the innocent-looking but rather tricky Lemma 4.1, saying that the process leaves the boundary $\partial\mathbb{U}$ in finite time with positive probability. We also prove Theorem 1.2 on the dimension being $1+\kappa/8$, where the full conformal invariance of the process targeted at all points is of immense help.

We end the paper with three open problems in Section 5.

2. Preliminaries

Notation.
$B(z,r)=\{w\in\mathbb{C}:|z-w|<r\}$, $\mathbb{U}=B(0,1)$, and $\mathbb{H}=\{w\in\mathbb{C}:\Im w>0\}$.

2.1. *Green function and Poisson kernel.* Let $\eta(z,\cdot;t)$ denote the law of 2D Brownian motion $(B_s, 0\le s\le t)$ starting from z. We can write

$$\eta(z,\cdot;t)=\int_{\mathbb{C}}\eta(z,w;t)dw,$$

where dw denotes the area measure and $\eta(z,w;t)$ is a measure on continuous curves from z to w. Define $\eta(z,w)=\int_0^\infty \eta(z,w;t)dt$. This is an infinite σ-finite measure. If D is a domain and $z,w\in D$, define $\eta_D(z,w)$ to be $\eta(z,w)$ restricted to curves stayed in D. If $z\neq w$ and D is a domain such that a Brownian motion in D eventually exits D, then the total mass $|\eta_D(z,w)|$ is finite and we define the *Green function*

$$G_D(z,w)=\pi|\eta_D(z,w)|.$$

In particular, when $D=\mathbb{U}$ and $z,w\in\mathbb{U}$, we have $G_{\mathbb{U}}(z,w)=\log|(1-\bar{z}w)/(z-w)|$. Just like planar Brownian motion, the Green function is also conformally invariant: if $\varphi:D\longrightarrow\varphi(D)$ is a conformal map and $z,w\in D$, then

$$G_{\varphi(D)}(\varphi(z),\varphi(w))=G_D(z,w). \tag{2.1}$$

Suppose that D is a connected domain with piecewise analytic boundary. Let B be a Brownian motion starting from $z\in D$ and stopped at the first exit time $\tau_D:=\inf\{t:B_t\notin D\}$. Denote by $\eta_D(z,\partial D)$ the law of $(B_t,0\le t\le\tau_D)$. We can write

$$\eta_D(z,\partial D)=\int_{\partial D}\eta_D(z,w)dw,$$

where dw is the length measure on ∂D and $\eta_D(z,w)$ is a measure on continuous curves from z to w. The measure $\eta_D(z,w)$ can be viewed as a measure on Brownian paths starting from z and restricted to to exit D at w. Define *Poisson kernel* $H_D(z,w)$ to be the total mass of $\eta_D(z,w)$.

Suppose that z,w are distinct boundary points. Define the measure on Brownian paths from z to w in D to be

$$\eta_D(z,w)=\lim_{\epsilon\to0}\frac{1}{\epsilon}\eta_D(z+\epsilon\mathbf{n}_z,w),$$

where $\mathbf{n}_z$ is the inward normal at z. The measure $\eta_D(z,w)$ is called Brownian excursion measure. Define the *boundary Poisson kernel* $H_D(z,w)$ to be the total mass of $\eta_D(z,w)$. From the conformal invariance of Brownian motion, we can derive the conformal covariance of the boundary Poisson kernel (see Lawler, 2005, Proposition 5.5): Suppose that $\varphi : D \longrightarrow \varphi(D)$ is a conformal map and $z,w \in \partial D$ and $\varphi(z), \varphi(w) \in \partial\varphi(D)$ are analytic boundary points, then

$$|\varphi'(z)\varphi'(w)|H_{\varphi(D)}(\varphi(z),\varphi(w)) = H_D(z,w). \tag{2.2}$$

Moreover, if $D = \mathbb{U}$ and $z,w \in \partial\mathbb{U}$, we have $H_{\mathbb{U}}(z,w) = 1/(\pi|z-w|^2)$. In particular, if $\theta = \arg(z/w) \in [0,2\pi)$, we have

$$H_{\mathbb{U}}(z,w) = \frac{1}{4\pi\sin^2(\theta/2)}. \tag{2.3}$$

2.2. *Chordal and radial SLE.* In this section, we review briefly the chordal and radial $\mathrm{SLE}_\kappa(\rho)$ processes and refer the reader to Werner (2004) and Lawler (2005) for a detailed introduction. The *chordal Loewner chain* with a continuous driving function $W : [0,\infty) \to \mathbb{R}$ is the solution for the following ODE: for $z \in \overline{\mathbb{H}}$,

$$\partial_t g_t(z) = \frac{2}{g_t(z) - W_t}, \quad g_0(z) = z.$$

This solution is well-defined up to the swallowing time

$$T(z) := \sup\left\{t : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\right\}.$$

For $t \geq 0$, define $K_t := \{z \in \overline{\mathbb{H}} : T(z) \leq t\}$, then $g_t(\cdot)$ is the unique conformal map from $\mathbb{H} \setminus K_t$ onto $\mathbb{H}$ with the expansion $g_t(z) = z + 2t/z + o(1/z)$ as $z \to \infty$.

Chordal SLE_κ is the chordal Loewner chain with driving function $W = \sqrt{\kappa}B$ where B is a one-dimensional Brownian motion. For $\kappa \in [0,4]$, the SLE_κ process is almost surely a continuous simple curve in $\mathbb{H}$ from 0 to ∞. Suppose γ is an SLE_κ curve in $\mathbb{H}$ from 0 to ∞, then it is conformal invariant: for any $c > 0$, the curve $c\gamma$ has the same law as γ (up to time change). Therefore, we could define chordal SLE in any simply connected domain. Suppose that D is a simply connected domain and $x, y \in \partial D$ are distinct boundary points. Define SLE_κ in D from x to y to be the image of SLE_κ in $\mathbb{H}$ from 0 to ∞ under any conformal map from $\mathbb{H}$ onto D sending the pair $(0,\infty)$ to (x,y). Chordal SLE_κ is reversible for $\kappa \in (0,8)$: suppose that γ is an SLE_κ in D from x to y, then the time-reversal of γ has the same law as an SLE_κ in D from y to x. Thus, we also call SLE_κ in D from x to y as SLE_κ in D with two end points (x,y).

The *radial Loewner chain* with a continuous driving function $W : [0,\infty) \to \partial\mathbb{U}$ is the solution for the following ODE: for $z \in \overline{\mathbb{U}}$,

$$\partial_t g_t(z) = g_t(z)\frac{W_t + g_t(z)}{W_t - g_t(z)}, \quad g_0(z) = z.$$

This solution is well-defined up to the swallowing time

$$T(z) := \sup\left\{t : \inf_{s\in[0,t]} |g_s(z) - W_s| > 0\right\}.$$

For $t \geq 0$, define $K_t := \{z \in \overline{\mathbb{U}} : T(z) \leq t\}$, then $g_t(\cdot)$ is the unique conformal map from $\mathbb{U} \setminus K_t$ onto $\mathbb{U}$ with the normalization: $g_t(0) = 0, g_t'(0) > 0$.

Radial SLE$_\kappa$ is the radial Loewner chain with driving function $W = \exp(i\sqrt{\kappa}B)$ where B is a one-dimensional Brownian motion. For $\kappa \in [0, 8)$, radial SLE$_\kappa$ is almost surely a continuous curve in $\mathbb{U}$ from 1 to the origin. *Radial SLE$_\kappa(\rho)$* with $W_0 = x \in \partial\mathbb{U}$ and force point $V_0 = y \in \partial\mathbb{U}$ is the radial Loewner chain with driving function W solving the following SDEs:

$$dW_t = i\sqrt{\kappa}B_t - \left(\frac{\kappa}{2}W_t + \frac{\rho}{2}W_t\frac{W_t + V_t}{W_t - V_t}\right)dt, \quad W_0 = x;$$
$$dV_t = V_t\frac{W_t + V_t}{W_t - V_t}dt, \quad V_0 = y.$$

The system has a unique solution up to the *collision time* $T := \inf\{t : W_t = V_t\}$.

We focus on the weight $\rho = \kappa - 6$ for the following reason: chordal SLE$_\kappa$ in $\mathbb{U}$ from x to y has the same law as radial SLE$_\kappa(\kappa-6)$ with starting point $W_0 = x$ and force point $V_0 = y$; see Schramm and Wilson (2005). Fix $\kappa \in [0, 8)$ and $\rho = \kappa - 6$. Define $\theta_t = \arg(W_t) - \arg(V_t) \in (0, 2\pi)$, then by Itô's formula, the process θ_t satisfies the SDE:

$$d\theta_t = \sqrt{\kappa}dB_t + \frac{\kappa - 4}{2}\cot(\theta_t/2)dt. \tag{2.4}$$

The collision time T is also the first time that θ_t hits 0 or 2π. Moreover, when $\kappa \in [0, 8)$, we have $\mathbb{E}[T] < \infty$.

Suppose that D is a proper simply connected domain. The *conformal radius* of D seen from $z \in D$ is $|\varphi'(z)|^{-1}$ where φ is any conformal map from D onto $\mathbb{U}$ sending z to the origin. We denote this conformal radius by $\mathrm{CR}(z; D)$. Define the inradius

$$\mathrm{inrad}(z; D) := \inf_{w \in \mathbb{C}\setminus D} |z - w|.$$

By Koebe's one quarter theorem and the Schwarz lemma (Lawler, 2005, Theorem 3.17, Lemma 2.1), we have that

$$\mathrm{inrad}(z; D) \le \mathrm{CR}(z; D) \le 4\mathrm{inrad}(z; D). \tag{2.5}$$

For any compact subset $K \subset \overline{\mathbb{U}}$, let D be the connected component of $\mathbb{U} \setminus K$ that contains z. Define the *capacity* of K seen from z to be $\mathrm{cap}(z; K) = -\log \mathrm{CR}(z; D)$. When z is the origin, we simply denote $\mathrm{CR}(0; D)$ and $\mathrm{cap}(0; K)$ by $\mathrm{CR}(D)$ and $\mathrm{cap}(K)$ respectively. One can check that the radial Loewner chain is parameterized by capacity seen from the origin.

2.3. *Overshoot estimate for subordinators.* Suppose that $(X(t), t \ge 0)$ is a right-continuous increasing process starting from 0 and taking values in $[0, \infty)$. We call X a subordinator if it has independent homogeneous increments on $[0, \infty)$. The Laplace transform of a subordinator has a nice expression: for $t > 0$ and $\lambda \ge 0$, we have $\mathbb{E}[\exp(\lambda X(t))] = \exp(-t\Phi(\lambda))$, where $\Phi : [0, \infty) \to [0, \infty)$. There exist a unique constant $d \ge 0$ and a unique measure Π on $(0, \infty)$, which is called the Lévy measure of X, with $\int(1 \wedge x)\Pi(dx) < \infty$ such that, for $\lambda \ge 0$,

$$\Phi(\lambda) = d\lambda + \int(1 - e^{-\lambda x})\Pi(dx).$$

Moreover, one has almost surely, for $t > 0$,

$$X(t) = dt + \sum_{s \le t}\Delta_s,$$

where $(\Delta_s, s \geq 0)$ is a Poisson point process with intensity Π. (More precisely, we have a Poisson point process $\{(\Delta_j, s_j) : j \in J\}$ with intensity $\Pi \otimes \mathsf{Lebesgue}$ on $(0,\infty) \times [0,\infty)$, the second coordinate being time, where J is a countable set, and we let $\Delta_s := \Delta_j$ when $s = s_j$, while $\Delta_s := 0$ otherwise.) Define the tail of the Lévy measure:

$$\overline{\Pi}(x) = \Pi((x,\infty)).$$

We introduce the inverse of X, and the processes of first-passage and last-passage of X: for $x > 0$,

$$L_x = \inf\{t : X(t) > x\}, \quad D_x = X(L_x), \quad G_x = X(L_x -).$$

A subordinator is a transient Markov process, its potential measure $U(dx)$ is called the renewal measure. It is defined as, for any nonnegative measurable function f,

$$\int_0^\infty f(x) U(dx) = \mathbb{E}\left[\int_0^\infty f(X(t))dt\right].$$

Proposition 2.1. *Bertoin (1999, Lemma 1.10). For every real numbers a, b, x such that $0 \leq a < x \leq a + b$, we have that*

$$\mathbb{P}[G_x \in da, D_x - G_x \in db] = U(da)\Pi(db).$$

Proposition 2.2. *Suppose that X is a subordinator with Lévy measure Π satisfying*

$$\int (e^{\lambda_0 x} - 1)\Pi(dx) < \infty, \quad \textit{for some } \lambda_0 > 0.$$

Then there exists a positive finite constant C (depending on Π and λ_0) such that

$$\mathbb{P}[D_x - x \geq y] \leq C e^{-\lambda_0 y}, \quad \textit{for all } x \geq 0, y \geq 0.$$

Proof: When $y \in [0,1]$, we could take $C \geq e^{\lambda_0}$. Thus we can suppose $y \geq 1$. We divide the event $\{D_x - x \geq y\}$ according to the values of G_x. For every $k \in \mathbb{Z}_+$, define

$$E_k = [G_x \leq x - k, D_x \geq x + y].$$

By Proposition 2.1, we have that

$$\mathbb{P}[E_k] \leq \overline{\Pi}(k+y) \leq \int_{u \geq k+y} e^{\lambda_0 u} e^{-\lambda_0 (k+y)} \Pi(du) \leq e^{-\lambda_0(k+y)} \int_{u \geq 1} e^{\lambda_0 u} \Pi(du).$$

Thus

$$\mathbb{P}[D_x - x \geq y] \leq \sum_{k \geq 0} \mathbb{P}[E_k] \leq e^{-\lambda_0 y} \frac{\int_{u \geq 1} e^{\lambda_0 u} \Pi(du)}{1 - e^{-\lambda_0}}.$$

So we can take

$$C = e^{\lambda_0} \vee \frac{\int_{u \geq 1} e^{\lambda_0 u} \Pi(du)}{1 - e^{-\lambda_0}}.$$

□

Remark 2.3. In the literature, people usually consider right-continuous subordinators. Whereas, the conclusions in this section also hold for left-continuous subordinators. Note that if X is a left-continuous subordinator, then X can be written as, for $t > 0$,

$$X(t) = dt + \sum_{s < t} \Delta_s,$$

where $(\Delta_s, s \geq 0)$ is a Poisson point process. Therefore, the proofs in this section can be modified for left-continuous subordinators without difficulty. In the later part of our paper, we will apply conclusions in this section for left-continuous subordinators.

3. Construction of the growth process CGE_κ

3.1. *The Poisson point process of SLE excursions.* Let $H_{\mathbb{U}}(x, y)$ be the boundary Poisson kernel for the unit disc $\mathbb{U}$ with distinct boundary points $x, y \in \mathbb{U}$ as introduced in Section 2.1. Denote by $\mu^{\#}_{\mathbb{U},\kappa}(x, y)$ the law of chordal SLE_κ in $\mathbb{U}$ with two end points x, y. For $\kappa \in [0, 8)$, define the *SLE_κ excursion measure* to be

$$\mu_{\mathbb{U},\kappa} = \int\int dxdy H_{\mathbb{U}}(x, y)\mu^{\#}_{\mathbb{U},\kappa}(x, y), \tag{3.1}$$

where dx, dy are length measures on $\partial\mathbb{U}$. Note that $\mu_{\mathbb{U},\kappa}$ is an infinite σ-finite measure. From the conformal invariance of SLE and the conformal covariance of boundary Poisson kernel (2.2), we can derive the conformal invariance of SLE excursion measure. For any Möbius transformation φ of $\mathbb{U}$, we define the measure $\varphi \circ \mu$ via $\varphi \circ \mu[A] := \mu[\gamma : \varphi(\gamma) \in A]$. Then we have the following conformal invariance.

Proposition 3.1. *The SLE excursion measure $\mu_{\mathbb{U},\kappa}$ is conformal invariant: for any Möbius transformation φ of $\mathbb{U}$, we have $\varphi \circ \mu_{\mathbb{U},\kappa} = \mu_{\mathbb{U},\kappa}$.*

We will construct a growth process from a Poisson point process of SLE excursions. The construction is not surprising if one is familiar with Sheffield and Werner (2012) and Werner and Wu (2013). To be self-contained, we briefly explain the construction. Let $(\gamma_t, t \geq 0)$ be a Poisson point process with intensity $\mu_{\mathbb{U},\kappa}$. More precisely, let $((\gamma_j, t_j), j \in J)$ be a Poisson point process with intensity $\mu_{\mathbb{U},\kappa} \otimes [0, \infty)$, and then arrange the excursions γ_j according to t_j: denote the excursion γ_j by γ_t if $t = t_j$ and γ_t is empty set if there is no t_j that equals t. There are only countably many excursions in $(\gamma_t, t \geq 0)$ that are not empty set. For $\kappa \in [0, 8)$, with probability one there is no γ_t passing through the origin. For each t such that γ_t is not the empty set, the curve γ_t separates $\mathbb{U}$ into two connected components, and we denote by U^0_t the one that contains the origin. Let f_t be the conformal map from U^0_t onto $\mathbb{U}$ normalized at the origin: $f_t(0) = 0, f'_t(0) > 0$. For $t > 0$, define the accumulated capacity to be

$$X_t = \sum_{s<t} \mathrm{cap}(\gamma_s).$$

Proposition 3.2. *For $t > 0$, the accumulated capacity X_t is almost surely finite if and only if $\kappa \in [0, 4)$.*

We will complete the proof of Proposition 3.2 in Section 3.3. Assuming Proposition 3.2, we can now construct the growth process for $\kappa \in [0, 4)$. For any fixed $T > 0$ and $r > 0$, let $t_1(r) < t_2(r) < \cdots < t_j(r)$ be the times t before T at which the distance between the two end points of γ_t is at least r. Define

$$\Psi^r_T = f_{t_j(r)} \circ \cdots \circ f_{t_1(r)}.$$

The map Ψ^r_T is a conformal map from some subset of $\mathbb{U}$ onto $\mathbb{U}$. By Proposition 3.2, we know that $X_T < \infty$ almost surely when $\kappa \in [0, 4)$. Then the conformal map

Ψ_T^r converges almost surely in the Carathéodory topology seen from the origin, as $r \to 0$; see Sheffield and Werner (2012, Section 4.3, Stability of Loewner chains). Define

$$\left(D_t^0 := \Psi_t^{-1}(\mathbb{U}), t \geq 0\right).$$

This is a decreasing sequence of simply connected domains containing the origin, and we call it *the growth process* CGE_κ *of SLE excursions targeted at the origin.* By the conformal invariance of the SLE excursion measure, we can derive the conformal invariance of CGE_κ.

Lemma 3.3. *For $\kappa \in [0,4)$, the law of the growth process $(D_t^0, t \geq 0)$ is conformally invariant under any Möbius transformation φ of $\mathbb{U}$ that preserves the origin.*

Proof: Let $(\hat{\gamma}_t, t \geq 0)$ be a Poisson point process with intensity $\mu_{\mathbb{U},\kappa}$, and define $\hat{f}_t$ and $\hat{\Psi}_t$ for each t as described above, and denote by $(\hat{D}_t^0, t \geq 0)$ the corresponding growth process targeted at the origin.

By Proposition 3.1, we know that the process $(\gamma_t := \varphi(\hat{\gamma}_t), t \geq 0)$ is also a Poisson point process with intensity $\mu_{\mathbb{U},\kappa}$. Define f_t and Ψ_t for each t, and denote by $(D_t^0, t \geq 0)$ be the corresponding growth process targeted at the origin. It is clear that

$$f_t = \varphi \circ \hat{f}_t \circ \varphi^{-1}, \quad \Psi_t = \circ_{s<t} f_s = \varphi \circ \hat{\Psi}_t \circ \varphi^{-1}.$$

Since $(\gamma_t, t \geq 0)$ has the same law as $(\hat{\gamma}_t, t \geq 0)$, the process $(D_t^0 = \varphi(\hat{D}_t^0), t \geq 0)$ has the same law as $(\hat{D}_t^0, t \geq 0)$ as desired. $\square$

We can construct CGE_κ targeted at any $z \in \mathbb{U}$ in the same way as above, except that we choose to normalize at z instead of normalizing at the origin. Another way to describe CGE_κ targeted at z would be $(\varphi(D_t^0), t \geq 0)$ where φ is any Möbius transformation of $\mathbb{U}$ that sends the origin to z. By Lemma 3.3, the choice of φ does not affect the law of $(\varphi(D_t^0), t \geq 0)$, thus CGE_κ targeted at z is well-defined.

Now we will describe the relation between two growth processes targeted at distinct points $z, w \in \mathbb{U}$. Let $(D_t^z, t \geq 0)$ (resp. $(D_t^w, t \geq 0)$) be CGE_κ processes targeted at z (resp. targeted at w), and define $T(z,w)$ (resp. $T(w,z)$) to be the first time t that $w \notin D_t^z$ (resp. $z \notin D_t^w$). We call $T(z,w)$ the disconnection time. The interesting property of these growth processes is that the two processes have the same law up to the disconnection time.

Proposition 3.4. *For $\kappa \in [0,4)$, and for any $z, w \in \mathbb{U}$, the law of $(D_t^z, t < T(z,w))$ is the same as the law of $(D_t^w, t < T(w,z))$.*

Proof: Recall a classical result about Poisson point processes (see Bertoin, 1996, Section 0.5): Let $(a_t, t \geq 0)$ be a Poisson point process with some intensity ν (defined in some metric space A). Let $\mathcal{F}_{t-} = \sigma(a_s, s < t)$. If $(\Phi_t, t \geq 0)$ is a process (with values on functions of A to A) such that for any $t > 0$, Φ_t is $\mathcal{F}_{t-}$-measurable, and that Φ_t preserves ν, then $(\Phi_t(a_t), t \geq 0)$ is still a Poisson point process with intensity ν.

Let $(\hat{\gamma}_t, t \geq 0)$ be a Poisson point process with intensity $\mu_{\mathbb{U},\kappa}$ and define $\mathcal{F}_{t-} = \sigma(\hat{\gamma}_s, s < t)$, let $\hat{f}_t^z$ and $\hat{\Psi}_t^z$ be the conformal maps as described above normalized at z. Let $(\hat{D}_t^z, t \geq 0)$ be the corresponding growth process targeted at z and $\hat{T}(z,w)$ be the first time t that $w \notin \hat{D}_t^z$. For each $t < \hat{T}(z,w)$, the domain $\hat{D}_t^z$ contains w, and let G_t be the conformal map from $\hat{D}_t^z$ onto $\mathbb{U}$ normalized at w: $G_t(w) = w$ and $G_t'(w) > 0$.

For each $t < \hat{T}(z,w)$, define $\varphi_t = G_t \circ (\hat{\Psi}_t^z)^{-1}$. For $t = \hat{T}(z,w)$, define $\varphi_t = \lim_{s\uparrow t} \varphi_s$. For $t > \hat{T}(z,w)$, define φ_t to be identity map. Note that $\hat{\Psi}_t^z = \circ_{s<t} \hat{f}_s^z$ and thus $\hat{\Psi}_t^z$ is $\mathcal{F}_{t-}$-measurable. Therefore, for all $t > 0$, φ_t is $\mathcal{F}_{t-}$-measurable. By Proposition 3.1 and the classical result of Poisson point process recalled at the beginning of the proof, we know that $(\gamma_t := \varphi_t(\hat{\gamma}_t), t \geq 0)$ is also a Poisson point process with intensity $\mu_{\mathbb{U},\kappa}$. For $(\gamma_t, t \geq 0)$, let $(D_t^w, t \geq 0)$ be the corresponding growth process targeted at w and let $T(w,z)$ be the first time t that $z \notin D_t^w$. By the construction, we have that

$$D_t^w = \hat{D}_t^z, \quad \text{for all } t < T(w,z).$$

Hence, in this coupling, we have $T(w,z) \leq \hat{T}(z,w)$. By symmetry, we have that $T(w,z) = \hat{T}(z,w)$ almost surely. In particular, this coupling implies that the two disconnection times have the same law, and the two growth processes $(D_t^z, t \geq 0)$ and $(D_t^w, t \geq 0)$ have the same law up to the disconnection time. □

Proposition 3.4 tells that, for any $z, w \in \mathbb{U}$, it is possible to couple the two growth processes targeted at z and w respectively to be identical up to the first time at which the points z, w are disconnected. Hence, it is possible to couple the growth processes $(D_t^z, t \geq 0)$ for all z in a fixed countable dense subset of $\mathbb{U}$ simultaneously in such a way that for any two points z and w, the above statement holds.

For such a coupling, we get a Markov process on domains $(D_t, t \geq 0)$: At $t = 0$, the domain is $\mathbb{U}$, and at time $t > 0$, it is the union of all the disjoint open subsets corresponding to the growth process targeted at all points z at time t. We call this Markov process *the conformal growth process of SLE excursions targeted at all points*, or CGE_κ. By construction, it is naturally conformal invariant. This completes the proof of Theorem 1.1.

In Subsections 3.2 and 3.3, we will calculate the Laplace transform of the accumulated capacity and complete the proof of Proposition 3.2.

3.2. *The Laplace transform of the capacity of chordal SLE.* In this section, we will calculate the Laplace transform of the capacity of chordal SLE_κ in $\mathbb{U}$. To this end, we need to recall some basic facts about hypergeometric functions. The hypergeometric function is defined for $|z| < 1$ by the power series

$$\phi(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \tag{3.2}$$

where $(q)_n$ is the Pochhammer symbol defined by $(q)_n = q(q+1)\cdots(q+n-1)$ for $n \geq 1$ and $(q)_n = 1$ for $n = 0$. The function in (3.2) is only well-defined for $c \notin \{0, -1, -2, -3...\}$. The hypergeometric function is a solution of Euler's hypergeometric differential equation

$$-ab\phi + (c - (a+b+1)z)\phi' + z(1-z)\phi'' = 0. \tag{3.3}$$

Note that, when $c \notin \mathbb{Z}$, the following function is also a solution to (3.3):

$$z^{1-c}\phi(1+a-c, 1+b-c; 2-c; z).$$

Proposition 3.5. *Fix $\kappa \in [0,8)$ and $\lambda \in (0, 1-\kappa/8)$. Suppose that $\gamma = \gamma^\theta$ is a chordal SLE_κ in $\mathbb{U}$ from $x \in \partial\mathbb{U}$ to $y \in \partial\mathbb{U}$, where $\theta = \arg(x) - \arg(y)$. Define three* *constants*

$$a = 1 - \frac{4}{\kappa} + \sqrt{\left(1-\frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}, \quad b = 1 - \frac{4}{\kappa} - \sqrt{\left(1-\frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}, \quad c = \frac{3}{2} - \frac{4}{\kappa}.$$

Assume that $c \notin \mathbb{Z}$. Define two functions f and g: for $u \in [0,1]$,

$$f(u) = \phi(a,b;c;u), \quad g(u) = u^{1-c}\phi(1+a-c, 1+b-c; 2-c; u).$$

Denote $u = \sin^2(\theta/4)$. Then, we have

$$\mathbb{E}[\exp(\lambda\, cap(\gamma^\theta))] = f(u) + \frac{(1-f(1))}{g(1)} g(u). \tag{3.4}$$

Proof: First, let us check the values of the functions f and g at the end points $u=0$ or $u=1$. Since $\kappa \in [0,8)$, $\lambda \in (0, 1-\kappa/8)$ and $c \notin \mathbb{Z}$, we have that

$$a \in (0,1), \quad b \in (1-8/\kappa, 0), \quad c \in (-\infty, 1) \setminus \mathbb{Z}.$$

Combining with Erdélyi et al. (1953, Page 104, Equation (46)), we have that (denoting by Γ the Gamma function)

$$f(0) = 1, \quad f(1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} = \frac{\cos\left(\pi\sqrt{\left(1-\frac{4}{\kappa}\right)^2 + \frac{8\lambda}{\kappa}}\right)}{\cos\left(\pi\left(1-\frac{4}{\kappa}\right)\right)} \in [-1,1);$$
$$g(0) = 0, \quad g(1) = \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)} \in (0,\infty).$$

Second, assuming the same notation as in Subsection 2.2, we know that γ has the same law as radial $\mathrm{SLE}_\kappa(\kappa-6)$ with $(W_0, V_0) = (x,y)$. Let $\theta_t = \arg(W_t) - \arg(V_t)$, we will argue that $e^{\lambda t} f(\sin^2(\theta_t/4))$ and $e^{\lambda t} g(\sin^2(\theta_t/4))$ are martingales up to the collision time T. Suppose that F is an analytic function defined on $(0,1)$. By the SDE (2.4) and Itô's formula, we know that $e^{\lambda t} F(\sin^2(\theta_t/4))$ is a local martingale if and only if

$$\lambda F(u) + \frac{3\kappa-8}{16}(1-2u)F'(u) + \frac{\kappa}{8}u(1-u)F''(u) = 0.$$

Since f and g are solutions to this ODE, we know that $e^{\lambda t} f(\sin^2(\theta_t/4))$ and $e^{\lambda t} g(\sin^2(\theta_t/4))$ are local martingales. Since f and g are finite at end points $u=0$ and $u=1$, and the collision time T has finite expectation, we may conclude that the processes $e^{\lambda t} f(\sin^2(\theta_t/4))$ and $e^{\lambda t} g(\sin^2(\theta_t/4))$ are martingales up to T.

Finally, we derive (3.4). Since $e^{\lambda t} f(\sin^2(\theta_t/4))$ and $e^{\lambda t} g(\sin^2(\theta_t/4))$ are martingales up to T which has finite expectation, Optional Stopping Theorem gives

$$\mathbb{E}\left[\exp(\lambda T) 1_{\{\theta_T = 0\}}\right] + f(1)\mathbb{E}\left[\exp(\lambda T) 1_{\{\theta_T = 2\pi\}}\right] = f(u);$$
$$g(1)\mathbb{E}\left[\exp(\lambda T) 1_{\{\theta_T = 2\pi\}}\right] = g(u).$$

These give (3.4) by noting that $\mathrm{cap}(\gamma) = T$. □

The martingale $e^{\lambda t} f(\sin^2(\theta_t/4))$ in the proof of Proposition 3.5 was studied in Schramm et al. (2009).

Remark 3.6. For $\kappa \in [0,8)$, suppose that γ is a chordal SLE_κ in $\mathbb{U}$ with distinct end points $x, y \in \partial\mathbb{U}$. For $\lambda \in \mathbb{R}$, define

$$F(\kappa, \lambda) = \mathbb{E}[\exp(\lambda \mathrm{cap}(\gamma))].$$

On the one hand, by Proposition 3.5, we see that when $c = 3/2 - 4/\kappa \notin \mathbb{Z}$ and $\lambda < 1 - \kappa/8$, the quantity $F(\kappa, \lambda)$ is finite. On the other hand, we know that $F(\kappa, \lambda)$ is continuous in (κ, λ) (for the continuity in κ; see for instance Kemppainen and Smirnov, 2017, Theorem 1.10). Therefore, the quantity $F(\kappa, \lambda)$ is finite for all $\kappa \in [0, 8)$ and $\lambda < 1 - \kappa/8$.

Since the boundary Poisson kernel $H_{\mathbb{U}}(x, y)$ blows up when $\theta = \arg(x) - \arg(y)$ is small, in order to understand the excursion measure $\mu_{\mathbb{U},\kappa}$, it will be important for us how the capacity $\mathrm{cap}(\gamma^\theta)$ behaves as $\theta \to 0$.

Proposition 3.7. *Fix $\kappa \in [0, 8)$ such that $c(\kappa) = \frac{3}{2} - \frac{4}{\kappa} \notin \mathbb{Z}$, and $\lambda \in (0, 1 - \kappa/8)$. As $\theta \to 0$, or equivalently, $u \to 0$, the Laplace transform of the capacity satisfies*

$$\mathbb{E}\big[\exp(\lambda \mathit{cap}(\gamma^\theta)) - 1\big] \asymp \begin{cases} u^{1-c} & \asymp \theta^{2(1-c)} \quad \text{if} \quad 8/3 < \kappa < 8\,, \\ u & \asymp \theta^2 \quad \text{if} \quad 0 < \kappa < 8/3,\ c \notin \mathbb{Z}_-\,, \end{cases} \tag{3.5}$$

with the constant factors implicit in $\asymp$ depending on κ and λ. Quite similarly, the expectation itself satisfies

$$\mathbb{E}\big[\mathit{cap}(\gamma^\theta)\big] \asymp \begin{cases} u^{1-c} & \asymp \theta^{2(1-c)} \quad \text{if} \quad 8/3 < \kappa < 8\,, \\ u \log(1/u) & \asymp \theta^2 \log(1/\theta) \quad \text{if} \quad \kappa = 8/3\,, \\ u & \asymp \theta^2 \quad \text{if} \quad 0 \le \kappa < 8/3\,, \end{cases} \tag{3.6}$$

with the constant factors implicit in $\asymp$ depending on κ.

Proof: For the functions given in Proposition 3.5 for the case $c \notin \mathbb{Z}$, it is easy to check that $f(u) - 1 = f(u) - f(0)$ decays like u as $u \to 0$ and $g(u)$ decays like u^{1-c} as $u \to 0$. This gives (3.5), with a phase transition at $c = 0$, that is, at $\kappa = 8/3$.

To get the expectation (3.6), one possibility would be to take the derivative of the Laplace transform $F(\kappa, \lambda)$ at $\lambda = 0$. However, our formula (3.4) is not particularly simple, hence this task is not obvious. Another approach could be to argue that, for small θ, it is very likely that $\mathrm{cap}(\gamma^\theta)$ is also small, hence $\exp(\lambda \mathrm{cap}(\gamma^\theta)) - 1$ and $\lambda \mathrm{cap}(\gamma^\theta)$ are likely to be close to each other, and hence it is not surprising if the $u \to 0$ asymptotics of their expectations are the same. However, a large portion of the expectations might come from when $\mathrm{cap}(\gamma^\theta)$ is large, hence this argument would also need some extra work. Finally, we have (3.4) and (3.5) only when $c(\kappa) \notin \mathbb{Z}$, which is an immediate drawback to start with. Therefore, we give the following separate and direct argument.

For $\kappa \in (0, 8)$ and $\theta < r < 1/4$, it follows immediately from the results of Alberts and Kozdron (2008) that

$$\mathbb{P}[\mathrm{diam}(\gamma^\theta) > r] \asymp (\theta/r)^{8/\kappa - 1}. \tag{3.7}$$

When the diameter is in $[r, 2r)$, the capacity is at most Cr^2. Moreover, if a curve γ^θ going from $\exp(-i\theta/2)$ to $\exp(i\theta/2)$ has diameter in $[r, 2r)$, and it also separates the center 0 from the point $1 - r/2$, then its capacity is at least cr^2. For SLE_κ from $\exp(-i\theta/2)$ to $\exp(i\theta/2)$, conditioned to have diameter in $[r, 2r)$, this separating event has a uniformly positive probability, hence (3.7) implies that

$$\mathbb{E}\big[\mathrm{cap}(\gamma^\theta) 1_{\{\mathrm{diam}(\gamma^\theta) \in [2^k\theta, 2^{k+1}\theta)\}}\big] \asymp (2^k\theta)^2 (2^{-k})^{8/\kappa - 1} = \theta^2 (2^k)^{3 - 8/\kappa}. \tag{3.8}$$

Summing this up over dyadic scales from around θ to around $1/4$, we get that

$$\mathbb{E}\big[\mathrm{cap}(\gamma^\theta) 1_{\{\mathrm{cap}(\gamma^\theta) < 1\}}\big] \asymp \begin{cases} \theta^{8/\kappa - 1} & \text{if} \quad \kappa > 8/3, \\ \theta^2 \log(1/\theta) & \text{if} \quad \kappa = 8/3, \\ \theta^2 & \text{if} \quad \kappa < 8/3. \end{cases}$$

Note that the last line holds even for $\kappa = 0$.

Now, $\mathrm{cap}(\gamma^\theta)$ is larger than $t \geq 1$ only if γ^θ comes closer than $\exp(-t)$ to 0. The probability of this has an exponential tail by the one-point estimate in the dimension upper bound (Rohde and Schramm, 2005, Lemma 6.3, Theorem 8.1), so this part of the probability space does not raise the expectation $\mathbb{E}[\mathrm{cap}(\gamma^\theta)]$ by more than a factor, and the proof of (3.6) is complete. □

3.3. *The Laplace transform of the accumulated capacity.* For $\kappa \in [0, 8)$, recall that $\mu_{\mathbb{U},\kappa}$ is the SLE excursion measure defined in (3.1). Let $(\gamma_t, t \geq 0)$ be a PPP with intensity $\mu_{\mathbb{U},\kappa}$ and define the accumulated capacity in the same way as before: $X_t = \sum_{s<t} \mathrm{cap}(\gamma_s)$. By Campbell's formula, we have that, for $\lambda \in \mathbb{R}$,

$$\mathbb{E}[\exp(\lambda X_t)] = \exp\left(t \int (e^{\lambda \mathrm{cap}(\gamma)} - 1)\mu_{\mathbb{U},\kappa}[d\gamma]\right). \tag{3.9}$$

In particular, the left hand side of (3.9) is finite if and only if the right hand side is finite. We will study the Laplace exponent

$$\Lambda_\kappa(\lambda) := \int (e^{\lambda \mathrm{cap}(\gamma)} - 1)\mu_{\mathbb{U},\kappa}[d\gamma] = \int\int dxdy H_{\mathbb{U}}(x,y)\mu^{\#}_{\mathbb{U},\kappa}(x,y)\left[e^{\lambda \mathrm{cap}(\gamma)} - 1\right]. \tag{3.10}$$

Proposition 3.8.

(1) *When $\kappa \in [0, 4)$, the Laplace exponent $\Lambda_\kappa(\lambda)$ is finite for $\lambda \in (0, 1 - \kappa/8)$ and infinite for $\lambda \geq 1 - \kappa/8$. If $\kappa \geq 4$, then $\Lambda_\kappa(\lambda)$ is infinite for all $\lambda > 0$.*

(2) *When $\kappa \in [0, 4)$, we have that $\mathbb{E}[X_t] < \infty$ for all t. When $\kappa \geq 4$, we have that $\mathbb{E}[X_t] = \infty$ for all $t > 0$.*

Proof: First, we show that $\Lambda_\kappa(\lambda)$ is finite when $\kappa \in [0, 4)$, $\lambda \in (0, 1 - \kappa/8)$, and $c := 3/2 - 4/\kappa \notin \mathbb{Z}$. Note that, in (3.10), the boundary Poisson kernel and the expectation $\mu^{\#}_{\mathbb{U},\kappa}(x,y)\left[e^{\lambda \mathrm{cap}(\gamma)} - 1\right]$ only depend on the angle difference of x, y. Assuming the same notation as in Proposition 3.5, we see that

$$\begin{aligned}
\Lambda_\kappa(\lambda) &= 4\pi \int_0^\pi \frac{d\theta}{4\pi \sin^2(\theta/2)} \mu^{\#}_{\mathbb{U},\kappa}(1, e^{i\theta})\left[e^{\lambda \mathrm{cap}(\gamma)} - 1\right] && \text{(by (2.3))}\\
&= \int_0^\pi \frac{d\theta}{\sin^2(\theta/2)} \left(f(\sin^2(\theta/4)) - 1 + \frac{1 - f(1)}{g(1)} g(\sin^2(\theta/4))\right) && \text{(by (3.4))}\\
&= \frac{1}{2}\int_0^{1/2} du\ u^{-3/2}(1-u)^{-3/2}\left(f(u) - 1 + \frac{1 - f(1)}{g(1)} g(u)\right). && \text{(set } u = \sin^2(\theta/4))
\end{aligned}$$

Using the decay rate (3.5) of Proposition 3.7, the exponent $\Lambda_\kappa(\lambda)$ is finite for $\lambda \in (0, 1 - \kappa/8)$ when $c < 1/2$ which is to say $\kappa < 4$, and infinite for every $\lambda > 0$ when $\kappa \geq 4$.

That is, $\Lambda_\kappa(\lambda)$ is finite for $\kappa \in [0, 4) \setminus \{8/3, 8/5, 8/7, ...\}$ and $\lambda \in (0, 1 - \kappa/8)$. It is infinite when $\lambda \geq 1 - \kappa/8$, since already the integrand, the right hand side of (3.4), explodes.

To extend this for $\kappa \in \{8/3, 8/5, 8/7, ...\}$, the continuity of $\kappa \mapsto F^\theta(\kappa, \lambda)$ mentioned in Remark 3.6 implies that the singularity in the integrand can be bounded from above by something integrable when $\lambda < 1 - \kappa/8$, and from below by something non-integrable when $\lambda \geq 1 - \kappa/8$.

For part (2) regarding $\mathbb{E}[X_t]$, we can do the analogous calculation, just using the decay rate (3.6) instead of (3.5). $\square$

Proof of Proposition 3.2: When $\kappa \in [0,4)$, we proved in Proposition 3.8 that $\mathbb{E}[X_t]$ is finite, and thus the accumulated capacity X_t is finite almost surely.

Next, we will argue that X_t diverges almost surely when $t > 0, \kappa \geq 4$. For $k \geq 1$, define M_k to be the number of excursions γ_s with $s < t$ such that $2^{-k} \leq \mathrm{cap}(\gamma_s) < 2^{-k+1}$. Since $(\gamma_s, s \geq 0)$ is a PPP with intensity $\mu_{\mathbb{U},\kappa}$, we know that M_k is a Poisson random variable with parameter

$$q_k := t\mu_{\mathbb{U},\kappa}[\gamma : 2^{-k} \leq \mathrm{cap}(\gamma) < 2^{-k+1}].$$

By part (2) of Proposition 3.8 , we have $\mathbb{E}[X_t] = \infty$ for $\kappa \geq 4$, thus $\sum_{k\geq 1} 2^{-k} q_k \geq \mathbb{E}[X_t]/2 = \infty$. Since $(M_k, k \geq 1)$ are independent Poisson random variables, we have

$$\begin{aligned}\mathbb{E}\left[\exp\left(-\sum_{k\geq 1} 2^{-k} M_k\right)\right] &= \Pi_{k\geq 1}\mathbb{E}\left[\exp\left(-2^{-k}M_k\right)\right]\\ &= \Pi_{k\geq 1}\exp\left(-q_k\left(1-e^{-2^{-k}}\right)\right) = 0.\end{aligned}$$

Therefore, when $\kappa \geq 4$, we almost surely have $X_t \geq \sum_{k\geq 1} 2^{-k} M_k = \infty$. $\square$

3.4. *Extremes of the conformal radii and Proof of Theorem 1.3.* Fix $\kappa \in [0,4)$, by Proposition 3.8, we know that the Laplace exponent $\Lambda_\kappa(\lambda)$ is finite for $\lambda \in (-\infty, 1-\kappa/8)$. In particular, this implies that it is differentiable on $(-\infty, 1-\kappa/8)$ (see Dembo and Zeitouni, 2010, Lemma 2.2.5). Moreover, by Strong Law of Large Numbers for subordinators (Bertoin (1996, Page 92)), we have almost surely that

$$\lim_{t\to\infty} \frac{X_t}{t} = \Lambda'_\kappa(0) \in (0,\infty),$$

which implies (1.2). To prove Theorem 1.3, we first summarize some basic properties of $\Lambda_\kappa(\lambda)$ and $\Lambda^*_\kappa(x)$ (see Dembo and Zeitouni, 2010, Lemmas 2.2.5, 2.2.20, and our Figure 1.2 in the Introduction):

(1) The Laplace exponent $\Lambda_\kappa(\lambda)$ is convex and smooth on $(-\infty, 1-\kappa/8)$.
(2) The Fenchel-Legendre transform $\Lambda^*_\kappa(x)$ is non-negative, convex, and smooth on $(0,\infty)$.
(3) We have $\Lambda^*_\kappa(x) = 0$ when $x = \Lambda'_\kappa(0)$, the function Λ^*_κ is increasing on $(\Lambda'_\kappa(0),\infty)$ and is decreasing on $(0,\Lambda'_\kappa(0))$.
(4) Since $\Lambda_\kappa(\lambda) \to -\infty$ as $\lambda \to -\infty$, we know that

$$\Lambda^*_\kappa(0) = +\infty, \quad \Lambda^*_\kappa(x) \uparrow +\infty \text{ as } x \downarrow 0.$$

(5) As $x \to +\infty$, we have

$$\lim_{x\to\infty} \frac{\Lambda^*_\kappa(x)}{x} = 1-\kappa/8.$$

Recall that α_{min} is defined through (1.4). From the above properties, we know that $2x - \Lambda^*_\kappa(x) = 0$ has a unique solution which is equal to $\alpha_{min} \in (0, \Lambda'_\kappa(0))$. We can complete the proof of Theorem 1.3 using the theory of large deviations:

Proof of Theorem 1.3: It suffices to give upper bound for $\Theta(\alpha)\cap B(0,1-\delta)$ for any $\delta>0$. Fix $\alpha\geq 0$ and assume $\beta>\alpha$. For $n\geq 1$, let $\mathcal{U}_n$ be the collection of open balls with centers in $e^{-n\beta}\mathbb{Z}^2\cap B(0,1-\delta/2)$ and radius $e^{-n\beta}$. For each ball $U\in\mathcal{U}_n$, denote by $z(U)$ the center of U. Define, for $u^-<u^+$,

$$\mathcal{U}_n(u^-,u^+)=\left\{U\in\mathcal{U}_n : u^-\leq \frac{-\log \mathrm{CR}\left(z(U);D_n^{z(U)}\right)}{n}\leq u^+\right\}.$$

By Cramér's theorem (see Dembo and Zeitouni, 2010, Theorem 2.2.3), for $u^-<u^+$, for any $U\in\mathcal{U}_n$, we have

$$\begin{aligned}\mathbb{P}[U\in\mathcal{U}_n(u^-,u^+)] &= \mathbb{P}\left[u^-n\leq -\log \mathrm{CR}\left(z(U);D_n^{z(U)}\right)\leq u^+n\right]\\ &\leq \exp\left(-n\left(\inf_{u^-\leq u\leq u^+}\Lambda_\kappa^*(u)+o(1)\right)\right),\end{aligned}\tag{3.11}$$

where the $o(1)$ term tends to zero as $n\to\infty$ uniformly in U. Define

$$\mathcal{C}_m(u^-,u^+)=\cup_{n\geq m}\mathcal{U}_n(u^-,u^+).$$

We claim that $\mathcal{C}_m(\alpha^-,\alpha^+)$ is a cover for $\Theta(\alpha)\cap B(0,1-\delta)$ for any $\alpha^-<\alpha<\alpha^+<\beta$ and any $m\geq 1$. Pick $\tilde\alpha^-\in(\alpha^-,\alpha)$, $\tilde\alpha^+\in(\alpha,\alpha^+)$. For any $z\in\Theta(\alpha)\cap B(0,1-\delta)$, since $\lim(-\log\mathrm{CR}(z;D_n^z))/n=\alpha$, we have that, for n large enough,

$$\exp(-n\tilde\alpha^-)\geq \mathrm{CR}(z;D_n^z)\geq \exp(-n\tilde\alpha^+).$$

Let w be the point in $e^{-n\beta}\mathbb{Z}^2$ that is the closest to z and denote by U the ball in $\mathcal{U}_n$ with center w. Since $\tilde\alpha^+<\beta$ and by (2.5), we know that w is contained in D_n^z. Moreover, for n large enough, by (2.5) and that $\beta>\alpha^+>\tilde\alpha^+>\tilde\alpha^->\alpha^-$, we have

$$\begin{aligned}\mathrm{CR}(w;D_n^w) &\geq \mathrm{inrad}(w;D_n)\geq \mathrm{inrad}(z;D_n)-e^{-n\beta}\\ &\geq \frac14\mathrm{CR}(z;D_n)-e^{-n\beta}\geq \frac14 e^{-n\tilde\alpha^+}-e^{-n\beta}\geq e^{-n\alpha^+}.\\ \mathrm{CR}(w;D_n^w) &\leq 4\mathrm{inrad}(w;D_n)\leq 4(\mathrm{inrad}(z;D_n)+e^{-n\beta})\\ &\leq 4(\mathrm{CR}(z;D_n)+e^{-n\beta})\leq 4(e^{-n\tilde\alpha^-}+e^{-n\beta})\leq e^{-n\alpha^-}.\end{aligned}$$

Therefore $z\in U\in\mathcal{U}_n(\alpha^-,\alpha^+)$. This implies that $\mathcal{C}_m(\alpha^-,\alpha^+)$ is a cover for $\Theta(\alpha)\cap B(0,1-\delta)$. We use these covers to bound s-Hausdorff measure of $\Theta(\alpha)\cap B(0,1-\delta)$. For $m\geq 1$, and $\alpha^-<\alpha<\alpha^+<\beta$, we have

$$\begin{aligned}&\mathbb{E}[\mathcal{H}_s(\Theta(\alpha)\cap B(0,1-\delta))]\\ &\leq \mathbb{E}\left[\sum_{U\in\mathcal{C}_m(\alpha^-,\alpha^+)}|\mathrm{diam}(U)|^s\right]\\ &\leq \sum_{n\geq m}\exp(2n\beta)\times\exp(-sn\beta)\times\exp\left(-n\left(\inf_{\alpha^-\leq u\leq\alpha^+}\Lambda_\kappa^*(u)+o(1)\right)\right)\\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\text{(By (3.11))}\\ &=\sum_{n\geq m}\exp\left(n\left(2\beta-s\beta-\inf_{\alpha^-\leq u\leq\alpha^+}\Lambda_\kappa^*(u)+o(1)\right)\right)\end{aligned}$$

If $s>2-\inf_{\alpha^-\leq u\leq\alpha^+}\Lambda_\kappa^*(u)/\beta$, then (taking $m\to\infty$) we have

$$\mathbb{E}[\mathcal{H}_s(\Theta(\alpha)\cap B(0,1-\delta))]=0.$$

This implies that

$$2 - \inf_{\alpha^- \le u \le \alpha^+} \Lambda^*_\kappa(u)/\beta \ge \dim(\Theta(\alpha)), \quad \text{almost surely.}$$

This holds for any $\beta > \alpha^+ > \alpha > \alpha^-$, thus by the continuity of Λ^*_κ, we have

$$2 - \Lambda^*_\kappa(\alpha)/\alpha \ge \dim(\Theta(\alpha)), \quad \text{almost surely.}$$

Finally, when $\alpha < \alpha_{min}$, we see that $\mathcal{H}_0(\Theta(\alpha) \cap B(0, 1-\delta)) = 0$ almost surely. This implies that $\Theta(\alpha) = \emptyset$ almost surely. □

4. Proof of Theorems 1.2 and 1.4

The following lemma is the key result of this section:

Lemma 4.1. *Fix $\kappa \in [0,4)$. Let $(D^0_t, t \ge 0)$ be CGE$_\kappa$ targeted at the origin. Then there exist constants $r_0 \in (0,1)$, $p_0 \in (0,1)$ and $t_0 > 0$ such that*

$$\mathbb{P}\left[D^0_{t_0} \subset B(0, r_0)\right] \ge p_0.$$

Proof: Since the closure of D^0_t is a compact subset of the unit disc, it is sufficient to show that there exist constants $p_0 \in (0,1)$ and $t_0 > 0$ such that

$$\mathbb{P}[\partial\mathbb{U} \cap \partial D^0_{t_0} = \emptyset] \ge p_0. \tag{4.1}$$

First, we argue that there exist $u, r, \delta > 0$ such that for any arc $I \subset \partial\mathbb{U}$ with length less than δ, we have

$$\mathbb{P}[I \cap \partial D^0_u = \emptyset] \ge r. \tag{4.2}$$

Let J be the collection of positive-length arcs of $\partial\mathbb{U}$ with both endpoints in $\{e^{i\theta} : \theta \in \mathbb{Q}\}$. Fix any $u > 0$; since $\partial D^0_u \cap \partial\mathbb{U}$ is a compact proper subset of $\partial\mathbb{U}$, we know that $\partial\mathbb{U} \setminus \partial D^0_u$ is a union of open arcs, thus

$$\sum_{I \in J} \mathbb{P}[I \cap \partial D^0_u = \emptyset] > 0.$$

Thus there exists $I_0 \in J$ such that $\delta := |I_0| > 0$ and

$$r := \mathbb{P}[I_0 \cap \partial D^0_u = \emptyset] > 0.$$

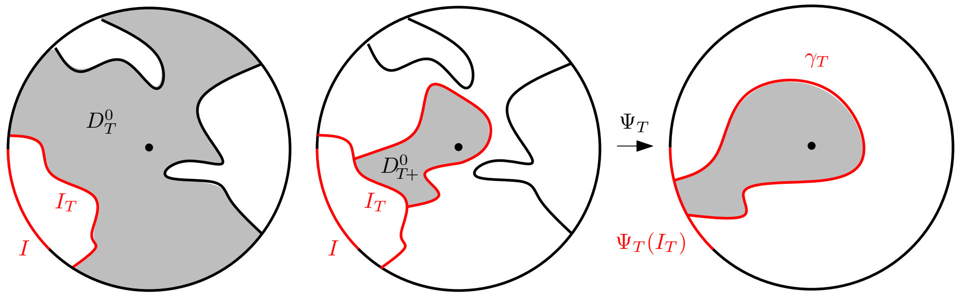
Since D^0_u is rotation invariant, we obtain (4.2).

For $\epsilon > 0$, define $E(\epsilon)$ to be the collection of excursions γ in $\mathbb{U}$ with the following property: if $x, y \in \partial\mathbb{U}$ are the two endpoints of γ, we require that the arc-length from x to y be less than ϵ and that γ disconnect the origin from the arc from y to x. Denote by $E(\gamma)$ the event that γ has this property. A standard SLE calculation (see, for instance, Schramm, 2001) shows that there is a universal constant $C < \infty$ such that

$$\begin{aligned} q(\epsilon) := \mu[E(\epsilon)] &= \int\int_{|x-y|\le\epsilon} dxdy H_{\mathbb{U}}(x,y) \mu^{\#}_{\mathbb{U},\kappa}(x,y)[E(\gamma)] \\ &\le C \int\int_{|x-y|\le\epsilon} dxdy |x-y|^{-2} |x-y|^{8/\kappa-1} \le C\epsilon^{8/\kappa-2}. \end{aligned}$$

In particular, $q(\epsilon) \to 0$ as $\epsilon \to 0$. Hence, with u, r, δ fixed above, we can choose $\epsilon_0 \in (0, \delta/2)$ such that

$$e^{-uq(\epsilon_0)} \ge 1 - r/2. \tag{4.3}$$



(A) Suppose that $I \cap \partial D_T^0 = \emptyset$, and let I_T be the connected component of $\partial D_T^0 \setminus \partial \mathbb{U}$ that disconnects I from the origin.

(B) Since $|I| \ge \delta$, the harmonic measure of I_T in D_T^0 seen from the origin is at least $\delta/2\pi$, thus the arc $\Psi_T(I_T)$ has length at least δ. Note that the distance between the two end points of γ_T is less than $\epsilon_0 \le \delta/2$. If the two end points of γ_T fall in $\Psi_T(I_T)$, then D_{T+}^0 will be disjoint of $\partial\mathbb{U}$.

FIGURE 4.3

Now let $(\gamma_t, t \ge 0)$ be a PPP with intensity $\mu_{\mathbb{U},\kappa}$ and let $(D_t^0, t \ge 0)$ be the corresponding growth process targeted at the origin. Let f_t, Ψ_t be the conformal maps defined in Section 3.1. Let $T = \inf\{t : \gamma_t \in E(\epsilon_0)\}$. We know that T has exponential law with parameter $q(\epsilon_0)$. Fix some arc I with length δ. Conditioned on the set $(\gamma_s, s < T)$ and on the event $E_1 = \{I \cap \partial D_T^0 = \emptyset\}$, let I_T be the connected component of $\partial D_T^0 \setminus \partial\mathbb{U}$ that disconnects I from the origin; see Figure 4.3(a). Recall that Ψ_T is the conformal map from D_T^0 onto $\mathbb{U}$ normalized at the origin. Consider the event E_2 that the two endpoints of γ_T fall in $\Psi_T(I_T)$. Since $|\Psi_T(I_T)| \ge \delta$ and $\gamma_T \in E(\epsilon_0)$, we know that the probability of E_2 is at least $\delta/(4\pi)$. Conditioned on $(\gamma_s, s \le T)$ and on the event $E_1 \cap E_2$, denoting $f_T \circ \Psi_T$ by Ψ_{T+}, we have that $\Psi_{T+}^{-1}(\mathbb{U}) \cap \partial\mathbb{U} = \emptyset$; see Figure 4.3(b). Therefore, for $t > T$,

$$\mathbb{P}\big[\partial\mathbb{U} \cap \partial D_t^0 = \emptyset \,\big|\, \sigma(\gamma_s, s < T), E_1\big] \ge \delta/(4\pi).$$

Thus

$$\mathbb{P}[\partial\mathbb{U} \cap \partial D_t^0 = \emptyset] \ge \mathbb{P}[t > T, I \cap \partial D_T^0 = \emptyset] \times \delta/(4\pi).$$

In order to show (4.1), we need to estimate $\mathbb{P}[t > T, I \cap \partial D_T^0 = \emptyset]$. We have

$$\begin{aligned} \mathbb{P}[t > T, I \cap \partial D_T^0 = \emptyset] &\ge \mathbb{P}[t > T > u, I \cap \partial D_u^0 = \emptyset] \\ &\ge \mathbb{P}[t > T > u] - (1 - r) && \text{(By (4.2))} \\ &= e^{-uq(\epsilon_0)} - e^{-tq(\epsilon_0)} - (1 - r) \\ &\ge r/2 - e^{-tq(\epsilon_0)}. && \text{(By (4.3))} \end{aligned}$$

We choose $t_0 > u$ large so that $e^{-t_0 q(\epsilon_0)} \le r/4$. Then we have

$$\mathbb{P}[\partial\mathbb{U} \cap \partial D_{t_0}^0 = \emptyset] \ge r\delta/(16\pi),$$

as desired. □

Lemma 4.2. *Assume the same notation as in Lemma 4.1. Then there exist constants $c, C \in (0, \infty)$ such that, for all $t > 0$,*

$$\mathbb{P}\left[D_t^0 \not\subset B(0, r_0)\right] \le Ce^{-ct}.$$

Proof: It is sufficient to prove that, for all $n \geq 1$, we have

$$\mathbb{P}\left[D^0_{nt_0} \not\subset B(0, r_0)\right] \leq (1-p_0)^n. \tag{4.4}$$

We will prove (4.4) by induction on n. Assume that (4.4) holds for n. Then, for $n+1$, we have

$$\mathbb{P}\left[D^0_{(n+1)t_0} \not\subset B(0, r_0)\right] \leq (1-p_0)^n \times \mathbb{P}\left[D^0_{(n+1)t_0} \not\subset B(0, r_0) \,\middle|\, D^0_{nt_0} \not\subset B(0, r_0)\right].$$

Let Ψ be the conformal map from $D^0_{nt_0}$ onto $\mathbb{U}$ normalized at the origin. Since $|\Psi(z)| \geq |z|$, we have

$$\Psi(B(0, r_0)) \supset B(0, r_0). \tag{4.5}$$

Thus

$$\begin{aligned}\mathbb{P}\left[D^0_{(n+1)t_0} \not\subset B(0, r_0)\right] &\leq (1-p_0)^n \times \mathbb{P}\left[D^0_{(n+1)t_0} \not\subset B(0, r_0) \,\middle|\, D^0_{nt_0} \not\subset B(0, r_0)\right] \\ &= (1-p_0)^n \times \mathbb{P}\left[\Psi\left(D^0_{(n+1)t_0}\right) \not\subset \Psi(B(0, r_0))\right] \\ &\leq (1-p_0)^n \times \mathbb{P}\left[D^0_{t_0} \not\subset B(0, r_0)\right] \\ &\qquad\qquad \left(\text{by } \Psi\left(D^0_{(n+1)t_0}\right) \overset{d}{=} D^0_{t_0} \text{ and } (4.5)\right) \\ &\leq (1-p_0)^{n+1},\end{aligned}$$

as desired. □

Lemma 4.3. *Assume the same notation as in Lemma 4.1. Then there exist constants $c, C \in (0, \infty)$ such that, for $r > 0$ and for all $t > 0$,*

$$\mathbb{P}\left[D^0_t \not\subset B(0, r)\right] \leq Ce^{-ct/\log(1/r)} \times \log(1/r).$$

In particular, this implies that

$$\mathbb{P}[T(0, r) > t] \leq Ce^{-ct/\log(1/r)} \times \log(1/r).$$

Proof: This will be somewhat similar to the previous lemma. First, it is enough to prove that, for any $n \in \mathbb{Z}_+$,

$$\mathbb{P}\left[D^0_t \subset B(0, r_0^n)\right] \geq \mathbb{P}\left[D^0_{t/n} \subset B(0, r_0)\right]^n, \tag{4.6}$$

because then

$$\mathbb{P}\left[D^0_t \not\subset B(0, r_0^n)\right] \leq n\,\mathbb{P}\left[D^0_{t/n} \not\subset B(0, r_0)\right],$$

and choosing $n = \lceil \log r / \log r_0 \rceil$ and using Lemma 4.2, we get the upper bound

$$\mathbb{P}\left[D^0_t \not\subset B(0, r)\right] \leq n\,Ce^{-c(t/n)},$$

which implies the conclusion.

We prove (4.6) by induction on n. More precisely, we claim that, for any $k \in \mathbb{Z}_+$ and $u > 0$,

$$\mathbb{P}\left[D^0_{(k+1)u} \subset B(0, r_0^{k+1}) \,\middle|\, D^0_{ku} \subset B(0, r_0^k)\right] \geq \mathbb{P}\left[D^0_u \subset B(0, r_0)\right], \tag{4.7}$$

and then (4.6) follows by taking $u = t/n$ and a telescoping product for $k = 0, 1, \ldots, n-1$.

Let Ψ be the conformal map from D^0_{ku} onto $\mathbb{U}$ normalized at the origin. We know that $\Psi(D^0_{(k+1)u})$ has the same law as D^0_u. To prove (4.7), it is then sufficient to show that, conditioned on $\{D^0_{ku} \subset B(0, r_0^k)\}$,

$$\Psi(B(0, r_0^{k+1})) \supset B(0, r_0). \tag{4.8}$$

On the event $\{D^0_{ku} \subset B(0, r^k_0)\}$, let ϕ_1 be the conformal map from D^0_{ku} onto $B(0, r^k_0)$ normalized at the origin; then $|\phi_1(z)| \geq |z|$. Let $\phi_2(z) = z/r^k_0$; then $\Psi = \phi_2 \circ \phi_1$. Thus

$$|\Psi(z)| \geq |z|/r^k_0,$$

which implies (4.8) and hence completes the proof. □

Corollary 4.4. *Assume the same notation as in Lemma 4.1. Then almost surely the growth process* $(D^0_t, t \geq 0)$ *is transient, i.e., the diameter of* D^0_t *goes to zero as* $t \to \infty$ *almost surely.*

Proof: For $n \geq 1$, set $r_n = e^{-\sqrt{n}}$. By Lemma 4.3, we have that

$$\mathbb{P}[D^0_n \not\subset B(0, r_n)] \leq C\sqrt{n}e^{-c\sqrt{n}}.$$

Thus

$$\sum_n \mathbb{P}[D^0_n \not\subset B(0, r_n)] < \infty.$$

By the Borel-Cantelli lemma, almost surely there is N such that

$$D^0_n \subset B(0, r_n), \quad \forall n \geq N.$$

This implies the conclusion. □

Proof of Theorem 1.2, Upper bound: For $n \geq 1$, define

$$\Gamma_n = \{z \in \Gamma : T(0, z) \leq n\}.$$

By Corollary 4.4, we see that $\Gamma = \cup_n \Gamma_n$, thus it is sufficient to show that, for $n \geq 1$, almost surely,

$$\dim(\Gamma_n) \leq 1 + \kappa/8. \tag{4.9}$$

For $m \geq 1$, let $\mathcal{U}_m$ be the collection of open balls with centers in $e^{-m}\mathbb{Z}^2 \cap \mathbb{U}$ and radius e^{-m}. Denote by $z(U)$ the center of $U \in \mathcal{U}_m$. For any $U \in \mathcal{U}_m$, suppose that $U \cap \Gamma_n \neq \emptyset$ and denote $z(U)$ by z; we will argue that this implies

$$\operatorname{inrad}(z; D^z_n) \leq e^{-m}. \tag{4.10}$$

There are two cases: $T(0, z) \leq n$ or $T(0, z) > n$. If $T(0, z) \leq n$, then $U \cap \Gamma_n \neq \emptyset$ implies $\operatorname{inrad}(z; D^z_{T(0,z)}) \leq e^{-m}$ which implies (4.10) since $T(0, z) \leq n$. If $T(0, z) > n$, then we know that $D^z_n = D^0_n$. Take $w \in U \cap \Gamma_n$. Since $T(0, w) \leq n$, we know that $w \notin D^0_n$, combining with $|z - w| < e^{-m}$, we obtain (4.10). Therefore, we have, for any $\lambda \in (0, 1 - \kappa/8)$

$$\begin{aligned}
\mathbb{P}[U \cap \Gamma_n \neq \emptyset] &\leq \mathbb{P}\left[\operatorname{inrad}(z; D^z_n) \leq e^{-m}\right] && (z = z(U)) \\
&\leq \mathbb{P}\left[\operatorname{CR}(z; D^z_n) \leq 4e^{-m}\right] && (\text{By } (2.5)) \\
&= \mathbb{P}\left[\operatorname{CR}(z; D^z_n)^{-\lambda} \geq (4e^{-m})^{-\lambda}\right] \\
&\leq (4e^{-m})^{\lambda}\mathbb{E}\left[\operatorname{CR}(z; D^z_n)^{-\lambda}\right] \\
&= 4^{\lambda}e^{-m\lambda}\exp(n\Lambda_\kappa(\lambda)).
\end{aligned}$$

We use $\{U \in \mathcal{U}_m : U \cap \Gamma_n \neq \emptyset\}$ to cover Γ_n and to bound s-Hausdorff measure of Γ_n: there is a constant C (only depending on κ, λ, n) such that

$$\mathbb{E}[\mathcal{H}_s(\Gamma_n)] \leq \sum_{U \in \mathcal{U}_m} \operatorname{diam}(U)^s \mathbb{P}[U \cap \Gamma_n \neq \emptyset] \leq Ce^{2m - ms - m\lambda}.$$

If $s > 2-\lambda$, taking $m \to \infty$, we have $\mathbb{E}[\mathcal{H}_s(\Gamma_n)] = 0$, this gives $2-\lambda \geq \dim(\Gamma_n)$ almost surely. This holds for any $\lambda \in (0, 1-\kappa/8)$, thus $1+\kappa/8 \geq \dim(\Gamma_n)$ almost surely. □

Proof of Theorem 1.2, Lower bound: Since Γ contains the conformal image of entire SLE_κ arcs, we just need to show that such a conformal map cannot have such a bad distortion that would ruin the dimension $1+\kappa/8$ proved in Beffara (2008) for SLE_κ in the upper half plane.

Suppose that $(\gamma_t, t \geq 0)$ is a PPP of SLE excursions and $(D_t^0, t \geq 0)$ is the corresponding growth process targeted at the origin. Let $t > 0$ be a time when γ_t is non-empty, and ϕ be the conformal map from $\mathbb{U}$ onto D_t^0 normalized at the origin. For any $r < 1$, we have some $M_r < \infty$ such that $1/M_r < |\phi'(z)| < M_r$ for all $|z| \leq r$. This implies that the diameter of every subset U of the closed ball $\overline{B(0,r)}$ is changed by at most some finite factor $\tilde{M}_r$, which implies that

$$\dim(\phi(\gamma_t \cap \overline{B(0,r)}) = \dim(\gamma_t \cap \overline{B(0,r)}). \tag{4.11}$$

Since we are dealing with $\kappa < 4$ only, the countable union of $\gamma_t \cap \overline{B(0, 1-1/n)}$ is all of γ_t except for its two endpoints. Thus, (4.11) implies that $\dim(\Gamma) \geq \dim \phi(\gamma_t) = 1+\kappa/8$. □

Proof of Theorem 1.4: By the conformal invariance of CGE_κ, it is equivalent to show that, for all $x > 0$,

$$\left| \mathbb{E}[T(0, e^{-x})] - \frac{x}{\Lambda_\kappa'(0)} \right| \leq C.$$

Let $(D_t^0, t \geq 0)$ be the growth process targeted at the origin. Define $X(t) = -\log \mathrm{CR}(D_t^0)$, which is the same as the accumulated capacity studied in Section 3.3. Define $\tau_x = \inf\{t : X(t) > x\}$, and $Y_x = X(\tau_x)$. It is clear that, for $\lambda < 1-\kappa/8$, the process $M_t = \exp(\lambda X_t - t\Lambda_\kappa(\lambda))$ is a martingale.

First, we argue that $(M_{t\wedge\tau_x})_{t\geq 0}$ is a uniformly integrable martingale. Pick $\beta > 1$ such that $\lambda\beta < 1-\kappa/8$. It is sufficient to show that $(M_{t\wedge\tau_x})_{t\geq 0}$ is uniformly bounded in L^β. We have

$$\mathbb{E}\left[M_{t\wedge\tau_x}^\beta\right] = \mathbb{E}[\exp(\lambda\beta X_{t\wedge\tau_x} - (t\wedge\tau_x)\beta\Lambda_\kappa(\lambda))] \leq \exp(\lambda\beta x)\mathbb{E}[\exp(\lambda\beta(Y_x - x))].$$

By Propositions 2.2 and 3.8, we know that $\mathbb{E}[\exp(\lambda\beta(Y_x - x))]$ is finite, thus

$$\sup_t \mathbb{E}\left[M_{t\wedge\tau_x}^\beta\right] < \infty,$$

as desired.

Second, we show that $|\mathbb{E}[\tau_x] - x/\Lambda_\kappa'(0)| \leq C$ for some $C < \infty$ only depending on κ. Since $(M_{t\wedge\tau_x})_{t\geq 0}$ is a uniformly integrable martingale, we can apply Optional Stopping Theorem and obtain

$$1 = \mathbb{E}\left[\exp\left(\lambda Y_x - \Lambda_\kappa(\lambda)\tau_x\right)\right]. \tag{4.12}$$

Differentiating (4.12) with respect to λ and setting $\lambda = 0$, we have

$$\mathbb{E}[\tau_x] = x/\Lambda_\kappa'(0) + \mathbb{E}[(Y_x - x)/\Lambda_\kappa'(0)].$$

By Propositions 2.2 and 3.8 again, we see that $\mathbb{E}[(Y_x - x)/\Lambda_\kappa'(0)]$ is uniformly bounded as desired.

Third, we argue that $T(0, e^{-x}) - \tau_x$ has exponentially decaying tail. Let Ψ be the conformal map from $D_{\tau_x}^0$ onto $\mathbb{U}$ normalized at the origin. Note that $(\Psi(D_{\tau_x+t}^0), t \geq$

0) has the same law as $(D_t^0, t \geq 0)$ and is independent of $(D_s^0, s \leq \tau_x)$. Let $\tilde{T}$ be an independent disconnection time, then, given $D_{\tau_x}^0$

$$\mathbb{P}[T(0, e^{-x}) \geq \tau_x + t] = \mathbb{P}\left[\tilde{T}\left(0, \Psi(e^{-x})\right) \geq t\right].$$

By the Growth Theorem (Lawler, 2005, Theorem 3.23), we have that, for any $z \in D_{\tau_x}^0$,

$$|z|\Psi'(0) \leq \frac{|\Psi(z)|}{(1 - |\Psi(z)|)^2}.$$

In particular, on the event $\{T(0, e^{-x}) > \tau_x\}$, we know that e^{-x} is still contained in $D_{\tau_x}^0$, and since $\Psi'(0) = \exp(Y_x) \geq e^x$, we have

$$1 \leq \frac{|\Psi(e^{-x})|}{(1 - |\Psi(e^{-x})|)^2},$$

thus

$$|\Psi(e^{-x})| \geq (3 - \sqrt{5})/2.$$

Combining with Lemma 4.3, we have that, for some constants c, C

$$\mathbb{P}[T(0, e^{-x}) \geq \tau_x + t] = \mathbb{P}\left[\tilde{T}\left(0, \Psi(e^{-x})\right) \geq t\right] \leq Ce^{-ct},$$

as desired.

Finally, we can complete the proof by noting that $T(0, e^{-x}) \geq \tau_{x-\log 4}$ by (2.5) and that $\tau_x - \tau_{x-\log 4}$ has exponentially decaying tail. □

5. Open questions

Even though we know that the Hausdorff dimension of the closure of $\cup_{t\geq 0}\partial D_t^0$ is $1+\kappa/8$, the dimension of a single ∂D_t^0 could be smaller; intuitively, this happens if the growing arcs form bottlenecks, producing shortcuts in ∂D_t^0. However, we do not expect this to happen.

Question 5.1 (Dimension of the boundary). Is the Hausdorff dimension of the closure of ∂D_t^0 almost surely equal to $1 + \kappa/8$?

One can view ∂D_t^0 as a Markov process on loops surrounding the origin. What is its stationary measure?

Question 5.2 (Stationary loop). Is ∂D_t^0 a continuous simple curve? Consider the rescaled loop around the origin:

$$\mathcal{L}_t^0 := \exp(t\Lambda_\kappa'(0))\, \partial D_t^0.$$

Show that it has a limiting distribution as $t \to \infty$, and identify this law.

Finally, possibly the most interesting question:

Question 5.3 (Discrete models). Identify the growth process CGE_κ for some values of κ as the scaling limit of some discrete models.

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