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Martingales and some generalizations arising from the supersymmetric hyperbolic sigma model

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Abstract. We introduce a family of real random variables (β, θ) arising from the supersymmetric nonlinear sigma model $H^{2|2}$ and containing the family β introduced by Sabot, Tarrès, and Zeng (Sabot et al., 2017) in the context of the vertex-reinforced jump process. Using this family we construct an exponential martingale generalizing the ones considered in Sabot and Zeng (2018+) and Disertori et al. (2017). Moreover, using the full supersymmetric nonlinear sigma model we also construct a generalization of the exponential martingale involving Grassmann variables.

1. Introduction and main results

The nonlinear supersymmetric hyperbolic sigma ($H^{2|2}$) model was introduced by Zirnbauer (1991) as a toy model for quantum diffusion. The corresponding measure can be better analyzed after passing to horospherical coordinates (u, s) as in Spencer

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and Zirnbauer (2004) (for the nonsupersymmetric version) and $(u, s, \overline{\psi}, \psi)$ as in Disertori et al. (2010) (cf. details below). In particular a phase transition in dimension $d \geq 3$ was proved, see Disertori et al. (2010) and Disertori and Spencer (2010).

The $H^{2|2}$ model has an interpretation as a random Schrödinger operator (Disertori and Spencer, 2010) and unexpectedly also as mixing measure and two point function for the vertex-reinforced jump process (Sabot and Tarrès, 2015; Bauerschmidt et al., 2018). This process was conceived by Werner and first developed by Davis and Volkov (2002, 2004).

More recently Sabot, Tarrès, and Zeng developed further the random Schrödinger operator interpretation (Sabot et al., 2017; Sabot and Zeng, 2018+). In particular they derived the explicit law for the random potential, and constructed two families of martingales in discrete time. One of them is the key ingredient to derive a characterization of recurrence/transience behavior of the vertex-reinforced jump process. Sabot and Zeng (2017) connected these families to certain continuous time martingales. Interesting formulas related to the work of Sabot, Tarrès, and Zeng appear also in Letac and Wesołowski (2017).

The above two families of discrete time martingales are only the first instances of an infinite hierarchy of martingales described in Disertori et al. (2017). All these martingales involve only the u components of the $H^{2|2}$ model. In this paper we extend these martingales to even larger families involving all the variables $(u, s, \overline{\psi}, \psi)$.

How this article is organized. In Sections 1 and 2 we consider only the marginal $\mu^W(du\, ds)$ of the full $H^{2|2}$ model obtained by integrating out the Grassmann variables $(\overline{\psi}, \psi)$. It is introduced in Section 1.1. The random variables u encode the asymptotics of local times for a time changed vertex reinforced jump process while the random variables s describe the corresponding fluctuations. For details see Merkl et al. (2018+).

In Section 1.2 we introduce a scaling transformation $\mathscr{S}$ for the variables (u, s). The effect of this scaling on the measure μ^W is formulated in Theorem 1.1. We provide two different proofs of it.

- The first proof, given in Section 2.1, is based on Lemma 2.2 which describes the ratio between the original and $\mathscr{S}$-transformed probability density of two supersymmetric sigma models with different parameters. Also for this lemma two different proofs are given.
  - The first proof, given in Section 2.2, is based on explicit computations on the quadratic form associated to the matrix A^W defined in equation (1.2).
  - An alternative proof, given in Appendix B.1, uses the description of the density of the supersymmetric sigma model in terms of 2×2 determinants connected to the linear algebra of Weyl spinors.

  Both these proofs are self-contained.
- The second proof of Theorem 1.1 uses conditioning on the u variables and a result from Disertori et al. (2017). It is given in Appendix B.2.

Theorem 1.1 is in turn the key ingredient to prove the martingale property, which extends Theorem 2.6 and Corollary 2.7 from Disertori et al. (2017) to test functions depending on (u, s) variables. Note that when the test function depends only on the u variable, we recover the martingales derived in Disertori et al. (2017). The martingale property on an infinite graph for the marginal μ^W is stated in

Section 1.3, while Section 1.2 contains some preliminary results in finite volume. All these results are proved in Section 2.

In Section 3 we extend the results of Sections 1 and 2 to the full $H^{2|2}$ supermeasure studied in Disertori et al. (2010), where Grassmann variables are included. In particular, this requires a generalization of the above mentioned scaling transformation $\mathscr{S}$ to a version including both, Grassmann and real-valued variables. The effect of this generalized scaling is given in Theorem 3.3, which is one of the main results of the paper. As a consequence, we introduce a generalization of the notion of martingale to a 'susy martingale', not to be confused with the notion of supermartingale in standard probability. Here the test functions may depend on Grassmann variables too. In particular when the test function depends only on the real variables u, s but not on the Grassmann variables, we recover the martingales described in Theorem 1.3 and Corollary 1.4.

1.1. *The nonlinear supersymmetric hyperbolic sigma model.* Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be a finite connected graph with vertex set $\tilde{V}$ and set of undirected edges $\tilde{E}$. We assume that $\tilde{G}$ has no direct loops and no parallel edges. We write $i \sim j$ if there is an edge between i and j. Let $\delta \in \tilde{V}$ be a distinguished vertex and set $V = \tilde{V} \setminus \{\delta\}$. Every edge $(i \sim j) \in \tilde{E}$ gets a weight $W_{ij} = W_{ji} > 0$. For convenience of notation, we set $W_{ij} = 0$ for all $i, j \in \tilde{V}$ with $i \not\sim j$. The euclidean scalar product is denoted by $\langle a, b\rangle = \sum_{i\in I} a_i b_i$, where $I = V$ or $I = \tilde{V}$, depending on the type of a and b. Let

$$\Omega_V := \left\{(u = (u_i)_{i\in\tilde{V}}, s = (s_i)_{i\in\tilde{V}}) \in \mathbb{R}^{\tilde{V}} \times \mathbb{R}^{\tilde{V}} : \ u_\delta = 0, s_\delta = 0\right\}. \tag{1.1}$$

We define the matrix $A^W(u) \in \mathbb{R}^{\tilde{V}\times\tilde{V}}$ by

$$A^W_{ij}(u) = \begin{cases} -W_{ij}e^{u_i+u_j} & \text{for } i \neq j, \\ \sum_{k\in\tilde{V}} W_{ik}e^{u_i+u_k} & \text{for } i = j. \end{cases} \tag{1.2}$$

Let $A^W_{VV}(u)$ denote its restriction to $V \times V$. We define $\rho^W : \Omega_V \to [0, \infty)$ by

$$\begin{aligned} \rho^W(u, s) &= \det A^W_{VV}(u) e^{-\frac{1}{2}\langle s, A^W(u)s\rangle} e^{-\frac{1}{2}\left\langle e^{-u}_{\tilde{V}}, A^W(u)e^{-u}_{\tilde{V}}\right\rangle} \\ &= \det A^W_{VV}(u) \prod_{(i\sim j)\in\tilde{E}} e^{-W_{ij}[\cosh(u_i-u_j)-1+\frac{1}{2}(s_i-s_j)^2 e^{u_i+u_j}]} \end{aligned} \tag{1.3}$$

where $e^{-u}_{\tilde{V}} = (e^{-u_i})_{i\in\tilde{V}}$ is a column vector. The last equality in (1.3) follows directly from

$$\begin{aligned} \left\langle e^{-u}_{\tilde{V}}, A^W(u)e^{-u}_{\tilde{V}}\right\rangle &= \sum_{i,j\in\tilde{V}} e^{-u_i} A^W_{ij}(u) e^{-u_j} = \sum_{i\in\tilde{V}}\sum_{k\in\tilde{V}} W_{ik}e^{u_k-u_i} - 2\sum_{(i\sim j)\in\tilde{E}} W_{ij} \\ &= 2\sum_{(i\sim j)\in\tilde{E}} W_{ij}[\cosh(u_i - u_j) - 1], \end{aligned} \tag{1.4}$$

where the first sum on the right-hand side of (1.4) comes from the diagonal terms in $A^W(u)$ and the second sum from the off-diagonal terms. Using the reference measure

$$\zeta(du_i ds_i) = e^{-u_i}\, du_i ds_i \tag{1.5}$$

on $\mathbb{R}^2$, the supersymmetric sigma model is described by the following probability measure on Ω_V:

$$\mu^W(du\,ds) = \rho^W(u,s) \prod_{i\in V} \frac{1}{2\pi} \zeta(du_i ds_i), \tag{1.6}$$

where we drop the Dirac measure located at $(u_\delta, s_\delta) = (0,0)$ in the notation. We denote the expectation with respect to μ^W by $\mathbb{E}_{\mu^W}$.
Notation. In the following, operations are frequently to be read componentwise, like $a^2 + b^2 = (a_i^2 + b_i^2)_{i\in\tilde{V}}$, $e^{-u}b/a = (e^{-u_i}b_i/a_i)_{i\in\tilde{V}}$, $\log a = (\log a_i)_{i\in\tilde{V}}$.

1.2. *Results in finite volume.* We set

$$\mathcal{G}_V = \{[a,b] \in (0,\infty)^{\tilde{V}} \times \mathbb{R}^{\tilde{V}} : (a_\delta, b_\delta) = (1,0)\}. \tag{1.7}$$

For the moment, one may read $[a_i, b_i]$ to be just the pair (a_i, b_i). However, any element of $\mathcal{G}_V$ can be identified with a family of matrices $[a_i, b_i]$, together with a group action described in Appendix A. For $[a,b] \in \mathcal{G}_V$ and $(u,s) \in \Omega_V \times \Omega_V$, we introduce the scaling transformation

$$\mathscr{S}_{[a,b]}(u,s) = \left(u_i + \log a_i, s_i - e^{-u_i}\frac{b_i}{a_i}\right)_{i\in\tilde{V}}, \tag{1.8}$$

$$\mathscr{S}_{[a,b]}^{-1}(u,s) = (\tilde{u},\tilde{s}) = (u_i - \log a_i, s_i + e^{-u_i}b_i)_{i\in\tilde{V}}. \tag{1.9}$$

We remark that in light cone coordinates $x_+ = e^u$, $y = se^u$ this corresponds to a scaling of x_+ and a translation of y. The scaling transformation arises naturally as a group action as is shown in Appendix A. We also need the following rescaling of the weights W:

$$W^a = (W_{ij}^a := a_i a_j W_{ij})_{i,j\in\tilde{V}}. \tag{1.10}$$

The same rescaling of weights was also used in Sabot et al. (2017). Denote by x_V the restriction of a vector $x \in \mathbb{R}^{\tilde{V}}$ to $\mathbb{R}^V$. Let

$$e_{VV}^{-u} = \operatorname{diag}(e^{-u_i}, i \in V) \tag{1.11}$$

denote the diagonal matrix in $\mathbb{R}^{V\times V}$ with entries e^{-u_i} on the diagonal. We consider the variables $\theta^{V,W}(u,s) = (\theta_i^{V,W}(u,s))_{i\in V}$ defined by

$$\theta^{V,W}(u,s) = e_{VV}^{-u} A_{VV}^W(u) s_V. \tag{1.12}$$

Componentwise, we have for $i \in V$

$$\theta_i^{V,W}(u,s) = \sum_{j\in\tilde{V}} W_{ij} e^{u_j}(s_i - s_j). \tag{1.13}$$

We need the random variables $\tilde{\beta}^{\tilde{V},W} = (\tilde{\beta}_i^{\tilde{V},W})_{i\in\tilde{V}}$ and their restriction $\beta^{V,W}$ to V defined by

$$\tilde{\beta}_i^{\tilde{V},W}(u) = \frac{1}{2}\sum_{j\in\tilde{V}} W_{ij} e^{u_j - u_i}, \qquad \beta^{V,W} = \tilde{\beta}_V^{\tilde{V},W} = (\tilde{\beta}_i^{\tilde{V},W})_{i\in V}. \tag{1.14}$$

These variables were introduced in Sabot et al. (2017). We drop the dependence on V, W, or both if there is no risk of confusion.

The following theorem describes the behavior of the supersymmetric sigma model μ^W with respect to the scaling transformation $\mathscr{S}_{[a,b]}$ and is a fundamental ingredient in this paper. Its extension to Grassmann variables is given in Theorem 3.3.

Theorem 1.1. *Let $[a,b] \in \mathcal{G}_V$. The image of μ^{W^a} under the map $\mathscr{S}_{[a,b]}$ is absolutely continuous with respect to μ^W with the following Radon-Nikodym derivative on Ω_V:*

$$\frac{d(\mathscr{S}_{[a,b]}\mu^{W^a})}{d\mu^W}(u,s) = \mathcal{L}^W(a,b)^{-1} e^{-\langle (a^2+b^2-1)_V, \beta^W(u)\rangle - \langle b_V, \theta^W(u,s)\rangle} \tag{1.15}$$

with the constant

$$\mathcal{L}^W(a,b) := \prod_{(i\sim j)\in\tilde{E}} e^{-W_{ij}(a_i a_j + b_i b_j - 1)} \cdot \prod_{j\in V} \frac{1}{a_j}. \tag{1.16}$$

In other words, for any measurable function $f : \Omega_V \to \mathbb{R}_0^+$, one has

$$\mathbb{E}_{\mu^W}\left[f(u,s) e^{-\langle (a^2+b^2-1)_V, \beta^W(u)\rangle - \langle b_V, \theta^W(u,s)\rangle}\right] = \mathcal{L}^W(a,b)\mathbb{E}_{\mu^{W^a}}[f \circ \mathscr{S}_{[a,b]}]. \tag{1.17}$$

In particular, $\mathcal{L}^W$ describes the joint Laplace transform of β^W and θ^W:

$$\mathcal{L}^W(a,b) = \mathbb{E}_{\mu^W}\left[e^{-\langle (a^2+b^2-1)_V, \beta^W(u)\rangle - \langle b_V, \theta^W(u,s)\rangle}\right]. \tag{1.18}$$

The special case $b = 0$ was proven as Theorem 3.1 in Disertori et al. (2017). For $a = \sqrt{1+\lambda}$ and $b = 0$ the Laplace transform $\mathcal{L}^W(a,b)$ in (1.18) equals the Laplace transform $\mathcal{L}^W(\lambda)$ given by formula (2.9) in Disertori et al. (2017).

1.3. *Results in infinite volume.* Let $G_\infty = (V_\infty, E_\infty)$ be an infinite graph with edge weights W_{ij}. We approximate G_∞ by finite graphs with wired boundary conditions $\tilde{G}_n = (\tilde{V}_n, \tilde{E}_n)$, where $\tilde{V}_n = V_n \cup \{\delta_n\}$, $V_n \uparrow V_\infty$, and

$$\tilde{E}_n = E_n \cup \{(i \sim \delta_n) : i \in V_n \text{ and } \exists j \in V_\infty \setminus V_n \text{ such that } (i \sim j) \in E_\infty\}. \tag{1.19}$$

We endow the edges of $\tilde{G}_n$ with the weights

$$W_{ij}^{(n)} = W_{ij} \quad \text{if } i \in V_n \text{ and } j \in V_n, \tag{1.20}$$

$$W_{i\delta_n}^{(n)} = W_{\delta_n i}^{(n)} = \sum_{j\in V_\infty\setminus V_n} W_{ij} \quad \text{for } i \in V_n, \quad \text{and} \quad W_{\delta_n\delta_n}^{(n)} = 0. \tag{1.21}$$

Let μ_n^W denote the $H^{2|2}$ measure defined in (1.6) for the graph $\tilde{G}_n$ with the weights $W_{ij}^{(n)}$.

Lemma 1.2 (Kolmogorov consistency)**.** *For $n \in \mathbb{N}$, the joint Laplace transform*

$$\mathcal{L}_n^W(a,b) = \mathbb{E}_{\mu_n^W}\left[e^{-\left\langle (a^2+b^2-1)_{V_n}, \beta^{V_n}\right\rangle - \left\langle b_{V_n}, \theta^{V_n}\right\rangle}\right] \tag{1.22}$$

of $\beta^{V_n} = (\beta_i)_{i\in V_n}$ and $\theta^{V_n} = (\theta_i)_{i\in V_n}$ satisfies the consistency relation

$$\mathcal{L}_n^W(a_{V_n}, b_{V_n}) = \mathcal{L}_{n+1}^W(a,b), \tag{1.23}$$

for all $[a,b] \in \mathcal{G}_{V_{n+1}}$ with $[a_i,b_i] = [1,0]$ for all $i \in \tilde{V}_{n+1} \setminus V_n$. In particular, the law of $(\beta^{V_n}, \theta^{V_n})$ with respect to μ_n^W agrees with the law of $(\beta^{V_{n+1}}, \theta^{V_{n+1}})|_{V_n}$ with respect to μ_{n+1}^W.

Consistency of the law of β was first observed by Sabot and Zeng (2018+); see also Lemma 2.4 in Disertori et al. (2017).

By Kolmogorov's consistency theorem, there is a probability space $(\Omega_\infty, \mathcal{F}_\infty, \mu_\infty^W)$ with random variables $\boldsymbol{\beta}_i, \boldsymbol{\theta}_i : \Omega_\infty \to \mathbb{R}$, $i \in V_\infty$, such that for all $n \in \mathbb{N}$ the law of

$$\left(\boldsymbol{\beta}^{(n)} = (\boldsymbol{\beta}_i)_{i\in V_n}, \boldsymbol{\theta}^{(n)} = (\boldsymbol{\theta}_i)_{i\in V_n}\right) \tag{1.24}$$

with respect to μ_∞^W agrees with the law of $(\beta^{V_n}, \theta^{V_n}) : \Omega_{V_n} \to \mathbb{R}^{V_n} \times \mathbb{R}^{V_n}$ with respect to μ_n^W. Moreover, by Lemma 2.3 in Disertori et al. (2017), for any finite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V \cup \{\delta\}$, there is a measurable function $f_V^W : \mathbb{R}^V \to \mathbb{R}^{\tilde{V}}$ such that

$$(u_i)_{i\in\tilde{V}} = f_V^W(\beta^V). \tag{1.25}$$

Using the definition (1.12) of θ^W, we have $s_V = A_{VV}^W(u)^{-1} e_{VV}^u \theta^V(u,s)$. Hence,

$$(s_i)_{i\in\tilde{V}} = g_V^W(\beta^V, \theta^V) \tag{1.26}$$

with the measurable function $g_V^W : \mathbb{R}^V \times \mathbb{R}^V \to \mathbb{R}^{\tilde{V}}$, $(\beta, \theta) \mapsto s = (s_i)_{i\in\tilde{V}}$ defined by $s_\delta = 0$ and $s_V = A_{VV}^W(f_V^W(\beta))^{-1} e_{VV}^{f_V^W(\beta)} \theta$. This allows us to couple the u and s-variables. We define

$$u^{(n)} = (u_i^{(n)})_{i\in\tilde{V}_n} = f_{V_n}^W(\boldsymbol{\beta}^{(n)}), \tag{1.27}$$

$$s^{(n)} = (s_i^{(n)})_{i\in\tilde{V}_n} = g_{V_n}^W(\boldsymbol{\beta}^{(n)}, \boldsymbol{\theta}^{(n)}), \tag{1.28}$$

$$u_i^{(n)} = s_i^{(n)} = 0 \quad \text{for } i \in V_\infty \setminus V_n. \tag{1.29}$$

We consider the following set of parameters

$$(-\infty, 0]^{(V_\infty)} = \{\alpha \in (-\infty, 0]^{V_\infty} : \alpha_i \neq 0 \text{ for only finitely many } i \in V_\infty\}. \tag{1.30}$$

For $\alpha \in (-\infty, 0]^{(V_\infty)}$ and $n \in \mathbb{N}$, we define $\alpha^{(n)} = (\alpha_i^{(n)})_{i\in\tilde{V}_n}$ by

$$\alpha_i^{(n)} = \alpha_i \quad \text{for } i \in V_n \qquad \text{and} \qquad \alpha_{\delta_n}^{(n)} = \sum_{j\in V_\infty\setminus V_n} \alpha_j. \tag{1.31}$$

Theorem 1.3. *For all $\alpha \in (-\infty, 0]^{(V_\infty)}$, the sequence $(M_\alpha^{(n)})_{n\in\mathbb{N}}$, defined by*

$$M_\alpha^{(n)} : (u^{(n)}, s^{(n)}) \mapsto \exp\left(\sum_{j\in\tilde{V}_n} \alpha_j^{(n)} e^{u_j^{(n)}} (1 + i s_j^{(n)})\right), \tag{1.32}$$

is a $\mathbb{C}$-valued martingale with respect to the filtration $\left(\mathcal{F}_n = \sigma(\boldsymbol{\beta}^{(n)}, \boldsymbol{\theta}^{(n)})\right)_{n\in\mathbb{N}}$.

Taking derivatives of the martingale $(M_\alpha^{(n)})_{n\in\mathbb{N}}$ at $\alpha = 0$, we obtain the following hierarchy of martingales.

Corollary 1.4. *For all $k \in \mathbb{N}$ and $j_1, \ldots, j_k \in V_\infty$,*

$$M_{j_1,\ldots,j_k}^{(n)} = \prod_{l=1}^k e^{u_{j_l}^{(n)}} (1 + i s_{j_l}^{(n)}), \quad n \in \mathbb{N}, \tag{1.33}$$

its real and imaginary part are martingales with respect to $\left(\mathcal{F}_n = \sigma(\boldsymbol{\beta}^{(n)}, \boldsymbol{\theta}^{(n)})\right)_{n\in\mathbb{N}}$.

In Disertori et al. (2017), we showed that the processes $(\mathbb{E}_{\mu_\infty^W}[M_\alpha^{(n)} | \sigma(u^{(n)})])_{n\in\mathbb{N}}$ and $(\mathbb{E}_{\mu_\infty^W}[M_{j_1,\ldots,j_k}^{(n)} | \sigma(u^{(n)})])_{n\in\mathbb{N}}$ are martingales. These facts are also immediate consequences of Theorem 1.3 and Corollary 1.4. The first two elements of the hierarchy also correspond to the martingales discovered in Sabot and Zeng (2018+).

2. The marginal $\mu^W(du\,ds)$

2.1. *Proof of Theorem 1.1.* Using the measure ζ introduced in formula (1.5), we consider the product

$$\zeta_V := \zeta^V \times \delta_{(0,0)} \tag{2.1}$$

composed of factors ζ indexed by V and one Dirac measure located at $(0,0) \in \mathbb{R}^2$ indexed by the special vertex δ.

Lemma 2.1. *For $[a,b] \in \mathcal{G}_V$, the image measure $\mathscr{S}_{[a,b]}\zeta_V$ of the measure ζ_V with respect to $\mathscr{S}_{[a,b]}$ is given by*

$$\mathscr{S}_{[a,b]}\zeta_V = \left(\prod_{i\in V} a_i\right)\zeta_V. \tag{2.2}$$

Proof: This is an immediate consequence of $e^{-\tilde{u}_i}\,d\tilde{u}_i = a_i e^{-u_i}\,du_i$ with $\tilde{u}_i = u_i - \log a_i$. □

Lemma 2.2 (Ratio of densities)**.** *For $[a,b] \in \mathcal{G}_V$ and $(u,s) \in \Omega_V$, one has*

$$\begin{aligned} &\frac{\rho^{W^a}(\mathscr{S}^{-1}_{[a,b]}(u,s))}{\rho^W(u,s)} \\ &= \prod_{(i\sim j)\in\tilde{E}} e^{W_{ij}(a_ia_j+b_ib_j-1)} \prod_{i\in V} \exp[-(a_i^2+b_i^2-1)\beta_i^W(u) - b_i\theta_i^W(u,s)]. \end{aligned} \tag{2.3}$$

This lemma is proven in Section 2.2, below.

Proof of Theorem 1.1: We abbreviate $c = (2\pi)^{-|V|}$. From (1.6), we know $d\mu^W = c\rho^W\,d\zeta_V$. Substituting W by W^a, this gives $d\mu^{W^a} = c\rho^{W^a}\,d\zeta_V$. We take now the image measure with respect to $\mathscr{S}_{[a,b]}$. The following calculation uses the description of $\mathscr{S}_{[a,b]}\zeta_V$ from Lemma 2.1 and in the last step the ratio of densities given in Lemma 2.2 together with the definition (1.16) of the constant $\mathcal{L}^W(a,b)$.

$$\begin{aligned} &d(\mathscr{S}_{[a,b]}\mu^{W^a}) = c(\rho^{W^a}\circ\mathscr{S}^{-1}_{[a,b]})\,d(\mathscr{S}_{[a,b]}\zeta_V) = c\frac{\rho^{W^a}\circ\mathscr{S}^{-1}_{[a,b]}}{\rho^W}\rho^W\prod_{i\in V} a_i\,d\zeta_V \\ &= \frac{\rho^{W^a}\circ\mathscr{S}^{-1}_{[a,b]}}{\rho^W}\prod_{i\in V} a_i\,d\mu^W = \mathcal{L}^W(a,b)^{-1}e^{-\langle(a^2+b^2-1)_V,\beta^W\rangle - \langle b_V,\theta^W\rangle}\,d\mu^W \end{aligned} \tag{2.4}$$

This implies the claim (1.15), which is written in (1.17) in a different notation. Taking the test function $f = 1$, (1.18) is a special case of (1.17). □

2.2. *Proof of Lemma 2.2.* We define the matrix $H^W_{\tilde{\beta}(u)} \in \mathbb{R}^{\tilde{V}\times\tilde{V}}$ by

$$(H^W_{\tilde{\beta}(u)})_{ij} = 2\tilde{\beta}_i(u)\delta_{ij} - W_{ij} \quad \text{for } i,j \in \tilde{V}. \tag{2.5}$$

Note that for all $i,j \in \tilde{V}$, one has

$$\begin{aligned} (H^W_{\tilde{\beta}(u)})_{ij} &= \left\{ \begin{array}{ll} -W_{ij} & \text{if } i \neq j, \\ 2\tilde{\beta}_i(u) = \sum_{k\in\tilde{V}} W_{ik}e^{u_k-u_i} & \text{if } i = j \end{array} \right\} \\ &= e^{-u_i-u_j}A^W_{ij}(u) = (e^{-u}A^W(u)e^{-u})_{ij}; \end{aligned} \tag{2.6}$$

recall that the graph $\tilde{G}$ has no direct loops and hence $W_{ii} = 0$ by the definition of the weights. Here and in the following, when calculating with matrices, we abbreviate $e^{\pm u} = \operatorname{diag}(e^{\pm u_i}, i \in \tilde{V})$. Thus, expressions like $e^{-u}s$ can be read in two equivalent ways, componentwise or as a matrix multiplication, both meaning the same object $(e^{-u_i}s_i)_{i\in\tilde{V}}$. We denote by $H^W_{\beta(u)} := (H^W_{\tilde{\beta}(u)})_{VV}$ the restriction to $V \times V$, i.e. $(H^W_{\beta(u)})_{ij} = 2\beta_i(u)\delta_{ij} - W_{ij}$ for $i, j \in V$, cf. (1.14).

Lemma 2.3. *For $(u, s) \in \Omega_V$, we have the relations*

$$2 \sum_{(i\sim j)\in\tilde{E}} W_{ij}[\cosh(u_i - u_j) - 1] = \left\langle e^{-u}_{\tilde{V}}, A^W(u)e^{-u}_{\tilde{V}} \right\rangle = \left\langle 1_{\tilde{V}}, H^W_{\tilde{\beta}(u)} 1_{\tilde{V}} \right\rangle, \tag{2.7}$$

$$\det A^W_{VV}(u) = \prod_{i\in V} e^{2u_i} \cdot \det H^W_{\beta(u)}, \quad \langle s, A^W(u)s\rangle = \left\langle e^u s, H^W_{\tilde{\beta}(u)} e^u s \right\rangle. \tag{2.8}$$

Proof: The claims follow from equation (1.4) and the relation (2.6) between $H^W_{\tilde{\beta}(u)}$ and $A^W(u)$. □

Lemma 2.4. *The matrix A^W is invariant with respect to the $\mathscr{S}$-operation in the following sense: For $[a, b] \in \mathcal{G}_V$, $(u, s) \in \Omega_V$, and $(\tilde{u}, \tilde{s}) = \mathscr{S}^{-1}_{[a,b]}(u, s) = (u - \log a, s + e^{-u}b)$, the following holds*

$$A^{W^a}(\tilde{u}) = A^W(u), \quad \textit{i.e.} \quad A^W = A^{W^a} \circ \mathscr{S}^{-1}_{[a,b]}. \tag{2.9}$$

Proof: For $i, j \in \tilde{V}$ with $i \neq j$, one has $A^{W^a}_{ij}(\tilde{u}) = a_i a_j W_{ij} e^{\tilde{u}_i + \tilde{u}_j} = W_{ij} e^{u_i+u_j} = A^W_{ij}(u)$. Since rows of both matrices $A^{W^a}(\tilde{u})$ and $A^W(u)$ sum up to 0, it follows also $A^{W^a}_{ii}(\tilde{u}) = A^W_{ii}(u)$. This proves the claim. □

Proof of Lemma 2.2: Substituting (2.9) into the definition (1.3) for ρ^{W^a}, we obtain

$$\rho^{W^a}(\mathscr{S}^{-1}_{[a,b]}(u, s)) = \rho^{W^a}(\tilde{u}, \tilde{s}) = \det A^W_{VV}(u) e^{-\frac{1}{2}\langle \tilde{s}, A^W(u)\tilde{s}\rangle} e^{-\frac{1}{2}\left\langle e^{-\tilde{u}}_{\tilde{V}}, A^W(u) e^{-\tilde{u}}_{\tilde{V}}\right\rangle}. \tag{2.10}$$

Inserting the definition of $\tilde{u}$ and $\tilde{s}$ in the exponents above and using (2.6), the facts $b_\delta = 0 = s_\delta$ and the definition (1.12) of θ^W, we obtain

$$\begin{aligned} \langle \tilde{s}, A^W(u)\tilde{s}\rangle &= \langle s, A^W(u)s\rangle + \langle b, e^{-u}A^W(u)e^{-u}b\rangle + 2\langle b, e^{-u}A^W(u)s\rangle \\ &= \langle s, A^W(u)s\rangle + \left\langle b, H^W_{\tilde{\beta}(u)} b\right\rangle + 2\langle b_V, \theta^W(u, s)\rangle, \end{aligned} \tag{2.11}$$

$$\left\langle e^{-\tilde{u}}_{\tilde{V}}, A^W(u)e^{-\tilde{u}}_{\tilde{V}}\right\rangle = \langle a, e^{-u}A^W(u)e^{-u}a\rangle = \left\langle a, H^W_{\tilde{\beta}(u)} a\right\rangle. \tag{2.12}$$

Using in the second equality (1.3) and (2.7), this implies

$$\begin{aligned} &\rho^{W^a}(\mathscr{S}^{-1}_{[a,b]}(u, s)) \\ &= \det A^W_{VV}(u) e^{-\frac{1}{2}\left(\langle s, A^W(u)s\rangle + \left\langle b, H^W_{\tilde{\beta}(u)} b\right\rangle\right) - \langle b_V, \theta^W(u,s)\rangle} e^{-\frac{1}{2}\left\langle a, H^W_{\tilde{\beta}(u)} a\right\rangle} \\ &= \rho^W(u, s) e^{-\frac{1}{2}\left(\left\langle a, H^W_{\tilde{\beta}(u)} a\right\rangle + \left\langle b, H^W_{\tilde{\beta}(u)} b\right\rangle - \left\langle 1_{\tilde{V}}, H^W_{\tilde{\beta}(u)} 1_{\tilde{V}}\right\rangle\right)} e^{-\langle b_V, \theta^W(u,s)\rangle}. \end{aligned} \tag{2.13}$$

Since $a_\delta^2 + b_\delta^2 - 1 = 0$, the first exponent in the last expression takes the form

$$
\begin{aligned}
&-\frac{1}{2}\left(\left\langle a, H^W_{\tilde\beta(u)} a\right\rangle + \left\langle b, H^W_{\tilde\beta(u)} b\right\rangle - \left\langle 1_{\tilde V}, H^W_{\tilde\beta(u)} 1_{\tilde V}\right\rangle\right)\\
&= \sum_{(i\sim j)\in\tilde E} W_{ij}(a_i a_j + b_i b_j - 1) - \sum_{i\in V}(a_i^2 + b_i^2 - 1)\beta_i^W(u). \qquad (2.14)
\end{aligned}
$$

This proves the claim. □

2.3. *Martingales.*

Proof of Kolmogorov consistency (Lemma 1.2): By Theorem 1.1, one has

$$\mathcal{L}_n^W(a_{V_n}, b_{V_n}) = \prod_{(i\sim j)\in\tilde E_n} e^{-W_{ij}^{(n)}(a_i a_j + b_i b_j - 1)} \cdot \prod_{j\in V_n} \frac{1}{a_j}, \qquad (2.15)$$

$$\mathcal{L}_{n+1}^W(a, b) = \prod_{(i\sim j)\in\tilde E_{n+1}} e^{-W_{ij}^{(n+1)}(a_i a_j + b_i b_j - 1)} \cdot \prod_{j\in V_{n+1}} \frac{1}{a_j}. \qquad (2.16)$$

Since $a_j = 1$ for $j \in V_{n+1} \setminus V_n$, one has

$$\prod_{j\in V_n} \frac{1}{a_j} = \prod_{j\in V_{n+1}} \frac{1}{a_j}. \qquad (2.17)$$

Consider $(i \sim j) \in \tilde E_{n+1}$.
Case $i, j \in V_n$: Then $(i \sim j) \in \tilde E_n$ and $W_{ij}^{(n)} = W_{ij}^{(n+1)}$. Consequently, one has $W_{ij}^{(n)}(a_i a_j + b_i b_j - 1) = W_{ij}^{(n+1)}(a_i a_j + b_i b_j - 1)$.
Case $i, j \in \tilde V_{n+1} \setminus V_n$: Then $[a_i, b_i] = [1, 0] = [a_j, b_j]$ and hence $a_i a_j + b_i b_j - 1 = 0$.
Case $i \in V_n$ and $j \in \tilde V_{n+1} \setminus V_n$: Then $[a_j, b_j] = [1, 0]$. For the given $i \in V_n$, we calculate

$$
\begin{aligned}
&\sum_{\substack{j\in\tilde V_{n+1}\setminus V_n:\\ (i\sim j)\in\tilde E_{n+1}}} W_{ij}^{(n+1)}(a_i a_j + b_i b_j - 1) = \left[W_{i\delta_{n+1}}^{(n+1)} + \sum_{j\in V_{n+1}\setminus V_n} W_{ij}\right](a_i - 1)\\
&= \sum_{j\in V_\infty\setminus V_n} W_{ij}(a_i - 1) = W_{i\delta_n}^{(n)}(a_i - 1) = W_{i\delta_n}^{(n)}(a_i a_{\delta_n} + b_i b_{\delta_n} - 1). \qquad (2.18)
\end{aligned}
$$

We conclude that the products over edge sets in (2.15) and (2.16) agree. The claim (1.23) follows. This identity holds in particular for $(a^2 + b^2 - 1, b)$ in a neighborhood of the origin. As a consequence, by analytic continuation, the characteristic function of $(\beta^{V_n}, \theta^{V_n})$ with respect to μ_n^W agrees with the characteristic function of $(\beta^{V_{n+1}}, \theta^{V_{n+1}})|_{V_n}$ with respect to μ_{n+1}^W. The claim follows. □

Proof of Theorem 1.3 (Generating martingale): By the definitions (1.27) and (1.28) of $u^{(n)}$ and $s^{(n)}$, it follows that $M_\alpha^{(n)}$ is $\mathcal{F}_n$-measurable. For $[a, b] \in \mathcal{G}_{V_{n+1}}$ with $[a_i, b_i] = [1, 0]$ for all $i \in \tilde V_{n+1} \setminus V_n$, we show

$$\mathbb{E}_{\mu_\infty^W}\left[M_\alpha^{(n+1)} \prod_{j\in V_n} e^{-(a_j^2 + b_j^2 - 1)\boldsymbol{\beta}_j - b_j\boldsymbol{\theta}_j}\right] = \mathbb{E}_{\mu_\infty^W}\left[M_\alpha^{(n)} \prod_{j\in V_n} e^{-(a_j^2 + b_j^2 - 1)\boldsymbol{\beta}_j - b_j\boldsymbol{\theta}_j}\right]. \qquad (2.19)$$

Note that for $j \in V_n$, one has $a_j > 0$ and $b_j \in \mathbb{R}$. So in particular, we prove the identity (2.19) for $a_j^2 + b_j^2 - 1$ and b_j belonging to a neighborhood of the origin, which implies the martingale property for $M_\alpha^{(n)}$.

We rewrite the claim in terms of expectations with respect to the supersymmetric sigma model on finite graphs. Let

$$\tilde{M}_\alpha^{(n)} : \Omega_{V_n} \to \mathbb{R}, \quad (u,s) \mapsto e^{\left\langle \alpha^{(n)}, e^u(1+is) \right\rangle}. \tag{2.20}$$

Using the definition of the variables $\boldsymbol{\beta}$ and $\boldsymbol{\theta}$, the identity (2.19) is equivalent to

$$\begin{aligned}&\mathbb{E}_{\mu_{n+1}^W}\left[\tilde{M}_\alpha^{(n+1)} \prod_{j \in V_n} e^{-(a_j^2+b_j^2-1)\beta_j^{V_{n+1}}(u) - b_j \theta_j^{V_{n+1}}(u,s)}\right] \\ =&\mathbb{E}_{\mu_n^W}\left[\tilde{M}_\alpha^{(n)} \prod_{j \in V_n} e^{-(a_j^2+b_j^2-1)\beta_j^{V_n}(u) - b_j \theta_j^{V_n}(u,s)}\right].\end{aligned} \tag{2.21}$$

Since $a_j^2 + b_j^2 - 1 = 0 = b_j$ for $j \in \tilde{V}_{n+1} \setminus V_n$, we rewrite the left-hand side of (2.21) using Theorem 1.1 as follows:

$$\begin{aligned}\text{lhs}(2.21) =&\mathbb{E}_{\mu_{n+1}^W}\left[\tilde{M}_\alpha^{(n+1)} \prod_{j \in V_{n+1}} e^{-(a_j^2+b_j^2-1)\beta_j^{V_{n+1}}(u) - b_j \theta_j^{V_{n+1}}(u,s)}\right] \\ =&\mathcal{L}_{n+1}^W(a,b)\mathbb{E}_{\mu_{n+1}^{W^a}}\left[\tilde{M}_\alpha^{(n+1)} \circ \mathscr{S}_{[a,b]}\right],\end{aligned} \tag{2.22}$$

where the last expectation is taken with respect to the supersymmetric sigma model on the graph $\tilde{G}_{n+1}$ with the rescaled weights $a_i a_j W_{ij}^{(n+1)}$. We calculate

$$\begin{aligned}\tilde{M}_\alpha^{(n+1)} \circ \mathscr{S}_{[a,b]} =& \exp\left(\left\langle \alpha^{(n+1)}, e^{u+\log a}(1 + i(s - e^{-u-\log a} b))\right\rangle\right) \\ =&e^{\left\langle a\alpha^{(n+1)}, e^u(1+is)\right\rangle} e^{-\left\langle \alpha^{(n+1)}, ib\right\rangle}.\end{aligned} \tag{2.23}$$

Note that $\left\langle \alpha^{(n+1)}, ib \right\rangle$ does not depend on u or s. Consequently, inserting the last expression into (2.22), we obtain

$$\text{lhs}(2.21) =\mathcal{L}_{n+1}^W(a,b) e^{-\left\langle \alpha^{(n+1)}, ib\right\rangle} \mathbb{E}_{\mu_{n+1}^{W^a}}\left[e^{\left\langle a\alpha^{(n+1)}, e^u(1+is)\right\rangle}\right]. \tag{2.24}$$

By Corollary 5.3 in Disertori et al. (2017),

$$\mathbb{E}_{\mu_{n+1}^{W^a}}\left[e^{\left\langle a\alpha^{(n+1)}, e^u(1+is)\right\rangle}\right] =e^{\left\langle a\alpha^{(n+1)}, 1_{\tilde{V}}\right\rangle} = e^{\left\langle \alpha^{(n+1)}, a\right\rangle}. \tag{2.25}$$

We conclude

$$\text{lhs}(2.21) = \mathcal{L}_{n+1}^W(a,b) e^{\left\langle \alpha^{(n+1)}, a - ib\right\rangle}. \tag{2.26}$$

The right-hand side of (2.21) can be obtained from the last expression by replacing $n+1$ by n. Thus, the claim (2.21) can be written as follows

$$\mathcal{L}_{n+1}^W(a,b) e^{\left\langle \alpha^{(n+1)}, a-ib\right\rangle} =\mathcal{L}_n^W(a_{V_n}, b_{V_n}) e^{\left\langle \alpha^{(n)}, a-ib\right\rangle}. \tag{2.27}$$

By Lemma 1.2, $\mathcal{L}_n^W(a_{V_n}, b_{V_n}) = \mathcal{L}_{n+1}^W(a,b)$. Furthermore, using $[a_{\delta_{n+1}}, b_{\delta_{n+1}}] = [1,0]$, we obtain

$$\begin{aligned}\left\langle \alpha^{(n+1)}, a - ib \right\rangle &= \sum_{j \in V_{n+1}} \alpha_j(a_j - ib_j) + \alpha_{\delta_{n+1}}^{(n+1)} = \sum_{j \in V_{n+1}} \alpha_j(a_j - ib_j) + \sum_{j \in V_\infty \setminus V_{n+1}} \alpha_j \\ &= \sum_{j \in V_n} \alpha_j(a_j - ib_j) + \sum_{j \in V_\infty \setminus V_n} \alpha_j = \left\langle \alpha^{(n)}, a - ib \right\rangle.\end{aligned} \tag{2.28}$$

This shows that (2.27) holds and finishes the proof of the martingale property. □

Proof of Corollary 1.4: By Theorem 1.3, $(M_\alpha^{(n)})_{n\in\mathbb{N}}$ is a martingale for all $\alpha \in (-\infty,0]^{(V_\infty)}$. The martingale property is equivalent to

$$\mathbb{E}_{\mu_\infty^W}[M_\alpha^{(n+1)}1_A] = \mathbb{E}_{\mu_\infty^W}[M_\alpha^{(n)}1_A] \tag{2.29}$$

for all $n \in \mathbb{N}_0$ and all events $A \in \mathcal{F}_n$. Taking left-sided derivatives at $\alpha = 0$, we get

$$\begin{aligned}
&\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M_\alpha^{(n)} = \partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} e^{\left\langle \alpha^{(n)}, e^{u^{(n)}}(1+is^{(n)})\right\rangle} = M_{j_1,\ldots,j_k}^{(n)} M_\alpha^{(n)},\\
&\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M_\alpha^{(n)}|_{\alpha=0} = M_{j_1,\ldots,j_k}^{(n)}.
\end{aligned} \tag{2.30}$$

Since $|\partial_{\alpha_{j_1}} \ldots \partial_{\alpha_{j_k}} M_\alpha^{(n)}| \le |M_{j_1,\ldots,j_k}^{(n)}|$ for all $\alpha \in (-\infty,0]^{(V_\infty)}$, we can interchange expectation and differentiation at $\alpha = 0$ in (2.29). This yields the martingale property for $M_{j_1,\ldots,j_k}^{(n)}$. □

The following are special cases of Corollary 1.4.

- Since $M_j^{(n)} = e^{u_j^{(n)}}\left(1 + is_j^{(n)}\right)$, we know that
$$\left(s_j^{(n)} e^{u_j^{(n)}}\right)_{n\in\mathbb{N}} \tag{2.31}$$
is a martingale.
- One has $M_{j,l}^{(n)} = e^{u_j^{(n)}+u_l^{(n)}}\left(1 - s_j^{(n)}s_l^{(n)} + i\left(s_j^{(n)} + s_l^{(n)}\right)\right)$. Hence,
$$\left(e^{u_j^{(n)}+u_l^{(n)}}\left(1 - s_j^{(n)}s_l^{(n)}\right)\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(e^{u_j^{(n)}+u_l^{(n)}}\left(s_j^{(n)} + s_l^{(n)}\right)\right)_{n\in\mathbb{N}} \tag{2.32}$$
are martingales. For $j = l$, this yields the martingales
$$\left(e^{2u_j^{(n)}}\left(1 - \left(s_j^{(n)}\right)^2\right)\right)_{n\in\mathbb{N}} \quad \text{and} \quad \left(2s_j^{(n)} e^{2u_j^{(n)}}\right)_{n\in\mathbb{N}}. \tag{2.33}$$
- One has
$$\begin{aligned}
M_{j,l,m}^{(n)} =& e^{u_j^{(n)}+u_l^{(n)}+u_m^{(n)}}\left(1 - s_j^{(n)}s_l^{(n)} - s_j^{(n)}s_m^{(n)} - s_l^{(n)}s_m^{(n)}\right.\\
&\left.+ i\left(s_j^{(n)} + s_l^{(n)} + s_m^{(n)} - s_j^{(n)}s_l^{(n)}s_m^{(n)}\right)\right).
\end{aligned} \tag{2.34}$$
Hence, the following are martingales:
$$\left(e^{u_j^{(n)}+u_l^{(n)}+u_m^{(n)}}\left(1 - s_j^{(n)}s_l^{(n)} - s_j^{(n)}s_m^{(n)} - s_l^{(n)}s_m^{(n)}\right)\right)_{n\in\mathbb{N}}, \tag{2.35}$$
$$\left(e^{u_j^{(n)}+u_l^{(n)}+u_m^{(n)}}\left(s_j^{(n)} + s_l^{(n)} + s_m^{(n)} - s_j^{(n)}s_l^{(n)}s_m^{(n)}\right)\right)_{n\in\mathbb{N}}, \tag{2.36}$$
$$\left(e^{3u_j^{(n)}}\left(1 - 3\left(s_j^{(n)}\right)^2\right)\right)_{n\in\mathbb{N}}, \quad \left(e^{3u_j^{(n)}}\left(3s_j^{(n)} - \left(s_j^{(n)}\right)^3\right)\right)_{n\in\mathbb{N}}. \tag{2.37}$$

3. Extension to Grassmann variables

We consider now the full supersymmetric $H^{2|2}$ model, studied in Disertori et al. (2010), including Grassmann variables. We start with some preliminaries in Sections 3.1 and 3.2. In the remaining part, we extend the scaling transformation, the Laplace transform, and the martingales introduced in the previous sections to include Grassmann variables.

3.1. *Grassmann algebras.* Let $\mathcal{V}$ be a finite dimensional $\mathbb{R}$-vector space. Let

$$\Lambda\mathcal{V} := \bigoplus_{n=0}^{\dim\mathcal{V}} \Lambda^n\mathcal{V}, \quad \Lambda\mathcal{V}_{\text{even}} := \bigoplus_{\substack{0\le n\le\dim\mathcal{V}\\ n \text{ even}}} \Lambda^n\mathcal{V}, \quad \Lambda\mathcal{V}_{\text{odd}} := \bigoplus_{\substack{0\le n\le\dim\mathcal{V}\\ n \text{ odd}}} \Lambda^n\mathcal{V} \tag{3.1}$$

be the Grassmann algebra generated by it, its even and its odd subspace, respectively. In particular, $\mathbb{R} = \Lambda^0\mathcal{V} \subseteq \Lambda\mathcal{V}$ and $\mathcal{V} = \Lambda^1\mathcal{V} \subseteq \Lambda\mathcal{V}$. The Grassmann product is bilinear and associative. Moreover, for all $w, w' \in \Lambda\mathcal{V}_{\text{odd}}$ it is anticommutative: $ww' = -w'w$. In particular, $w^2 = 0$. Let body : $\Lambda\mathcal{V} \to \Lambda^0\mathcal{V} = \mathbb{R}$ be the projection to the 0th component and soul : $\Lambda\mathcal{V} \to \bigoplus_{n=1}^{\dim\mathcal{V}} \Lambda^n\mathcal{V}$, $\text{soul}(w) = w - \text{body}(w)$, denote the projection to the nilpotent part. The subset of positive even elements is defined by

$$\Lambda\mathcal{V}^+_{\text{even}} = \{a \in \Lambda\mathcal{V}_{\text{even}} : \text{body}(a) > 0\}. \tag{3.2}$$

As a generalization of (A.1), for $a \in \Lambda\mathcal{V}^+_{\text{even}}, b \in \Lambda\mathcal{V}_{\text{even}}, \overline{w}, w \in \Lambda\mathcal{V}_{\text{odd}}$, we set

$$[a, b, \overline{w}, w] := \begin{pmatrix} a & b & \overline{w} & w \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.3}$$

The set of matrices, cf. (A.2),

$$\boldsymbol{\mathcal{G}}(\mathcal{V}) := \{[a, b, \overline{w}, w] : a \in \Lambda\mathcal{V}^+_{\text{even}}, b \in \Lambda\mathcal{V}_{\text{even}}, \overline{w}, w \in \Lambda\mathcal{V}_{\text{odd}}\} \tag{3.4}$$

endowed with matrix multiplication forms a group, non-Abelian except in trivial cases, with the neutral element $[1, 0, 0, 0]$. In other words,

$$[a, b, \overline{w}, w] \cdot [a', b', \overline{w}', w'] = [aa', b + ab', \overline{w} + a\overline{w}', w + aw'], \tag{3.5}$$

$$[a, b, \overline{w}, w]^{-1} = [a^{-1}, -ba^{-1}, -\overline{w}a^{-1}, -wa^{-1}]; \tag{3.6}$$

cf. (A.3) and (A.4). Note that a^{-1} is well-defined because $\text{body}(a) > 0$.

We take again a finite graph $\tilde{G} = (\tilde{V}, \tilde{E})$ with $\tilde{V} = V \cup \{\delta\}$ as in Subsection 1.1. We define the cartesian power of the group $\boldsymbol{\mathcal{G}}(\mathcal{V})$ with one component pinned to the neutral element:

$$\begin{aligned} &\boldsymbol{\mathcal{G}}(\mathcal{V})_V := \\ &\left\{[a, b, \overline{w}, w] := ([a_i, b_i, \overline{w}_i, w_i])_{i\in\tilde{V}} \in \boldsymbol{\mathcal{G}}(\mathcal{V})^{\tilde{V}} : \; [a_\delta, b_\delta, \overline{w}_\delta, w_\delta] = [1, 0, 0, 0]\right\}. \end{aligned} \tag{3.7}$$

3.2. *Superfunctions and superexpectation.* Let

$$\mathcal{A}(\mathcal{V}) = \mathcal{A}_V(\mathcal{V}) = C^\infty(\Omega_V, \Lambda\mathcal{V}) = C^\infty(\Omega_V, \mathbb{R}) \otimes \Lambda\mathcal{V} \tag{3.8}$$

be the Grassmann algebra over $\mathcal{V}$ with coefficients being smooth real-valued functions $f \in C^\infty(\Omega_V, \mathbb{R})$, $(u, s) \mapsto f(u, s)$. Elements of $\mathcal{A}(\mathcal{V})$ are called superfunctions.

Assume that the vector space $\mathcal{V}$ has a basis $(\overline{\psi}_i, \psi_i)_{i\in V}$. Moreover, we set

$$\overline{\psi}_\delta = \psi_\delta = 0. \tag{3.9}$$

Then, $\overline{\psi}_i, \psi_i \in \mathcal{V} \subseteq \Lambda\mathcal{V}_{\text{odd}}$, $i \in \tilde{V}$, implies $\psi_i\overline{\psi}_j = -\overline{\psi}_j\psi_i$, $\psi_i\psi_j = -\psi_j\psi_i$, and $\overline{\psi}_i\overline{\psi}_j = -\overline{\psi}_j\overline{\psi}_i$ for all $i, j \in \tilde{V}$. To describe a superfunction in $\mathcal{A}(\mathcal{V})$, the following abbreviations are useful:

$$\mathcal{I}_V = \{(i_1, \ldots, i_n) \in V^n : n \in \mathbb{N}_0, i_1 < \ldots < i_n\} \tag{3.10}$$

with respect to some fixed linear order $<$ of the vertex set V. For $I = (i_1, \ldots, i_n) \in \mathcal{I}_V$, we set

$$\psi_I = \psi_{i_1} \cdots \psi_{i_n} \tag{3.11}$$

and similarly for $\overline{\psi}_I$. By convention, $\overline{\psi}_\emptyset = \psi_\emptyset = 1$. Thus, a superfunction $f \in \mathcal{A}(\mathcal{V})$ can be uniquely written as

$$f(u, s, \overline{\psi}, \psi) = \sum_{I,J \in \mathcal{I}_V} f_{IJ}(u, s)\overline{\psi}_I \psi_J \tag{3.12}$$

with coefficients $f_{IJ} \in C^\infty(\Omega_V, \mathbb{R})$. Here $f_{\emptyset\emptyset}$ is the body of f and $f - f_{\emptyset\emptyset}$ its nilpotent part. An element $f \in \mathcal{A}(\mathcal{V})$ is even if $f_{IJ} = 0$ whenever $|I| + |J|$ is odd; f is odd if $f_{IJ} = 0$ whenever $|I| + |J|$ is even. Let $\mathcal{A}(\mathcal{V})_{\text{even}} = C^\infty(\Omega_V, \Lambda\mathcal{V}_{\text{even}})$ and $\mathcal{A}(\mathcal{V})_{\text{odd}} = C^\infty(\Omega_V, \Lambda\mathcal{V}_{\text{odd}})$ denote the set of even and odd elements of $\mathcal{A}(\mathcal{V})$, respectively, and let $\mathcal{A}(\mathcal{V})^+_{\text{even}} = \{f \in \mathcal{A}(\mathcal{V})_{\text{even}} : \text{body}(f) > 0\}$. Smooth functions (like exp) of elements in $\mathcal{A}(\mathcal{V})_{\text{even}}$ are understood as power series in the nilpotent part.

In analogy to the parameter dependent W^a in formula (1.10) we will consider a further generalization of the supersymmetric sigma model $H^{2|2}$ from Disertori et al. (2010) involving parameters that depend on Grassmann variables. Our parameters belong to another Grassmann algebra $\Lambda\mathcal{V}'$ with another finite-dimensional $\mathbb{R}$-vector space $\mathcal{V}'$. Both vector spaces $\mathcal{V}$ and $\mathcal{V}'$ are viewed as subspaces of their direct sum $\mathcal{V}'' = \mathcal{V} \oplus \mathcal{V}'$. The corresponding Grassmann algebras are related by $\Lambda\mathcal{V}'' = \Lambda\mathcal{V} \otimes_{\text{a}} \Lambda\mathcal{V}'$, where the subscript "a" means that the Grassmann product is extended to be anticommuting on odd elements. In particular, $\Lambda\mathcal{V} = \Lambda\mathcal{V} \otimes \mathbb{R} \subseteq \Lambda\mathcal{V}''$ and $\Lambda\mathcal{V}' = \mathbb{R} \otimes \Lambda\mathcal{V}' \subseteq \Lambda\mathcal{V}''$.

We will consider superfunctions $f \in \mathcal{A}(\mathcal{V}'')$. Each such function can be represented as in (3.12) with coefficients $f_{IJ} \in \mathcal{A}(\mathcal{V}')$. In the following, we consider coupling constants $W_{ij} \in \Lambda\mathcal{V}'^+_{\text{even}}$ for all $(i \sim j) \in \tilde{E}$ and $W_{ij} = 0$ whenever $(i \sim j) \notin \tilde{E}$. We define the superdensity $\boldsymbol{\rho}^W \in \mathcal{A}(\mathcal{V}'')^+_{\text{even}}$ by

$$\begin{aligned}\boldsymbol{\rho}^W(u, s, \overline{\psi}, \psi) =& e^{-\frac{1}{2}\langle s, A^W(u)s\rangle} e^{-\langle \overline{\psi}, A^W(u)\psi\rangle} e^{-\frac{1}{2}\langle e^{-u}_{\tilde{V}}, A^W(u)e^{-u}_{\tilde{V}}\rangle} \\ =& \frac{e^{-\langle \overline{\psi}, A^W(u)\psi\rangle}}{\det A^W_{VV}(u)} \rho^W(u, s)\end{aligned} \tag{3.13}$$

with the matrix $A^W(u) \in \mathbb{R}^{\tilde{V}\times\tilde{V}}$ defined in (1.2) and the density ρ^W defined in (1.3). Note that since $\text{body}(W_{ij}) > 0$ one has $\text{body}(\det A^W_{VV}(u)) > 0$. As Lemma 3.1 below shows, ρ^W is the marginal of $\boldsymbol{\rho}^W$. Therefore we use the same symbol writing the supersymmetric variant with the corresponding bold symbol. This convention will also be used below for other quantities like ζ, μ^W, and $\mathcal{L}^W$. In the following, we use the Grassmann "derivative" ∂_η with respect to any Grassmann variable η. It is defined by

$$\partial_\eta(\eta\phi_1 + \phi_2) = \phi_1 \tag{3.14}$$

for any superfunctions ϕ_1 and ϕ_2 that do not contain η. In particular, it fulfills $\partial_\eta\eta = 1$ and the anticommuting product rule $\partial_\eta(\phi_1\phi_2) = (\partial_\eta\phi_1)\phi_2 + (-1)^\sigma\phi_1\partial_\eta\phi_2$, where $\sigma = 0$ if ϕ_1 is even and $\sigma = 1$ if ϕ_1 is odd. Grassmann derivatives anticommute with each other. Let

$$\boldsymbol{d\zeta}_V = \boldsymbol{d\zeta}_V[u, s, \overline{\psi}, \psi] := \prod_{i\in V} \frac{1}{2\pi}\zeta(du_i ds_i)\partial_{\overline{\psi}_i}\partial_{\psi_i} = \prod_{i\in V} \frac{e^{-u_i}}{2\pi} du_i ds_i \partial_{\overline{\psi}_i}\partial_{\psi_i} \quad (3.15)$$

be the supersymmetric reference measure, where we suppress again the Dirac measure $\delta_{(0,0)}(du_\delta\, ds_\delta)$ in the notation. With these notions the supersymmetric sigma model is given by

$$\boldsymbol{\mu}^W(du\, ds\, \partial_{\overline{\psi}}\, \partial_\psi) := \boldsymbol{d\zeta}_V[u, s, \overline{\psi}, \psi] \circ \boldsymbol{\rho}^W(u, s, \overline{\psi}, \psi), \quad (3.16)$$

where the symbol $\circ$ means that the partial derivatives $\partial_{\overline{\psi}}$ and ∂_ψ act not only on the superdensity $\boldsymbol{\rho}^W(u, s, \overline{\psi}, \psi)$, but also on the test function as follows:

$$\int \boldsymbol{d\mu}^W f = \int_{\Omega_V} \boldsymbol{d\zeta}_V(\boldsymbol{\rho}^W f) \quad (3.17)$$

for any $f \in \mathcal{A}(\mathcal{V}'')$ for which the integral is defined. In particular, it is also well-defined for the constant function $f = 1$ because of the fast decay of the functions body$[\exp(-\frac{1}{2}\left\langle s, A^W(u)s\right\rangle)]$ and body$[\exp(-W_{ij}\cosh(u_i - u_j)]$, cf. (2.7). Note that the superintegral $\int \boldsymbol{d\mu}^W f$ with integrable arguments $f \in \mathcal{A}(\mathcal{V}'')$ takes values in $\Lambda\mathcal{V}'$.

Lemma 3.1. *The probability measure μ^W defined in (1.6) is the marginal of the supermeasure $\boldsymbol{\mu}^W$ defined in (3.16) in the following sense. In the special case when the weights W_{ij} are real-valued and the superfunction f is an ordinary function $f = f(u, s)$, i.e. does not depend on any Grassmann variables, we have the real-valued integral*

$$\int \boldsymbol{d\mu}^W f = \int d\mu^W f. \quad (3.18)$$

Proof: Since f is an ordinary function, the Grassmann part in $\int \boldsymbol{d\mu}^W f$ is reduced to

$$\prod_{i\in V} \partial_{\overline{\psi}_i}\partial_{\psi_i} e^{-\left\langle\overline{\psi}, A^W(u)\psi\right\rangle} = \det A^W_{VV}(u). \quad (3.19)$$

Therefore, the definition (3.13) of $\boldsymbol{\rho}^W$ yields

$$\prod_{i\in V} \partial_{\overline{\psi}_i}\partial_{\psi_i}\boldsymbol{\rho}^W(u, s, \overline{\psi}, \psi) = \rho^W(u, s). \quad (3.20)$$

The result follows. □

3.3. *Super scaling transformation.* We generalize now the definition (A.11) of the scaling transformation $\mathscr{S}_{[a,b]} : \Omega_V \to \Omega_V$ to the present setup involving Grassmann parameters. Take a superparameter $[a, b, \overline{\chi}, \chi] \in \boldsymbol{\mathcal{G}}(\mathcal{V}')_V$; recall that $[a_\delta, b_\delta, \overline{\chi}_\delta, \chi_\delta] = [1, 0, 0, 0]$ by (3.7). In order to find an analogue to equation (1.17), we consider a generalization of the pull-back

$$\mathscr{S}^*_{[a,b]} f := f \circ \mathscr{S}_{[a,b]}, \quad f : \Omega_V \to \mathbb{R} \quad (3.21)$$

to a supertransformation $\mathscr{S}^*_{[a,b,\overline{\chi},\chi]} : \mathcal{A}(\mathcal{V}'') \to \mathcal{A}(\mathcal{V}'')$ defined as follows. Take a general element

$$f(u,s,\overline{\psi},\psi) = \sum_{I,J\in\mathcal{I}_V} f_{IJ}(u,s)\overline{\psi}_I\psi_J \in \mathcal{A}(\mathcal{V}'') \tag{3.22}$$

with coefficients $f_{IJ} \in \mathcal{A}(\mathcal{V}')$. In the following, for any even u', s', we interpret $f_{IJ}(u', s')$ again as power series in the nilpotent part of u' and s'. We set

$$(\mathscr{S}^*_{[a,b,\overline{\chi},\chi]}f)(u,s,\overline{\psi},\psi) = \sum_{I,J\in\mathcal{I}_V} f_{IJ}(u',s')\overline{\psi}'_I\psi'_J \in \mathcal{A}(\mathcal{V}''), \tag{3.23}$$

where the expressions for $u' = u'(u), s' = s'(u,s), \overline{\psi}' = \overline{\psi}'(u,\overline{\psi}), \psi' = \psi'(u,\psi)$ are given by the following formula, to be read componentwise

$$[e^{-u'}, s', \overline{\psi}', \psi'] = [e^{-u}, s, \overline{\psi}, \psi] \cdot [a, b, \overline{\chi}, \chi]^{-1}. \tag{3.24}$$

This means that the explicit expressions for u', s', $\overline{\psi}'$, and ψ' are given by

$$\begin{aligned} u'_i &= u_i + \log a_i, \quad s'_i = s_i - e^{-u_i} b_i a_i^{-1}, \\ \overline{\psi}'_i &= \overline{\psi}_i - e^{-u_i}\overline{\chi}_i a_i^{-1}, \quad \psi'_i = \psi_i - e^{-u_i}\chi_i a_i^{-1} \end{aligned} \tag{3.25}$$

for all $i \in V$. Note that $[e^{-u'_\delta}, s'_\delta, \overline{\psi}'_\delta, \psi'_\delta] = [1,0,0,0]$, and that u'_i and s'_i are even superfunctions in $\mathcal{A}(\mathcal{V}'')$.

Note that $\mathscr{S}^*_\cdot$ is a group operation, i.e. for all $v, v' \in \mathcal{G}(\mathcal{V})_V$,

$$\mathscr{S}^*_{[1,0,0,0]} = \mathrm{id}, \quad \mathscr{S}^*_{v\cdot v'} = \mathscr{S}^*_v \mathscr{S}^*_{v'}, \quad \mathscr{S}^*_{v^{-1}} = (\mathscr{S}^*_v)^{-1}. \tag{3.26}$$

We will need the following transformation formula for the supermeasure $d\boldsymbol{\zeta}_V$ with respect to $\mathscr{S}^*_\cdot$.

Lemma 3.2. *For $v = [a,b,\overline{\chi},\chi] \in \mathcal{G}(\mathcal{V}')_V$ and for any compactly supported (or sufficiently fast decaying) test superfunction $f \in \mathcal{A}(\mathcal{V}'')$, one has*

$$\int d\boldsymbol{\zeta}_V \mathscr{S}^*_v f = \prod_{j\in V} a_j \int d\boldsymbol{\zeta}_V f. \tag{3.27}$$

Proof: Using $(\mathscr{S}^*_v)^{-1}(e^{-u_i}) = e^{-(u_i - \log a_i)}$ and using the supertransformation formula described in Lemma C.1 in Appendix C, we calculate

$$\begin{aligned} \int d\boldsymbol{\zeta}_V \mathscr{S}^*_v f =& (2\pi)^{-|V|} \int \prod_{i\in V} du_i ds_i \partial_{\overline{\psi}_i}\partial_{\psi_i} \left((\mathscr{S}^*_v f)(u,s,\overline{\psi},\psi) \prod_{i\in V} e^{-u_i} \right) \\ =& (2\pi)^{-|V|} \int \prod_{i\in V} du_i ds_i \partial_{\overline{\psi}_i}\partial_{\psi_i} \mathscr{S}^*_v \left(f(u,s,\overline{\psi},\psi) \prod_{i\in V} e^{-(u_i - \log a_i)} \right) \\ =& (2\pi)^{-|V|} \int \prod_{i\in V} du_i ds_i \partial_{\overline{\psi}_i}\partial_{\psi_i} f(u,s,\overline{\psi},\psi) \prod_{i\in V} e^{-(u_i - \log a_i)}. \end{aligned} \tag{3.28}$$

The claim follows. □

3.4. *Grassmann-Laplace transform.* In analogy to the definition (1.12) of $\theta^{V,W}$, we define odd superfunctions $\overline{\phi}^{V,W}(u,\overline{\psi})$ and $\phi^{V,W}(u,\psi)$ by

$$\overline{\phi}^{V,W}(u,\overline{\psi}) = e^{-u}_{VV} A^W_{VV}(u)\overline{\psi}_V, \quad \phi^{V,W}(u,\psi) = e^{-u}_{VV} A^W_{VV}(u)\psi_V. \tag{3.29}$$

Here, the restriction $\psi_V = (\psi_i)_{i\in V}$ should not be confused with the product ψ_I, $I \in \mathcal{I}_V$, defined in (3.11). Componentwise, we have for $i \in V$

$$\overline{\phi}_i^{V,W}(u,\overline{\psi}) = \sum_{j\in\tilde{V}} W_{ij}e^{u_j}(\overline{\psi}_i - \overline{\psi}_j), \quad \phi_i^{V,W}(u,\psi) = \sum_{j\in\tilde{V}} W_{ij}e^{u_j}(\psi_i - \psi_j), \tag{3.30}$$

cf. (1.13). As for β and θ, we will drop the dependence on V, W, or both if there is no risk of confusion.

Our goal is to derive a generalization of Theorem 1.1 including Grassmann variables. In the following, we abbreviate for $[a,b,\overline{\chi},\chi] \in \boldsymbol{\mathcal{G}}(\mathcal{V}')_V$

$$\begin{aligned} \varpi^V =& \varpi^{V,W} = (\beta^V, \theta^V, \overline{\phi}^V, \phi^V), \\ \pi^V_{[a,b,\overline{\chi},\chi]} =& (a^2+b^2+2\overline{\chi}\chi - 1, b, \overline{\chi}, \chi)_V, \end{aligned} \tag{3.31}$$

which fulfill $\varpi^V, \pi^V_{[a,b,\overline{\chi},\chi]} \in (\mathcal{A}(\mathcal{V}'')_{\mathrm{even}} \times \mathcal{A}(\mathcal{V}'')_{\mathrm{even}} \times \mathcal{A}(\mathcal{V}'')_{\mathrm{odd}} \times \mathcal{A}(\mathcal{V}'')_{\mathrm{odd}})^V$. We use the following generalization of the Euclidean scalar product:

$$\begin{aligned} &\left\langle \pi^V_{[a,b,\overline{\chi},\chi]}, \varpi^V \right\rangle \\ =& \langle (a^2+b^2+2\overline{\chi}\chi-1)_V, \beta^W\rangle + \langle b_V, \theta^W\rangle + \langle \overline{\chi}_V, \phi^W\rangle + \langle \overline{\phi}^W, \chi_V\rangle. \end{aligned} \tag{3.32}$$

Note the reversed order of factors in the last product, which causes a sign change due to anticommutativity.

Theorem 3.3. *For $[a,b,\overline{\chi},\chi] \in \boldsymbol{\mathcal{G}}(\mathcal{V}')_V$, the joint Grassmann-Laplace transform of β^W, θ^W, ϕ^W, and $\overline{\phi}^W$ is well-defined and given by*

$$\int \boldsymbol{d\mu}^W e^{-\left\langle \pi^V_{[a,b,\overline{\chi},\chi]}, \varpi^V\right\rangle} = \boldsymbol{\mathcal{L}}^W(a,b,\overline{\chi},\chi) \tag{3.33}$$

with the constant

$$\boldsymbol{\mathcal{L}}^W(a,b,\overline{\chi},\chi) = \prod_{(i\sim j)\in\tilde{E}} e^{-W_{ij}(a_ia_j+b_ib_j+\overline{\chi}_i\chi_j+\overline{\chi}_j\chi_i-1)} \cdot \prod_{j\in V}\frac{1}{a_j} \in \Lambda\mathcal{V}'_{\mathrm{even}}. \tag{3.34}$$

Moreover, for every compactly supported (or not too fast increasing[1] in u and s) test superfunction $f \in \mathcal{A}(\mathcal{V}'')$ it holds

$$\int \boldsymbol{d\mu}^W f e^{-\left\langle \pi^V_{[a,b,\overline{\chi},\chi]}, \varpi^V\right\rangle} = \boldsymbol{\mathcal{L}}^W(a,b,\overline{\chi},\chi) \int \boldsymbol{d\mu}^{W^a} \boldsymbol{\mathscr{S}}^*_{[a,b,\overline{\chi},\chi]} f, \tag{3.35}$$

where $W^a = (W^a_{ij} := a_ia_jW_{ij})_{i,j\in\tilde{V}}$ with $W^a_{ij} \in \Lambda\mathcal{V}'^{+}_{\mathrm{even}}$.

Note that equation (3.35) is the analogue of (1.17). We remark that in the special case $b=0$, $\overline{\chi}=0=\chi$, which was already treated in Theorem 2.1 in Disertori et al. (2017), $a^2+b^2+2\overline{\chi}\chi-1$ just reduces to a^2-1, which was called λ in the citation. If we want the Laplace parameters $a^2+b^2+2\overline{\chi}\chi-1$ and b to be real-valued, this enforces the parameters a *not* to be real-valued but to take values in the even part of a Grassmann algebra. This is why we have to allow Grassmann algebra-valued weights $W^a_{ij} \in \Lambda\mathcal{V}'_{\mathrm{even}}$ rather than only real-valued weights.

[1] A sufficient condition is given in (3.52), below.

Proof of Theorem 3.3: We abbreviate again $v = [a, b, \overline{\chi}, \chi]$. Using Lemma 3.2, we obtain

$$\int \boldsymbol{d\mu}^{W^a}\, \mathscr{S}_v^* f = \int \boldsymbol{d\zeta}_V\, (\boldsymbol{\rho}^{W^a} \mathscr{S}_v^* f) = \int \boldsymbol{d\zeta}_V\, (\mathscr{S}_v^*((\mathscr{S}_{v^{-1}}^* \boldsymbol{\rho}^{W^a})\, f))$$
$$= \prod_{j\in V} a_j \int \boldsymbol{d\zeta}_V\, ((\mathscr{S}_{v^{-1}}^* \boldsymbol{\rho}^{W^a})\, f). \tag{3.36}$$

The condition given in (3.52) below ensures sufficiently fast decay for the body of the measure $\boldsymbol{d\mu}^{W^a}\, \mathscr{S}_v^* f$ to make the integral well-defined. In particular, this holds for the constant function $f = 1$. Note that

$$(\mathscr{S}_{v^{-1}}^* f)(u, s, \overline{\psi}, \psi) = f(u - \log a, s + e^{-u} b, \overline{\psi} + e^{-u}\overline{\chi}, \psi + e^{-u}\chi). \tag{3.37}$$

By Lemma 2.4, one has $A^{W^a}(u - \log a) = A^W(u)$ for $a = (a_i)_{i\in\tilde{V}} \in (\mathbb{R}_0^+)^{\tilde{V}}$ with $a_\delta = 1$. Since the entries of the matrix $A^W(u)$ are smooth functions of $W_{ij} e^{u_i+u_j}$, this identity remains true if we replace a_i, $i \in V$, by even elements of the Grassmann algebra $\Lambda\mathcal{V}'$ with $\mathrm{body}(a_i) > 0$. Consequently (cf. (2.9)),

$$\mathscr{S}_{v^{-1}}^* A^{W^a} = A^W. \tag{3.38}$$

The definition (3.13) allows us to rewrite $\boldsymbol{\rho}^{W^a}$ as follows:

$$\boldsymbol{\rho}^{W^a}(u, s, \overline{\psi}, \psi) = e^{-\frac{1}{2}\langle s, A^{W^a}(u) s\rangle} e^{-\langle \overline{\psi}, A^{W^a}(u)\psi\rangle} e^{-\frac{1}{2}\langle e_{\tilde{V}}^{-u}, A^{W^a}(u) e_{\tilde{V}}^{-u}\rangle}. \tag{3.39}$$

Using (3.38) and the expression (2.6) for $H_{\tilde{\beta}(u)}^W$, we calculate

$$\mathscr{S}_{v^{-1}}^*(\langle \overline{\psi}, A^{W^a}(u)\psi\rangle) = \langle \overline{\psi} + e^{-u}\overline{\chi}, A^W(u)(\psi + e^{-u}\chi)\rangle$$
$$= \langle \overline{\psi}, A^W(u)\psi\rangle + \left\langle \overline{\phi}^W(u, \overline{\psi}), \chi_V \right\rangle + \langle \overline{\chi}_V, \phi^W(u, \psi)\rangle + \left\langle \overline{\chi}, H_{\tilde{\beta}(u)}^W \chi \right\rangle. \tag{3.40}$$

As in (2.11) and (2.12), we obtain

$$\mathscr{S}_{v^{-1}}^* \left\langle s, A^{W^a}(u) s\right\rangle = \langle s, A^W(u) s\rangle + \left\langle b, H_{\tilde{\beta}(u)}^W b\right\rangle + 2\left\langle b_V, \theta^W(u, s)\right\rangle, \tag{3.41}$$

$$\mathscr{S}_{v^{-1}}^* \left\langle e_{\tilde{V}}^{-u}, A^{W^a}(u) e_{\tilde{V}}^{-u}\right\rangle = \left\langle a, H_{\tilde{\beta}(u)}^W a\right\rangle. \tag{3.42}$$

Combining the above identities and relation (2.7), we find

$$\mathscr{S}_{v^{-1}}^* \boldsymbol{\rho}^{W^a}(u, s, \overline{\psi}, \psi)$$
$$= \boldsymbol{\rho}^W(u, s, \overline{\psi}, \psi) e^{-\frac{1}{2}\left(\left\langle a, H_{\tilde{\beta}(u)}^W a\right\rangle + \left\langle b, H_{\tilde{\beta}(u)}^W b\right\rangle + 2\left\langle \overline{\chi}, H_{\tilde{\beta}(u)}^W \chi\right\rangle - \left\langle 1_{\tilde{V}}, H_{\tilde{\beta}(u)}^W 1_{\tilde{V}}\right\rangle\right)}$$
$$\cdot\, e^{-\langle b_V, \theta^W(u,s)\rangle} e^{-\left\langle \overline{\phi}^W(u, \overline{\psi}), \chi_V\right\rangle - \langle \overline{\chi}_V, \phi^W(u,\psi)\rangle}. \tag{3.43}$$

Using $a_\delta^2 + b_\delta^2 + 2\overline{\chi}_\delta \chi_\delta - 1 = 0$, we rewrite the first exponent in the last expression as follows

$$-\frac{1}{2}\left(\left\langle a, H_{\tilde{\beta}(u)}^W a\right\rangle + \left\langle b, H_{\tilde{\beta}(u)}^W b\right\rangle + 2\left\langle \overline{\chi}, H_{\tilde{\beta}(u)}^W \chi\right\rangle - \left\langle 1_{\tilde{V}}, H_{\tilde{\beta}(u)}^W 1_{\tilde{V}}\right\rangle\right) \tag{3.44}$$
$$= \sum_{(i\sim j)\in\tilde{E}} W_{ij}(a_i a_j + b_i b_j + \overline{\chi}_i \chi_j + \overline{\chi}_j \chi_i - 1) - \sum_{i\in V}(a_i^2 + b_i^2 + 2\overline{\chi}_i \chi_i - 1)\beta_i^W.$$

Substituting this in (3.43) and the result in (3.36), claim (3.35) follows. Formula (3.33) is the special case of (3.35) for f being the constant 1. □

3.5. *Ward identities.* To use symmetries of the supersymmetric sigma model, we consider cartesian coordinates $x = (x_i)_{i\in\tilde V}$, $y = (y_i)_{i\in\tilde V}$, $z = (z_i)_{i\in\tilde V}$, $\xi = (\xi_i)_{i\in\tilde V}$, and $\eta = (\eta_i)_{i\in\tilde V}$ defined by

$$x_i = \sinh u_i - \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}, \quad y_i = s_i e^{u_i}, \quad \xi_i = e^{u_i}\overline{\psi}_i, \quad \eta_i = e^{u_i}\psi_i, \tag{3.45}$$

$$z_i = \sqrt{1 + x_i^2 + y_i^2 + 2\xi_i\eta_i} = \cosh u_i + \left(\frac{1}{2}s_i^2 + \overline{\psi}_i\psi_i\right)e^{u_i}. \tag{3.46}$$

In particular, $x_\delta = y_\delta = \xi_\delta = \eta_\delta = 0$ and $z_\delta = 1$. Let

$$\mathcal{S}_{\rm cart}(x, y, \xi, \eta) = -\sum_{(i\sim j)\in\tilde E} W_{ij}(-1 - x_i x_j - y_i y_j + z_i z_j - \xi_i\eta_j + \eta_i\xi_j) \tag{3.47}$$

and define

$$\int \boldsymbol{d\mu}^W_{\rm cart} f := \int \prod_{i\in V}\frac{dx_i dy_i}{2\pi}\,\partial_{\xi_i}\partial_{\eta_i}\left(\prod_{i\in V}\frac{1}{z_i}\cdot e^{\mathcal{S}_{\rm cart}(x,y,\xi,\eta)} f(x,y,\xi,\eta)\right) \tag{3.48}$$

for any compactly supported or sufficiently fast decaying test function f.

Let $\mathcal{V}_{\rm cart}$ denote the $\mathbb{R}$-vector space with basis $(\xi_i, \eta_i)_{i\in V}$. Let $\mathbb{S}_{\rm susy}(\Omega_V, \xi, \eta)$ denote the space of superfunctions of the form

$$\begin{aligned} f_{\rm cart}:\quad & \Omega_V \to \mathcal{A}(\mathcal{V}_{\rm cart}) \\ & (x, y) \mapsto f_{\rm cart}(x, y, \xi, \eta) = \textstyle\sum_{I,J\in\mathcal{I}_V} f_{IJ}(x, y)\xi_I\eta_J, \end{aligned} \tag{3.49}$$

where the coefficients f_{IJ} are Schwartz functions and

$$\xi_I = \prod_{i\in I}\xi_i, \quad \eta_J = \prod_{j\in J}\eta_j. \tag{3.50}$$

After doing the change of coordinates given in (3.45), we obtain the test function in horospherical coordinates $f_{\rm hor} : \Omega_V \to \mathcal{A}(\mathcal{V})$,

$$\begin{aligned}(u, s) \mapsto & f_{\rm hor}(u, s, \overline{\psi}, \psi) \\ =& f_{\rm cart}\left(x(u, s, \overline{\psi}, \psi), y(u, s, \overline{\psi}, \psi), \xi(u, s, \overline{\psi}, \psi), \eta(u, s, \overline{\psi}, \psi)\right). \end{aligned} \tag{3.51}$$

These notions can be directly extended to superfunctions involving parameters that depend on Grassmann variables by considering $f_{\rm cart}, \mathcal{S}_{\rm cart} : \Omega_V \to \mathcal{A}(\mathcal{V}_{\rm cart}) \otimes_{\rm a} \Lambda\mathcal{V}'$. Lemma 5.1 of Disertori et al. (2017), which is based on Disertori et al. (2010), implies that for any superfunction $f_{\rm cart}(x, y, \xi, \eta)$ with the property

$$e^{\mathcal{S}_{\rm cart}} f_{\rm cart} \in \mathbb{S}_{\rm susy}(\Omega_V, \xi, \eta) \otimes_{\rm a} \Lambda\mathcal{V}', \tag{3.52}$$

one has

$$\int \boldsymbol{d\mu}^W_{\rm cart} f_{\rm cart} = \int \boldsymbol{d\mu}^W f_{\rm hor}, \tag{3.53}$$

where we recall that all components of W are now even elements in the Grassmann algebra with positive body.

Lemma 3.4 (Ward identities)**.** *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function and $\tau = (\tau_i)_{i\in V} \in (\Lambda\mathcal{V}'_{\rm odd})^V$. If $f(\langle\alpha, x + z + iy\rangle + \langle\tau, \xi + i\eta\rangle)e^{\mathcal{S}_{\rm cart}} \in \mathbb{S}_{\rm susy}(\Omega_V, \xi, \eta) \otimes_{\rm a} \Lambda\mathcal{V}'$, then the following identity holds*

$$\int \boldsymbol{d\mu}^W_{\rm cart} f(\langle\alpha, x + z + iy\rangle + \langle\tau, \xi + i\eta\rangle) = f(\langle\alpha, 1\rangle). \tag{3.54}$$

Proof: Let $\varphi \in \mathbb{R}$. We define $\xi^\varphi = (\xi_j^\varphi)_{j\in\tilde{V}}$, $\eta^\varphi = (\eta_j^\varphi)_{j\in\tilde{V}}$ by

$$\begin{pmatrix} \xi_j^\varphi \\ \eta_j^\varphi \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}. \tag{3.55}$$

Note that $\mathcal{S}_{\text{cart}}(x, y, \xi, \eta) = \mathcal{S}_{\text{cart}}(x, y, \xi^\varphi, \eta^\varphi)$. Furthermore, the supertransformation $(x, y, \xi, \eta) \mapsto (x, y, \xi^\varphi, \eta^\varphi)$ has super Jacobian 1 and hence leaves the reference supermeasure $dx\, dy\, \partial_\xi \partial_\eta$ invariant. The assumption $f(\langle\alpha, x + z + iy\rangle + \langle\tau, \xi + i\eta\rangle)e^{\mathcal{S}_{\text{cart}}} \in \mathbb{S}_{\text{susy}}(\Omega_V, \xi, \eta) \otimes_{\text{a}} \Lambda\mathcal{V}'$ assures that all expectations in the following calculations exist and are finite and justifies that we can exchange the order of integration in (3.57), below. It follows

$$\begin{aligned} \text{lhs}(3.54) &= \int \boldsymbol{d\mu}_{\text{cart}}^W f(\langle\alpha, x + z + iy\rangle + \langle\tau, \xi^\varphi + i\eta^\varphi\rangle) \\ &= \int \boldsymbol{d\mu}_{\text{cart}}^W f(\langle\alpha, x + z + iy\rangle + e^{-i\varphi}\langle\tau, \xi + i\eta\rangle). \end{aligned} \tag{3.56}$$

Consequently,

$$\begin{aligned} \text{lhs}(3.54) &= \frac{1}{2\pi}\int_0^{2\pi} \int \boldsymbol{d\mu}_{\text{cart}}^W f(\langle\alpha, x + z + iy\rangle + e^{-i\varphi}\langle\tau, \xi + i\eta\rangle)\, d\varphi \\ &= \int \boldsymbol{d\mu}_{\text{cart}}^W \frac{1}{2\pi}\int_0^{2\pi} f(\langle\alpha, x + z + iy\rangle + e^{-i\varphi}\langle\tau, \xi + i\eta\rangle)\, d\varphi. \end{aligned} \tag{3.57}$$

Note that

$$g(r) := \frac{1}{2\pi}\int_0^{2\pi} f(\langle\alpha, x + z + iy\rangle + e^{-i\varphi} r)\, d\varphi - f(\langle\alpha, x + z + iy\rangle) \tag{3.58}$$

is an analytic superfunction of $r \in \Lambda\mathcal{V}'_{\text{even}}$, which vanishes for all $r \in \mathbb{R}$ by the mean value theorem for holomorphic functions. Consequently, using that $g(r)$ for $r \in \Lambda\mathcal{V}'_{\text{even}}$ is defined as a Taylor series in the nilpotent part of $\mathbb{R}$, we obtain $g(r) = 0$ for all $r \in \Lambda\mathcal{V}'_{\text{even}}$. This yields

$$\text{lhs}(3.54) = \int \boldsymbol{d\mu}_{\text{cart}}^W f(\langle\alpha, x + z + iy\rangle). \tag{3.59}$$

The claim (3.54) follows from Lemma 5.2 of Disertori et al. (2017), which is again based on Disertori et al. (2010). □

Corollary 3.5 (Ward identity for exp)**.** *For all $\alpha \in (-\infty, 0]^{\tilde{V}}$ and $\tau = (\tau_i)_{i\in V} \in (\Lambda\mathcal{V}'_{\text{odd}})^V$, one has*

$$\int \boldsymbol{d\mu}^W e^{\langle\alpha, e^u(1+is)\rangle + \langle\tau, e^u(\overline{\psi}+i\psi)\rangle} = e^{\langle\alpha, 1\rangle}, \tag{3.60}$$

using the abbreviation $e^u(1 + is) = (e^{u_j}(1 + is_j))_{j\in\tilde{V}}$.

Proof: We apply Lemma 3.4 to the function $f = \exp$. Note that since $\text{body}(x_j + z_j) = \text{body}(e^{u_j}) > 0$ and $\alpha_j \leq 0$ the assumption $e^{\langle\alpha, x+z+iy\rangle + \langle\tau, \xi+i\eta\rangle} e^{\mathcal{S}_{\text{cart}}} \in \mathbb{S}_{\text{susy}}(\Omega_V, \xi, \eta) \otimes_{\text{a}} \Lambda\mathcal{V}'$ is satisfied. Using (3.45) and (3.46), we find $x_j + z_j + iy_j = e^{u_j}(1 + is_j)$ and $\xi_j + i\eta_j = e^{u_j}(\overline{\psi}_j + i\psi_j)$ for $j \in \tilde{V}$. This proves the claim. □

3.6. *Susy martingales.* Consider an infinite graph $G_\infty = (V_\infty, E_\infty)$. As described in Section 1.3, we approximate this infinite graph by finite graphs with wired boundary conditions $\tilde{G}_n = (\tilde{V}_n = V_n \cup \{\delta_n\}, \tilde{E}_n)$ with $V_n \uparrow V_\infty$. Let $\mathcal{V}_\infty$ be a vector space with a basis denoted by $(\overline{\psi}_i, \psi_i)_{i\in V_\infty}$. Let $\mathcal{V}_n \subseteq \mathcal{V}_\infty$ be the subspace generated by $(\overline{\psi}_i, \psi_i)_{i\in V_n}$. We set $\overline{\psi}_{\delta_n} = \psi_{\delta_n} = 0$. Let $\pi_n : \Omega_{V_{n+1}} \to \Omega_{V_n}$ be the projection $((u_i, s_i)_{i\in V_{n+1}}, (u_{\delta_{n+1}}, s_{\delta_{n+1}}) = (0,0)) \mapsto ((u_i, s_i)_{i\in V_n}, (u_{\delta_n}, s_{\delta_n}) = (0,0))$. Identifying $f \in \mathcal{A}_{V_n}(\mathcal{V}_n)$ (cf. (3.8)) with $f \circ \pi_n \in \mathcal{A}_{V_{n+1}}(\mathcal{V}_{n+1})$, we view $\mathcal{A}_{V_n}(\mathcal{V}_n)$ as a subset of $\mathcal{A}_{V_{n+1}}(\mathcal{V}_{n+1})$.

In order to have Grassmann parameters available, we consider another vector space $\mathcal{V}'_\infty$ together with a filtration of finite-dimensional subspaces $\mathcal{V}'_1 \subseteq \mathcal{V}'_2 \subseteq \mathcal{V}'_3 \subseteq \ldots$, $\bigcup_{n=1}^\infty \mathcal{V}'_n = \mathcal{V}'_\infty$. For $i, j \in V_\infty$, we take weights $W_{ij} = W_{ji} \in (\Lambda\mathcal{V}'_\infty)_{\text{even}}$ such that $W_{ij} \in (\Lambda\mathcal{V}'_n)^+_{\text{even}}$ whenever $i \sim j$ is an edge in $\tilde{G}_n$ for some n and $W_{ij} = 0$ whenever i and j are not connected by an edge in the infinite graph G_∞. The edges of $\tilde{G}_n$ are given the weights $W^{(n)}_{ij}$ defined as in (1.20) and (1.21). Let $\boldsymbol{\mu}^W_n$ denote the supersymmetric sigma model with Grassmann variables defined in (3.16) for the graph $\tilde{G}_n$ with weights $W^{(n)}_{ij}$.

Let $n \in \mathbb{N}$. Recall the definition (3.31) of ϖ^{V_n} and $\pi^{V_n}_{[a,b,\overline{\chi},\chi]}$ for $[a, b, \overline{\chi}, \chi] \in \mathcal{G}(\mathcal{V}'_n)_{V_n}$. We consider the joint Grassmann-Laplace transform

$$\boldsymbol{\mathcal{L}}^W_n(a, b, \overline{\chi}, \chi) = \int \boldsymbol{d\mu}^W_n \, e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n} \right\rangle}. \tag{3.61}$$

Test functions. Following the discussion above eq. (3.53) we will consider the space $\mathcal{T}_n$ of test functions $f \in \mathcal{A}_{V_n}(\mathcal{V}_n) \otimes_{\text{a}} \Lambda\mathcal{V}'_n$ such that $e^{\mathcal{S}_{\text{cart}}} f_{\text{cart}} \in \mathbb{S}_{\text{susy}}(\Omega_{V_n}, \xi, \eta) \otimes_{\text{a}} \Lambda\mathcal{V}'_n$.

Functions of $\beta, \theta, \overline{\phi}, \phi$. Let $\mathcal{U}_n$ be a vector space with basis $(\overline{\phi}_i, \phi_i)_{i\in V_n}$. In analogy to the definition (3.8) of $\mathcal{A}(\mathcal{V})$, we denote by $\mathcal{B}_{V_n}(\mathcal{U}_n) = C^\infty(\mathbb{R}^{V_n} \times \mathbb{R}^{V_n}, \Lambda\mathcal{U}_n)$ the Grassmann algebra over $\mathcal{U}_n$ where the coefficients are given by smooth real-valued functions $f_{IJ} \in C^\infty(\mathbb{R}^{V_n} \times \mathbb{R}^{V_n}, \mathbb{R})$, $(\beta, \theta) \mapsto f_{IJ}(\beta, \theta)$. If we insert the functions $\beta = \beta^{V_n}(u)$, $\theta = \theta^{V_n}(u, s)$, $\overline{\phi} = \overline{\phi}^{V_n}(u, \overline{\psi})$, and $\phi = \phi^{V_n}(u, \psi)$, cf. formulas (1.14), (1.12), and (3.29), in the representation

$$f(\beta, \theta, \overline{\phi}, \phi) = \sum_{I,J\in\mathcal{I}_{V_n}} f_{IJ}(\beta, \theta)\overline{\phi}_I \phi_J \in \mathcal{B}_{V_n}(\mathcal{U}_n), \tag{3.62}$$

the superfunction in horospherical coordinates can be written as

$$f_{\text{hor}}(u, s, \overline{\psi}, \psi) = f(\varpi^{V_n}(u, s, \overline{\psi}, \psi)) = \sum_{I,J\in\mathcal{I}_{V_n}} \tilde{f}_{IJ}(u, s)\overline{\psi}_I \psi_J. \tag{3.63}$$

Again, these definitions extend directly to functions involving Grassmann-dependent parameters $\mathcal{B}_{V_n}(\mathcal{U}_n) \otimes_{\text{a}} \Lambda\mathcal{V}'_n$.

One may wish to define a susy analogue of infinite volume measures for functions of the real and Grassmann variables $\beta, \theta, \phi, \overline{\phi}$. The next lemma gives an analogue of Kolmogorov consistency. In the same spirit as in formula (1.24) in Mitter and Scoppola (2008) it would allow to define an infinite-volume expectation functional for test superfunctions depending only on finitely many supervariables.

Lemma 3.6 (Consistency).
For $n \in \mathbb{N}$ *and* $[a, b, \overline{\chi}, \chi] \in \mathcal{G}(\mathcal{V}'_{n+1})_{V_{n+1}}$ *with* $[a_i, b_i, \overline{\chi}_i, \chi_i] = [1, 0, 0, 0]$ *for all*

$i \in \tilde{V}_{n+1} \setminus V_n$, one has

$$\boldsymbol{\mathcal{L}}_n^W(a_{V_n}, b_{V_n}, \overline{\chi}_{V_n}, \chi_{V_n}) = \boldsymbol{\mathcal{L}}_{n+1}^W(a, b, \overline{\chi}, \chi). \tag{3.64}$$

Consequently, for any superfunction $f \in \mathcal{B}_{V_n}(\mathcal{U}_n) \otimes_{\mathrm{a}} \Lambda\mathcal{V}_n'$ such that $f_{\mathrm{hor}} \in \mathcal{T}_n$ one has

$$\int \boldsymbol{d\mu}_n^W f(\varpi^{V_n}) = \int \boldsymbol{d\mu}_{n+1}^W f((\varpi^{V_{n+1}})|_{V_n}). \tag{3.65}$$

Informally, this means that the (super-)law of $\varpi^{V_n} = (\beta^{V_n}, \theta^{V_n}, \overline{\phi}^{V_n}, \phi^{V_n})$ with respect to $\boldsymbol{\mu}_n^W$ agrees with the (super-)law of $\varpi^{V_{n+1}}|_{V_n} = (\beta^{V_{n+1}}, \theta^{V_{n+1}}, \overline{\phi}^{V_{n+1}}, \phi^{V_{n+1}})|_{V_n}$ with respect to $\boldsymbol{\mu}_{n+1}^W$.

Proof: Using the expression (3.34) for the Grassmann-Laplace transform, the proof of (3.64) is in complete analogy with the proof of Lemma 1.2, using Theorem 3.3 as the analogue of Theorem 1.1 and replacing expressions of the form $a_i a_j + b_i b_j - 1$ originating from formula (1.16) by expressions $a_i a_j + b_i b_j + \overline{\chi}_i \chi_j + \overline{\chi}_j \chi_i - 1$, appearing in formula (3.34).

To prove (3.65), we consider first the special case $f(\varpi^{V_n}) = e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n} \right\rangle}$. We claim $f_{\mathrm{hor}} \in \mathcal{T}_n$. Indeed note that replacing u in $\beta(u)$ with $u = u(x, y, \xi, \eta)$ we can write (cf. Lemma 2.3)

$$\mathcal{S}_{\mathrm{cart}}(x, y, \xi, \eta) = -\frac{1}{2}\left\langle 1_{\tilde{V}_n}, H_{\tilde{\beta}} 1_{\tilde{V}_n} \right\rangle - \frac{1}{2}\left\langle y, H_{\tilde{\beta}} y \right\rangle - \left\langle \xi, H_{\tilde{\beta}} \eta \right\rangle, \tag{3.66}$$

$$\begin{aligned} -\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n} \right\rangle &= -\mathcal{S}_{\mathrm{cart}} + C_W(a, b, \overline{\chi}, \chi) \qquad (3.67)\\ &\quad - \frac{1}{2}\left\langle a, H_{\tilde{\beta}} a \right\rangle - \frac{1}{2}\left\langle (y+b), H_{\tilde{\beta}}(y+b) \right\rangle - \left\langle (\xi + \overline{\chi}), H_{\tilde{\beta}}(\eta + \chi) \right\rangle \end{aligned}$$

where

$$C_W(a, b, \overline{\chi}, \chi) := \sum_{(i \sim j) \in \tilde{E}_n} W_{ij}\left[1 - a_i a_j - b_i b_j - \overline{\chi}_i \chi_j - \overline{\chi}_j \chi_i\right] \tag{3.68}$$

is a constant in $(\mathcal{V}_n')_{\mathrm{even}}$. Letting $c := \min\{\mathrm{body}(a_j^2) : j \in \tilde{V}_n\} > 0$ we have

$$e^{\mathcal{S}_{\mathrm{cart}}} f_{\mathrm{cart}}(x, y, \xi, \eta) = e^{c\mathcal{S}_{\mathrm{cart}}(x, y+b, \xi+\overline{\chi}, \eta+\chi)} e^{F(x,y,\xi,\eta)} e^{C_W(a,b,\overline{\chi},\chi)} \tag{3.69}$$

where $\mathrm{body}F(x, y, \xi, \eta) \le 0$, and all derivatives of F of any order in x, y, ξ, η are algebraic functions of these variables without singularities. Hence $e^{\mathcal{S}_{\mathrm{cart}}} f_{\mathrm{cart}} \in \mathbb{S}_{\mathrm{susy}}(\Omega_{V_n}, \xi, \eta) \otimes_{\mathrm{a}} \Lambda\mathcal{V}_n'$.

For the special case $f(\varpi^{V_n}) = e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n} \right\rangle}$ claim (3.65) reads

$$\int \boldsymbol{d\mu}_n^W e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n} \right\rangle} = \int \boldsymbol{d\mu}_{n+1}^W e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}|_{V_n} \right\rangle}. \tag{3.70}$$

This formula is just another way of writing equation (3.64). For the remainder of this proof, we consider $c := a^2 + b^2 + 2\overline{\chi}\chi - 1$, b, $\overline{\chi}$, χ rather than a, b, $\overline{\chi}$, χ as our list of independent variables, viewing $a = \sqrt{c - b^2 - 2\overline{\chi}\chi + 1}$ as a function of $(c, b, \overline{\chi}, \chi)$. This makes sense as long as $\mathrm{body}(c - b^2) > -1$. We take all iterated Grassmann derivatives of the form $\prod_{k=1}^m \partial_{\chi_{i_k}} \prod_{\overline{k}=1}^{\overline{m}} \partial_{\overline{\chi}_{\overline{i}_{\overline{k}}}}$ with $i_k, \overline{i}_{\overline{k}} \in V_n$ in equation

(3.70). Afterwards, we set $\chi = 0$ and $\overline{\chi} = 0$. For $I, J \in \mathcal{I}_{V_n}$, we obtain

$$\begin{aligned}
&\int \boldsymbol{d\mu}_n^W \overline{\phi}_I^{V_n} \phi_J^{V_n} e^{-\langle c_{V_n}, \beta^{V_n}\rangle - \langle b_{V_n}, \theta^{V_n}\rangle} \\
=&\int \boldsymbol{d\mu}_{n+1}^W \overline{\phi}_I^{V_{n+1}} \phi_J^{V_{n+1}} e^{-\langle c_{V_n}, \beta^{V_{n+1}}|_{V_n}\rangle - \langle b_{V_n}, \theta^{V_{n+1}}|_{V_n}\rangle} \qquad (3.71)
\end{aligned}$$

for any Grassmann monomial g. Note that the identity (3.71) holds in particular for all real b, c in a neighborhood of the origin.

For a general function assume first the weights W_{ij} take only real values. Then, β and θ take only real values because the integration variables u and s take real values. Hence, using the uniqueness theorem for Laplace transforms and the representation (3.62) of the superfunction f, the claim (3.65) follows under our additional assumption $W_{ij} \in \mathbb{R}$; note that the hypothesis $f \in \mathcal{B}_{V_n}(\mathcal{U}_n) \otimes_{\mathrm{a}} \Lambda\mathcal{V}'_n$ with $f_{\mathrm{hor}} \in \mathcal{T}_n$ provides the necessary integrability. Because both sides of the claim (3.65) are analytic superfunctions in the weights W_{ij}, the claim follows also in the general case. □

We remark that in the above proof, it is essential to allow the scaling parameters a to take values in the even part of a Grassmann algebra rather than taking only real values, because we have written $a = \sqrt{c - b^2 - 2\overline{\chi}\chi + 1}$ with real c and b and Grassmann variables $\overline{\chi}$ and χ.

For $\alpha \in (-\infty, 0]^{(V_\infty)}$ we use again the definition of $\alpha^{(n)}$ given in formula (1.31). On the contrary, given $\tau = (\tau_i)_{i\in V_\infty}$ such that $\tau_i \in (\Lambda\mathcal{V}'_n)_{\mathrm{odd}}$ for all $n \in \mathbb{N}$ and $i \in V_n$, we denote by $\tau^{(n)}$ the restriction of τ to V_n. Note that $\Lambda\mathcal{V}'_n \subseteq \Lambda\mathcal{V}'_{n+1}$.

The following theorem is an extension of the martingale property stated in Theorem 1.3.

Theorem 3.7. *For $n \in \mathbb{N}$, $\alpha \in (-\infty, 0]^{(V_\infty)}$, and $\tau = (\tau_i)_{i\in V_\infty}$ as above, let*

$$M_{\alpha,\tau}^{(n)} = M_{\alpha,\tau}^{(n)}(u, s, \overline{\psi}, \psi) = e^{\langle \alpha^{(n)}, e^u(1+is)\rangle + \langle \tau_{V_n}, e^u(\overline{\psi} + i\psi)\rangle}. \qquad (3.72)$$

For any test superfunction $g \in \mathcal{B}_{V_n}(\mathcal{U}_n) \otimes_{\mathrm{a}} \Lambda\mathcal{V}'_n$ with $g_{\mathrm{hor}} \in \mathcal{T}_n$, one has

$$\int \boldsymbol{d\mu}_{n+1}^W M_{\alpha,\tau}^{(n+1)} g(\varpi^{V_{n+1}}|_{V_n}) = \int \boldsymbol{d\mu}_n^W M_{\alpha,\tau}^{(n)} g(\varpi^{V_n}). \qquad (3.73)$$

Note that in (3.72) we need a definition for α_{δ_n} because $e^{u_{\delta_n}}(1 + is_{\delta_n}) = 1$. In contrast to this, $e^{u_{\delta_n}}(\overline{\psi}_{\delta_n} + i\psi_{\delta_n}) = 0$, hence no definition of τ_{δ_n} is needed.

Proof of Theorem 3.7: The proof is in complete analogy to the proof of Theorem 1.3, with an extended set of variables.

We consider first the special case $g(\varpi^{V_n}) = e^{-\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n}\rangle}$ with $[a, b, \overline{\chi}, \chi] \in \boldsymbol{\mathcal{G}}(\mathcal{V}'_n)_{V_n}$. Note that with this choice $g_{\mathrm{hor}} \in \mathcal{T}_n$. Now, set $[a_i, b_i, \overline{\chi}_i, \chi_i] = [1, 0, 0, 0]$ for $i \in \tilde{V}_{n+1} \setminus V_n$. The fact $\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}|_{V_n}\right\rangle = \left\langle \pi^{V_{n+1}}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}\right\rangle$ and equation (3.35) from Theorem 3.3 yield

$$\begin{aligned}
&\int \boldsymbol{d\mu}_{n+1}^W M_{\alpha,\tau}^{(n+1)} e^{-\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}|_{V_n}\rangle} = \int \boldsymbol{d\mu}_{n+1}^W M_{\alpha,\tau}^{(n+1)} e^{-\langle \pi^{V_{n+1}}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}\rangle} \\
=&\boldsymbol{\mathcal{L}}_{n+1}^W(a, b, \overline{\chi}, \chi) \int \boldsymbol{d\mu}_{n+1}^{W^a} \boldsymbol{\mathscr{S}}^*_{[a,b,\overline{\chi},\chi]} M_{\alpha,\tau}^{(n+1)}. \qquad (3.74)
\end{aligned}$$

The following calculation is analogous to formula (2.23):

$$\begin{aligned}\mathscr{S}^*_{[a,b,\overline{\chi},\chi]} M^{(n+1)}_{\alpha,\tau} =& \exp\left(\left\langle \alpha^{(n+1)}, e^{u+\log a}(1+i(s-e^{-u-\log a}b))\right\rangle\right)\cdot \\ & \exp\left(\left\langle \tau^{(n+1)}, e^{u+\log a}(\overline{\psi}-e^{-u-\log a}\overline{\chi}+i(\psi-e^{-u-\log a}\chi))\right\rangle\right) \\ =& e^{\left\langle a\alpha^{(n+1)}, e^u(1+is)\right\rangle + \left\langle a\tau^{(n+1)}, e^u(\overline{\psi}+i\psi)\right\rangle} e^{-\left\langle \alpha^{(n+1)}, ib\right\rangle - \left\langle \tau^{(n+1)}, \overline{\chi}+i\chi\right\rangle}.\end{aligned} \tag{3.75}$$

Inserting this in (3.74) and using the Ward identity from Corollary 3.5, we obtain the following analog of the calculation from formula (2.24) to (2.26):

$$\begin{aligned}&\int \boldsymbol{d\mu}^W_{n+1} M^{(n+1)}_{\alpha,\tau} e^{-\left\langle \pi^{V_{n+1}}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}\right\rangle} = \boldsymbol{\mathcal{L}}^W_{n+1}(a,b,\overline{\chi},\chi)\cdot \\ &\quad e^{-\left\langle \alpha^{(n+1)}, ib\right\rangle - \left\langle \tau^{(n+1)}, \overline{\chi}+i\chi\right\rangle} \int \boldsymbol{d\mu}^{W^a}_{n+1} e^{\left\langle a\alpha^{(n+1)}, e^u(1+is)\right\rangle + \left\langle a\tau^{(n+1)}, e^u(\overline{\psi}+i\psi)\right\rangle} \\ &= \boldsymbol{\mathcal{L}}^W_{n+1}(a,b,\overline{\chi},\chi) e^{-\left\langle \alpha^{(n+1)}, ib\right\rangle - \left\langle \tau^{(n+1)}, \overline{\chi}+i\chi\right\rangle} e^{\left\langle a\alpha^{(n+1)}, 1\right\rangle} \\ &= \boldsymbol{\mathcal{L}}^W_{n+1}(a,b,\overline{\chi},\chi) e^{\left\langle \alpha^{(n+1)}, a-ib\right\rangle - \left\langle \tau^{(n+1)}, \overline{\chi}+i\chi\right\rangle}.\end{aligned} \tag{3.76}$$

In the same way, replacing $n+1$ by n yields

$$\int \boldsymbol{d\mu}^W_n M^{(n)}_{\alpha,\tau} e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n}\right\rangle} = \boldsymbol{\mathcal{L}}^W_n((a,b,\overline{\chi},\chi)_{V_n}) e^{\left\langle \alpha^{(n)}, a-ib\right\rangle - \left\langle \tau^{(n)}, \overline{\chi}+i\chi\right\rangle}. \tag{3.77}$$

The consistency result from Lemma 3.6 can be written in the form $\boldsymbol{\mathcal{L}}^W_{n+1}(a,b,\overline{\chi},\chi) = \boldsymbol{\mathcal{L}}^W_n((a,b,\overline{\chi},\chi)_{V_n})$. Identity (2.28) states $\left\langle \alpha^{(n+1)}, a-ib\right\rangle = \left\langle \alpha^{(n)}, a-ib\right\rangle$. Finally, using $\overline{\chi}_j = \chi_j = 0$ for all $j \in \tilde{V}_{n+1} \setminus V_n$, we obtain

$$\begin{aligned}\left\langle \tau^{(n+1)}, \overline{\chi}+i\chi\right\rangle &= \sum_{j\in V_{n+1}} \tau^{(n+1)}_j (\overline{\chi}_j + i\chi_j) \\ &= \sum_{j\in V_n} \tau^{(n)}_j (\overline{\chi}_j + i\chi_j) = \left\langle \tau^{(n)}, \overline{\chi}+i\chi\right\rangle.\end{aligned} \tag{3.78}$$

It follows that

$$\int \boldsymbol{d\mu}^W_{n+1} M^{(n+1)}_{\alpha,\tau} e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_{n+1}}|_{V_n}\right\rangle} = \int \boldsymbol{d\mu}^W_n M^{(n)}_{\alpha,\tau} e^{-\left\langle \pi^{V_n}_{[a,b,\overline{\chi},\chi]}, \varpi^{V_n}\right\rangle}. \tag{3.79}$$

Using the same argument as in the proof of Lemma 3.6, replacing the supermeasure $\boldsymbol{d\mu}^W_k$, $k \in \{n, n+1\}$, by $\boldsymbol{d\mu}^W_k M^{(k)}_{\alpha,\tau}$, the claim (3.73) follows for any superfunction $g \in \mathcal{B}_{V_n}(\mathcal{U}_n) \otimes_{\mathrm{a}} \Lambda\mathcal{V}'_n$ with $g_{\mathrm{hor}} \in \mathcal{T}_n$. ☐

Corollary 3.8. *For $n, k, m \in \mathbb{N}$ and $j_1, \ldots, j_k, l_1, \ldots, l_m \in V_{n+1}$, let*

$$M^{(n)}_{j_1,\ldots,j_k,l_1,\ldots,l_m} = \prod_{p=1}^k e^{u^{(n)}_{j_p}}(1+is^{(n)}_{j_p}) \prod_{q=1}^m e^{u^{(n)}_{l_q}}(\overline{\psi}_{l_q} + i\psi^{(n)}_{l_q}). \tag{3.80}$$

For any superfunction $g \in \mathcal{B}_{V_n}(\mathcal{U}_n)$ with $g(\varpi^{V_n}) \in \mathcal{P}_s(n)$, one has

$$\int \boldsymbol{d\mu}^W_{n+1} M^{(n+1)}_{j_1,\ldots,j_k,l_1,\ldots,l_m} g(\varpi^{V_{n+1}}|_{V_n}) = \int \boldsymbol{d\mu}^W_n M^{(n)}_{j_1,\ldots,j_k,l_1,\ldots,l_m} g(\varpi^{V_n}). \tag{3.81}$$

The same holds for the real and imaginary part of $M^{(n)}_{j_1,\ldots,j_k,l_1,\ldots,l_m}$.

Proof: In analogy to Corollary 1.4 the proof follows directly from the Taylor expansion of formula (3.73) with respect to α and τ. ☐

Appendix A. Group structure of scaling

Recall the definition of the set $\mathcal{G}_V$ in (1.7). To describe its group structure it is now convenient to encode any pair $(a,b) \in \mathbb{R}^+ \times \mathbb{R}$ with

$$[a,b] := \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}. \tag{A.1}$$

The set of matrices

$$\mathcal{G} := \{[a,b] : a > 0, b \in \mathbb{R}\} \tag{A.2}$$

endowed with matrix multiplication forms a non-Abelian group. Its group operation can be written in the following form:

$$[a'', b''] = [a,b] \cdot [a', b'] = [aa', b + ab']. \tag{A.3}$$

The group $\mathcal{G}$ has the neutral element $[1,0]$; the inverse is given by

$$[a,b]^{-1} = [1/a, -b/a]. \tag{A.4}$$

We endow $\mathcal{G}$ with the Lebesgue measure in the (a,b)-coordinates $\lambda(da\, db) = da\, db$. We introduce coordinates $(u,s) \in \mathbb{R}^2$ of $\mathcal{G}$ by

$$a = e^{-u} \qquad \text{and} \qquad b = s. \tag{A.5}$$

In these coordinates the Lebesgue measure $da\, db$ takes the form of the measure ζ from formula (1.5):

$$da\, db = \zeta(du\, ds). \tag{A.6}$$

Right operation on $\mathcal{G}$. Note that this measure λ is *not* a Haar measure on $\mathcal{G}$. We define the right operations

$$\mathcal{R}_{v'} : \mathcal{G} \to \mathcal{G}, v \mapsto v'' = v \cdot v' \qquad \text{for } v' \in \mathcal{G}. \tag{A.7}$$

Under $\mathcal{R}_{v'}$, using the notation $v'' = [a'', b''] = [e^{-u''}, s'']$, the measure λ scales as follows:

$$\mathcal{R}_{[a',b']}[\lambda](da''\, db'') = \frac{1}{a'}\, da''\, db'' = \frac{1}{a'}\, \zeta(du''\, ds''). \tag{A.8}$$

Cartesian power of $\mathcal{G}$. With the above identification of $[a,b]$ in terms of 2×2-matrices, the definition (1.7) of $\mathcal{G}_V$ reads as follows:

$$\mathcal{G}_V := \{[a,b] := ([a_i, b_i])_{i \in \tilde{V}} \in \mathcal{G}^{\tilde{V}} :\ [a_\delta, b_\delta] = [1,0]\}. \tag{A.9}$$

In particular, the group operation $\cdot : \mathcal{G}_V \times \mathcal{G}_V \to \mathcal{G}_V$ is understood componentwise. The set $\mathcal{G}_V$ can be identified with the set Ω_V, defined in (1.1), via the componentwise coordinate change to (u,s)-coordinates

$$\iota : \mathcal{G}_V \to \Omega_V, \quad [a,b] \mapsto (-\log a, b). \tag{A.10}$$

$\mathscr{S}$-operation as right operation. Using the identification ι, the $\mathscr{S}$-operation (1.8) can be written as right operation with inverse elements $[a,b] \in \mathcal{G}_V$:

$$\mathscr{S}_{[a,b]} : \Omega_V \to \Omega_V, \quad \mathscr{S}_{[a,b]} = \iota \circ \mathcal{R}_{[a,b]^{-1}} \circ \iota^{-1} = \iota \circ \mathcal{R}_{[1/a,-b/a]} \circ \iota^{-1}. \tag{A.11}$$

Note that $[a_\delta, b_\delta] = [1,0]$ implies $\mathscr{S}_{[a,b]}(u,s) \in \Omega_V$. The map $\mathscr{S} : \mathcal{G}_V \times \Omega_V \to \Omega_V$, $\mathscr{S}([a,b],(u,s)) = \mathscr{S}_{[a,b]}(u,s)$, is a group action. Indeed, for $v_1, v_2, v \in \mathcal{G}_V$ it holds

$$\mathscr{S}_{v_1}(\mathscr{S}_{v_2}(\iota(v))) = \iota((v \cdot v_2^{-1}) \cdot v_1^{-1}) = \iota(v \cdot (v_1 \cdot v_2)^{-1}) = \mathscr{S}_{v_1 \cdot v_2}(\iota(v)). \tag{A.12}$$

Moreover, for the neutral element $[1,0] \in \mathcal{G}_V$ the map $\mathscr{S}_{[1,0]}$ is the identity. Consequently, $\mathscr{S}_{[a,b]}$ is invertible for $[a,b] \in \mathcal{G}_V$ with the inverse $\mathscr{S}_{[a,b]}^{-1} = \mathscr{S}_{[a,b]^{-1}}$.

Appendix B. Alternative proofs

B.1. *Second proof of Lemma 2.2.* We can represent the density ρ^W of the supersymmetric sigma model as follows. Recall the bijection ι introduced in (A.10).

Lemma B.1. *For $(u,s) = \iota(v) \in \Omega_V$ with $v = [a,b] \in \mathcal{G}_V$, the density ρ^W defined in (1.3) can be written as follows:*

$$\rho^W(u,s) = \det A_{VV}^W(u) \exp\left(\sum_{(i\sim j)\in\tilde{E}} \frac{W_{ij}}{2} \det\left(\frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) \right) \tag{B.1}$$

Proof: Let $(u,s) = \iota(v) \in \Omega_V$. It suffices to prove for all $(i \sim j) \in \tilde{E}$

$$-\left[\cosh(u_i - u_j) - 1 + \frac{1}{2}(s_i - s_j)^2 e^{u_i + u_j} \right] = \frac{1}{2} \det\left(\frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right). \tag{B.2}$$

For $i \in \tilde{V}$, $v_i = [e^{-u_i}, s_i] = [a_i, b_i]$, we calculate

$$\frac{v_i v_i^t}{a_i} = e^{u_i} \begin{pmatrix} e^{-u_i} & s_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-u_i} & 0 \\ s_i & 1 \end{pmatrix} = \begin{pmatrix} e^{-u_i} + s_i^2 e^{u_i} & s_i e^{u_i} \\ s_i e^{u_i} & e^{u_i} \end{pmatrix}. \tag{B.3}$$

Consequently, the claim (B.2) follows from

$$\begin{aligned} \det\left(\frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) &= (e^{-u_i} - e^{-u_j} + s_i^2 e^{u_i} - s_j^2 e^{u_j})(e^{u_i} - e^{u_j}) - (s_i e^{u_i} - s_j e^{u_j})^2 \\ &= 2 - 2\cosh(u_i - u_j) - (s_i - s_j)^2 e^{u_i + u_j}. \end{aligned} \tag{B.4}$$

□

To deal with determinants of differences of 2×2-matrices, we need the following elementary lemma, which is motivated by the linear algebra of spinors. Let

$$\varepsilon = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{B.5}$$

Lemma B.2. *For all $v_i = [a_i, b_i], v_j = [a_j, b_j] \in \mathcal{G}$, one has*

$$\det\left(\frac{v_i v_i^t}{a_i} - \frac{v_j v_j^t}{a_j} \right) = 2 - \frac{\|v_i^t \varepsilon v_j\|^2}{a_i a_j}, \tag{B.6}$$

where $\|\cdot\|$ means the euclidean norm of 2×2-matrices.

Proof: The bilinear form $\operatorname{trace}(A\varepsilon B^t\varepsilon)$ on 2×2-matrices $A, B \in \mathbb{R}^{2\times 2}$ is symmetric. Indeed, using $\varepsilon^t = -\varepsilon$,

$$\operatorname{trace}(B\varepsilon A^t\varepsilon) = \operatorname{trace}((B\varepsilon A^t\varepsilon)^t) = \operatorname{trace}(\varepsilon A\varepsilon B^t) = \operatorname{trace}(A\varepsilon B^t\varepsilon). \tag{B.7}$$

The corresponding quadratic form is given by

$$\operatorname{trace}(A\varepsilon A^t\varepsilon) = -2\det A. \tag{B.8}$$

It follows

$$\det(A-B) = \det A + \det B + \operatorname{trace}(A\varepsilon B^t\varepsilon). \tag{B.9}$$

Taking now $A = a_i^{-1}v_iv_i^t$ and $B = a_j^{-1}v_jv_j^t = B^t$, which fulfill $\det A = a_i^{-2}(\det v_i)^2 = 1 = \det B$, we obtain

$$\begin{aligned}\det\left(\frac{v_iv_i^t}{a_i} - \frac{v_jv_j^t}{a_j}\right) =& \det(A-B) = \det A + \det B + \operatorname{trace}(A\varepsilon B^t\varepsilon)\\ =& 2 + \operatorname{trace}(A\varepsilon B\varepsilon) = 2 + \frac{1}{a_ia_j}\operatorname{trace}(v_iv_i^t\varepsilon v_jv_j^t\varepsilon).\end{aligned} \tag{B.10}$$

Using $\varepsilon^t = -\varepsilon$ again, we rewrite the last trace as follows:

$$\operatorname{trace}(v_iv_i^t\varepsilon v_jv_j^t\varepsilon) = \operatorname{trace}(v_i^t\varepsilon v_jv_j^t\varepsilon v_i) = -\operatorname{trace}(v_i^t\varepsilon v_j(v_i^t\varepsilon v_j)^t) = -\|v_i^t\varepsilon v_j\|^2. \tag{B.11}$$

Substituting this into (B.10), the claim (B.6) follows. □

Second proof of Lemma 2.2: We take $v = [a,b]$, $v' = [a',b']$, and $v'' = [a'',b'']$ in $\mathcal{G}_V$ with $v'' = v' \cdot v$ and set $(u,s) = \iota(v')$, $(\tilde{u},\tilde{s}) = \iota(v'')$. By (1.9), we have $\mathscr{S}^{-1}_{[a,b]}(u,s) = \iota(v'')$. Since $A^{W^a}(\tilde{u}) = A^W(u)$ as stated in Lemma 2.4, it follows

$$\det A^{W^a}_{VV}(\tilde{u}) = \det A^W_{VV}(u). \tag{B.12}$$

Using Lemma B.1 and this fact, we obtain

$$\begin{aligned}&\frac{\rho^{W^a}(\mathscr{S}^{-1}_{[a,b]}(u,s))}{\rho^W(u,s)} = \frac{\rho^{W^a}(\iota([a'',b'']))}{\rho^W(\iota([a',b']))} = \\ &\exp\left(\sum_{(i\sim j)\in\tilde{E}}\frac{W_{ij}}{2}\left[a_ia_j\det\left(\frac{v_i''(v_i'')^t}{a_i''} - \frac{v_j''(v_j'')^t}{a_j''}\right) - \det\left(\frac{v_i'(v_i')^t}{a_i'} - \frac{v_j'(v_j')^t}{a_j'}\right)\right]\right)\end{aligned} \tag{B.13}$$

We apply Lemma B.2 to $v' = [a',b']$ and $v'' = [a'',b'']$ as follows, using $a_i'' = a_ia_i'$ for $i \in \tilde{V}$:

$$\begin{aligned}&a_ia_j\det\left(\frac{v_i''(v_i'')^t}{a_i''} - \frac{v_j''(v_j'')^t}{a_j''}\right) - \det\left(\frac{v_i'(v_i')^t}{a_i'} - \frac{v_j'(v_j')^t}{a_j'}\right)\\ =& 2a_ia_j - \frac{a_ia_j}{a_i''a_j''}\|(v_i'')^t\varepsilon v_j''\|^2 - 2 + \frac{\|(v_i')^t\varepsilon v_j'\|^2}{a_i'a_j'}\\ =& 2(a_ia_j - 1) + \frac{1}{a_i'a_j'}\left(\|(v_i')^t\varepsilon v_j'\|^2 - \|(v_i'')^t\varepsilon v_j''\|^2\right).\end{aligned} \tag{B.14}$$

Note that

$$(v_i')^t \varepsilon v_j' = \begin{pmatrix} 0 & -a_i' \\ a_j' & b_j' - b_i' \end{pmatrix} \quad \text{and} \tag{B.15}$$

$$(v_i'')^t \varepsilon v_j'' = \begin{pmatrix} 0 & -a_i'' \\ a_j'' & b_j'' - b_i'' \end{pmatrix} = \begin{pmatrix} 0 & -a_i a_i' \\ a_j a_j' & b_j' - b_i' + a_j' b_j - a_i' b_i \end{pmatrix}. \tag{B.16}$$

We calculate the last parenthesis in (B.14), writing $\langle \cdot, \cdot \rangle$ for the euclidean scalar product of matrices:

$$\|(v_i')^t \varepsilon v_j'\|^2 - \|(v_i'')^t \varepsilon v_j''\|^2 = \left\langle (v_i')^t \varepsilon v_j' + (v_i'')^t \varepsilon v_j'', (v_i')^t \varepsilon v_j' - (v_i'')^t \varepsilon v_j'' \right\rangle \tag{B.17}$$

$$= \left\langle \begin{pmatrix} 0 & -a_i'(1+a_i) \\ a_j'(1+a_j) & -2(b_i'-b_j') - (a_i' b_i - a_j' b_j) \end{pmatrix}, \begin{pmatrix} 0 & -a_i'(1-a_i) \\ a_j'(1-a_j) & a_i' b_i - a_j' b_j \end{pmatrix} \right\rangle$$

$$= -a_i' a_j' \left(\frac{a_i'}{a_j'}(a_i^2 - 1) + \frac{a_j'}{a_i'}(a_j^2 - 1) + 2(b_i' - b_j') \left(\frac{b_i}{a_j'} - \frac{b_j}{a_i'} \right) + a_i' a_j' \left(\frac{b_i}{a_j'} - \frac{b_j}{a_i'} \right)^2 \right).$$

This yields

$$\begin{aligned} \text{l.h.s. in (B.14)} =& 2(a_i a_j + b_i b_j - 1) - \left(\frac{a_i'}{a_j'}(a_i^2 + b_i^2 - 1) + \frac{a_j'}{a_i'}(a_j^2 + b_j^2 - 1) \right) \\ & - 2\frac{b_i}{a_j'}(b_i' - b_j') - 2\frac{b_j}{a_i'}(b_j' - b_i'). \end{aligned} \tag{B.18}$$

Multiplying this with $W_{ij}/2$, summing the result over $(i \sim j) \in \tilde{E}$, and using the symmetry $W_{ij} = W_{ji}$, we obtain

$$\begin{aligned} & \sum_{(i \sim j) \in \tilde{E}} \frac{W_{ij}}{2} \left[a_i a_j \det \left(\frac{v_i''(v_i'')^t}{a_i''} - \frac{v_j''(v_j'')^t}{a_j''} \right) - \det \left(\frac{v_i'(v_i')^t}{a_i'} - \frac{v_j'(v_j')^t}{a_j'} \right) \right] \\ = & \sum_{(i \sim j) \in \tilde{E}} W_{ij} \left[a_i a_j + b_i b_j - 1 \right] - \sum_{i,j \in \tilde{V}} W_{ij} \left[\frac{1}{2} \frac{a_i'}{a_j'}(a_i^2 + b_i^2 - 1) + \frac{b_i}{a_j'}(b_i' - b_j') \right]. \end{aligned} \tag{B.19}$$

Next, we rewrite the definitions (1.14) of β_i^W and (1.13) of θ_i^W, $i \in V$, in the following form, using $[a', b'] = [e^{-u}, s]$:

$$\beta_i^W = \frac{1}{2} \sum_{j \in \tilde{V}} W_{ij} e^{u_j - u_i} = \frac{1}{2} \sum_{j \in \tilde{V}} W_{ij} \frac{a_i'}{a_j'}, \tag{B.20}$$

$$\theta_i^W = \sum_{j \in \tilde{V}} W_{ij} e^{u_j} (s_i - s_j) = \sum_{j \in \tilde{V}} W_{ij} \frac{1}{a_j'} (b_i' - b_j'). \tag{B.21}$$

Since $a_\delta^2 + b_\delta^2 - 1 = 0$, $b_\delta = 0$, we obtain

$$\text{l.h.s. in (B.19)} = \sum_{(i \sim j) \in \tilde{E}} W_{ij} \left[a_i a_j + b_i b_j - 1 \right] - \sum_{i \in V} \left[(a_i^2 + b_i^2 - 1)\beta_i^W + b_i \theta_i^W \right], \tag{B.22}$$

Substituting this into (B.13), the claim (2.3) follows. □

B.2. *Proof of Theorem 1.1 by conditioning.* Our second proof of Theorem 1.1 uses the known transformation behavior of $\mu^{W^a}(du\,ds)$ with respect to $\mathscr{S}_{[a,0]}$ from Disertori et al. (2017) and the fact that conditionally on u, the s-variables are jointly Gaussian. The following lemma describes the conditional distribution of θ^W given β^W.

Lemma B.3. *Conditioned on β^W, the random vector $\theta^W \in \mathbb{R}^V$ is normally distributed with mean 0 and covariance matrix*

$$H^W_{\beta(u)} = e^{-u}_{VV} A^W_{VV}(u) e^{-u}_{VV}. \tag{B.23}$$

Proof: By definition, conditioned on u, the vector s_V is centered Gaussian with covariance matrix A^{-1}, where $A := A^W_{VV}(u)$. Since u is a function of β^W by Lemma 2.3 of Disertori et al. (2017), we have conditioned on β^W that $\theta^W = e^{-u}_{VV} A s_V$ is also centered Gaussian with covariance matrix $(e^{-u}_{VV}A)A^{-1}(e^{-u}_{VV}A)^t = e^{-u}_{VV} A e^{-u}_{VV}$. The representation (B.23) follows from (2.6). □

Proof of Theorem 1.1 by conditioning: To prove (1.17), by the monotone class theorem, it suffices to consider test functions of the form $f(u,s) = g(u)h(s)$ with measurable functions $g, h : \mathbb{R}^{\tilde{V}} \to \mathbb{R}^+_0$. We calculate

$$\mathbb{E}_{\mu^{W^a}}\left[f \circ \mathscr{S}_{[a,b]}\right] = \mathbb{E}_{\mu^{W^a}}\left[g(u+\log a)h(s - e^{-(u+\log a)}b)\right]. \tag{B.24}$$

The behavior of the supersymmetric sigma model μ^{W^a} with rescaled weights with respect to the shift $u \mapsto u + \log a$ in the u variables was studied in Disertori et al. (2017). Using Theorem 3.1 of that paper with $\lambda = a^2 - 1$ yields

$$\begin{aligned}
&\mathbb{E}_{\mu^{W^a}}\left[g(u+\log a)h(s - e^{-(u+\log a)}b)\right] \\
=&\mathcal{L}^W(a,0)^{-1}\mathbb{E}_{\mu^W}\left[g(u)h(s-e^{-u}b)e^{-\left\langle (a^2-1)_V, \beta^W(u)\right\rangle}\right] \\
=&\mathcal{L}^W(a,0)^{-1}\mathbb{E}_{\mu^W}\left[g(u)\mathbb{E}_{\mu^W}[h(s-e^{-u}b)|u]e^{-\left\langle (a^2-1)_V, \beta^W(u)\right\rangle}\right]
\end{aligned} \tag{B.25}$$

with the constant $\mathcal{L}^W(a,0)$ given in (1.16); recall that β^W is a function of u. By the definition of the supersymmetric sigma model, cf. (1.6) and (1.3), conditioned on u the vector s_V is centered Gaussian with covariance matrix $A^W_{VV}(u)^{-1}$ and $s_\delta = 0$. Consequently, abbreviating $c = (2\pi)^{-|V|/2}\sqrt{\det A^W_{VV}(u)}$ and $\sigma(ds) = \delta_0(ds_\delta)\prod_{i\in V} ds_i$, the conditional expectation in (B.25) is μ^W-a.s. given by

$$\begin{aligned}
&\mathbb{E}_{\mu^W}[h(s-e^{-u}b)|u] = c\int_{\mathbb{R}^{\tilde{V}}} h(s-e^{-u}b)e^{-\frac{1}{2}\left\langle s, A^W(u)s\right\rangle}\,\sigma(ds) \\
=&c\int_{\mathbb{R}^{\tilde{V}}} h(s)e^{-\frac{1}{2}\left\langle s+e^{-u}b, A^W(u)(s+e^{-u}b)\right\rangle}\,\sigma(ds) \\
=&c\int_{\mathbb{R}^{\tilde{V}}} h(s)e^{-\frac{1}{2}\left\langle s, A^W(u)s\right\rangle - \frac{1}{2}\left\langle e^{-u}b, A^W(u)e^{-u}b\right\rangle - \left\langle e^{-u}b, A^W(u)s\right\rangle}\,\sigma(ds).
\end{aligned} \tag{B.26}$$

Using $b_\delta = 0$ and (2.6), we obtain

$$\left\langle e^{-u}b, A^W(u)e^{-u}b\right\rangle = \left\langle b_V, H^W_{\beta(u)}b_V\right\rangle = 2\left\langle b_V^2, \beta^W(u)\right\rangle - \sum_{i,j\in V} W_{ij}b_ib_j. \tag{B.27}$$

Similarly, using the definition (1.12) of θ^W, we obtain

$$\left\langle e^{-u}b, A^W(u)s\right\rangle = \left\langle b_V, e^{-u}_{VV}A^W_{VV}(u)s_V\right\rangle = \left\langle b_V, \theta^W(u,s)\right\rangle. \tag{B.28}$$

Inserting (B.27) and (B.28) into (B.26) yields

$$\begin{aligned}
&\mathbb{E}_{\mu^W}[h(s-e^{-u}b)|u] \\
=&\prod_{i,j\in V} e^{\frac{1}{2}W_{ij}b_ib_j}\cdot e^{-\left\langle b_V^2,\beta^W(u)\right\rangle} c\int_{\mathbb{R}^{\tilde V}} h(s)e^{-\frac{1}{2}\left\langle s,A^W(u)s\right\rangle - \left\langle b_V,\theta^W(u,s)\right\rangle}\,\sigma(ds) \\
=&\mathcal{L}^W(a,0)\mathcal{L}^W(a,b)^{-1}\cdot e^{-\left\langle b_V^2,\beta^W(u)\right\rangle}\mathbb{E}_{\mu^W}\left[h(s)e^{-\left\langle b_V,\theta^W(u,s)\right\rangle}\,\middle|\, u\right].
\end{aligned} \tag{B.29}$$

Inserting the above in (B.25) yields the claim (1.17). Equality (1.18) follows from (1.17) applied to the function $f(u,s)=1$. □

Appendix C. Coordinate transformations for superfunctions

We abbreviate $\boldsymbol{x} = (u,s,\overline{\psi},\psi) = (u_i,s_i,\overline{\psi}_i,\psi_i)_{i\in V}$ and $d\boldsymbol{x} = \prod_{i\in V} du_i\, ds_i\, \partial_{\overline{\psi}_i}\partial_{\psi_i}$.

Lemma C.1. *For $v\in\mathcal{G}(\mathcal{V}')_V$ and any compactly supported (or sufficiently fast decaying) superfunction f, one has*

$$\int d\boldsymbol{x}\ (\mathscr{S}_v^* f)(\boldsymbol{x}) = \int d\boldsymbol{x}\ f(\boldsymbol{x}). \tag{C.1}$$

Proof: Consider a supermatrix

$$M = \begin{pmatrix} A & \Sigma \\ \Gamma & B \end{pmatrix} \tag{C.2}$$

where A,B have even entries, Σ,Γ have odd entries, and A and B are invertible. Its superdeterminant is defined by

$$\operatorname{sdet} M = \frac{\det(A-\Sigma B^{-1}\Gamma)}{\det B}. \tag{C.3}$$

It plays an analogous role in Berezin's supertransformation formula as the ordinary determinant plays in the classical transformation formula; cf. Theorem 2.1 in Berezin (1987).

For $v=[a,b,\overline{\chi},\chi]$, the change of coordinates generating $\mathscr{S}_v^*$ is given by

$$\boldsymbol{x}'(\boldsymbol{x}) = (u',s',\overline{\psi}',\psi') = (u+\log a, s-e^{-u}ba^{-1}, \overline{\psi}-e^{-u}\overline{\chi}a^{-1}, \psi-e^{-u}\chi a^{-1}). \tag{C.4}$$

This transformation has the super Jacobi matrix given by

$$\frac{\partial\boldsymbol{x}'}{\partial\boldsymbol{x}} = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial s} & \frac{\partial u'}{\partial\overline{\psi}} & \frac{\partial u'}{\partial\psi} \\ \frac{\partial s'}{\partial u} & \frac{\partial s'}{\partial s} & \frac{\partial s'}{\partial\overline{\psi}} & \frac{\partial s'}{\partial\psi} \\ \frac{\partial\overline{\psi}'}{\partial u} & \frac{\partial\overline{\psi}'}{\partial s} & \frac{\partial\overline{\psi}'}{\partial\overline{\psi}} & \frac{\partial\overline{\psi}'}{\partial\psi} \\ \frac{\partial\psi'}{\partial u} & \frac{\partial\psi'}{\partial s} & \frac{\partial\psi'}{\partial\overline{\psi}} & \frac{\partial\psi'}{\partial\psi} \end{pmatrix} = \begin{pmatrix} A & \mathbf{0} \\ \Gamma & \mathbf{1} \end{pmatrix} \tag{C.5}$$

with

$$A = \begin{pmatrix} 1 & 0 \\ e^{-u}ba^{-1} & 1 \end{pmatrix},\quad \mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},\quad \Gamma = \begin{pmatrix} e^{-u}\overline{\chi}a^{-1} & 0 \\ e^{-u}\chi a^{-1} & 0 \end{pmatrix}. \tag{C.6}$$

Here $e^{-u}ba^{-1}$ is the diagonal matrix with the entries $e^{-u_i}b_i a_i^{-1}$. This super Jacobi matrix has the superdeterminant sdet $\frac{\partial \boldsymbol{x}'}{\partial \boldsymbol{x}} = 1$. Consequently, the inverse supertransformation has the superdeterminant sdet $\frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}'} = 1$, as well. We obtain

$$\int d\boldsymbol{x}\, (\mathscr{S}_v^* f)(\boldsymbol{x}) = \int d\boldsymbol{x}\, f(\boldsymbol{x}'(\boldsymbol{x})) = \int d\boldsymbol{x}'\, f(\boldsymbol{x}')\ \mathrm{sdet}\, \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{x}'} = \int d\boldsymbol{x}'\, f(\boldsymbol{x}'). \tag{C.7}$$

□

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References

R. Bauerschmidt, T. Helmuth and A. Swan. Dynkin isomorphism and Mermin–Wagner theorems for hyperbolic sigma models and recurrence of the two-dimensional vertex-reinforced jump process. *ArXiv Mathematics e-prints* (2018). arXiv: 1802.02077.

F. A. Berezin. *Introduction to superanalysis*, volume 9 of *Mathematical Physics and Applied Mathematics*. D. Reidel Publishing Co., Dordrecht (1987). ISBN 90-277-1668-4. MR914369.

B. Davis and S. Volkov. Continuous time vertex-reinforced jump processes. *Probab. Theory Related Fields* **123** (2), 281–300 (2002). MR1900324.

B. Davis and S. Volkov. Vertex-reinforced jump processes on trees and finite graphs. *Probab. Theory Related Fields* **128** (1), 42–62 (2004). MR2027294.

M. Disertori, F. Merkl and S. W. W. Rolles. A supersymmetric approach to martingales related to the vertex-reinforced jump process. *ALEA Lat. Am. J. Probab. Math. Stat.* **14** (1), 529–555 (2017). MR3663098.

M. Disertori and T. Spencer. Anderson localization for a supersymmetric sigma model. *Comm. Math. Phys.* **300** (3), 659–671 (2010). MR2736958.

M. Disertori, T. Spencer and M. R. Zirnbauer. Quasi-diffusion in a 3D supersymmetric hyperbolic sigma model. *Comm. Math. Phys.* **300** (2), 435–486 (2010). MR2728731.

G. Letac and J. Wesołowski. Multivariate reciprocal inverse Gaussian distributions from the Sabot-Tarrès-Zeng integral. *ArXiv Mathematics e-prints* (2017). arXiv: 1709.04843.

F. Merkl, S. W. W. Rolles and P. Tarrès. Convergence of vertex-reinforced jump processes to an extension of the supersymmetric hyperbolic nonlinear sigma model (2018+). To appear in *Prob. Theory and Related Fields.*

P. K. Mitter and B. Scoppola. The global renormalization group trajectory in a critical supersymmetric field theory on the lattice $\mathbb{Z}^3$. *J. Stat. Phys.* **133** (5), 921–1011 (2008). MR2461190.

C. Sabot and P. Tarrès. Edge-reinforced random walk, vertex-reinforced jump process and the supersymmetric hyperbolic sigma model. *J. Eur. Math. Soc. (JEMS)* **17** (9), 2353–2378 (2015). MR3420510.

C. Sabot, P. Tarrès and X. Zeng. The vertex reinforced jump process and a random Schrödinger operator on finite graphs. *Ann. Probab.* **45** (6A), 3967–3986 (2017). MR3729620.

C. Sabot and X. Zeng. Hitting times of interacting drifted Brownian motions and the vertex reinforced jump process. *ArXiv Mathematics e-prints* (2017). arXiv: 1704.05394.

C. Sabot and X. Zeng. A random Schrödinger operator associated with the vertex reinforced jump process on infinite graphs (2018+). To appear in *J. Amer. Math. Soc.*

T. Spencer and M. R. Zirnbauer. Spontaneous symmetry breaking of a hyperbolic sigma model in three dimensions. *Comm. Math. Phys.* **252** (1-3), 167–187 (2004). MR2104878.

M. R. Zirnbauer. Fourier analysis on a hyperbolic supermanifold with constant curvature. *Comm. Math. Phys.* **141** (3), 503–522 (1991). MR1134935.