ALEA, Lat. Am. J. Probab. Math. Stat. **16**, 389–398 (2019)
DOI: 10.30757/ALEA.v16-14

A simple proof of a Kramers' type law for self-stabilizing diffusions in double-wells landscape

Julian Tugaut

Univ Lyon, Université Jean Monnet,
CNRS UMR 5208, Institut Camille Jordan,
Maison de l'Université, 10 rue Tréfilerie,
CS 82301,
42023 Saint-Étienne Cedex 2,
France
E-mail address: tugaut@math.cnrs.fr
URL: http://tugaut.perso.math.cnrs.fr

Abstract. The present paper is devoted to the study of a McKean-Vlasov diffusion oftype self-stabilizing. We obtain this model by taking the hydrodynamical limit of a mean-field system of particles. The main question that we study is the exit-time. We take a confining potential with two wells : $a < 0$ and $b > 0$. We start with a deterministic condition $x_0 > 0$ and we show that the first time that this diffusion leaves an interval of the form $(d; +\infty)$ (d verifying some assumptions) satisfies a Kramers'type law. In other words, this time is exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H\right\}$ as the diffusion coefficient σ goes to 0, H being the exit cost. Incidentally, we also prove that the solution of the granular media equation is trapped (for the 2-Wasserstein distance) in a ball centered around δ_b during a time at least exponentially equivalent to $\exp\left\{\frac{2}{\sigma^2}H\right\}$.

1. Introduction

We are interested in a McKean-Vlasov diffusion of the following so-called "self-stabilizing" type:

$$X_t = X_0 + \sigma B_t - \int_0^t \nabla V\left(X_s\right) ds - \alpha \int_0^t \left(X_s - \mathbb{E}[X_s]\right) ds\,, \tag{1.1}$$

V being a double-wells landscape and α being positive. The exact assumptions on the confining potential V will be given subsequently.

Received by the editors April 2nd, 2018; accepted January 28th, 2019.
2010 *Mathematics Subject Classification.* 60F10, 60J60, 60H10, 82C22.
Key words and phrases. Self-stabilizing diffusion, Exit-time, Large deviations, Granular media equation.

In this paper, we focus on the one-dimensional case.

We take $X_0 := x_0 \in \mathbb{R}$. We consider an open domain $\mathcal{D} \subset \mathbb{R}$ which contains x_0 and we introduce $\tau_{\mathcal{D}}(\sigma) := \inf\{t \geq 0 \;:\; X_t \notin \mathcal{D}\}$, the first exit-time of X from the domain $\mathcal{D}$. The subject of this article is to study the exit-time $\tau_{\mathcal{D}}(\sigma)$ as σ goes to 0.

More precisely, we aim to establish a Kramers'type law:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}(H-\delta)\right] \leq \tau_{\mathcal{D}}(\sigma) \leq \exp\left[\frac{2}{\sigma^2}(H+\delta)\right]\right\} = 1\,,$$

for any $\delta > 0$. Here, H corresponds to the so-called "exit cost" (which is explained subsequently).

The natural framework to study $\tau_{\mathcal{D}}(\sigma)$ as σ goes to 0 is the one of the large deviations for stochastic processes. Freidlin and Wentzell theory solves the exit-problem for time-homogeneous diffusions. Let us briefly recall this theory. We refer the reader to Freidlin and Wentzell (1998); Dembo and Zeitouni (2010) for a complete review. We look at the diffusion

$$x_t^\sigma = x_0 + \sigma\beta_t - \int_0^t \nabla U\left(x_s^\sigma\right) ds\,.$$

U is a $\mathcal{C}^\infty$-continuous function from $\mathbb{R}$ to $\mathbb{R}$ and β is a Brownian motion in $\mathbb{R}$. Let a_0 be a minimizer of U and $\mathcal{G}$ be a domain which contains a_0.

We also consider the deterministic path $\Psi_t(x_0) := x_0 - \int_0^t \nabla U\left(\Psi_s(x_0)\right) ds$. Then, for any $T, \delta > 0$:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\sup_{t\in[0;T]} ||x_t^\sigma - \Psi_t(x_0)|| > \delta\right\} = 0\,. \tag{1.2}$$

We consider the following assumptions on the domain $\mathcal{G}$:

(1) The unique stable equilibrium point in $\mathcal{G}$ of the ordinary differential equation $\dot{\varphi}_t = -\nabla U\left(\varphi_t\right)$ is at $a_0 \in \mathcal{G}$ and $\varphi_0 \in \mathcal{G}$ implies $\varphi_t \in \mathcal{G}$ for any $t \geq 0$. Moreover, $\lim\limits_{t\to\infty}\varphi_t = a_0$.

(2) All the trajectories of the deterministic system $\dot{\varphi}_t = -\nabla U\left(\varphi_t\right)$ starting from $\varphi_0 \in \partial\mathcal{G}$ converges to a_0 as t goes to infinity.

(3) $H := \inf\limits_{\partial\mathcal{G}} U - U(a_0) < \infty$. This quantity H will be denoted as "exit-cost".

Under these assumptions, for any $\delta > 0$, the following Kramers'type law holds:

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H-\delta\right)\right] < \tau_{\mathcal{G}}(\sigma) < \exp\left[\frac{2}{\sigma^2}\left(H+\delta\right)\right]\right\} = 1\,.$$

We end the introduction by giving an essential definition which is of crucial interest in large deviations for stochastic processes.

Definition 1.1. Let $\mathcal{G}$ be a subset of $\mathbb{R}$ and let U be a potential on $\mathbb{R}$. For all $x \in \mathbb{R}$, we consider the dynamical system

$$\psi_t(x) = x - \int_0^t \nabla U\left(\psi_s(x)\right) ds\,.$$

We say that the domain $\mathcal{G}$ is stable by $-\nabla U$ if the orbit $\{\psi_t(x)\,;\, t \in \mathbb{R}_+\}$ is included in $\mathcal{G}$ for all $x \in \mathcal{G}$.

2. Assumptions and notations

We take similar assumptions on the confining potential as the ones in Herrmann et al. (2008); Herrmann and Tugaut (2010a).

Assumption 2.1. *The potential V satisfies the following hypotheses:*

- *The coefficient ∇V is locally Lipschitz, that is, for each $R > 0$ there exists $K_R > 0$ such that*
$$|\nabla V(x) - \nabla V(y)| \leq K_R |x - y| \ ,$$
for $x, y \in \{z \in \mathbb{R} \ : \ |z| < R\}$.
- *The function V is continuously differentiable.*
- *The potential V is convex at infinity: $\lim_{x \to \pm\infty} V''(x) = +\infty$.*
- *There exist $r \in \mathbb{N}$ and $C > 0$ such that $|\nabla V(x)| \leq C\left(1 + |x|^{2r-1}\right)$ and $r \geq 2$.*
- *The potential V has two local minima located at $a < 0$ and $b > 0$ and has a local maximum located at 0.*
- *$V''(a) > 0$, $V''(b) > 0$ and $V''(0) < 0$.*
- *V'' is convex.*

An example of such potential is $x \mapsto \frac{x^4}{4} - \frac{x^2}{2}$ with $a = -1 < 0$ and $b = 1 > 0$.

The procedure of the present work can be applied with $x_0 > 0$ or $x_0 < 0$ (the case $x_0 = 0$ will be discussed in the conclusion of the paper). However, all the notations will be different if we start at the right or at the left. As a consequence, we choose a side of start.

Assumption 2.2. *We assume that $x_0 > 0$.*

We consider an additional assumption on the interaction.

Assumption 2.3. *We have $\alpha < \frac{V''(b)}{\sqrt{2}}$.*

Since the initial law is a Dirac measure at point $x_0 \in \mathbb{R}$ (so that any moment at time 0 exists), we know that there exists a unique strong solution X to Equation (1.1), see Herrmann et al. (2008, Theorem 2.13). Moreover: $\sup_{t \in \mathbb{R}_+} \mathbb{E}\left\{||X_t||^{2p}\right\} < \infty$ for any $p \in \mathbb{N}^*$ where $\mathbb{N}^*$ is the set of positive integers.

We now give the hypotheses on the domain $\mathcal{D}$.

Assumption 2.4. *We consider the dynamical system*
$$\varphi_t = x_0 - \int_0^t \nabla V\left(\varphi_s\right) ds \, .$$
The orbit $\{\varphi_t \ ; \ t \geq 0\}$ is included into the domain $\mathcal{D}$. Moreover, $\lim_{t \to \infty} \varphi_t = b \in \mathcal{D}$.

We point out that we do not assume that the domain $\mathcal{D}$ is stable by $-\nabla V$.

We now explain why Assumption 2.4 is essential (and natural). Since the dynamical system and the diffusion are close when the noise is small (see Limit (1.2)), if there exists $T > 0$ such that $\varphi_T \notin \mathcal{D}$, one can easily prove that X_T is not on the domain $\mathcal{D}$ with a probability which converges to 1 as σ goes to 0. As a consequence, there exists $T > 0$ such that $\mathbb{P}\left(\tau_{\mathcal{D}}(\sigma) \leq T\right) \longrightarrow 1$ as σ goes to 0.

The main trick to obtain the Kramers' type law is to compare X with two time-homogeneous diffusions (let us point out that we do not use coupling like it has been done in Tugaut (2018) because it requires some convexity of $V+F*\delta_b$). However, to get the Kramers'type law for these two diffusions, we need the domain $\mathcal{D}$ to satisfy some hypothesis of stability since it is necessary in the classical Freidlin-Wentzell theory.

Assumption 2.5. *From now on, we introduce the potential*

$$W_b(x) := V(x) + \frac{\alpha}{2}(x-b)^2 \ . \tag{2.1}$$

The open domain $\mathcal{D}$ is stable by the vector field $-\nabla W_b$.

Indeed, as σ goes to 0, the law of X_t will be close (as t is sufficiently large) to δ_b so that X_t has a drift close to $x \mapsto -\nabla V(x) - \alpha(x-b)$. As a consequence, the effective potential W_b appears naturally in the small-noise limit.

We consider a last hypothesis on the domain $\mathcal{D}$.

Assumption 2.6. *There exists $\rho > 0$ such that for all $x \in \mathcal{D}$, we have*

$$\langle x-b\,;\, \nabla W_b(x)\rangle \geq \rho\,|x-b|^2 \ .$$

Hypothesis 2.6 allows us to control the moments (the ones which intervene in the drift) of $\mathcal{L}(X_t)$. This assumption seems strong and looks like a convexity assumption. However, it is not. Indeed, it is possible that potential W_b is not convex on $\mathcal{D}$ but satisfies Assumption 2.6.

We finish the introduction by giving an example of domain which satisfies Assumptions 2.4–2.6. To do so, we separate in two different cases.

Example 2.7 (First case). We know that b is a local minimum of W_b. Indeed, $W_b'(b) = V'(b) + \alpha(b-b) = 0$ and $W_b''(b) = V''(b) + \alpha > 0$. We assume that the potential W_b does not have any other critical point. (For example, if α is larger than $\sup_{\mathbb{R}} -V''$, we know that W_b is convex.)

In this case, any interval of the form $(d;+\infty)$ with $d < b$ satisfies the assumptions. Also, any interval of the form $(-\infty;e)$ with $e > b$ satisfies the assumptions.

We point out that this first case can be seen as the easy one since the only condition on the interval is to contain b and to be different from $\mathbb{R}$.

We also point out that Tugaut (2018) does not solve this first case if W_b is not convex.

As an example, we can consider $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $\alpha > \frac{1}{4}$. We remark that $\frac{V''(1)}{\sqrt{2}} = \sqrt{2} > \sup_{\mathbb{R}} -V'' = 1 > \dfrac{1}{4}$.

We now present the second case.

Example 2.8 (Second case). Let us assume that W_b admits another local minimum, denoted as y_0. Since V'' is convex, we know that there exists a unique local maximum that will be denoted as c. Then, we can easily prove that any interval of the form $(d;+\infty)$ with $d \in]c;b[$ satisfies the assumptions.

As an example, we can consider $V(x) := \frac{x^4}{4} - \frac{x^2}{2}$ and $\alpha < \frac{1}{4}$.

We can remark that the intermediate case (when there are exactly two critical points) is not discussed. In fact, it can be solved like the second case.

We also point out that we do not need either $W_b(y_0) \leq W_b(b)$ nor $W_b(y_0) > W_b(b)$. Indeed, this does not have influence in the procedure that we present here. However, it is important to describe the limit of the law of X_t. We know, thanks to Tugaut (2013a,b) that $\mathcal{L}(X_t)$ will converge to an invariant probability measure. But, if $W_b(y_0) \leq W_b(b)$, we also know (see Tugaut, 2014) that there is no invariant probability measure near δ_b in the small-noise limit. As a consequence, the state b is not stable.

3. Main results

From now on, we always consider intervals of the form $(d;+\infty)$ with $d < b$. If we are in the first case, we put $c := -\infty$. We always assume $d > c$.

Definition 3.1. We define the exit-time $\tau_{\mathcal{D}}(\sigma) := \inf\{t > 0 \ : \ X_t \notin \mathcal{D}\}$ and its associated exit-cost $H_d := \inf_{x\in\partial\mathcal{D}} W_b(x) - W_b(b) = W_b(d) - W_b(b)$.

Theorem A: *Under Hypotheses 2.1–2.6, we have the following limit for any $\delta > 0$:*

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H_d-\delta\right)\right]\leq\tau_{\mathcal{D}}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H_d+\delta\right)\right]\right\}=1\,. \tag{3.1}$$

We now specify if we are in the first case.

Theorem B: *Under Hypotheses 2.1–2.6, if W_b does not admit any other critical point than b, we have the following limit for any $\delta > 0$:*

$$\lim_{\sigma\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\sigma^2}\left(H_e-\delta\right)\right]\leq\tau_{(-\infty;e)}(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H_e+\delta\right)\right]\right\}=1\,, \tag{3.2}$$

where $e > b$ and $H_e := W_b(e) - W_b(b)$.

The proof of Theorem B is similar to the one of Theorem A so it is left to the reader.

Incidentally, we also prove the following result which can be linked to the rate of convergence of $\mathcal{L}(X_t)$. Indeed, if we start from $x_0 \in (0;+\infty)$, we know - in the setting of the second case if moreover $W_b(y_0) \leq W_b(b)$ - that the law of X_t will converge towards an invariant probability near δ_a (in the small-noise limit). A remaining question is the rate of convergence. We here provide a partial answer by showing that this convergence does not occur before a time $\exp\left\{\frac{2}{\sigma^2}\left(H_c-\delta\right)\right\}$, for any $\delta > 0$, where $H_c := W_b(c) - W_b(b)$.

Proposition C: *Under Hypotheses 2.1–2.6, for any $\kappa, \delta > 0$, if σ is sufficiently small, we have:*

$$\sup_{T_\kappa\leq t\leq\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]}\mathbb{E}[|X_t-b|^2]<\kappa^2\,, \tag{3.3}$$

T_κ being a positive constant which does not depend on σ. And $H_c := W_b(c) - W_b(b)$.

We now give an immediate corollary if we are in the first case.

Corollary D: *Under Hypotheses 2.1–2.6, if W_b does not admit any other critical point than b, for any $\kappa, \delta > 0$, if σ is sufficiently small, we have:*

$$\sup_{T_\kappa\leq t\leq\exp\left[\frac{2K}{\sigma^2}\right]}\mathbb{E}[|X_t-b|^2]<\kappa^2\,, \tag{3.4}$$

T_κ being a positive constant which does not depend on σ.

4. Proof of Proposition C

Lemma 4.1. *Under Assumptions 2.1-2.6, we have:*

$$\frac{d}{dt}\mathbb{E}\left[|X_t-b|^2\right]\leq-2\lambda\mathbb{E}\left[|X_t-b|^2\right]+K\sqrt{\mathbb{P}\left(X_t\leq\xi\right)}+\sigma^2\,,$$

λ, K being positive constants and ξ being any positive real such that $\xi < x_0$ and $\xi < b$.

The proof is similar to the one of Tugaut (2018, Lemma 4.1).

Proof: By Itô formula, we have:

$$|X_t - b|^2 = |x_0 - b|^2 + 2\sigma \int_0^t (X_s - b)\, dW_s$$
$$- 2\int_0^t (X_s - b)\, V'(X_s) ds - 2\alpha \int_0^t (X_s - b)\,(X_s - \mathbb{E}[X_s])\, ds + \sigma^2 t\,.$$

However, $\mathbb{E}\left\{(X_t - b)\,(X_t - \mathbb{E}\left[X_t\right])\right\} = \mathrm{Var}\left(X_t\right) \geq 0$.

We take the expectation then we take the derivative. We thus obtain:

$$\frac{d}{dt}\mathbb{E}\left[|X_t - b|^2\right] \leq - 2\mathbb{E}\left[(X_t - b)\, V'(X_t)\right] + \sigma^2\,.$$

However,

$$(X_t - b)\, V'(X_t) = (X_t - b)\, V'(X_t)\mathbb{1}_{X_t > \xi} + (X_t - b)\, V'(X_t)\mathbb{1}_{X_t \leq \xi}\,.$$

Since V'' is convex and since $\xi > 0$, for any $x > \xi$, we have, $(x - b)V'(x) \geq \lambda(\xi)(x - b)^2$. Consequently, we have:

$$\frac{d}{dt}\mathbb{E}\left[|X_t - b|^2\right] \leq - 2\lambda(\xi)\mathbb{E}\left[|X_t - b|^2\right] + \sigma^2$$
$$+ 2\mathbb{E}\left\{\left[\lambda(\xi)\,|X_t - b|^2 - (X_t - b)\, V'(X_t)\right]\mathbb{1}_{X_t \leq \xi}\right\}$$

According to Assumption 2.1, we have $|V'(X_t)| \leq C\left(1 + |X_t|^{2r-1}\right)$ so that

$$\left|\lambda(\xi)\,|X_t - b|^2 - (X_t - b)\, V'(X_t)\right| \leq C'(\xi)\left(1 + |X_t|^{2r}\right)\,,$$

$C'(\xi)$ being a positive constant. Cauchy-Schwarz inequality yields

$$\frac{d}{dt}\mathbb{E}\left[|X_t - b|^2\right] \leq - 2\lambda(\xi)\mathbb{E}\left[|X_t - b|^2\right] + \sigma^2$$
$$+ 2C''(\xi)\left(1 + \sqrt{\mathbb{E}\left[|X_t^{4r}|\right]}\right)\sqrt{\mathbb{P}\left(X_t \leq \xi\right)}$$

The uniform boundedness of the moments (see Herrmann et al. (2008)) implies the existence of a positive constant $K(\xi)$ such that

$$\frac{d}{dt}\mathbb{E}\left[|X_t - b|^2\right] \leq -2\lambda(\xi)\mathbb{E}\left[|X_t - b|^2\right] + \sigma^2 + K(\xi)\sqrt{\mathbb{P}\left(X_t \leq \xi\right)}\,,$$

which achieves the proof.

□

Remark 4.2. For any $T > 0$, we have $\lim_{\sigma \to 0} \mathbb{P}\left\{\tau_{(\xi;+\infty)}(\sigma) \leq T)\right\} = 0$.

This is a classical result from the large deviations theory for stochastic processes, see Limit (1.2) and Dembo and Zeitouni (2010, page 221).

Lemma 4.3. *For any $\kappa > 0$, there exist some positive constants σ_0 and $T_\kappa(\sigma_0)$ such that for any $\sigma < \sigma_0$, there exists $T_\kappa(\sigma)$ satisfying $T_\kappa(\sigma) \leq T_\kappa(\sigma_0)$ and $\mathbb{E}\left[\left|X_{T_\kappa(\sigma)} - b\right|^2\right] \leq \kappa^2$.*

It is a straightforward consequence of Lemma 4.1 and Remark 4.2 so we do not give the proof.

In particular, we have

$$\left|\mathbb{E}\left[X_{T_\kappa(\sigma)}\right] - b\right| \leq \kappa\,.$$

We now introduce the two diffusions $X^{+,\kappa}$ and $X^{-,\kappa}$ by

$$X_t^{\pm,\kappa} = X_{T_\kappa(\sigma)} + \sigma\left(B_t - B_{T_\kappa(\sigma)}\right) - \alpha\int_{T_\kappa(\sigma)}^t \left(X_s^{\pm,\kappa} - (b\pm\kappa)\right)ds \tag{4.1}$$

From now on, κ is arbitrarily small. By $b_\kappa^\pm$, we denote the positive critical point (close to b) of the potential $x\mapsto V(x)+\frac{\alpha}{2}\left(x-(b\pm\kappa)\right)^2$. By a simple computation, we get:

$$b_\kappa^\pm = b \pm \frac{\alpha}{V''(b)+\alpha}\kappa + o(\kappa)\,.$$

Now, if κ is small enough, we know that the Freidlin-Wentzell theory may be applied to Diffusion $X^{\pm,\kappa}$ and domain $(d;+\infty)$ (for any $d>c$). So, we deduce that $\tau_{(d;+\infty)}^\pm(\sigma) := \inf\left\{t\geq T_\kappa(\sigma)\,:\,X_t^{\pm,\kappa}\leq d\right\}$ satisfies a Kramers'type law. In particular, we have

$$\lim_{\sigma\to 0}\mathbb{P}\left(\tau_{(d;+\infty)}^\pm(\sigma)\leq \exp\left[\frac{2}{\sigma^2}\left(H_\kappa^\pm(d)-\delta\right)\right]\right) = 0\,,$$

for any $\delta>0$. Here, $H_\kappa^\pm(d) := V(d)-V\left(b_\kappa^\pm\right)+\frac{\alpha}{2}\left(d-b_\kappa^\pm\right)^2$. However, by proceeding like in Lemma 4.1, we obtain

$$\frac{d}{dt}\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right] \leq -2\rho(\kappa)\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right] + \sigma^2 + K'\sqrt{\mathbb{P}\left(X_t^{\pm,\kappa}\notin\mathcal{D}\right)}\,,$$

$K'>0$ and $\lim_{\kappa\to 0}\rho(\kappa)=\rho$.

Consequently, there exists $T'_\kappa(\sigma) > T_\kappa(\sigma)$ which is uniform with respect to $\sigma\in\,]0;1]$ such that:

$$\sup_{t\in\left[T'_\kappa(\sigma);\exp\left[\frac{2}{\sigma^2}\left(H_\kappa^\pm(d)-\delta\right)\right]\right]}\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right] < \left(\frac{V''(b)}{2(V''(b)+\alpha)}\right)^2\kappa^2\,,$$

if σ is small enough. In particular, with d converging to c and κ arbitrarily small, we obtain

$$\sup_{t\in\left[T'_\kappa(\sigma);\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]\right]}\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right] < \left(\frac{V''(b)}{2(V''(b)+\alpha)}\right)^2\kappa^2\,,$$

if σ is small enough.

Definition 4.4. We now introduce

$$\mathcal{T}(\kappa,\sigma) := \inf\left\{t\geq T_\kappa(\sigma)\,:\,\mathbb{E}\left[\left|X_t-b\right|^2\right]>\kappa^2\right\}\,.$$

We will prove that $\mathcal{T}(\kappa,\sigma)\geq\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]$ for any $\delta>0$, if σ is small enough.

We can remark that for any $t\in[T_\kappa(\sigma);\mathcal{T}(\kappa,\sigma)]$: $X_t^{-,\kappa}\leq X_t\leq X_t^{+,\kappa}$. This implies :

$$\mathbb{E}\left[\left|X_t-b\right|^2\right]\leq\mathbb{E}\left[\left|X_t^{+,\kappa}-b\right|^2\right]+\mathbb{E}\left[\left|X_t^{-,\kappa}-b\right|^2\right]\,.$$

However, for any $t\in\left[T'_\kappa(\sigma);\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]\right]$, we have

$$\begin{aligned}\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b\right|^2\right] &\leq \mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right]+\left|b_\kappa^\pm-b\right|^2\\ &+2\left|b_\kappa^\pm-b\right|\sqrt{\mathbb{E}\left[\left|X_t^{\pm,\kappa}-b_\kappa^\pm\right|^2\right]}\\ &\leq \left(\frac{V''(b)}{2(V''(b)+\alpha)}\right)^2\kappa^2+\left(\frac{\alpha}{V''(b)+\alpha}\right)^2\kappa^2\\ &+2\frac{\alpha}{V''(b)+\alpha}\frac{V''(b)}{2(V''(b)+\alpha)}\kappa^2++o(\kappa^2)\\ &\leq \left(\frac{V''(b)+2\alpha}{2V''(b)+2\alpha}\right)^2\kappa^2+o(\kappa^2)\,.\end{aligned}$$

Consequently, if κ is small enough, we have:

$$\sup_{t\in[T'_\kappa(\sigma);\min\left\{\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right];\mathcal{T}(\kappa,\sigma)\right\}]}\mathbb{E}\left[|X_t-b|^2\right]\leq 2\left(\frac{V''(b)+2\alpha}{2V''(b)+2\alpha}\right)^2\kappa^2+o(\kappa^2)\,.$$

As $V''(b)>\sqrt{2}\alpha$, $2\left(\frac{V''(b)+2\alpha}{2V''(b)+2\alpha}\right)^2$ is strictly less than 1. If we were assuming $\mathcal{T}(\kappa,\sigma)<\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]$, we would obtain an absurdity. As a consequence, for any $t\in[T'_\kappa(\sigma);\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]]$, we have $\mathbb{E}\left[|X_t-b|^2\right]\leq\kappa^2$.

A remaining question is what happens as $t\in[T_\kappa(\sigma);T'_\kappa(\sigma)]$. In fact, by proceeding like in Tugaut (2018) (Theorem A and Lemma 4.1 applied to a domain included in $\mathcal{D}$ and satisfying the assumptions in Tugaut (2018)), we can prove that there exists $h>0$ such that $\sup_{T_\kappa(\sigma)\leq t\leq e^{\frac{2h}{\sigma^2}}}\mathbb{E}\left[|X_t-b|^2\right]<\kappa^2$. By taking σ small enough such that $e^{\frac{2h}{\sigma^2}}>T'_\kappa(\sigma)$ (which is possible since $T'_\kappa(\sigma)$ is uniformly bounded with respect to $\sigma\leq 1$), the proof is achieved.

5. Proof of Theorem A

We now prove Theorem A. First, as $T_\kappa(\sigma)$ is uniformly bounded with respect to $\sigma\in(0;1]$, we deduce that the probability of exiting $(d;+\infty)$ before $T_\kappa(\sigma)$ goes to 0 as σ goes to 0.

The main idea now is to compare the exit-time of X with the ones of $X^{\pm,\kappa}$. We have

$$\sup_{T_\kappa\leq t\leq\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]}\mathbb{E}\left[|X_t-b|^2\right]<\kappa^2\,,$$

for σ sufficiently small.

Consequently, for any $t\in\left[T_\kappa;\exp\left[\frac{2}{\sigma^2}(H_c-\delta)\right]\right]$, we have $X_t^{-,\kappa}\leq X_t\leq X_t^{+,\kappa}$. As a consequence, if we put $\tau(\sigma):=\inf\{t\geq 0\,:\,X_t\leq d\}$, we have

$$\inf\left\{\tau_\kappa^-(\sigma);e^{\frac{2}{\sigma^2}(H_c-\delta)}\right\}\leq\inf\left\{\tau(\sigma);e^{\frac{2}{\sigma^2}(H_c-\delta)}\right\}\leq\inf\left\{\tau_\kappa^+(\sigma);e^{\frac{2}{\sigma^2}(H_c-\delta)}\right\}\,.$$

However, a Kramers'type law is satisfied by $\tau_\kappa^\pm(\sigma)$. So, for any $\xi>0$, we have

$$\lim_{\sigma\to 0}\mathbb{P}\left(\exp\left[\frac{2}{\sigma^2}\left(H_\kappa^-(d)-\xi\right)\right]\leq\tau(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H_\kappa^+(d)+\xi\right)\right]\right)=1\,.$$

Consequently, by taking κ sufficiently small, we obtain that for any $\delta > 0$, we have

$$\lim_{\sigma\to 0}\mathbb{P}\left(\exp\left[\frac{2}{\sigma^2}\left(H(d)-\delta\right)\right]\leq\tau(\sigma)\leq\exp\left[\frac{2}{\sigma^2}\left(H(d)+\delta\right)\right]\right)=1\,.$$

This achieves the proof.

6. Conclusion

In this paper, a Kramers'type law has been obtained without assuming a global convexity assumption. The hypotheses on the domain and on the confining potential are natural and not technical. Incidentally, we have derive an inequality which shows that the first moment of the law of the McKean-Vlasov diffusion is trapped into a ball during a time which goes to infinity as the coefficient diffusion goes to 0.

However, there are still some remaining questions. First, we can wonder if the ideas developed in this paper can be applied with an interacting potential F with a degree more than 2. In other words, can we apply this procedure to obtain a Kramers'type law for the diffusion:

$$X_t = X_0+\sigma B_t-\int_0^t\nabla V\left(X_s\right)ds-\int_0^t\left[\nabla F*\mathcal{L}\left(X_s\right)\right]\left(X_s\right)ds\,, \tag{6.1}$$

where F is more general, typically with $\deg(F)=2n$?

We think that the idea to apply the procedure is to consider the drift $-\nabla V - \nabla F*\nu$ for any ν such that $\mathbb{W}_{2n}(\nu;\delta_b)=\kappa$, then to show that the diffusion X is trapped between two time-homogeneous diffusions.

Another question is about the general dimensional case. We consider that this question is even more difficult. The idea probably is to generalize Lemma 4.1 in order to better understand the control of the moments of X_t by the exit-time and reciprocally.

A question which is challenging is the one of the exit from the interval $(c;+\infty)$. Indeed, in the procedure described previously, we assume that the interval has the form $(d;+\infty)$ with $c<d<b$. However, in order to be able to find the basins of attraction of each invariant probability (we know that there are three such invariant probabilities if σ is small enough thanks to Herrmann and Tugaut, 2010a,b, 2012), we will need to extend to c. To obtain the limit

$$\lim_{\sigma\to 0}\mathbb{P}\left(\tau_{(c;+\infty)}(\sigma)\leq\exp\left(\frac{2}{\sigma^2}\left(H_c-\delta\right)\right)\right)=0$$

is an easy task but the other inequality will be more difficult and we do believe that potential theory for giving the expectation of $\tau_{(c;+\infty)}(\sigma)$ will be helpful.

One can also wonder what happens if the initial measure is not a Dirac measure. This is strongly linked to the question of the characterization of the basins of attractions. We think that the procedure developed in this paper can be applied under some minor modifications.

Finally, we discuss about the initial condition. If $x_0=0$, we will either go to a either go to b. Consequently, we think that this remains to the previous question.

References

A. Dembo and O. Zeitouni. *Large deviations techniques and applications*, volume 38 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin (2010). MR2571413.

M. I. Freidlin and A. D. Wentzell. *Random perturbations of dynamical systems*, volume 260 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, second edition (1998). ISBN 0-387-98362-7. MR1652127.

S. Herrmann, P. Imkeller and D. Peithmann. Large deviations and a Kramers' type law for self-stabilizing diffusions. *Ann. Appl. Probab.* **18** (4), 1379–1423 (2008). MR2434175.

S. Herrmann and J. Tugaut. Non-uniqueness of stationary measures for self-stabilizing processes. *Stochastic Process. Appl.* **120** (7), 1215–1246 (2010a). MR2639745.

S. Herrmann and J. Tugaut. Stationary measures for self-stabilizing processes: asymptotic analysis in the small noise limit. *Electron. J. Probab.* **15**, no. 69, 2087–2116 (2010b). MR2745727.

S. Herrmann and J. Tugaut. Self-stabilizing processes: uniqueness problem for stationary measures and convergence rate in the small-noise limit. *ESAIM Probab. Stat.* **16**, 277–305 (2012). MR2956576.

J. Tugaut. Convergence to the equilibria for self-stabilizing processes in double-well landscape. *Ann. Probab.* **41** (3A), 1427–1460 (2013a). MR3098681.

J. Tugaut. Self-stabilizing processes in multi-wells landscape in $\mathbb{R}^d$-convergence. *Stochastic Process. Appl.* **123** (5), 1780–1801 (2013b). MR3027901.

J. Tugaut. Self-stabilizing processes in multi-wells landscape in $\mathbb{R}^d$-invariant probabilities. *J. Theoret. Probab.* **27** (1), 57–79 (2014). MR3174216.

J. Tugaut. Exit-problem of McKean-Vlasov diffusions in double-well landscape. *J. Theoret. Probab.* **31** (2), 1013–1023 (2018). MR3803922.