Functional convergence for moving averages with heavy tails and random coefficients

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Abstract. We study functional convergence of sums of moving averages with random coefficients and heavy-tailed innovations. Under some standard moment conditions and the assumption that all partial sums of the series of coefficients are a.s. bounded between zero and the sum of the series we obtain functional convergence of the corresponding partial sum stochastic process in the space $D[0,1]$ of càdlàg functions with the Skorohod $M_2$ topology.

1. Introduction

Let $(Z_i)_{i \in \mathbb{Z}}$ be a sequence of i.i.d. regularly varying random variables with index of regular variation $\alpha \in (0, 2)$. This means that

$$P(|Z_i| > x) = x^{-\alpha} L(x), \quad x > 0,$$

where $L$ is a slowly varying function at $\infty$. Regular variation implies $E|Z_i|^\beta < \infty$ for every $\beta \in (0, \alpha)$. We study the moving average process with random coefficients, defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},$$

where $(C_i)_{i \geq 0}$ is a sequence of random variables independent of $(Z_i)$, such that the series in (1.2) is a.s. convergent. One sufficient condition for that is

$$\sum_{j=0}^{\infty} |C_j|^{\alpha-\epsilon} < \infty \quad \text{a.s. for some } \epsilon > 0 \quad (1.3)$$
We will use the following moment condition on the sequence \((C_j)\):

\[
\sum_{j=0}^{\infty} \mathbb{E}|C_j|^\delta < \infty \quad \text{for some } \delta < \alpha, \ 0 < \delta \leq 1.
\] (1.4)

This condition also implies the a.s. convergence of the series in (1.2), since

\[
\mathbb{E}|X_i|^\delta \leq \sum_{j=0}^{\infty} \mathbb{E}|C_j|^\delta \mathbb{E}|Z_i-j|^\delta = \mathbb{E}|Z_1|^\delta \sum_{j=0}^{\infty} \mathbb{E}|C_j|^\delta < \infty.
\]

Beside condition (1.4) we will require some other moment conditions, which will be specified in Section 3. We also impose the following (usual) regularity conditions on \(Z_1\):

\[
\mathbb{E}Z_1 = 0, \quad \text{if } \alpha \in (1, 2),
\] (1.5)

\[
Z_1 \text{ is symmetric, if } \alpha = 1.
\] (1.6)

Let \((a_n)\) be a sequence of positive real numbers such that

\[
n \mathbb{P}(|Z_1| > a_n) \to 1,
\] (1.7)
as \(n \to \infty\). Regular variation of \(Z_i\) can be expressed in terms of vague convergence of measures on \(E = \mathbb{R} \setminus \{0\}\): for \(a_n\) as in (1.7) and as \(n \to \infty\),

\[
n \mathbb{P}(a_n^{-1}Z_i \in \cdot) \xrightarrow{v} \mu(\cdot),
\] (1.8)

with the measure \(\mu\) on \(E\) given by

\[
\mu(dx) = \left(p 1_{(0,\infty)}(x) + r 1_{(-\infty,0)}(x)\right) \alpha|x|^{-\alpha-1} dx,
\] (1.9)

where

\[
p = \lim_{x \to \infty} \frac{\mathbb{P}(Z_i > x)}{\mathbb{P}(|Z_i| > x)} \quad \text{and} \quad r = \lim_{x \to \infty} \frac{\mathbb{P}(Z_i \leq -x)}{\mathbb{P}(|Z_i| > x)}.
\] (1.10)

When the coefficients \(C_i\) are deterministic, Basrak and Krizmanić (2014) obtained functional convergence of the partial sum process of \(X_i\’s\) with respect to the Skorohod \(M_2\) topology on \(D[0,1]\). More precisely, they showed that under the condition on the coefficients \(C_i\):

\[
0 \leq \sum_{i=0}^{s} C_i \sum_{i=0}^{\infty} C_i \leq 1, \quad \text{for every } s = 0, 1, 2, \ldots,
\] (1.11)

the following

\[
\frac{1}{a_n} \sum_{i=1}^{n-j} X_i \overset{d}{\to} \left( \sum_{j=0}^{\infty} C_j \right)V(\cdot),
\] (1.12)

holds in \(D[0,1]\), where \(V(\cdot)\) is an \(\alpha\)-stable Lévy process and \(D[0,1]\) is the space of real-valued right continuous functions on \([0,1]\) with left limits.

Recall here that if at least two coefficients are nonzero, then the convergence in (1.12) cannot hold with respect to the more usual Skorohod \(J_1\) topology on \(D[0,1]\), but if all the coefficients are nonnegative, then the convergence in (1.12) holds in the \(M_1\) topology, see Avram and Taqqu (1992). The aim of this article is to obtain the functional convergence with respect to the \(M_2\) topology as in (1.12) when the coefficients \(C_i\) are random variables. Limit theory for moving averages with random coefficients, but without the time component, have already been studied, see Kulik (2006). These processes can represent various stochastic models, such as
solutions to stochastic recurrence equations and stochastic integrals (usually with some predictability assumption instead of the independence between the coefficients $C_j$ and the noise variables $Z_j$, see Hult and Samorodnitsky, 2008).

The Skorohod $M_2$ topology on $D[0,1]$ is defined using completed graphs and their parametric representations (see Section 12.11 in Whitt, 2002 for details). Here we give only a characterization of the $M_2$ topology using the Hausdorff metric on the spaces of graphs, since it will be convenient for our purposes. For $x \in D[0,1]$ the completed graph of $x$ is the set

$$\Gamma_x = \{(t,z) \in [0,1] \times \mathbb{R} : z = \lambda x(t-) + (1-\lambda)x(t) \text{ for some } \lambda \in [0,1]\},$$

where $x(t-)$ is the left limit of $x$ at $t$. Besides the points of the graph $\{(t,x(t)) : t \in [0,1]\}$, the graph $\Gamma_x$ also contains the vertical line segments joining $(t,x(t))$ and $(t,x(t-))$ for all discontinuity points $t$ of $x$. Now, for $x_1, x_2 \in D[0,1]$ define

$$d_{M_2}(x_1,x_2) = \left( \sup_{a \in \Gamma_{x_1}} \inf_{b \in \Gamma_{x_2}} d(a,b) \right) \vee \left( \sup_{a \in \Gamma_{x_2}} \inf_{b \in \Gamma_{x_1}} d(a,b) \right),$$

where $d$ is the metric on $\mathbb{R}^2$ defined by $d((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| \vee |y_1 - y_2|$ for $(x_i, y_i) \in \mathbb{R}^2$, $i = 1, 2$, where $a \vee b = \max\{a,b\}$. The metric $d_{M_2}$ induces the $M_2$ topology. This topology is weaker than the more frequently used $M_1$ and $J_1$ topologies. Therefore the $M_1$ convergence implies the $M_2$ convergence, but the converse does not hold in general. For instance, take the moving average process with heavy-tailed innovations $Z_i$ and deterministic coefficients $C_0 = 1$, $C_1 = -1$, $C_2 = 1$, and $C_i = 0$ for $i \geq 3$, i.e.

$$X_i = Z_i - Z_{i-1} + Z_{i-2}, \quad i \in \mathbb{Z}.$$ 

Since the condition (1.11) is satisfied, the $M_2$ convergence in relation (1.12) holds. Clusters of large values in the sequence $(X_n)$ contain positive and negative values, which means that the corresponding partial sum processes have jumps of opposite signs within temporal clusters of large values, and this precludes the $M_1$ convergence. The detailed proof of this fact for the moving average process $(X_i)$ defined above is given in Appendix.

The paper is organized as follows. In Section 2 we obtain functional convergence for finite order moving average processes, and then in Section 3 we extend this result to infinite order moving averages. A technical result needed for establishing functional convergence for infinite order moving averages when $\alpha \in [1,2)$ is given in Appendix.

### 2. Finite order MA processes

Let $C_0, C_1, \ldots, C_q$ (for some fixed $q \in \mathbb{N}$) be random variables satisfying

$$0 \leq \sum_{i=0}^s C_i \left/ \sum_{i=0}^q C_i \right. \leq 1 \text{ a.s. for every } s = 0, 1, \ldots, q. \quad (2.1)$$

Put $C = \sum_{i=0}^q C_i$. Observe that condition (2.1) implies that $C, \sum_{i=0}^s C_i$ and $\sum_{i=s}^q C_i$ are a.s. of the same sign for every $s = 0, 1, \ldots, q$. Also note that condition (2.1) is satisfied if the $C_j$’s are all nonnegative or all nonpositive.
Let \((X_t)\) be a moving average process defined by
\[
X_t = \sum_{i=0}^{q} C_i Z_{t-i}, \quad t \in \mathbb{Z},
\]
and let the corresponding partial sum process be
\[
V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} X_i, \quad t \in [0, 1],
\]
where the normalizing sequence \((a_n)\) satisfies \((1.7)\).

**Theorem 2.1.** Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \(\alpha \in (0, 2)\), such that \((1.5)\) and \((1.6)\) hold. Assume \(C_0, C_1, \ldots, C_q\) are random variables, independent of \((Z_i)\), that satisfy \((2.1)\). Then
\[
V_n(\cdot) \overset{d}{\to} \tilde{C} V(\cdot), \quad n \to \infty,
\]
in \(D[0, 1]\) endowed with the \(M_2\) topology, where \(V\) is an \(\alpha\)-stable Lévy process with characteristic triple \((0, \mu, b)\), with \(\mu\) as in \((1.9)\) and
\[
b = \begin{cases} 
0, & \alpha = 1, \\
(p-r) \frac{\alpha}{1-\alpha}, & \alpha \in (0, 1) \cup (1, 2),
\end{cases}
\]
and \(\tilde{C}\) is a random variable, independent of \(V\), such that \(\tilde{C} \overset{d}{=} C\).

As in Basrak and Krizmanić (2014) one can prove the following lemma (with the notation \(C_i = 0\) for \(i < 0\)).

**Lemma 2.2.**
(i) For \(k < q\) it holds
\[
\sum_{i=1}^{k} \frac{C_i Z_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{k-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{\infty} C_s - \sum_{u=k-q}^{q-1} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{\infty} C_s - \sum_{u=0}^{q-k-1} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{\infty} C_s.
\]
(ii) For \(k \geq q\) it holds
\[
\sum_{i=1}^{k} \frac{C_i Z_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} = \sum_{u=0}^{q-1} \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{\infty} C_s - \sum_{u=0}^{q-1} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{\infty} C_s =: H_n(k) - G_n.
\]
(iii) For \(q \leq k \leq n - q\) it holds
\[
\sum_{i=1}^{k} \frac{C_i Z_i}{a_n} - \sum_{i=1}^{k+q} \frac{X_i}{a_n} = - \sum_{u=0}^{q-1} \frac{Z_{-u}}{a_n} \sum_{s=u+1}^{\infty} C_s - \sum_{u=0}^{q-1} \frac{Z_{k+u}}{a_n} \sum_{s=0}^{q-u} C_s =: -G_n - T_n(k).
\]
Proof: (Theorem 2.1) Since the random variables $Z_i$ are i.i.d. and regularly varying, it is known that

$$
\frac{|nt|}{a_n} Z_i - \frac{|nt|}{a_n} E\left( Z_i 1_{\{|Z_i| \leq a_n\}}\right), \quad t \in [0,1],
$$

converges in distribution, as $n \to \infty$, in $D[0,1]$ with the $M_1$ topology to an $\alpha$–stable Lévy process with characteristic triple $(0, \mu, 0)$ (see Theorem 3.4 in Basrak et al., 2012). By Karamata’s theorem, as $n \to \infty$,

$$
n E\left( \frac{Z_i}{a_n} 1_{\{|Z_i| \leq a_n\}}\right) \to (p-r) \frac{\alpha}{1-\alpha}, \quad \text{if } \alpha < 1,
$$

$$
n E\left( \frac{Z_i}{a_n} 1_{\{|Z_i| > a_n\}}\right) \to (p-r) \frac{\alpha}{\alpha-1}, \quad \text{if } \alpha > 1,
$$

with $p$ and $r$ as in (1.10). Therefore conditions (1.5) and (1.6), Corollary 12.7.1 in Whitt (2002) (which gives a sufficient condition for addition to be continuous in the $M_1$ topology) and the continuous mapping theorem yield that $V_n^Z(\cdot) \overset{d}{\to} V(\cdot)$, as $n \to \infty$, in $D[0,1]$ with the $M_1$ topology, where

$$
V_n^Z(t) := \sum_{i=1}^{\lfloor nt \rfloor} \frac{Z_i}{a_n}, \quad t \in [0,1],
$$

and $V$ is an $\alpha$–stable Lévy process with characteristic triple $(0, \mu, b)$.

It is well known that the space $D[0,1]$ equipped with the Skorohod $J_1$ topology is a Polish space (i.e. metrizable as a complete separable metric space), see Billingsley (1968), Section 14. The same holds for the $M_1$ topology, since it is topologically complete (see Whitt, 2002, Section 12.8) and separability remains preserved in the weaker topology. Therefore by Corollary 5.18 in Kallenberg (1997), we can find a random variable $\tilde{C}$, independent of $V$, such that $\tilde{C} \overset{d}{=} C$. This and the fact that $C$ is independent of $V_n^Z$, by an application of Theorem 3.29 in Kallenberg (1997), imply

$$
(B(\cdot), V_n^Z(\cdot)) \overset{d}{=} (\tilde{B}(\cdot), V(\cdot)), \quad \text{as } n \to \infty, \quad (2.3)
$$
in $D([0,1], \mathbb{R}^2)$ with the product $M_1$ topology, where $B(t) = C$ and $\tilde{B}(t) = \tilde{C}$ for $t \in [0,1]$.

Let $g: D([0,1], \mathbb{R}^2) \to D[0,1]$ be a function defined by

$$
g(x) = x_1 x_2, \quad x = (x_1, x_2) \in D([0,1], \mathbb{R}^2),
$$

where $(x_1 x_2)(t) = x_1(t) x_2(t)$ for $t \in [0,1]$. Let

$$
D_1 = \{ u \in D([0,1]) : \text{Disc}(u) = \emptyset \},
$$

and

$$
D_2 = \{ (u, v) \in D([0,1], \mathbb{R}^2) : \text{Disc}(u) = \emptyset \},
$$

where $\text{Disc}(u)$ is the set of discontinuity points of $u$. Then by Theorem 13.3.2 in Whitt (2002) the function $g$ is continuous on the set $D_2$ (with the Skorohod $M_1$ topology on $D[0,1]$ and product $M_1$ topology on $D([0,1], \mathbb{R}^2)$). Hence $\text{Disc}(g) \subseteq D_2^*$, and

$$
P((\tilde{B}, V) \in \text{Disc}(g)) \leq P((\tilde{B}, V) \in D_2^*) \leq P(\tilde{B} \in D_1^*) = 0.$$

This allows us to apply the continuous mapping theorem (see for instance Theorem 3.1 in Resnick, 2007) to relation (2.3) which yields \( g(B, V_n^Z) \overset{d}{\to} g(\tilde{B}, V) \), i.e.
\[
CV_n^Z(\cdot) \overset{d}{\to} CV(\cdot), \quad \text{as } n \to \infty,
\]
in \([0,1]\) with the \( M_1 \) topology. Using the fact that \( M_1 \) convergence implies \( M_2 \) convergence, we obtain
\[
CV_n^Z(\cdot) \overset{d}{\to} CV(\cdot), \quad \text{as } n \to \infty,
\]
in \((D[0,1], d_{M_2})\) as well. If we can show that for every \( \epsilon > 0 \)
\[
\lim_{n \to \infty} P[d_{M_2}(CV_n^Z, V_n) > \epsilon] = 0,
\]
an application of Slutsky’s theorem (see for instance Theorem 3.4 in Resnick, 2007) will imply \( V_n(\cdot) \overset{d}{\to} CV(\cdot) \), as \( n \to \infty \), in \((D[0,1], d_{M_2})\).

Fix \( \epsilon > 0 \) and let \( n \in \mathbb{N} \) be large enough, i.e. \( n > \max\{2q, 2q/\epsilon\} \). By the definition of the metric \( d_{M_2} \) we have
\[
d_{M_2}(CV_n^Z, V_n) = \left( \sup_{a \in \Gamma_{CV_n^Z}} \inf_{b \in \Gamma_{V_n}} d(a, b) \right) \vee \left( \sup_{a \in \Gamma_{V_n}} \inf_{b \in \Gamma_{CV_n^Z}} d(a, b) \right)
\]
and therefore
\[
P[d_{M_2}(V_n^Z, V_n) > \epsilon] \leq P(Y_n > \epsilon) + P(T_n > \epsilon).
\]
(2.5)

In order to estimate the first term on the right hand side of (2.5) note that
\[
\{Y_n > \epsilon\} \subseteq \{\exists a \in \Gamma_{CV_n^Z} \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{V_n}\}
\]
\[
\subseteq \{\exists k \in \{1, \ldots, q - 1\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon\}
\[
\cup \{\exists k \in \{q, \ldots, n - q\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon\}
\[
\cup \{\exists k \in \{n - q + 1, \ldots, n\} \text{ such that } |CV_n^Z(k/n) - V_n(k/n)| > \epsilon\}
\]
\[
=: A_{n}^{Y} \cup B_{n}^{Y} \cup C_{n}^{Y},
\]
(2.6)

where the second inclusion above follows from the fact that the paths of \( V_n \) and \( CV_n^Z \) are constant on the intervals of the form
\[
\left[\frac{j}{n}, \frac{j + 1}{n}\right), \quad j = 0, 1, \ldots, n - 1.
\]

More precisely, if there is a point \( a = (t_a, x_a) \in \Gamma_{CV_n^Z} \) such that \( d(a, \Gamma_{V_n}) > \epsilon \), then necessarily \( t_a \in [i/n, (i + 1)/n) \) for some \( i = 1, \ldots, n \). If \( a \) lies on a horizontal part of the completed graph, then \( x_a = CV_n^Z(i/n) \) and
\[
|CV_n^Z(i/n) - V_n(i/n)| \geq d(a, \Gamma_{V_n}) > \epsilon.
\]
Alternatively, if \( a \) lies on a vertical part of the completed graph, then \( x_a \in [CV_n^Z((i-1)/n), CV_n^Z(i/n)) \), and one can similarly conclude that
\[
|CV_n^Z(k/n) - V_n(k/n)| > \epsilon.
\]
for some \( k = 1, \ldots, n \) (in fact \( k = i \) or \( k = i - 1 \); see Basrak and Krizmanić, 2014 for details). Moreover, if \( q \leq k \leq n - q \), from \( q/n < \epsilon/2 \) it follows similarly that

\[
\left| CV^Z_n \left( \frac{k}{n} \right) - V_n \left( \frac{(k + q)}{n} \right) \right| > \epsilon.
\]

By Lemma 2.2 (i) we obtain

\[
P(A_Y^n) \leq \sum_{k=1}^{q-1} P \left( \left| \sum_{i=1}^{k} \frac{CZ_i}{a_n} - \sum_{i=1}^{k} \frac{X_i}{a_n} \right| > \epsilon \right)
\]

\[
\leq \sum_{k=1}^{q-1} \left[ P \left( \sum_{u=0}^{k-1} \frac{|Z_{k-u}|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right) + P \left( \sum_{u=k-q}^{q} \frac{|Z_u|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right) \right]
\]

\[
+ P \left( \sum_{u=0}^{q-k-1} \frac{|Z_{u+k}|}{a_n} \sum_{s=u+1}^{q} |C_s| > \frac{\epsilon}{3} \right)
\]

\[
\leq 3(q-1)(2q-1) P \left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)} \right),
\]

where \( C_* = \sum_{s=0}^{q} |C_s| \). For an arbitrary \( M > 0 \) it holds that

\[
P \left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)} \right) = P \left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)}, C_* > M \right) + P \left( \frac{|Z_0|}{a_n} C_* > \frac{\epsilon}{3(2q-1)}, C_* \leq M \right)
\]

\[
\leq P \left( C_* > M \right) + P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q-1)M} \right).
\]

By the regular variation property we observe

\[
\lim_{n \to \infty} P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q-1)M} \right) = 0,
\]

and hence from (2.7) we get

\[
\limsup_{n \to \infty} P(A_Y^n) \leq P \left( C_* > M \right).
\]

Letting \( M \to \infty \) we conclude

\[
\lim_{n \to \infty} P(A_Y^n) = 0.
\]

Next, using Lemma 2.2 (ii) and (iii), for an arbitrary \( M > 0 \) we obtain

\[
P(B_Y^n \cap \{ C_* \leq M \}) = P \left( \exists k \in \{ q, \ldots, n - q \} \text{ such that } |H_n(k) - G_n| > \epsilon \right.
\]

\[
\text{and } |-G_n - T_n(k)| > \epsilon, C_* \leq M \bigg) 
\]

\[
\leq P \left( |G_n| > \frac{\epsilon}{2}, C_* \leq M \right) + \sum_{k=q}^{n-q} P \left( |H_n(k)| > \frac{\epsilon}{2} \text{ and } |T_n(k)| > \frac{\epsilon}{2}, C_* \leq M \right).
\]
Note that
\[ P \left( |G_n| > \frac{\epsilon}{2}, C_* \leq M \right) \leq P \left( C_* \sum_{u=0}^{q-1} \frac{|Z_{-u}|}{a_n} > \frac{\epsilon}{2}, C_* \leq M \right) \]
\[ \leq P \left( \sum_{u=0}^{q-1} \frac{|Z_{-u}|}{a_n} > \frac{\epsilon}{2M} \right) \]
\[ \leq q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right). \]

Similarly
\[ P \left( |H_n(k)| > \frac{\epsilon}{2} \right. \left. \text{and } |T_n(k)| > \frac{\epsilon}{2}, C_* \leq M \right) \]
\[ \leq P \left( \sum_{u=0}^{q-1} \frac{|Z_{k-u}|}{a_n} > \frac{\epsilon}{2M} \right. \left. \text{and } \sum_{u=1}^{q} \frac{|Z_{k+u}|}{a_n} > \frac{\epsilon}{2M} \right) \]
\[ = P \left( \sum_{u=0}^{q-1} \frac{|Z_{k-u}|}{a_n} > \frac{\epsilon}{2M} \right) P \left( \sum_{u=1}^{q} \frac{|Z_{k+u}|}{a_n} > \frac{\epsilon}{2M} \right) \]
\[ \leq \left[ q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right) \right]^2, \]
where the equality above holds since the random variables \( Z_i \) are independent. Therefore
\[ P(B_n^Y \cap \{ C_* \leq M \}) \leq q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right) + \sum_{k=q}^{n-q} \left[ q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right) \right]^2 \]
\[ \leq q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right) + \frac{q^2}{n} \left[ q P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM} \right) \right]^2 \]
and an application of the regular variation property yields
\[ \lim_{n \to \infty} P(B_n^Y \cap \{ C_* \leq M \}) = 0. \]

Thus
\[ \limsup_{n \to \infty} P(B_n^Y \cap \{ C_* \leq M \}) \leq \limsup_{n \to \infty} P(B_n^Y \cap \{ C_* > M \}) \leq P(C_* > M), \]
and letting again \( M \to \infty \) we conclude
\[ \lim_{n \to \infty} P(B_n^Y) = 0. \]  \hspace{1cm} (2.9)

In a similar manner as in (2.7), but using (ii) from Lemma 2.2 instead of (i) we get
\[ \lim_{n \to \infty} P(C_n^Y) = 0. \]  \hspace{1cm} (2.10)

From relations (2.6), (2.8), (2.9) and (2.10) we obtain
\[ \lim_{n \to \infty} P(Y_n > \epsilon) = 0. \]  \hspace{1cm} (2.11)

It remains to estimate the second term on the right hand side of (2.5). For each \( k \geq q \), set \( V_k^{Z,\min} = \min\{ CV_n^Z((k-q)/n), CV_n^Z(k/n) \} \) and \( V_k^{Z,\max} = \max\{ \).
$CV^Z_n((k-q)/n), CV^Z_n(k/n))$. From the definition of $T_n$, the Hausdorff metric and the number $n$ it follows

\[ \{ T_n > \epsilon \} \subseteq \{ \exists a \in \Gamma_{V_n} \text{ such that } d(a, b) > \epsilon \text{ for every } b \in \Gamma_{CV^Z_n} \} \]

\[ \subseteq \{ \exists k \in \{1, \ldots, 2q - 1\} \text{ such that } |V_n(k/n) - CV^Z_n(k/n)| > \epsilon \} \]

\[ \cup \{ \exists k \in \{2q, \ldots, n\} \text{ such that } d(V_n(k/n), [V_k^{Z,\min}, V_k^{Z,\max}]) > \epsilon \} \]

\[ =: A^T_n \cup B^T_n, \quad (2.12) \]

where $d$ is the Euclidean metric on $\mathbb{R}$. The argument behind the second inclusion in (2.12) is similar to the one given after (2.6). Indeed, assume there is a point $a = (t_a, x_a) \in \Gamma_{V_n}$ such that

\[ d(a, \Gamma_{CV^Z_n}) > \epsilon. \quad (2.13) \]

Then necessarily $t_a \in [i/n, (i + 1)/n)$ for some $i = 1, \ldots, n$. The case $i \leq 2q - 1$ is covered by the same argument used to obtain (2.6) and the set $A^Y_n$. Therefore, we may assume $i \geq 2q$. From (2.13) we immediately obtain

\[ d(a, (i/n, CV^Z_n(i/n))) > \epsilon \quad \text{and} \quad d(a, ((i-q)/n, CV^Z_n((i-q)/n))) > \epsilon. \quad (2.14) \]

Suppose first that $x_a = V_n(i/n)$ for some $i = 2q, \ldots, n$. Recall that $q/n < \epsilon/2$. Since $\max\{|t_a - i/n|, |t_a - (i-q)/n|\} \leq (q+1)/n < \epsilon$, from (2.14) we conclude that

\[ \text{If } x_a \in [V_n((i-1)/n), V_n(i/n)) \text{ (in this case } t_a = i/n), \text{ relation } (2.14) \text{ again implies } \quad d(V_n(i/n), [V_i^{Z,\min}, V_i^{Z,\max}]) > \epsilon. \]

\[ \text{If } x_a \in [V_n((i-1)/n), V_n(i/n)) \text{ (in this case } t_a = i/n), \text{ relation } (2.14) \text{ again implies } \quad d(V_n((i-1)/n), [V_i^{Z,\min}, V_i^{Z,\max}]) > \epsilon. \]

Thus we obtain

\[ \max\{ d(V_n(i/n), [V_i^{Z,\min}, V_i^{Z,\max}]), d(V_n((i-1)/n), [V_{i-1}^{Z,\min}, V_{i-1}^{Z,\max}]) \} > \epsilon. \]

Finally we conclude that there exists $k \in \{2q, \ldots, n\}$ such that

\[ \text{If } x_a \in [V_n((i-1)/n), V_n(i/n)) \text{ (in this case } t_a = i/n), \text{ relation } (2.14) \text{ again implies } \quad d(V_n(k/n), [V_k^{Z,\min}, V_k^{Z,\max}]) > \epsilon. \]

Using Lemma 2.2 (i) and (ii), one could similarly as before for the set $A^Y_n$ obtain

\[ \lim_{n \to \infty} P(A^T_n) = 0. \quad (2.15) \]

Note that $P(B^T_n)$ is bounded above by

\[ P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} > V_k^{Z,\max} + \epsilon \right) \]

\[ + \ P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } \sum_{i=1}^{k} \frac{X_i}{a_n} < V_k^{Z,\min} - \epsilon \right). \]

In the sequel we consider only the first of these two probabilities, since the other one can be handled in a similar manner. The first probability using Lemma 2.2 can be bounded by

\[ P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } G_n - H_n(k) > \epsilon \text{ and } G_n + T_n(k-q) > \epsilon \right) \]

\[ \leq \ P \left( G_n > \frac{\epsilon}{2} \right) \]

\[ + \ P \left( \exists k \in \{2q, \ldots, n\} \text{ such that } H_n(k) < -\frac{\epsilon}{2} \text{ and } T_n(k-q) > \frac{\epsilon}{2} \right). \]
From the calculations yielding (2.9) we conclude that $P(G_n > \epsilon/2) \to 0$ as $n \to \infty$. The second term is bounded by

$$P(C_* > M) + \sum_{k=2q}^{n} P\left(H_n(k) < -\frac{\epsilon}{2} \text{ and } T_n(k-q) > \frac{\epsilon}{2}, C_* \leq M\right)$$  \hspace{1cm} (2.16)

for an arbitrary $M > 0$. Note that

$$H_n(k) = \frac{Z_{k-u}}{a_n} \sum_{s=u+1}^{q} C_s \text{ and } T_n(k-q) = \frac{Z_{k-u}}{a_n} \sum_{s=0}^{u} C_s.$$  

Therefore for a fixed $k \in \{2q, \ldots, n\}$, on the event $\{H_n(k) < -\epsilon/2 \text{ and } T_n(k-q) > \epsilon/2, C_* \leq M\}$ there exist $i, j \in \{0, \ldots, q-1\}$ such that

$$\frac{Z_{k-i}}{a_n} \sum_{s=i+1}^{q} C_s < -\frac{\epsilon}{2q} \text{ and } \frac{Z_{k-i}}{a_n} \sum_{s=0}^{j} C_s > \frac{\epsilon}{2q}.$$  

From (2.1) it follows that the sums $\sum_{s=0}^{j} C_s$ and $\sum_{s=i+1}^{q} C_s$ are a.s. of the same sign and their absolute values are bounded by $C_*$. Hence if these sums are positive we obtain $Z_{k-i}M/a_n < -\epsilon/(2q)$ and $Z_{k-j}M/a_n > \epsilon/(2q)$, while if they are negative we obtain $Z_{k-i}M/a_n > \epsilon/(2q)$ and $Z_{k-j}M/a_n < -\epsilon/(2q)$. Note that the case $i = j$ is not possible since then we would have $Z_{k-i} < 0$ and $Z_{k-i} > 0$. From this, using the stationarity of the sequence $(Z_i)$, we conclude that the expression in (2.16) is bounded by

$$P(C_* > M) + n P\left(\exists i, j \in \{0, \ldots, q-1\}, i \neq j \text{ s.t. } M \frac{Z_{k-i}}{a_n} < -\frac{\epsilon}{2q} \text{ and } M \frac{Z_{k-j}}{a_n} > \frac{\epsilon}{2q}\right)$$

$$\leq P(C_* > M) + n \left(\frac{q}{2}\right)^2 P\left(\frac{|Z_0|}{a_n} > \frac{\epsilon}{2qM}\right)^2,$$

which tends to 0 if we first let $n \to \infty$ and then $M \to \infty$. Together with relations (2.12) and (2.15) this implies

$$\lim_{n \to \infty} P(T_n > \epsilon) = 0.$$  \hspace{1cm} (2.17)

Now from (2.5), (2.11) and (2.17) we obtain

$$\lim_{n \to \infty} P[d_{M_*}(CV_n^Z, V_n) > \epsilon] = 0,$$  \hspace{1cm} (2.18)

and finally we conclude that $V_n(\cdot) \overset{d}{\to} \tilde{C}V(\cdot)$, as $n \to \infty$, in $(D[0,1], d_{M_*})$. This concludes the proof. \hfill \Box

3. Infinite order MA processes

Let $(X_i)$ be a moving average process defined by

$$X_i = \sum_{j=0}^{\infty} C_j Z_{i-j}, \quad i \in \mathbb{Z},$$

where $(Z_i)$ is an i.i.d. sequence of regularly varying random variables with index $\alpha \in (0, 2)$, such that $EZ_i = 0$ if $\alpha \in (1, 2)$ and $Z_i$ is symmetric if $\alpha = 1$. Let $\{C_i, i =$
0, 1, 2, \ldots \} be a sequence of random variables, independent of \((Z_i)\), satisfying
\[
\sum_{i=0}^{\infty} E|C_i|^{\delta} < \infty \quad \text{for some } \delta < \alpha, \; 0 < \delta \leq 1, \quad (3.1)
\]
and
\[
0 \leq \sum_{i=0}^{s} C_i \bigg/ \sum_{i=0}^{\infty} C_i \leq 1 \text{ a.s. for every } s = 0, 1, 2, \ldots \quad (3.2)
\]
Let \(C = \sum_{i=0}^{\infty} C_i\). Condition (3.1) implies \(C\) is a.s. finite, and ensures that the series in the definition of \(X_i\) above converges almost surely. Define further the corresponding partial sum stochastic process \(V_n\) as in \((2.2)\). Beside the above stated conditions, we require also the following conditions: for \(\alpha \in (0, 1)\)
\[
\sum_{i=0}^{\infty} E|C_i|^{\gamma} < \infty \quad \text{for some } \gamma \in (\alpha, 1), \quad (3.3)
\]
and for \(\alpha \in [1, 2)\)
\[
\lim_{n \to \infty} (\ln n)^{1+\eta} E \left[ \left( \sum_{i=n}^{\infty} |C_i| \right)^{\eta-\delta} \sum_{j=n}^{\infty} |C_j|^{\delta} \right] = 0 \quad \text{for some } \eta > \alpha. \quad (3.4)
\]
The latter condition is borrowed from Avram and Taqqu \((1992)\), where they studied \(M_1\) functional convergence of sums of moving averages with deterministic coefficients. Since in the case \(\alpha \in (1, 2)\) we will also need that the series \(\sum_{i=0}^{\infty} E|C_i|\) converges, we assume \(\delta = 1\) in \((3.1)\) if \(\alpha > 1\).

For a deterministic sequence \((C_j)\) condition \((3.3)\) is not needed since it is implied by \((3.1)\). The latter in general does not hold when the coefficients \(C_j\) are random. It can easily be seen by the following example. Take \(\epsilon > 0\) such that \(\delta + \epsilon < \gamma\). Let \(S = \sum_{j=1}^{\infty} j^{-(1+\delta+\epsilon)} < \infty\) and \(S_k = S^{-1} \sum_{j=1}^{k} j^{-(1+\delta+\epsilon)}\), \(k \in \mathbb{N}\) (with \(S_0 = 0\)). Taking \(P\) to be the Lebesgue measure on the Borel subsets of \((0, 1)\) and
\[
C_i(\omega) = i \mathbb{I}_{(S_{i-1}, S_i]}(\omega), \quad \omega \in (0, 1), \; i \in \mathbb{N},
\]
we obtain
\[
\sum_{i=1}^{\infty} E|C_i|^{\delta} = S^{-1} \sum_{i=1}^{\infty} i^{\delta}(S_i - S_{i-1}) = S^{-1} \sum_{i=1}^{\infty} \frac{1}{i^{1+\epsilon}} < \infty,
\]
and
\[
\sum_{i=1}^{\infty} E|C_i|^{\gamma} = S^{-1} \sum_{i=1}^{\infty} i^{\gamma}(S_i - S_{i-1}) = S^{-1} \sum_{i=1}^{\infty} \frac{1}{i^{1+\delta+\epsilon-\gamma}} = \infty,
\]
since \(1 + \delta + \epsilon - \gamma < 1\).

\textbf{Theorem 3.1.} Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \(\alpha \in (0, 2)\). Suppose that conditions \((1.5)\) and \((1.6)\) hold. Let \(\{C_i, i = 0, 1, 2, \ldots\}\) be a sequence of random variables, independent of \((Z_i)\), such that \((3.1)\) and \((3.2)\) hold. Assume also \((3.3)\) holds if \(\alpha \in (0, 1)\), and \((3.4)\) if \(\alpha \in [1, 2)\). Then
\[
V_n(\cdot) \overset{d}{\to} \tilde{CV}(\cdot), \quad n \to \infty,
\]
We have to show \( \epsilon > b \). and note that the probability in (3.5) is bounded above by

\[
\text{Let } \alpha \in (0, 1) \cup (1, 2), \text{ and assume conditions (3.1) and (3.3) hold. Then for every } \epsilon > 0
\]

\[
\lim_{q \to \infty} \limsup_{n \to \infty} P \left[ \left( 2 \sum_{j=q+1}^{\infty} |C_j| \right) \sum_{i=1}^{n} \frac{|Z_{i-q}|}{a_n} + \sum_{i=-\infty}^{0} \frac{|Z_{i-q}|}{a_n} \sum_{j=1}^{n} |C_{q-i+j}| > \epsilon \right] = 0.
\]

**Proof:** Let

\[
D_{i}^{n,q} = \begin{cases} 
2 \sum_{j=q+1}^{\infty} |C_j|, & i = 1, \ldots, n, \\
\sum_{j=1}^{n} |C_{q-i+j}|, & i \leq 0.
\end{cases}
\]

We have to show

\[
\lim_{q \to \infty} \limsup_{n \to \infty} P \left( \sum_{i=-\infty}^{n} \frac{D_{i}^{n,q} |Z_{i-q}|}{a_n} > \epsilon \right) = 0. \tag{3.5}
\]

Let

\[
Z_{i,n}^\leq = \frac{Z_i}{a_n} 1\{ \frac{|Z_i|}{a_n} \leq 1 \} \quad \text{and} \quad Z_{i,n}^\geq = \frac{Z_i}{a_n} 1\{ \frac{|Z_i|}{a_n} > 1 \},
\]

and note that the probability in (3.5) is bounded above by

\[
P \left( \sum_{i=-\infty}^{n} D_{i}^{n,q} |Z_{i-q,n}^\leq| > \epsilon \right) + P \left( \sum_{i=-\infty}^{n} D_{i}^{n,q} |Z_{i-q,n}^\geq| > \epsilon \right). \tag{3.6}
\]

Using Markov’s inequality, the triangle inequality \( |\sum_{i=1}^{\infty} a_i|^s \leq \sum_{i=1}^{\infty} |a_i|^s \) with \( s \in (0, 1) \), the fact that \((C_i)\) is independent of \((Z_i)\) and the stationarity of the sequence \((Z_i)\), for the first term in (3.6) we obtain

\[
P \left( \sum_{i=-\infty}^{n} D_{i}^{n,q} |Z_{i-q,n}^\leq| > \epsilon \right) \leq \left( \frac{\epsilon}{2} \right)^{-\gamma} E \left( \sum_{i=-\infty}^{n} D_{i}^{n,q} |Z_{i-q,n}^\leq|^\gamma \right)
\]

\[
\leq \left( \frac{\epsilon}{2} \right)^{-\gamma} E \left( \sum_{i=-\infty}^{n} (D_{i}^{n,q})^\gamma |Z_{i-q,n}^\leq|^\gamma \right)
\]

\[
\leq \left( \frac{\epsilon}{2} \right)^{-\gamma} E |Z_{i,n}^\leq|^\gamma \sum_{i=-\infty}^{n} E(D_{i}^{n,q})^\gamma.
\]
Similarly, note that \( E[|C_j|^\gamma] \), for \( j = q + 1, q + 2, \ldots \), appears in the sum \( \sum_{i=-\infty}^{0} n \sum_{j=q+1}^{\infty} E[|C_{q-i+j}|^\gamma] \) at most \( n \) times, and hence

\[
P \left( \sum_{i=-\infty}^{n} D_i^{n,q} |Z_{i-q,n}| > \frac{\epsilon}{2} \right) \leq \left( \frac{\epsilon}{2} \right)^{-\gamma} n E[Z_{i,n}]^{\gamma} \left( 2^\gamma n \sum_{j=q+1}^{\infty} E[|C_j|^\gamma] + n \sum_{j=q+1}^{\infty} E[|C_j|^\gamma] \right)
\]

Again by triangle inequality we have

\[
\sum_{i=-\infty}^{n} E[D_i^{n,q}] \leq 2^\gamma n \sum_{j=q+1}^{\infty} E[|C_j|^\gamma] + \sum_{i=-\infty}^{0} n \sum_{j=q+1}^{\infty} E[|C_{q-i+j}|^\gamma],
\]

If \( (C_i) \), conditions (3.1) and (3.3) imply (3.5).

Similarly,

\[
P \left( \sum_{i=-\infty}^{n} D_i^{n,q} |Z_{i-q,n}| > \frac{\epsilon}{2} \right) \leq (2^\delta + 1) \left( \frac{\epsilon}{2} \right)^{-\delta} n E[Z_{i,n}]^{\delta} \sum_{j=q+1}^{\infty} E[|C_j|^\delta].
\]

By Karamata’s theorem and (1.7), as \( n \to \infty \),

\[
n E[Z_{i,n}]^\gamma = \frac{E(|Z_1|^\gamma 1_{|Z_1| \leq a_n})}{a_n P(|Z_1| > a_n)} \cdot n P(|Z_1| > a_n) \to \frac{\alpha}{\gamma - \alpha} < \infty
\]

and

\[
n E[Z_{i,n}]^\delta = \frac{E(|Z_1|^\delta 1_{|Z_1| > a_n})}{a_n P(|Z_1| > a_n)} \cdot n P(|Z_1| > a_n) \to \frac{\alpha}{\alpha - \delta} < \infty.
\]

From this and relations (3.7) and (3.8) we conclude that

\[
\limsup_{n \to \infty} P \left( \sum_{i=-\infty}^{n} D_i^{n,q} |Z_{i-q,n}| > \epsilon \right) \leq M \left( \sum_{j=q+1}^{\infty} E[|C_j|^\gamma] + \sum_{j=q+1}^{\infty} E[|C_j|^\delta] \right),
\]

where \( M = (2^\gamma + 1)(\epsilon/2)^{-\gamma} \alpha/(\gamma - \alpha) + (2^\delta + 1)(\epsilon/2)^{-\delta} \alpha/(\alpha - \delta) < \infty \). Now letting \( q \to \infty \), conditions (3.1) and (3.3) imply (3.5). \( \square \)

**Lemma 3.3.** Let \( \alpha \in (1, 2) \). Assume conditions (3.1) and (3.4) hold, and \( EZ_1 = 0 \). If \( (q_n) \) is a sequence of positive integers tending to infinity, then

\[
\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{[nt]} C_{q_n}^n Z_{i-q_n} \right| > \epsilon \right) = 0 \quad \text{for every } \epsilon > 0,
\]

where \( C_{q_n}^n = \sum_{j=q_n+1}^{\infty} C_j \).

**Proof:** Let \( \tilde{Z}_{i,n}^\leq = Z_{i,n}^\leq - EZ_{i,n}^\leq \) and \( \tilde{Z}_{i,n}^> = Z_{i,n}^> + EZ_{i,n}^> \), and note that \( Z_{i,n}^\leq/a_n = \tilde{Z}_{i,n}^\leq + \tilde{Z}_{i,n}^> \), \( EZ_{i,n}^\leq = 0 \) and also \( EZ_{i,n}^> = EZ_{i,n}^> + EZ_{i,n}^\leq = E(Z_i/a_n) = 0 \). Thus

\[
P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{[nt]} C_{q_n}^n Z_{i-q_n} \right| > \epsilon \right)
\]
maximal inequality by C is a martingale (with respect to the filtration and thus condition (3.4) yields
\( \lim_{n \to \infty} \sum_{i=1}^{k} C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq} \geq \frac{\epsilon}{2} \) + \( \lim_{n \to \infty} \sum_{i=1}^{k} C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq} \geq \frac{\epsilon}{2} \)
\[ \leq \text{P} \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq} \right| \right) + \text{P} \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq} \right| \right) \]
\[ =: I_1 + I_2. \]
Since \( C_{q_{n-1}}^{\alpha} \) is independent of \( (Z_{i}) \) and \( E Z_{i-q_{n}}^{\leq} = 0 \), it follows that \( \sum_{i=1}^{k} C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq} \) is a martingale (with respect to the filtration \( (\mathcal{F}_k) \)), where the \( \sigma \)-field \( \mathcal{F}_k \) is generated by \( C_{i}, i \geq 0 \) and \( Z_{j-q_{n}}, j \leq k-q_{n} \). Hence by Markov’s inequality and Doob’s maximal inequality
\[ E \left( \sup_{1 \leq k \leq n} |S_k| \right) \leq \left( \frac{k}{\kappa - 1} \right)^{\kappa} E|S_n|^\kappa, \]
which holds for \( \kappa > 1 \) and \( (S_k)_k \) a martingale (see Durrett, 1996, p. 251) we obtain
\[ I_1 \leq 2^{\left( \frac{\epsilon}{2} \right)^{-\eta}} \left( \frac{\eta}{\eta - 1} \right)^{\eta} \sum_{i=1}^{n} E|C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq}|^{\eta}, \]
with \( \eta \) as in (3.4). Note that \( (C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq})_{i} \) is a martingale difference sequence, and hence by the Bahr- Esseen inequity
\[ E \left| \sum_{j=1}^{n} Y_j \right|^{\kappa} \leq 2^{\left( \frac{\kappa}{\kappa - 1} \right)^{\eta} \sum_{i=1}^{n} E|C_{q_{n-1}}^{\alpha}|^{\eta}, \]
which holds for \( \kappa \in [1, 2] \) and \( (Y_j)_j \) a martingale-difference sequence (see Chatterji, 1969, Lemma 1) we have
\[ I_1 \leq 2^{\left( \frac{\epsilon}{2} \right)^{-\eta}} \left( \frac{\eta}{\eta - 1} \right)^{\eta} \sum_{i=1}^{n} E|C_{q_{n-1}}^{\alpha} Z_{i-q_{n-1}}^{\leq}|^{\eta} \]
\[ = 2^{\left( \frac{\epsilon}{2} \right)^{-\eta}} \left( \frac{\eta}{\eta - 1} \right)^{\eta} n E|Z_{1-n}^{\leq}|^{\eta} E|C_{q_{n-1}}^{\alpha}|^{\eta}. \]
Using the inequality \( |a - b|^n \leq 2^n (|a|^n + |b|^n) \) and a special case of Jensen’s inequality
\[ (E|Y|^\kappa)^{\epsilon} \leq E|Y|^\epsilon \]
(which holds for \( \kappa \geq 1 \) we have
\[ E|Z_{1-n}^{\leq}|^{\eta} \leq 2^{\eta} E|Z_{1-n}^{\leq}|^{\eta} + (E|Z_{1-n}^{\leq}|)^{\eta} \leq 2^{\eta+1} E|Z_{1-n}^{\leq}|^{\eta}, \]
and hence
\[ I_1 \leq 2^{2+2\eta} \epsilon^{-\eta} \left( \frac{\eta}{\eta - 1} \right)^{\eta} n E|Z_{1-n}^{\leq}|^{\eta} E|C_{q_{n-1}}^{\alpha}|^{\eta}. \]
Note that
\[ E|C_{q_{n-1}}^{\alpha}|^{\eta} = E((C_{q_{n-1}}^{\alpha})^{\eta - \delta} \cdot |C_{q_{n-1}}^{\alpha}|^{\delta}) \leq E \left[ \left( \sum_{i=q_{n+1}}^{\infty} |C_i| \right)^{\eta - \delta} \sum_{j=q_{n+1}}^{\infty} |C_j|^{\delta} \right], \]
and thus condition (3.4) yields \( \lim_{n \to \infty} E|C_{q_{n-1}}^{\alpha}|^{\eta} = 0 \). This and the fact that \( \lim_{n \to \infty} n E|Z_{1-n}^{\leq}|^{\eta} = \alpha/(\eta - \alpha) \) (which holds by Karamata’s theorem) allows us to conclude that \( \lim_{n \to \infty} I_1 = 0. \)
For $I_2$ by Markov’s inequality we obtain

$$I_2 \leq P \left( \sum_{i=1}^{n} |C''_{q_n} \tilde{Z}_{i-q_n,n}| > \frac{\epsilon}{2} \right) \leq \left( \frac{\epsilon}{2} \right)^{-1} E|\tilde{Z}_{1,n}^\prec| \sum_{i=1}^{n} E|C''_{q_n}|.$$ 

Since $\tilde{Z}_{i,n}^\prec = Z_{i,n}^\prec - EZ_{i,n}^\prec$, it holds that

$$E|\tilde{Z}_{1,n}^\prec| \leq E|Z_{1,n}^\prec| + |EZ_{1,n}^\prec| \leq 2E|Z_{1,n}^\prec|.$$ (3.11)

Therefore

$$I_2 \leq 4\epsilon^{-1} nE|Z_{1,n}^\prec| \sum_{j=q_n+1}^{\infty} E|C|,$$

yielding $\lim_{n \to \infty} I_2 = 0$, since by Karamata’s theorem $\lim_{n \to \infty} nE|Z_{1,n}^\prec| = \alpha/(\alpha-1)$ and we assumed (3.1) holds with $\delta = 1$ in this case. This completes the proof of the lemma. $\Box$

Lemma 3.4. Let $\alpha \in (1, 2)$. Assume conditions (3.1) and (3.4) hold, and $EZ_1 = 0$. If $(q_n)$ is a sequence of positive integers tending to infinity, such that $\ln n = O(\ln q_n)$, then

$$\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} C_j Z_{i-j,n} \right) > \epsilon \right) = 0 \quad \text{for every } \epsilon > 0.$$

Proof: Note that

$$P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} C_j Z_{i-j,n} \right| > \epsilon \right) \leq P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} C_j \tilde{Z}_{i-j,n} \right| > \epsilon \right) + P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} C_j \tilde{Z}_{i-j,n} \right| > \epsilon \right)$$

$$=: I_3 + I_4.$$ 

Let

$$W_n(t) = \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} C_j \tilde{Z}_{i-j,n}, \quad t \in [0, 1].$$

Take $0 \leq t_1 < t_2 \leq 1$, and consider (for $\rho > 0$)

$$P(|W_n(t_2) - W_n(t_1)| > \rho) = P \left( \left| \sum_{i=1}^{\lfloor nt_2 \rfloor} \sum_{j=q_n+1}^{\infty} C_j \tilde{Z}_{i-j,n} \right| > \rho \right)$$

$$= P \left( \left| \sum_{i=-\infty}^{\lfloor nt_2 \rfloor - 1} \bar{D}_{i-q_n}^{n,t_1,t_2} \tilde{Z}_{i-q_n,n} \right| > \rho \right),$$ (3.12)
where

\[
\tilde{D}_{i-q_n}^{n,t_1,t_2} = \begin{cases} 
q_n-i+\lfloor nt_2 \rfloor & i \leq \lfloor nt_1 \rfloor - 1, \\
\sum_{j=q_n-i+\lfloor nt_1 \rfloor +1}^{q_n-i+\lfloor nt_2 \rfloor} C_j, & i = \lfloor nt_1 \rfloor, \ldots, \lfloor nt_2 \rfloor - 1,
\end{cases}
\]

and the last equality in (3.12) follows by standard changes of variables and order of summation. The sequence \((\tilde{D}_{i-q_n}^{n,t_1,t_2} \tilde{Z}_{i-q_n,n})\) is a martingale difference sequence, and hence the Bahr-Esseen inequality (which holds also for infinite sums, by the Fatou lemma) and Markov’s inequality imply

\[
P\left( \left\| \sum_{i=-\infty}^{\lfloor nt_2 \rfloor-1} \tilde{D}_{i-q_n}^{n,t_1,t_2} \tilde{Z}_{i-q_n,n} \right\| > \rho \right) \leq \rho^{-\eta} \left( \frac{\eta}{\eta-1} \right)^{\eta} \sum_{i=-\infty}^{\lfloor nt_2 \rfloor-1} \mathbb{E} \left| \tilde{D}_{i-q_n}^{n,t_1,t_2} \tilde{Z}_{i-q_n,n} \right|^{\eta} \\
= \rho^{-\eta} \left( \frac{\eta}{\eta-1} \right)^{\eta} \mathbb{E} \left| \tilde{Z}_{t_1,n} \right|^{\eta} \sum_{i=-\infty}^{\lfloor nt_2 \rfloor-1} \mathbb{E} \left| \tilde{D}_{i-q_n}^{n,t_1,t_2} \right|^{\eta},
\]

with \(\eta\) as in (3.4). With the same argument as in (3.10) we obtain

\[
\sum_{i=\lfloor nt_1 \rfloor}^{\lfloor nt_2 \rfloor-1} \mathbb{E} \left| \tilde{D}_{i-q_n}^{n,t_1,t_2} \right|^{\eta} \\
\leq \sum_{i=\lfloor nt_1 \rfloor}^{\lfloor nt_2 \rfloor-1} \mathbb{E} \left( \sum_{s=q_n+1}^{\infty} |C_s| \right)^{\eta-\delta} \sum_{j=q_n+1}^{\infty} |C_j|^{\delta} \\
= (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor) \mathbb{E} \left( \sum_{s=q_n+1}^{\infty} |C_s| \right)^{\eta-\delta} \sum_{j=q_n+1}^{\infty} |C_j|^{\delta},
\]

(3.13)

and

\[
\sum_{i=-\infty}^{\lfloor nt_1 \rfloor-1} \mathbb{E} \left| \tilde{D}_{i-q_n}^{n,t_1,t_2} \right|^{\eta} \\
\leq \mathbb{E} \left( \sum_{s=q_n+1}^{\infty} |C_s| \right)^{\eta-\delta} \sum_{i=-\infty}^{\lfloor nt_1 \rfloor-1} \sum_{j=q_n-i+\lfloor nt_1 \rfloor +1}^{q_n-i+\lfloor nt_2 \rfloor} |C_j|^{\delta} \\
\leq (\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor) \mathbb{E} \left( \sum_{s=q_n+1}^{\infty} |C_s| \right)^{\eta-\delta} \sum_{j=q_n+1}^{\infty} |C_j|^{\delta},
\]

(3.14)

where the last inequality follows from the fact that every \(|C_j|^{\delta}\), for \(j \geq q_n + 1\), appears in the sum \(\sum_{i=-\infty}^{\lfloor nt_1 \rfloor-1} \sum_{j=q_n-i+\lfloor nt_1 \rfloor +1}^{q_n-i+\lfloor nt_2 \rfloor} |C_j|^{\delta}\) at most \(\lfloor nt_2 \rfloor - \lfloor nt_1 \rfloor\) times. Therefore

\[
P\left( \left\| \sum_{i=-\infty}^{\lfloor nt_2 \rfloor-1} \tilde{D}_{i-q_n}^{n,t_1,t_2} \tilde{Z}_{i-q_n,n} \right\| > \rho \right)
\]
Since by (3.9) and Karamata’s theorem $\sup_n \{nE[Z_{1,n}^\leq]n\} < \infty$, and $(|nt_2| - |nt_1|)/n \leq 2(t_2 - t_1)$ for large $n$, it follows that

$$P\left(\left|\sum_{i=-\infty}^{nt_2-1} D_{i-q_n}^{n,t_1,t_2} Z_{i-q_n,n}^{\leq}\right| > \rho\right) \leq M\rho^{-\eta}(t_2 - t_1)E\left[\left(\sum_{s=q_n+1}^{\infty} |C_s|\right)^{\eta-\delta} \sum_{j=q_n+1}^{\infty} |C_j|^\delta\right],$$

for some constant $M$ independent of $n$. Now by Theorem 2 in Avram and Taqqu (1989) and the arguments in the proof of Proposition 4 in Avram and Taqqu (1992) we conclude that

$$P\left(\sup_{0 \leq t \leq 1} \left|\sum_{i=-\infty}^{nt-1} D_{i-q_n}^{n,0,t} Z_{i-q_n,n}^{\leq}\right| > \rho\right) \leq M'\rho^{-\eta}E\left[\left(\sum_{s=q_n+1}^{\infty} |C_s|\right)^{\eta-\delta} \sum_{j=q_n+1}^{\infty} |C_j|^\delta\right]$$

for some constant $M'$ independent of $n$. From this and condition (3.4), since $\ln n = O(\ln q_n)$, it follows that

$$\lim_{n \to \infty} I_3 = \lim_{n \to \infty} P\left(\sup_{0 \leq t \leq 1} \left|\sum_{i=-\infty}^{nt-1} D_{i-q_n}^{n,0,t} Z_{i-q_n,n}^{\leq}\right| > \frac{\epsilon}{2}\right) = 0. \quad (3.15)$$

Further, note that

$$P\left(\sup_{0 \leq t \leq 1} \left|\sum_{i=1}^{nt} \sum_{j=q_n+1}^{\infty} C_j Z_{i-j,n}^{>}\right| > \frac{\epsilon}{2}\right) \leq P\left(\sum_{i=1}^{n} \sum_{j=q_n+1}^{\infty} |C_j Z_{i-j,n}^{>}| > \frac{\epsilon}{2}\right)
\leq \left(\frac{\epsilon}{2}\right)^{-1} \sum_{i=1}^{n} \sum_{j=q_n+1}^{\infty} |E[C_j Z_{i-j,n}^{>}]|
= \left(\frac{\epsilon}{2}\right)^{-1} |E[Z_{1,n}^{>}]| \sum_{i=1}^{n} \sum_{j=q_n+1}^{\infty} E[C_j]
= \left(\frac{\epsilon}{2}\right)^{-1} nE[Z_{1,n}^{>}] \sum_{j=q_n+1}^{\infty} E[C_j].$$

By (3.11) and Karamata’s theorem $\sup_n \{nE[Z_{1,n}^{>}]\} < \infty$, and hence condition (3.1) (with $\delta = 1$) implies

$$\lim_{n \to \infty} I_4 = 0. \quad (3.16)$$

Now from (3.15) and (3.16) we get the conclusion of the lemma. \hfill \Box

We are now ready to prove Theorem 3.1.

Proof: (Theorem 3.1) For $q \in \mathbb{N}$ define

$$X_i^q = \sum_{j=0}^{q-1} C_j Z_{i-j} + C'_q Z_{i-q}, \quad i \in \mathbb{Z},$$
where \( C_q' = \sum_{i=q}^{\infty} C_i \), and
\[
V_{n,q}(t) = \sum_{i=1}^{\lfloor nt \rfloor} X_i^q \frac{q_i}{a_n}, \quad t \in [0, 1].
\]

Now we treat separately the cases \( \alpha \in (0,1), \alpha \in (1,2) \) and \( \alpha = 1 \).

Case \( \alpha \in (0,1) \). Fix \( q \in \mathbb{N} \). Since the coefficients \( C_0, \ldots, C_{q-1}, C_q' \) satisfy condition (2.1), an application of Theorem 2.1 to a finite order moving average process \( (X_i^q)_i \) yields that, as \( n \to \infty \),
\[
V_{n,q}(\cdot) \overset{d}{\to} \widetilde{CV}(\cdot)
\]
in \((D[0,1], d_{M_2})\). If we show that for every \( \epsilon > 0 \)
\[
\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P}[d_{M_2}(V_{n,q}, V_n) > \epsilon] = 0,
\]
then by a generalization of Slutsky’s theorem (see for instance Theorem 3.5 in Resnick, 2007) it will follow \( V_n(\cdot) \overset{d}{\to} \widetilde{CV}(\cdot) \), as \( n \to \infty \), in \((D[0,1], d_{M_2})\). Since the Skorohod \( M_2 \) metric on \( D[0,1] \) is bounded above by the uniform metric on \( D[0,1] \), it suffices to show that
\[
\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) = 0.
\]
Recalling the definitions, we have
\[
\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) \leq \lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sum_{i=1}^{n} \frac{|X_i^q - X_i|}{a_n} > \epsilon \right).
\]
Put \( C''_q = C_q' - C_q = \sum_{j=q+1} C_j \) and observe
\[
\sum_{i=1}^{n} |X_i^q - X_i| = \sum_{i=1}^{n} \left| \sum_{j=0}^{q-1} C_j Z_i-j + C_q' Z_i-q - \sum_{j=0}^{\infty} C_j Z_i-j \right|
\]
\[
= \sum_{i=1}^{n} \left| C''_q Z_i-q - \sum_{j=q+1}^{\infty} C_j Z_i-j \right|
\]
\[
\leq \sum_{i=1}^{n} \left[ |C''_q| |Z_i-q| + \sum_{j=q+1}^{\infty} |C_j| |Z_i-j| \right]
\]
\[
\leq \left( 2 \sum_{j=q+1}^{\infty} |C_j| \right) \sum_{i=1}^{n} |Z_i-q| + \sum_{i=-\infty}^{0} |Z_i-q| \sum_{j=1}^{n} |C_{i-1+j}|.
\]
Lemma 3.2 now implies
\[
\lim_{q \to \infty} \limsup_{n \to \infty} \mathbb{P} \left( \sum_{i=1}^{n} \frac{|X_i^q - X_i|}{a_n} > \epsilon \right) = 0,
\]
which means that \( V_n(\cdot) \overset{d}{\to} \widetilde{CV}(\cdot) \), as \( n \to \infty \), in \((D[0,1], d_{M_2})\).
Case $\alpha \in (1, 2)$. Let $(q_n)$ be a sequence of positive integers such that $q_n = \lfloor n^{1/10} \rfloor$. We first show that $\lim_{n \to \infty} P[d_{M_2}(V_{n,q_n}, V_n) > \epsilon] = 0$ for every $\epsilon > 0$. For this, similar to the case $\alpha \in (0, 1)$, it suffices to show that

$$\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} |V_{n,q_n}(t) - V_n(t)| > \epsilon \right) = 0.$$ 

Recalling the definitions, we have

$$V_{n,q_n}(t) - V_n(t) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} (X_i^{q_n} - X_i) = \frac{1}{a_n} \sum_{i=1}^{\lfloor nt \rfloor} \left( C_{q_n} Z_{i-q_n} + \sum_{j=q_n+1}^{\infty} C_j Z_{i-j} \right),$$

and hence

$$P \left( \sup_{0 \leq t \leq 1} |V_{n,q_n}(t) - V_n(t)| > \epsilon \right) \leq P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \frac{C_{q_n} Z_{i-q_n}}{a_n} \right| > \frac{\epsilon}{2} \right) + P \left( \sup_{0 \leq t \leq 1} \left| \sum_{i=1}^{\lfloor nt \rfloor} \sum_{j=q_n+1}^{\infty} \frac{C_j Z_{i-j}}{a_n} \right| > \frac{\epsilon}{2} \right).$$

Applying Lemma 3.3 and Lemma 3.4 we conclude

$$\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} |V_{n,q_n}(t) - V_n(t)| > \epsilon \right) = 0.$$

Thus, in order to have $V_n(\cdot) \overset{d}{\to} \tilde{C}V(\cdot)$ in $D[0,1]$ with the $M_2$ topology, according to Slutsky’s theorem (see Resnick, 2007, Theorem 3.4), it remains to show $V_{n,q_n}(\cdot) \overset{d}{\to} \tilde{C}V(\cdot)$ in $(D[0,1], d_{M_2})$ as $n \to \infty$. Note that we cannot simply use Theorem 2.1 as we did in the case $\alpha \in (0, 1)$, since now $q_n$ depends on $n$. By careful analysis of the proof of Theorem 2.1 we see that relations that have to be checked, in order that the statement of Theorem 2.1 remains valid if we replace $q$ by $q_n$, are (2.8), (2.9) and (2.17) (with $C_q = \sum_{s=0}^{\infty} |C_s|$). Hence we have to establish the following relations

$$\lim_{n \to \infty} (q_n - 1)(2q_n - 1) P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{3(2q_n - 1)M} \right) = 0,$$

$$\lim_{n \to \infty} \left[ q_n P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2q_n M} \right) + nq_n^2 \left( P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2q_n M} \right) \right)^2 \right] = 0,$$

$$\lim_{n \to \infty} \left( \frac{q_n}{2} \right)^2 \left( P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{2q_n M} \right) \right)^2 = 0,$$

for arbitrary $\epsilon > 0$ and $M > 0$. For all of this, taking into consideration relation (1.1), i.e. the regular variation property of $Z_0$, it suffices to show

$$\lim_{n \to \infty} nq_n^2 \left( P \left( \frac{|Z_0|}{a_n} > \frac{\epsilon}{q_n} \right) \right)^2 = 0,$$

which holds by Lemma 4.1 in Appendix. Therefore we conclude $V_n(\cdot) \overset{d}{\to} \tilde{C}V(\cdot)$ in $(D[0,1], d_{M_2})$.

Case $\alpha = 1$. Since $Z_i$ is symmetric, note that $\tilde{Z}_{i,n}^{\leq} = Z_{i,n}^{\leq}$ and $\tilde{Z}_{i,n}^{\geq} = Z_{i,n}^{\geq}$. We proceed as in the case $\alpha \in (1, 2)$ to obtain $\lim_{n \to \infty} I_1 = 0$ and $\lim_{n \to \infty} I_3 = 0$ (with
the notation form the proofs of Lemma 3.3 and Lemma 3.4). For \( I_2 \), by Markov’s inequality and the triangle inequality
\[
|\sum_{i=1}^{n} a_i| \leq \sum_{i=1}^{n} |a_i|^{\delta}
\]
we obtain
\[
I_2 \leq P \left( \sum_{i=q_n}^{n} |C_{q_n} Z_{i-q_n,n}| > \epsilon \right) \leq \left( \frac{\epsilon}{2} \right)^{-\delta} n E|Z_{1,n}|^{\delta} \sum_{i=q_n+1}^{\infty} E|C_j|^{\delta}.
\]
By Karamata’s theorem
\[
\lim_{n \to \infty} n E|Z_{1,n}|^{\delta} = (1 - \delta)^{-1}
\]
and hence from (3.1) we have \( \lim_{n \to \infty} I_2 = 0 \). Similarly we obtain \( \lim_{n \to \infty} I_4 = 0 \). This all allows us to conclude
\[
\lim_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} |V_{n,q_n}(t) - V_n(t)| > \epsilon \right) = 0.
\]
As before, Lemma 4.1 from Appendix and the modified proof of Theorem 2.1 (with \( q \) replaced by \( q_n \)) imply
\[
V_{n,q_n}(\cdot) \xrightarrow{d} CV(\cdot) \quad \text{in} \quad \mathbb{D}[0,1], d_{M_2}.
\]
Now the statement of the theorem follows by an application of Slutsky’s theorem. \( \square \)

**Remark 3.5.** When the sequence of coefficients \((C_j)\) is deterministic, condition (3.4) is not needed. This was shown by Basrak and Krizmanić (2014), but their proof contains an error (i.e. they used Lemma 2 from Avram and Taqqu (1992), but the conditions needed to use this lemma were not fulfilled). Therefore in the proposition below we improve the proof of Theorem 3.1 in Basrak and Krizmanić (2014) in the case \( \alpha \in [1,2) \), thus showing that condition (3.4) can be dropped if all coefficients of the moving average process are deterministic.

**Proposition 3.6.** Let \((Z_i)_{i \in \mathbb{Z}}\) be an i.i.d. sequence of regularly varying random variables with index \( \alpha \in [1,2) \). Suppose that conditions (1.5) and (1.6) hold. Let \( \{C_i, i = 0, 1, 2, \ldots\} \) be a sequence of real numbers satisfying
\[
\sum_{j=0}^{\infty} |C_j|^{\delta} < \infty \quad \text{for some} \quad \delta < \alpha, \quad 0 < \delta \leq 1,
\]
and
\[
0 \leq \sum_{i=0}^{s} C_i \left/ \sum_{i=0}^{\infty} C_i \right. \leq 1 \quad \text{for every} \quad s = 0, 1, 2, \ldots.
\]
Then
\[
V_n(\cdot) \xrightarrow{d} CV(\cdot), \quad n \to \infty,
\]
in \( \mathbb{D}[0,1] \) endowed with the \( M_2 \) topology, where \( C = \sum_{j=0}^{\infty} C_j \), \( V \) is an \( \alpha \)-stable Lévy process with characteristic triple \((0,\mu,b)\), with \( \mu \) as in (1.9) and
\[
b = \begin{cases} 0, & \alpha = 1, \\ (p - r) \frac{\alpha}{1 - \alpha}, & \alpha \in (1,2). \end{cases}
\]

**Proof:** Fix \( q \in \mathbb{N} \), and define
\[
X_i^q = \sum_{j=0}^{q-1} C_j Z_{i-j} + C_q' Z_{i-q}, \quad i \in \mathbb{Z},
\]
where \( C_q' = \sum_{j=q}^{\infty} C_j \), and
\[
V_{n,q}(t) = \sum_{i=1}^{[nt]} \frac{X_i^q}{a_n}, \quad t \in [0,1].
\]
Since the coefficients $C_0, \ldots, C_{q-1}, C_q$ satisfy condition (2.1), an application of Theorem 2.1, adjusted to deterministic coefficients $C_j$, yields $V_{n,q}(\cdot) \overset{d}{\to} CV(\cdot)$ in $(D[0,1],d_{M_2})$ as $n \to \infty$ (see also Theorem 2.1 in Basrak and Krizmanić, 2014). Therefore, in order to have $V_n(\cdot) \overset{d}{\to} CV(\cdot)$ in $D[0,1]$ with the $M_2$ topology, by a generalization of Slutsky’s theorem we have to show that for every $\epsilon > 0$

$$\lim_{q \to \infty} \limsup_{n \to \infty} P[d_{M_2}(V_{n,q}, V_n) > \epsilon] = 0.$$ 

As before it suffices to show that

$$\lim_{q \to \infty} \limsup_{n \to \infty} P \left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) = 0. \quad (3.20)$$

As in the proof of Theorem 3.1 we have

$$P \left( \sup_{0 \leq t \leq 1} |V_{n,q}(t) - V_n(t)| > \epsilon \right) \leq P \left( \sup_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} C''_q Z_{i-q} \right| > \frac{\epsilon}{2} \right) + P \left( \sup_{1 \leq k \leq n} \left| \sum_{j=q+1}^{\infty} \frac{C_j Z_{i-j}}{a_n} \right| > \frac{\epsilon}{2} \right), \quad (3.21)$$

where $C''_q = \sum_{i=q+1}^{\infty} C_i$. Take $\tau > 0$ such that

$$\begin{cases}
\alpha - \tau > \delta, & \text{if } \alpha = 1, \\
\alpha - \tau > 1, & \text{if } \alpha \in (1, 2).
\end{cases}$$

Condition (3.18) implies $\sum_{j=0}^{\infty} |C_j|^{|\alpha-\tau|} < \infty$. Similarly $\sum_{j=0}^{\infty} |C_j| < \infty$. This implies that for large $q$ we have $|C''_q| < 1$, which allows us to apply Lemma 2 in Avram and Taqqu (1992) to the first term on the right-hand side of (3.21), to obtain (for large $q$)

$$P \left( \sup_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} C''_q Z_{i-q} \right| > \frac{\epsilon}{2} \right) \leq M \left( \frac{\epsilon}{2} \right)^{-\alpha + \tau} \frac{1}{n} \sum_{i=1}^{n} |C''_q|^{|\alpha-\tau|} = M \left( \frac{\epsilon}{2} \right)^{-\alpha + \tau} |C''_q|^{|\alpha-\tau|} \quad (3.22)$$

where $M$ is a constant independent of $n$ and $q$. Using the following inequalities

$$\begin{cases}
|\sum_{i=1}^{m} a_i|^s \leq \sum_{i=1}^{m} |a_i|^s, & \text{if } s \leq 1, \\
|\sum_{i=1}^{m} a_i|^s \leq \sum_{i=1}^{m} |a_i|^s, & \text{if } s > 1 \text{ and } |\sum_{i=1}^{m} a_i| < 1,
\end{cases} \quad (3.23)$$

we have

$$|C''_q|^{|\alpha-\tau|} \leq \begin{cases}
\sum_{j=q+1}^{\infty} |C_j|^{|\alpha-\tau|}, & \text{if } \alpha = 1, \\
\sum_{j=q+1}^{\infty} |C_j|, & \text{if } \alpha \in (1, 2),
\end{cases}$$

yielding $\lim_{q \to \infty} |C''_q|^{|\alpha-\tau|} = 0$. Now from (3.22) we obtain

$$\lim_{q \to \infty} \limsup_{n \to \infty} P \left( \sup_{1 \leq k \leq n} \left| \frac{1}{a_n} \sum_{i=1}^{k} C''_q Z_{i-q} \right| > \frac{\epsilon}{2} \right) = 0. \quad (3.24)$$
Note that the second term on the right-hand side of (3.21) is bounded above by

\[
P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} \frac{C_{i,j}^+ Z_{i-j}}{a_n} \right| > \frac{\epsilon}{4} \right) + P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} \frac{C_{i,j}^- Z_{i-j}}{a_n} \right| > \frac{\epsilon}{4} \right),
\]

where \( C_{i,j}^+ = C_j \vee 0 \) and \( C_{i,j}^- = (-C_j) \vee 0 \). In the sequel we consider only the first of these two probabilities since the other one can be handled in the same manner.

Assume first \( \alpha \in (1, 2) \). Recall \( Z_{i,n}^\leq \) and \( Z_{i,n}^> \) from the proof of Lemma 3.3, and note

\[
P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} \frac{C_{i,j}^+ Z_{i-j}}{a_n} \right| > \frac{\epsilon}{4} \right) \leq P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j}^{\leq} \right| > \frac{\epsilon}{8} \right) + P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j,n}^> \right| > \frac{\epsilon}{8} \right)
\]

(3.25)

Since the coefficients \( C_{i,j}^+ \) are nonnegative, the moving average processes

\[
Y_{i,n,q}^{\leq} := \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j,n}^{\leq}, \quad i = 1, 2, \ldots,
\]

are associated, as nondecreasing functions of independent random variables (see Esary et al., 1967). Thus the sequence \( (\sum_{i=1}^{k} Y_{i,n,q}^{\leq})_k \) is a demimartingale (see Section 2.1 in Prakasa Rao, 2012), and hence by Markov’s inequality and the maximal inequality for demimartingales

\[
E \left( \sup_{1 \leq k \leq n} |S_k| \right)^\kappa \leq \left( \frac{\kappa}{\kappa-1} \right)^\kappa E|S_n|^\kappa,
\]

which holds for \( \kappa > 1 \) and \( (S_k)_k \) a demimartingale (see for example Corollary 2.4 in Wang et al., 2010) we obtain

\[
P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j,n}^{\leq} \right| > \frac{\epsilon}{8} \right) \leq \left( \frac{\epsilon}{8} \right)^{-(\alpha + \tau)} \left( \frac{\alpha + \tau}{\alpha + \tau - 1} \right)^{\alpha + \tau} E \left| \sum_{i=1}^{n} Y_{i,n,q}^{\leq} \right|^{\alpha + \tau},
\]

(3.26)

and similarly

\[
P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j,n}^> \right| > \frac{\epsilon}{8} \right) \leq \left( \frac{\epsilon}{8} \right)^{-(\alpha - \tau)} \left( \frac{\alpha - \tau}{\alpha - \tau - 1} \right)^{\alpha - \tau} E \left| \sum_{i=1}^{n} Y_{i,n,q}^> \right|^{\alpha - \tau},
\]

(3.27)

where \( Y_{i,n,q}^> := \sum_{j=q+1}^{\infty} C_{i,j}^+ Z_{i-j,n}^> \). By standard changes of variables and order of summation we have

\[
\sum_{i=1}^{n} Y_{i,n,q}^{\leq} = \sum_{i=-\infty}^{n-1} \left( \sum_{j=q+1+(-i)\vee 0}^{q+n-i} C_{j}^+ Z_{i-j,n}^{\leq} \right).
\]
Note that \((\sum_{j=q+1+(-i) \sqrt{0}}^{q+n-i} C_j^+) Z_{-q,n}^\leq i\) is a martingale difference sequence, and thus by the Bahr-Esseen inequality we obtain
\[
\mathbb{E}\left[ \sum_{i=1}^{n} Y_{i,n,q}^\leq \right]^{\alpha+\tau} \leq 2 \sum_{i=-\infty}^{n-1} \left( \sum_{j=q+1+(-i) \sqrt{0}}^{q+n-i} C_j^+ \right)^{\alpha+\tau} \mathbb{E}[Z_{-q,n}^\leq |^{\alpha+\tau}].
\]

Noting that for large \(q\), \(\sum_{j=q+1+(-i) \sqrt{0}}^{q+n-i} C_j^+ < 1\), the second inequality in (3.23) yields that (for large \(q\))
\[
\mathbb{E}\left[ \sum_{i=1}^{n} Y_{i,n,q}^\leq \right]^{\alpha+\tau} \leq 2|Z_{1,n}^\leq |^{\alpha+\tau} \sum_{j=q+1}^{\infty} C_j^+.
\]

Now note that every \(C_j^+\), for \(j \geq q+1\), appears in the sum \(\sum_{i=-\infty}^{n-1} \sum_{j=q+1+(-i) \sqrt{0}}^{q+n-i} C_j^+\) at most \(n\) times, and therefore
\[
\mathbb{E}\left[ \sum_{i=1}^{n} Y_{i,n,q}^\leq \right]^{\alpha+\tau} \leq 2n|Z_{1,n}^\leq |^{\alpha+\tau} \sum_{j=q+1}^{\infty} C_j^+.
\]

Similarly we obtain
\[
\mathbb{E}\left[ \sum_{i=1}^{n} Y_{i,n,q}^\geq \right]^{\alpha-\tau} \leq 2n|Z_{1,n}^\geq |^{\alpha-\tau} \sum_{j=q+1}^{\infty} C_j^+.
\]

Jensen’s inequality, as in (3.9), yields
\[
\mathbb{E}[Z_{1,n}^\leq ]^{\alpha+\tau} \leq 2^{\alpha+\tau+1} \mathbb{E}[Z_{1,n}^\leq ]^{\alpha+\tau},
\]

and similarly
\[
\mathbb{E}[Z_{1,n}^\geq ]^{\alpha-\tau} \leq 2^{\alpha-\tau+1} \mathbb{E}[Z_{1,n}^\geq ]^{\alpha-\tau}.
\]

Collecting all these facts, from (3.26) and (3.27) we obtain, for large \(q\),
\[
P\left( \sup_{1 \leq k \leq n} \left\{ \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_j^+ Z_{i-j,n}^\leq \right\} > \left(\frac{\epsilon}{8}\right) \right) \leq 2^{\alpha+\tau+2} \left(\frac{\epsilon}{8}\right)^{-(\alpha+\tau)} \left(\frac{\alpha + \tau}{\alpha + \tau - 1}\right)^{\alpha+\tau} \sum_{j=q+1}^{\infty} C_j^+ \quad (3.28)
\]

and
\[
P\left( \sup_{1 \leq k \leq n} \left\{ \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_j^+ Z_{i-j,n}^\geq \right\} > \left(\frac{\epsilon}{8}\right) \right) \leq 2^{\alpha-\tau+2} \left(\frac{\epsilon}{8}\right)^{-(\alpha-\tau)} \left(\frac{\alpha - \tau}{\alpha - \tau - 1}\right)^{\alpha-\tau} \sum_{j=q+1}^{\infty} C_j^+. \quad (3.29)
\]

From (3.25), (3.28) and (3.29) we see that for some positive constant \(M'\) the following inequality holds for large \(q\)
\[
P\left( \sup_{1 \leq k \leq n} \left\{ \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_j^+ Z_{i-j,n}^\leq a_n \right\} > \left(\frac{\epsilon}{4}\right) \right) \leq M'(n|Z_{1,n}^\leq |^{\alpha+\tau} + n|Z_{1,n}^\geq |^{\alpha-\tau}) \sum_{j=q+1}^{\infty} C_j^+.
\]
By Karamata’s theorem \( nE[ Z_{1,n}^{-} ]^{\alpha + \tau} \to \alpha / \tau \) and \( nE[ Z_{1,n}^{-} ]^{\alpha - \tau} \to \alpha / \tau \), as \( n \to \infty \). Therefore, since \( \sum_{j=q+1}^{\infty} C_{j}^{+} \leq \sum_{j=q+1}^{\infty} |C_{j}| \to 0 \) as \( q \to \infty \), we have

\[
\lim_{q \to \infty} \lim_{n \to \infty} \sup_{1 \leq k \leq n} \left( \frac{\sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{j}^{+} Z_{i-j}}{a_{n}} > \frac{\epsilon}{4} \right) = 0.
\]

Hence we conclude

\[
\lim_{q \to \infty} \lim_{n \to \infty} \sup_{1 \leq k \leq n} \left( \frac{\sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{j} Z_{i-j}}{a_{n}} > \frac{\epsilon}{2} \right) = 0. \tag{3.30}
\]

Now, from (3.21), (3.24) and (3.30) follows (3.20), which means that \( V_{n} (\cdot) \overset{d}{\to} CV (\cdot) \) in \( D [0,1] \) with the \( M_{2} \) topology.

Assume now \( \alpha = 1 \). Relation (3.26) holds also in this case, but for (3.27) we need a different argument since \( \alpha - \tau < 1 \), and thus we can not use the maximal inequality for demimartingales. By Markov’s inequality and the first inequality in (3.23) we have

\[
P \left( \sup_{1 \leq k \leq n} \left| \sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{j}^{+} \tilde{Z}_{i-j,n} \right| > \frac{\epsilon}{8} \right) \leq P \left( \sum_{i=1}^{n} \sum_{j=q+1}^{\infty} C_{j}^{+} |\tilde{Z}_{i-j,n}| > \frac{\epsilon}{8} \right)
\]

\[
\leq \left( \frac{\epsilon}{8} \right)^{-(\alpha - \tau)} E \left( \sum_{i=1}^{n} \sum_{j=q+1}^{\infty} C_{j}^{+} |\tilde{Z}_{i-j,n}| \right)^{\alpha - \tau},
\]

\[
\leq \left( \frac{\epsilon}{8} \right)^{-(\alpha - \tau)} E|\tilde{Z}_{1,n}|^{\alpha - \tau} \sum_{i=1}^{n} \sum_{j=q+1}^{\infty} (C_{j}^{+})^{\alpha - \tau}
\]

\[
\leq \left( \frac{\epsilon}{8} \right)^{-(\alpha - \tau)} nE|\tilde{Z}_{1,n}|^{\alpha - \tau} \sum_{j=q+1}^{\infty} |C_{j}|^{\alpha - \tau}.
\]

From the symmetry of \( Z_{1} \), Karamata’s theorem and (1.7) we obtain, as \( n \to \infty \),

\[
nE|\tilde{Z}_{1,n}|^{\alpha - \tau} = nE|\tilde{Z}_{1,n}|^{\alpha - \tau} = \frac{E(|Z_{1}|^{\alpha - \tau} 1(|Z_{1}| > a_{n}))}{a_{n}^{\alpha - \tau} P(|Z_{1}| > a_{n})} \cdot n P(|Z_{1}| > a_{n}) \to \frac{\alpha}{\tau},
\]

Therefore, since \( \lim_{q \to \infty} \sum_{j=q+1}^{\infty} |C_{j}|^{\alpha - \tau} = 0 \), we have

\[
\lim_{q \to \infty} \lim_{n \to \infty} \sup_{1 \leq k \leq n} \left( \frac{\sum_{i=1}^{k} \sum_{j=q+1}^{\infty} C_{j}^{+} \tilde{Z}_{i-j,n}}{a_{n}} > \frac{\epsilon}{8} \right) = 0,
\]

and as in the case \( \alpha \in (1, 2) \) it follows that \( V_{n} (\cdot) \overset{d}{\to} CV (\cdot) \) in \( D [0,1] \) with the \( M_{2} \) topology. This completes the proof. \( \square \)

4. Appendix

We provide a technical result used in the proof of Theorem 3.1 and an example of moving average process with heavy-tailed innovations and deterministic coefficients for which the \( M_{1} \) convergence of the corresponding partial sum process does not hold (but for which the \( M_{2} \) convergence holds).
Lemma 4.1. Let \( Z_1 \) be a regularly varying random variable with index \( \alpha \in [1, 2) \) and \( (a_n) \) a sequence of positive real numbers such that (1.7) holds. Let \( q_n = [n^{1/10}] \), \( n \in \mathbb{N} \). Then

\[
\lim_{n \to \infty} nq_n^2 \left[ P \left( |Z_1| > \frac{a_n}{q_n} \right) \right]^2 = 0.
\]

Proof: By (1.1) and (1.7) we have

\[
\lim_{n \to \infty} na_n^{-\alpha} L(a_n) = 1. \tag{4.1}
\]

Since \( L \) is a slowly varying function, it holds that for all \( s > 0 \) and \( t \in \mathbb{R} \), as \( x \to \infty \), \( x^s[L(x)]^t \to \infty \) and \( x^{-\alpha}[L(x)]^t \to 0 \) (Bingham et al., 1989, Proposition 1.3.6). Hence \( a_n^{2-\alpha} L(a_n) \to \infty \) as \( n \to \infty \), and since by (4.1)

\[
\lim_{n \to \infty} \frac{n}{a_n^2} a_n^{-\alpha} L(a_n) = 1,
\]

it follows that \( n/a_n^2 \to 0 \) as \( n \to \infty \). This yields

\[
\frac{a_n}{q_n} = \sqrt{\frac{a_n^2}{n} \cdot \frac{\sqrt{n}}{q_n}} \to \infty \quad \text{as } n \to \infty,
\]

since by the definition of the sequence \( (q_n) \), \( \sqrt{n}/q_n \to \infty \). Thus for \( u > 0 \), \( M_n(u) := (a_n/q_n)^{-u}[L(a_n/q_n)]^2 \to 0 \) as \( n \to \infty \).

From (1.1) we obtain

\[
nq_n^2 \left[ P \left( |Z_1| > \frac{a_n}{q_n} \right) \right]^2 = nq_n^2 \left( \frac{a_n}{q_n} \right)^{-2\alpha} \left[ L \left( \frac{a_n}{q_n} \right) \right]^2 = nq_n^2 \left( \frac{a_n}{q_n} \right)^{-2\alpha + u} M_n(u). \]

By (4.1) we have

\[
a_n^\alpha \geq KnL(a_n)
\]

for some positive constant \( K \) independent of \( n \), and hence taking some \( v > 0 \) such that \( u + v < 2\alpha \) we obtain

\[
nq_n^2 \left[ P \left( |Z_1| > \frac{a_n}{q_n} \right) \right]^2 = nq_n^{2+2\alpha-u}(a_n^\alpha)^{-2+(u+v)/\alpha} a_n^{-\alpha} M_n(u)
\leq K^{-2+(u+v)/\alpha} q_n^{2+2\alpha-u} n^{-1+(u+v)/\alpha} a_n^{-\alpha} \left[ L(a_n) \right]^{-2+(u+v)/\alpha} M_n(u)
= q_n^{2+2\alpha-u} n^{-1+(u+v)/\alpha} M_n(u, v)
\leq n^{2+2\alpha-u)/10-1+(u+v)/\alpha} M_n(u, v). \tag{4.2}
\]

where

\[
M_n(u, v) := K^{-2+(u+v)/\alpha} a_n^{-v}[L(a_n)]^{-2+(u+v)/\alpha} M_n(u) \to 0 \quad \text{as } n \to \infty.
\]

Now let \( u = 1/5 \) and \( v = 1/5 \), and note that for this choice of \( u \) and \( v \) it holds that

\[
\frac{2 + 2\alpha - u}{10} - 1 + \frac{u + v}{\alpha} \leq -\frac{1}{50} < 0.
\]

Therefore from (4.2) we obtain

\[
\lim_{n \to \infty} nq_n^2 \left[ P \left( |Z_1| > \frac{a_n}{q_n} \right) \right]^2 = 0.
\]
Further, we claim that on the event
\[ M \leq \V \leq \omega_3 \] for some \( \epsilon > 0 \), where
\[ \omega_3(x) = \sup_{t \leq t' \leq t_2} M(x(t_1), x(t_2)) \]
\((x \in D[0, 1], \delta > 0)\) and
\[ M(x_1, x_2, x_3) = \left\{ \begin{array}{ll} 0, & \text{if } x_2 \in [x_1, x_3], \\ \min\{|x_2 - x_1|, |x_3 - x_2|\}, & \text{otherwise}, \end{array} \right. \]
Note that \( M(x_1, x_2, x_3) \) is the distance from \( x_2 \) to \([x_1, x_3]\), and \( \omega_3(x) \) is the \( M_1 \) oscillation of \( x \). To show (4.3) we use, with appropriate modifications, the procedure of Avram and Taqqu (2009) in the proof of their Theorem 1.

Let \( i' = i'(n) \) be the index at which \( \max_{1 \leq i \leq n-1} |Z_i| \) is obtained. Fix \( \epsilon > 0 \) and introduce the events
\[ A_{n,e} = \{|Z_i'| > \epsilon a_n\} = \left\{ \max_{1 \leq i \leq n-1} |Z_i| > \epsilon a_n \right\} \]
and
\[ B_{n,e} = \{|Z_i'| > \epsilon a_n \text{ and } \exists l \neq 0, -i' - 1 \leq l \leq 1, \text{ such that } |Z_{i'+l}| > \epsilon a_n/4 \}. \]
Using the facts that \((Z_i)\) is an i.i.d. sequence and \( n \Pr(Z_1 > \lambda a_n) \to \lambda^{-\alpha} \) as \( n \to \infty \) for \( \lambda > 0 \) (which follows from the regular variation property of \( Z_1 \) and (1.7)) we get
\[ \lim_{n \to \infty} \Pr(A_{n,e}) = 1 - e^{-\epsilon^{-\alpha}} \] \( (4.4) \)
and
\[ \lim_{n \to \infty} \sup \Pr(B_{n,e}) \leq \frac{\epsilon^{-2\alpha}}{4^{-\alpha}} \] \( (4.5) \)
(see Example 5.1 in Krizmanić, 2014).

On the event \( A_{n,e} \setminus B_{n,e} \) one has \( |Z_{i'}| > \epsilon a_n \) and \( |Z_{i'+l}| \leq \epsilon a_n/4 \) for every \( l \neq 0, -i' - 1 \leq l \leq 1 \), and hence
\[ \left| V_n\left(\frac{i'}{n}\right) - V_n\left(\frac{i'-1}{n}\right) \right| = \left| X_{i'-1} \right| a_n = \frac{|Z_{i'} - Z_{i'-1} + Z_{i'-2}|}{a_n} > \frac{\epsilon}{2} \] \( (4.6) \)
and
\[ \left| V_n\left(\frac{i'+1}{n}\right) - V_n\left(\frac{i'}{n}\right) \right| = \left| X_{i'+1} \right| a_n = \frac{|Z_{i'+1} - Z_{i'} + Z_{i'-1}|}{a_n} > \frac{\epsilon}{2}. \] \( (4.7) \)
Further, we claim that on the event \( A_{n,e} \setminus B_{n,e} \) it also holds that
\[ V_n\left(\frac{i'}{n}\right) \notin \left[ V_n\left(\frac{i'-1}{n}\right), V_n\left(\frac{i'+1}{n}\right) \right]. \] \( (4.8) \)
If this is not the case, then
\[ V_n \left( \frac{i'-1}{n} \right) \leq V_n \left( \frac{i'}{n} \right) \leq V_n \left( \frac{i'+1}{n} \right), \]
i.e.
\[ 0 \leq \frac{X_{i'}}{a_n} \leq \frac{X_{i'} + X_{i'+1}}{a_n}. \]
Therefore
\[ \frac{X_{i'}}{a_n} = \frac{Z_{i'} - Z_{i'-1} + Z_{i'-2}}{a_n} \geq 0 \quad \text{and} \quad \frac{X_{i'+1}}{a_n} = \frac{Z_{i'+1} - Z_{i'} + Z_{i'-1}}{a_n} \geq 0, \]
and this implies
\[ \frac{Z_{i'}}{a_n} > \frac{\epsilon}{2} \quad \text{and} \quad \frac{Z_{i'}}{a_n} < -\frac{\epsilon}{2}, \]
which is not possible. Thus relation (4.8) holds, and it implies
\[ M \left( V_n \left( \frac{i'-1}{n} \right), V_n \left( \frac{i'}{n} \right), V_n \left( \frac{i'+1}{n} \right) \right) = \min \left\{ \left| V_n \left( \frac{i'}{n} \right) - V_n \left( \frac{i'-1}{n} \right) \right|, \left| V_n \left( \frac{i'+1}{n} \right) - V_n \left( \frac{i'}{n} \right) \right| \right\}. \]
Taking into account (4.6) and (4.7) we obtain
\[ \omega_{2/n}(V_n) = \sup_{0 \leq t_2 - t_1 \leq 2/n} M(V_n(t_1), V_n(t), V_n(t_2)) \]
\[ \geq M \left( V_n \left( \frac{i'-1}{n} \right), V_n \left( \frac{i'}{n} \right), V_n \left( \frac{i'+1}{n} \right) \right) > \frac{\epsilon}{2} \]
on the event \( A_{n,\epsilon} \setminus B_{n,\epsilon} \). Therefore, since \( \omega_\delta \) is nondecreasing in \( \delta \), it holds that
\[ \lim \inf_{n \to \infty} P(A_{n,\epsilon} \setminus B_{n,\epsilon}) \leq \lim \inf_{n \to \infty} P(\omega_{2/n}(V_n) > \epsilon/2) \]
\[ \leq \lim_{\delta \to 0} \lim \sup_{n \to \infty} P(\omega_\delta(V_n) > \epsilon/2). \quad (4.9) \]
Since \( x^{2\alpha}(1 - e^{-x^{-\alpha}}) \) tends to infinity as \( x \to \infty \), we can find \( \epsilon > 0 \) such that \( \epsilon^{2\alpha}(1 - e^{-\epsilon^{-\alpha}}) > 4^\alpha \), i.e.
\[ 1 - e^{-\epsilon^{-\alpha}} > \frac{\epsilon^{-2\alpha}}{4^{-\alpha}}. \]
For this \( \epsilon \), by relations (4.4) and (4.5), it holds that
\[ \lim_{n \to \infty} P(A_{n,\epsilon}) > \lim \sup_{n \to \infty} P(B_{n,\epsilon}), \]
i.e.
\[ \lim \inf_{n \to \infty} P(A_{n,\epsilon} \setminus B_{n,\epsilon}) \geq \lim_{n \to \infty} P(A_{n,\epsilon}) - \lim \sup_{n \to \infty} P(B_{n,\epsilon}) > 0. \]
Therefore by (4.9) we obtain
\[ \lim \sup_{\delta \to 0} \lim \sup_{n \to \infty} P(\omega_\delta(V_n) > \epsilon/2) > 0 \]
and relation (4.3) holds, which means that \( V_n \) does not converge in distribution in \( D[0,1] \) endowed with the \( M_1 \) topology.
Acknowledgements

The author would like to thank the anonymous referee for the careful reading of the manuscript and helpful remarks which improved the paper.

References


