



Trajectories in random minimal transposition factorizations

Valentin Féray and Igor Kortchemski

Universität Zürich

E-mail address: valentin.feray@math.uzh.ch

URL: <http://user.math.uzh.ch/feray>

CNRS & CMAP, École polytechnique

E-mail address: igor.kortchemski@math.cnrs.fr

URL: <http://http://igor-kortchemski.perso.math.cnrs.fr>

Abstract. We study random typical minimal factorizations of the n -cycle, which are factorizations of $(1, \dots, n)$ as a product of $n-1$ transpositions, chosen uniformly at random. Our main result is, roughly speaking, a local convergence theorem for the trajectories of finitely many points in the factorization. The main tool is an encoding of the factorization by an edge and vertex-labelled tree, which is shown to converge to Kesten’s infinite Bienaymé-Galton-Watson tree with Poisson offspring distribution, uniform i.i.d. edge labels and vertex labels obtained by a local exploration algorithm.

1. Introduction

1.1. *Background and informal description of the results.* We are interested in the combinatorial structure of typical minimal factorizations of the n -cycle as $n \rightarrow \infty$. Specifically, for an integer $n \geq 1$, we let \mathfrak{S}_n be the symmetric group acting on $[n] := \{1, 2, \dots, n\}$ and we denote by \mathfrak{T}_n be the set of all transpositions of \mathfrak{S}_n . Let $(1, 2, \dots, n)$ be the n -cycle which maps i to $i+1$ for $1 \leq i \leq n-1$. The elements of the set

$$\mathfrak{M}_n := \{(\tau_1, \dots, \tau_{n-1}) \in \mathfrak{T}_n^{n-1} : \tau_1 \tau_2 \cdots \tau_{n-1} = (1, 2, \dots, n)\}$$

are called *minimal factorizations of $(1, 2, \dots, n)$ into transpositions* (it is indeed easy to see that at least $n-1$ transpositions are required to factorize a n -cycle). In the sequel, the elements of \mathfrak{M}_n will be simply called minimal factorizations of size n . It is known since Dénes (1959) that $|\mathfrak{M}_n| = n^{n-2}$ and bijective proofs were

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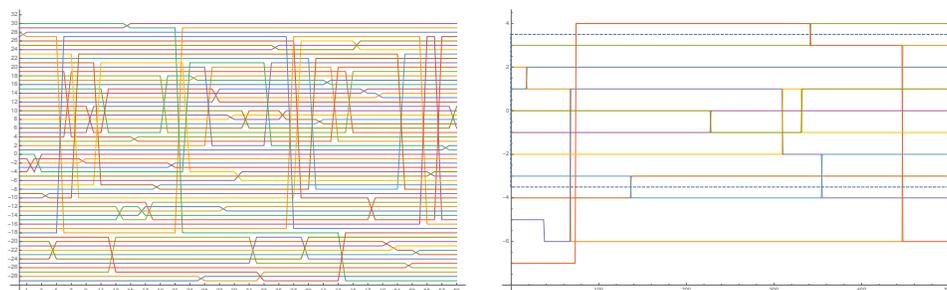


FIGURE 1.1. Left: a simulation of the 60 trajectories $(X_i^{(60)})_{-29 \leq i \leq 30}$ of a uniform minimal factorization of the 60-cycle. Right: a simulation of all the trajectories $(X_i^{(500)})_{i \in \mathcal{I}_3^{(500)}}$ which cross the dashed rectangle $[0, 500] \times [-3, 3]$; here $\mathcal{I}_3^{(500)} = \{-7, -5, -4, -3, -2, -1, 0, 1, 2, 3\}$.

later given by Moszkowski (1989), Goulden and Pepper (1993), Goulden and Yong (2002).

This work is a companion paper of Féray and Kortchemski (2018) from the same authors: both papers investigate the asymptotic properties of a uniform random minimal factorization $\mathcal{F}^{(n)}$ of n , but the questions of interest and the methods are in some sense orthogonal. One important motivation is the work of Angel et al. (2007), who studied uniform random factorization of the reverse permutation by using only nearest-neighbor transpositions (where the reverse permutation ρ is defined by $\rho(i) = n + i - 1$ for $1 \leq i \leq n$, and such factorizations are usually referred to as *sorting networks*). Further motivation for studying minimal factorizations is given in Féray and Kortchemski (2018, Section 1.1).

Throughout the paper, we write $\mathcal{F}^{(n)} = (t_1^{(n)}, \dots, t_{n-1}^{(n)})$ for a uniform random element in \mathfrak{M}_n . We may view $\mathcal{F}^{(n)}$ as a random permutation-valued process starting at id_n and ending at $(1, 2, \dots, n)$ by considering the partial products

$$(t_1^{(n)} t_2^{(n)} \dots t_k^{(n)})_{1 \leq k \leq n-1}.$$

Since we view the partial products as a process indexed by k , the argument k will be referred to as the time. Each partial product acts on the *space* $\{1, \dots, n\}$. The adjectives “global” and “local” below refer to the space.

In Féray and Kortchemski (2018), we considered the global geometry of these partial products and described the one dimensional marginals of this process, i.e., the various possible limiting behaviour for the partial product taken at time K_n (with K_n tending to $+\infty$), depending on the behavior of K_n/\sqrt{n} as $n \rightarrow \infty$.

In this paper we are interested in some *local* properties, namely in the trajectories of a given element i (and a fixed number of neighbouring elements) when applying successively the transpositions $t_k^{(n)}$ ($1 \leq k \leq n-1$). We prove in Theorem 1.1 below that these trajectories converge to some random integer-valued step function, after some renormalization in time, but *without* any renormalization in space. Some combinatorial consequences of this result are also discussed.

This global/local opposition between [Féray and Kortchemski \(2018\)](#) and the current paper is also reflected in the tools that we use. In [Féray and Kortchemski \(2018\)](#), the main tool is to code the random permutation obtained by the partial product taken at a fixed time by a bitype biconditioned Bienaymé–Galton–Watson tree and to study its scaling limits. Here, we code the whole minimal factorization by a random tree, and study its local limit.

This *discrete* limit behaviour of the trajectories is in contrast with the model of random sorting networks introduced by [Angel et al. \(2007\)](#): in the latter case, the limiting trajectories are random sine curves, as was conjectured by [Angel et al. \(2007\)](#) and recently proved by [Dauvergne \(2018\)](#); see also [Angel et al. \(2019\)](#); [Gorin and Rahman \(2019+\)](#) for some local limit results with no space renormalization and a different time-renormalization.

Convention: we shall always multiply permutations from left to right, that is $\tau_1 \dots \tau_{n-1}$ is the permutation obtained by first applying τ_1 then τ_2 , and so on (we warn the reader that this is opposite to the standard convention for compositions of functions).

1.2. *Main result: local convergence of the trajectories.* In order to “zoom-in” around a neighborhood of 1, it is convenient to work with the n -cycle

$$(-\lfloor (n-1)/2 \rfloor, \dots, 0, 1, \dots, \lfloor n/2 \rfloor).$$

To this end, for every integer $1 \leq a \leq n-1$, we set $\tilde{a} = a$ if $a \leq n/2$ and $\tilde{a} = i - n$ otherwise. If $\tau = (a, b) \in \mathfrak{S}_n$, we set $\tilde{\tau} = (\tilde{a}, \tilde{b})$. Finally, we set $\tilde{\mathcal{F}}^{(n)} = (\tilde{t}_1^{(n)}, \dots, \tilde{t}_{n-1}^{(n)})$. Now, for $|i| \leq n/2$, we define the *trajectory* $X_i^{(n)}$ of i in $\tilde{\mathcal{F}}^{(n)}$ by $X_i^{(n)}(0) = i$ and

$$X_i^{(n)}(k) = \tilde{t}_1^{(n)} \tilde{t}_2^{(n)} \dots \tilde{t}_k^{(n)}(i), \quad 1 \leq k \leq n-1, \quad (1.1)$$

We also set $X_i^{(n)}(n) = X_i^{(n)}(n-1)$. See the left part of [Fig. 1.1](#) for an illustration.

For every $A \geq 1$, denote by $\mathcal{I}_A^{(n)}$ the indices of all the trajectories of $\tilde{\mathcal{F}}^{(n)}$ that enter in the rectangle $[0, n] \times [-A, A]$; formally,

$$\mathcal{I}_A^{(n)} = \{|i| \leq n/2 : \text{there exists } 0 \leq k \leq n-1, |X_i^{(n)}(k)| \leq A\}.$$

We will consider the vector of trajectories of all i in $\mathcal{I}_A^{(n)}$. Since this is a random set, let us clarify the underlying topology first.

We denote by $\mathbb{D}([0, 1])$ the set of all real-valued left-continuous functions with right-hand limits (càdlàg functions for short) on $[0, 1]$, equipped with Skorokhod J_1 topology (see [Jacod and Shiryaev, 2003](#), Chap VI for background, but topologies should not be an issue since we will only work with step functions). For fixed $I \subset \mathbb{Z}$, $|I| < \infty$, we equip $\mathbb{D}([0, 1])^I$ with the product topology. We shall work in the space

$$\bigsqcup_{I \subset \mathbb{Z}, |I| < \infty} \mathbb{D}([0, 1])^I,$$

which is the disjoint union of these topologies, that is $(f_i^{(n)})_{i \in I^{(n)}}$ tends to $(f_i)_{i \in I}$ if and only if $I^{(n)} = I$ for n large enough and $f_i^{(n)}$ tends to f_i in the Skorokhod topology for every i in I .

We show that the trajectories of $\tilde{\mathcal{F}}^{(n)}$ converge locally in distribution as $n \rightarrow \infty$ in the following sense.

of all transpositions moving $i \in \mathbb{Z}$ (transpositions are here again identified with two-element sets), and let $\widetilde{\mathbb{M}}_i^{(n)}$ be the set $\{1 \leq j \leq n-1 : \widetilde{\tau}_1^{(n)} \cdots \widetilde{\tau}_{j-1}^{(n)}(i) \neq \widetilde{\tau}_1^{(n)} \cdots \widetilde{\tau}_j^{(n)}(i)\}$ of indices of all transpositions that affect the trajectory of $i \in \mathbb{Z}$. The following results will be deduced from Theorem 1.1 and from the construction of the limiting trajectories.

Corollary 1.2. *With the above notation, the following assertions hold.*

- (i) For every $k \geq 1$, $(\#\widetilde{\mathbb{T}}_i^{(n)}, \#\widetilde{\mathbb{M}}_i^{(n)})_{|i| \leq k}$ converges in distribution to a random vector whose one dimensional marginal distributions are $1 + \text{Poisson}(1)$ random variables;
- (ii) As $n \rightarrow \infty$, $\mathbb{P}(X_1^{(n)}(k) \geq 1 \text{ for every } 0 \leq k \leq n-1) \rightarrow 1 - 1/e$;
- (iii) As $n \rightarrow \infty$, $(\#\widetilde{\mathbb{T}}_1^{(n)}, \#\widetilde{\mathbb{M}}_1^{(n)})$ converges in distribution to a vector of two independent $1 + \text{Poisson}(1)$ random variables.
- (iv) For every $i, j \geq 1$:

$$\mathbb{P}\left(\#\widetilde{\mathbb{T}}_1^{(n)} = i, \#\widetilde{\mathbb{T}}_2^{(n)} = j\right) \xrightarrow{n \rightarrow \infty} e^{-2} \left(\frac{i+j-2}{(i+j-1)!} + \frac{i+j-1}{i!j!} - \frac{i+j-1}{(i+j)!} \right).$$

The event/statistics considered in items (ii), (iii) and (iv) above are somewhat arbitrary, the purpose is to show on specific examples how our construction allows the explicit computation of some limiting probabilities. More generally, the proof of Corollary 1.2 gives a means to determine the law of the limiting random vector in (i). The computation becomes however quickly cumbersome.

As for the results, it is quite surprising that $\#\widetilde{\mathbb{T}}_1^{(n)}$ and $\#\widetilde{\mathbb{M}}_1^{(n)}$ are asymptotically independent. We do not have a simple explanation of this fact, especially since the random sets $\widetilde{\mathbb{T}}_1^{(n)}$ and $\widetilde{\mathbb{M}}_1^{(n)}$ are *not* asymptotically independent (in particular, their smallest element is the same). Also note that $\#\widetilde{\mathbb{T}}_1^{(n)}$ and $\#\widetilde{\mathbb{M}}_1^{(n)}$ are *not* independent for fixed n , even though the fact that they have the same distribution stems from a symmetry property, see Theorem 1.3 below. A conjecture on the joint distribution of $(\#\widetilde{\mathbb{T}}_1^{(n)}, \#\widetilde{\mathbb{M}}_1^{(n)})$ at fixed n is given at the end of the introduction (Conjecture 1.4). Finally, we have not recognized any standard bivariate distribution for the limiting probability distribution in (iv).

1.4. *Symmetries.* Finally, we are interested in symmetry properties that are satisfied by $\mathcal{F}^{(n)}$. Indeed, some of these symmetries are not visible in the limiting object, and may therefore help to compute limiting distributions. As a concrete example, the limiting Poisson distribution for $\#\widetilde{\mathbb{M}}_i^{(n)}$ given in Corollary 1.2(i) is proved as a consequence of an equidistribution result for $\#\widetilde{\mathbb{M}}_i^{(n)}$ and $\#\widetilde{\mathbb{T}}_i^{(n)}$ for fixed n .

Our result is the following distributional identity (for convenience, we state it with the objects $\mathbb{T}^{(n)}, \mathbb{M}^{(n)}$ defined exactly as $\widetilde{\mathbb{T}}^{(n)}, \widetilde{\mathbb{M}}^{(n)}$ when replacing $\widetilde{\mathcal{F}}^{(n)}$ with $\mathcal{F}^{(n)}$). The proof is based on a bijection of Goulden and Yong (2002) between Cayley trees and minimal factorizations.

Theorem 1.3. *For every $k \geq 1$, it holds that*

$$\begin{aligned} & (\#\mathbb{T}_1^{(n)}, \#\mathbb{M}_1^{(n)}, \#\mathbb{T}_2^{(n)}, \#\mathbb{M}_2^{(n)}, \dots, \#\mathbb{T}_k^{(n)}, \#\mathbb{M}_k^{(n)}) \\ & \stackrel{(d)}{=} (\#\mathbb{M}_n^{(n)}, \#\mathbb{T}_1^{(n)}, \#\mathbb{M}_1^{(n)}, \#\mathbb{T}_2^{(n)}, \dots, \#\mathbb{M}_{k-1}^{(n)}, \#\mathbb{T}_k^{(n)}) \\ & \stackrel{(d)}{=} (\#\mathbb{M}_k^{(n)}, \#\mathbb{T}_k^{(n)}, \dots, \#\mathbb{M}_2^{(n)}, \#\mathbb{T}_2^{(n)}, \#\mathbb{M}_1^{(n)}, \#\mathbb{T}_1^{(n)}). \end{aligned}$$

1.5. *A conjecture.* We conclude this Introduction with a conjectural formula for the bivariate probability generating polynomial of $(\#\mathbb{T}_1^{(n)}, \#\mathbb{M}_1^{(n)})$ for a *fixed value* of n . Proving this conjecture would give an alternate proof of Corollary 1.2 (iii), that is of the asymptotic distribution of this pair of statistics.

Conjecture 1.4. *Fix $n \geq 2$. Then*

$$\mathbb{E}[x^{\#\mathbb{T}_1^F} y^{\#\mathbb{M}_1^F}] = xy \left(\frac{n-2+x+y}{n} \right)^{n-2}.$$

This conjecture has been numerically checked until $n = 8$. Note that the right-hand side is known to be the bivariate probability generating polynomial of the degrees of the vertices labelled 1 and 2 in a uniform random Cayley tree (observe that this is different however from that of $(\#\mathbb{T}_1^{(n)}, \#\mathbb{T}_2^{(n)})$, i.e. the degrees of the vertices labelled 1 and 2 in $\text{Tree}(\mathcal{F}^{(n)})$). None of the bijections between minimal factorizations and trees we are aware of explain this fact.

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2. Local convergence of the factorization tree

TABLE 2.1. Table of the main notation and symbols appearing in Section 2.

\mathfrak{M}_n	The set of all minimal factorizations of $(1, 2, \dots, n)$.
$\text{Tree}(F)$	The edge-labelled vertex-labelled non-plane tree associated with a minimal factorization F .
\widetilde{F}	The minimal factorization obtained from F by subtracting n to all edge-labels larger than $n/2$ when $F \in \mathfrak{M}_n$.
\mathcal{T}_n	A BGW tree with Poisson(1) offspring distribution conditioned on having n vertices.
\mathcal{T}_∞	The BGW tree with Poisson(1) offspring distribution conditioned to survive.
$\mathcal{E}(F)$	The pointed edge-labelled non-plane tree associated with a minimal factorization F .
τ'	When τ is an edge and vertex-labelled tree, τ' is the vertex-labelled tree obtained from τ by forgetting the edge labels.

The goal of this section is to establish Theorem 1.1, which is a local limit theorem for the trajectories of a random uniform minimal factorization \mathcal{F}_n . The main tool is the encoding of a minimal factorization F_n as a vertex and edge-labelled tree $\text{Tree}(\widetilde{F}_n)$, which was presented in the Introduction (see Fig. 1.2).

After providing background on trees and local convergence (Section 2.1), we will see that the tree $\text{Tree}(\mathcal{F}_n)$ without vertex labels is a Bienaymé-Galton-Watson (BGW) tree with a uniform edge-ordering. The local limit of such a tree follows from standard results in the random tree literature (Section 2.2). We then explain (Section 2.3) how to reconstruct the *vertex labels* by a local labelling algorithm, which can also be run on infinite trees (Section 2.4). Continuity and locality properties of the labelling algorithm yield a local limit result for the vertex and edge-labelled tree $\text{Tree}(\widetilde{F}_n)$ (Theorem 2.12) Theorem 1.1 follows easily (Section 2.5).

2.1. Preliminaries on trees.

2.1.1. *Plane trees.* We use Neveu's formalism [Neveu \(1986\)](#) to define (rooted) plane trees: let $\mathbb{N} = \{1, 2, \dots\}$ be the set of all positive integers, and consider the set of labels $\mathbb{U} = \bigcup_{n \geq 0} \mathbb{N}^n$ with the convention $\mathbb{N}^0 = \{\emptyset\}$. For $u = (u_1, \dots, u_n) \in \mathbb{U}$, we denote by $|u| = n$ the length of u ; if $n \geq 1$, we define $pr(u) = (u_1, \dots, u_{n-1})$ and for $i \geq 1$, we let $ui = (u_1, \dots, u_n, i)$; more generally, for $v = (v_1, \dots, v_m) \in \mathbb{U}$, we let $uv = (u_1, \dots, u_n, v_1, \dots, v_m) \in \mathbb{U}$ be the concatenation of u and v . A plane tree is a nonempty subset $\tau \subset \mathbb{U}$ such that (i) $\emptyset \in \tau$; (ii) if $u \in \tau$ with $|u| \geq 1$, then $pr(u) \in \tau$; (iii) if $u \in \tau$, then there exists an integer $k_u(\tau) \geq 0$ such that $ui \in \tau$ if and only if $1 \leq i \leq k_u(\tau)$. Observe that plane trees are rooted at \emptyset by definition.

We will view each vertex u of a tree τ as an individual of a population for which τ is the genealogical tree. The vertex \emptyset is called the root of the tree and for every $u \in \tau$, $k_u(\tau)$ is the number of children of u (if $k_u(\tau) = 0$, then u is called a leaf, otherwise, u is called an internal vertex), $|u|$ is its generation, $pr(u)$ is its parent and more generally, the vertices $u, pr(u), pr \circ pr(u), \dots, pr^{|u|}(u) = \emptyset$ are its ancestors. To simplify, we will sometimes write k_u instead of $k_u(\tau)$. A plane tree is said to be locally finite if all its vertices have a finite number of children. Finally, if τ is a tree and h is a nonnegative integer, we let $[[\tau]]_h = \{u \in \tau : |u| \leq h\}$ denote the tree obtained from τ by keeping the vertices in the first h generations.

2.1.2. *Non-plane trees.* In parallel to plane trees, since $\text{Tree}(F_n)$ is by definition a non-plane tree, we will need to consider non-plane trees. By definition, a non-plane tree is a connected graph without cycles. A non-plane tree with a distinguished vertex is called pointed. Equivalently, a pointed non-plane tree is an equivalence class of plane trees under permutation of the order of the children of its vertices, the root of the plane tree being the distinguish vertex. If τ is a plane tree, we denote by $\text{Shape}(\tau)$ the corresponding non-plane tree (informally, we forget the planar structure of τ , and point it at the root vertex).

2.1.3. *Labelled trees.* In this work, we consider trees which are:

- *edge labelled*, in the sense that all edges carry real-valued labels,
- *vertex labelled*, in the sense that some vertices (potentially none, potentially all) carry integer values.

Edge labels, as well as vertex labels, will always be assumed to be distinct. We do not impose any relation between vertex and edge labellings. In the figures, we shall use framed labels for vertex labels to avoid confusion with edge labels.

To simplify notation, we say that a tree is **E**-labelled, **V**-labelled or **EV**-labelled if it is respectively edge labelled, vertex labelled, or edge and vertex-labelled.

2.1.4. *Local convergence for labelled trees.* Let τ be a locally finite (potentially infinite) non-plane tree, pointed, **EV**-labelled in the sense defined above. As for plane trees, for every integer $h \geq 1$, we denote by $[[\tau]]_h$ the finite non-plane, pointed, **EV**-labelled tree obtained from τ by keeping only the vertices at distance at most h from the pointed vertex (together with the edges between them and their labels). We say that $[[\tau]]_h$ is a *labelled ball*, see Figure 2.3 for an example.

We consider a topology on the set of all locally finite, non-plane, pointed, **EV**-labelled trees such that $\tau_n \rightarrow \tau$ if and only if for each $h \geq 1$, the **V**-labelled ball $[[\tau_n]]_h$ and $[[\tau]]_h$ coincide for n large enough and the edge labels of $[[\tau_n]]_h$ tends to that of $[[\tau]]_h$. It is easy to construct a metric d_{loc} for such a topology and to see that the resulting metric space is Polish. Similar metrics can be defined for other families of trees (plane, unlabelled or only **E** or **V**-labelled, etc.) and will be also denoted by d_{loc} .

Finally, if τ is an **EV**-labelled tree and $\alpha > 0$, we denote by $\alpha \cdot \tau$ the **EV**-labelled tree obtained by multiplying all the *edge labels* by α (vertex labels are not modified!), and we denote by τ' the **V**-labelled tree obtained from τ by forgetting the edge labels.

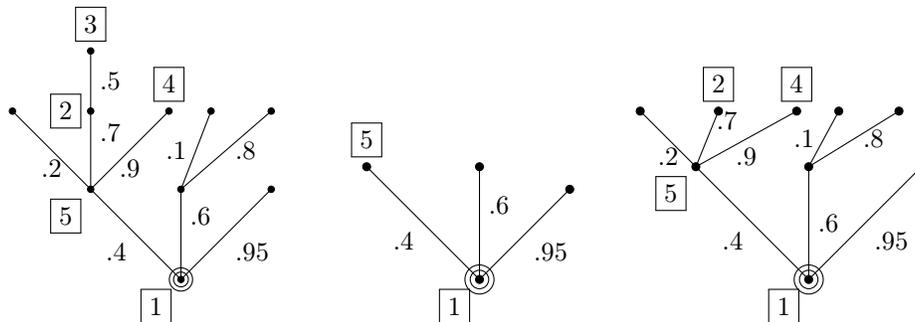


FIGURE 2.3. From left to right: a pointed **EV**-labelled tree τ (considered as non-plane), and its labelled balls $[[\tau]]_1$ and $[[\tau]]_2$.

2.1.5. *Random BGW trees.* Let μ be a probability measure on \mathbb{Z}_+ (called the offspring distribution) such that $\mu(0) > 0$, $\mu(0) + \mu(1) < 1$ (to avoid trivial cases). When $\sum_{i \geq 0} i\mu(i) \leq 1$, the BGW measure with offspring distribution

μ is a probability measure BGW^μ on the set of all plane finite trees such that $\text{BGW}^\mu(\tau) = \prod_{u \in \tau} \mu(k_u)$ for every finite tree τ . For every integer $n \geq 1$, we denote by BGW_n^μ the conditional probability measure BGW^μ given the set \mathbb{A}_n of all plane trees with n vertices, so that

$$\text{BGW}_n^\mu(\tau) = \frac{\text{BGW}^\mu(\tau)}{\text{BGW}^\mu(\mathbb{A}_n)}, \quad \tau \in \mathbb{A}_n$$

(we always implicitly restrict ourselves to values of n such that $\text{BGW}^\mu(\mathbb{A}_n) > 0$). This is the distribution of a BGW random tree with offspring distribution μ , conditioned on having n vertices.

2.1.6. *The infinite BGW tree.* Let $\mu = (\mu_i)_{i \geq 0}$ be a critical offspring distribution (meaning that $\sum i\mu_i = 1$) with $\mu_1 \neq 1$. The conditioned random tree BGW_n^μ is known to have a local limit \mathcal{T}_∞ , which we now present (see [Janson \(2012, Section 5\)](#) and [Lyons and Peres \(2016\)](#) for a formal definition of \mathcal{T}_∞). Let $\bar{\mu}$ be the size-biased distribution of μ defined by $\bar{\mu}_k = k\mu_k$ for $k \geq 0$. In \mathcal{T}_∞ , there are two types of nodes: normal nodes and special, with the root being special. Normal nodes have offspring according to independent copies of μ , while special nodes have offspring according to independent copies of $\bar{\mu}$. Moreover, all children of a normal node are normal; among all the children of a special node one is selected uniformly at random and is special, while all other children are normal. The tree \mathcal{T}_∞ has a unique infinite path (called the spine) formed by all the special vertices (see [Fig. 2.4](#) for an illustration of this construction).

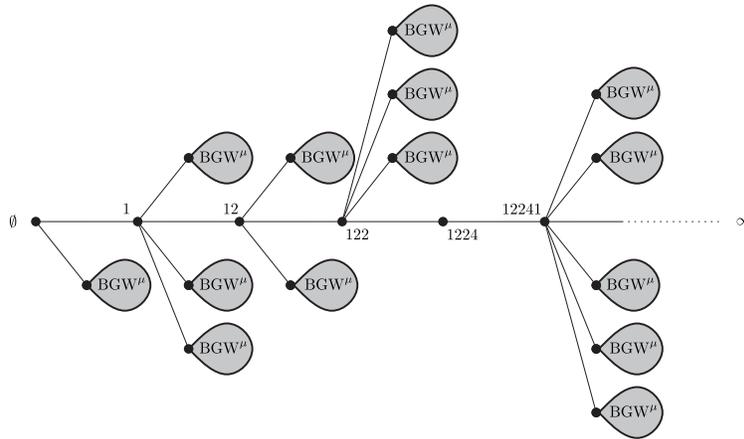


FIGURE 2.4. An illustration of \mathcal{T}_∞ .

In this paper, we shall use the particular case where μ is a $\text{Poisson}(1)$ offspring distribution. Moreover, we consider trees with edge labels. Specifically, let \mathcal{T}_n a BGW tree with $\text{Poisson}(1)$ offspring distribution conditioned on having n vertices, let \mathcal{T}_n^e be the tree obtained from \mathcal{T}_n by labelling its edges in a uniform way, using once each integer from 1 to $n-1$. Similarly, let \mathcal{T}_∞^u be the above constructed random tree \mathcal{T}_∞ , with edges labelled by i.i.d random variables following the uniform distribution on $[0, 1]$. Finally, we recall that $\frac{1}{n}\text{Shape}(\mathcal{T}_n^e)$ is the tree obtained by

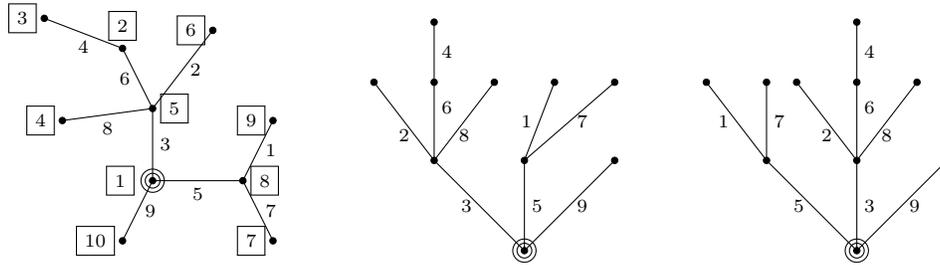


FIGURE 2.5. Construction of the trees associated with the factorization $F_n = ((8, 9) (5, 6) (1, 5) (2, 3) (1, 8) (2, 5) (7, 8) (4, 5) (1, 10))$ of $(1, 2, \dots, 10)$. On the left, we have the pointed non-plane EV-labelled tree $\text{Tree}(F_n)$. On the middle and on the right, we have represented $\mathcal{E}(F_n)$, which is obtained from $\text{Tree}(F_n)$ by forgetting the vertex labels. To emphasize that $\mathcal{E}(F_n)$ is non plane, and we have represented the *same* tree $\mathcal{E}(F_n)$ with two different embeddings.

multiplying the edge labels of \mathcal{T}_n^e by $\frac{1}{n}$ (there are no vertex labels for the moment). The following result is an adaptation to our need of standard local convergence result for BGW trees.

Proposition 2.1. *The convergence*

$$\frac{1}{n} \text{Shape}(\mathcal{T}_n^e) \xrightarrow[n \rightarrow \infty]{(d)} \text{Shape}(\mathcal{T}_\infty^u)$$

holds in distribution in the set of all locally finite, non-plane pointed, E-labelled trees equipped with d_{loc} .

Proof: The convergence $\mathcal{T}_n \rightarrow \mathcal{T}_\infty$ of plane non-labelled trees in distribution in the local topology is a classical result in random tree theory, see, e.g., [Janson \(2012\)](#); [Abraham and Delmas \(2014\)](#). The map Shape is clearly continuous, implying that $\text{Shape}(\mathcal{T}_n) \rightarrow \text{Shape}(\mathcal{T}_\infty)$ locally in distribution. We therefore only need to justify that, for each $h \geq 1$, the edge labels at height at most h on the left-hand side jointly converge to those on the right-hand side. This is however obvious, since, on the left-hand side, we have a uniform labeling with the numbers $1/n, 2/n, \dots, (n-1)/n$, while on the right we have independent uniform label in $[0, 1]$. \square

2.2. Forgetting the vertex labels yields a random BGW tree. In this section, as an intermediate step to study $\text{Tree}(\mathcal{F}_n)$ and $\text{Tree}(\widetilde{\mathcal{F}}_n)$, we consider a variant without vertex labels. Namely, for a minimal factorization F_n , we introduce $\mathcal{E}(F_n)$, the pointed E-labelled non-plane tree obtained from $\text{Tree}(F_n)$ by pointing the vertex with label 1 and forgetting other vertex labels. See Fig. 2.5 for an example. As shown by [Moszkowski \(1989\)](#), the map \mathcal{E} is a bijection. Since the proof is short and elegant, we include it here.

Proposition 2.2 (Moszkowski). *The map \mathcal{E} is a bijection from \mathfrak{M}_n to the set of all pointed E-labelled non-plane trees with n vertices (where edges are labelled from 1 to $n-1$).*

Proof: Let T be a non-plane EV-labelled tree with n vertices (with vertices labelled from 1 to n and edges from 1 to $n - 1$). We associate with the tree T the sequence $\mathcal{S}(T)$ of the $n - 1$ transpositions $(a_1, b_1) \dots (a_{n-1}, b_{n-1})$, where a_i and b_i are the vertex-labels of the extremities of the edge labelled i . It is known since the work of Dénes (1959) that the function \mathcal{S} is a bijection between EV-labelled trees with n vertices and minimal factorizations of *all* cyclic permutations of length n .

Observe that if we forget the vertex-labels of T except 1, then there is a unique way to relabel the $n - 1$ other vertices to get a factorization of the cycle $(1, 2, \dots, n)$. Indeed, relabelling these labels amounts to conjugating the associated cyclic permutation by a permutation fixing 1 and there is always a unique way to conjugate a cyclic permutation by a permutation fixing 1 to get the cycle $(1, 2, \dots, n)$. \square

Recall that we denote by \mathcal{F}_n is a uniform random minimal factorization in \mathfrak{M}_n . It turns out that the law of $\mathcal{E}(\mathcal{F}_n)$ can be related to a BGW tree as follows. As in the previous section, let \mathcal{T}_n^e be the plane E-labelled tree obtained from a Poisson(1) BGW tree \mathcal{T}_n by labelling its edges from 1 to $n - 1$ in a uniform way. Building on Proposition 2.2, we can now prove the following result.

Proposition 2.3. *The random trees $\mathcal{E}(\mathcal{F}_n)$ and $\text{Shape}(\mathcal{T}_n^e)$ both follow the uniform distribution on the set of all pointed non-plane E-labelled trees with n vertices (with edge label set $\llbracket n - 1 \rrbracket$).*

Proof: Since \mathcal{E} is a bijection from the set \mathfrak{M}_n to the set of all pointed non-plane E-labelled trees with n vertices, it is immediate that $\mathcal{E}(\mathcal{F}_n)$ follows the uniform distribution of the latter set.

Then, note that there is a natural bijection ϕ between pointed non-plane E-labelled trees (with edge label set $\llbracket n - 1 \rrbracket$) and non-plane V-labelled trees (with vertex set $\llbracket n \rrbracket$, also known as Cayley trees): label the pointed vertex by 1, and then label every other vertex v with $\ell(v^{\leftarrow}) + 1$, where $\ell(v^{\leftarrow})$ is the label of the edge adjacent to v closest to the pointed vertex.

By construction, $\phi(\text{Shape}(\mathcal{T}_n^e))$ is obtained from \mathcal{T}_n by labelling the root with 1 and other vertices uniformly with numbers from 1 to n (and forgetting the root and the planar structure). It is well-known (see the second proof of Theorem 3.17 in van der Hofstad, 2017) that this is a uniform random non-plane V-labelled tree with n vertices. Applying ϕ^{-1} , it follows that $\text{Shape}(\mathcal{T}_n^e)$ is a uniform random pointed non-plane E-labelled trees with n vertices. This completes the proof. \square

By combining the local convergence of $\text{Shape}(\mathcal{T}_n^e)$ (Proposition 2.1) with Proposition 2.3, we get the following:

Proposition 2.4. *The convergence*

$$\frac{1}{n} \cdot \mathcal{E}(\mathcal{F}_n) \xrightarrow[n \rightarrow \infty]{(d)} \text{Shape}(\mathcal{T}_\infty^u) \quad (2.1)$$

holds in distribution in the set of all locally finite, non-plane, pointed, E-labelled trees equipped with d_{loc} .

We insist on the fact that the above proposition is a convergence of *edge-labelled* trees: indeed, the objects in (2.1) do not carry vertex labels. The difficulty is now to insert the vertex labels in the above proposition. This is not as easy as for the edge labels, since in $\text{Tree}(\mathcal{F}_n)$, the vertex labels are determined by the edge labels (otherwise \mathcal{E} would not be a bijection). We will see in the next sections that

the vertex labels can actually be recovered by a local exploration algorithm. This enables to define a labelling procedure on infinite tree \mathcal{T}_∞^u , giving a limit for the EV-labelled tree $\text{Tree}(\widetilde{\mathcal{F}}_n)$.

2.3. The labelling algorithm. Consider a finite E-labelled tree τ , either plane, or non-plane pointed.

If $k \geq 1$ and τ has at least k vertices, we define a procedure $\text{Find}_k(\tau, \ell)$ which successively assigns values $1, 2, \dots, k$ to k vertices of τ as follows. The value 1 is given to the pointed vertex. Assume that i has been assigned and that we want to assign $i + 1$. Among all edges adjacent to i in τ , denote by e_1 the one with the smallest label, and set $\ell_1 = \ell_{e_1}$. Call j_1 the other extremity of e_1 and consider all edges e adjacent to j_1 with $\ell_e > \ell_1$.

- If there are none, then assign value $i + 1$ to j_1 .
- Otherwise, among all such edges, denote by e_2 the one with smallest label, and set $\ell_2 = \ell_{e_2}$ (in particular $\ell_2 > \ell_1$). Call j_2 the other extremity of e_2 . We then iterate the process: if there is no edge adjacent to j_2 with label greater than ℓ_2 , then we assign value $i + 1$ to the vertex j_2 . Otherwise, among all edges adjacent to j_2 with labels greater than ℓ_2 , we consider the one with smallest label, and so on.

We stop the procedure when k has been assigned. Formally, this is described in Algorithm 1.

Algorithm 1 Assign values $1, 2, \dots, k$ to k vertices of τ

```

value(root)  $\leftarrow$  1
 $i \leftarrow$  1
while  $i \leq k$  do
  CurrentVertex  $\leftarrow$  vertex with value  $i$ 
  HigherVertices  $\leftarrow$  all neighbours of CurrentVertex
  while HigherVertices  $\neq \emptyset$  do
     $x \leftarrow$  minimal value of  $\ell_{\{\text{CurrentVertex}, v\}}$  for  $v$  in HigherVertices
    CurrentVertex  $\leftarrow$  vertex  $v$  in HigherVertices such that  $\ell_{\{\text{CurrentVertex}, v\}} = x$ 
    HigherVertices  $\leftarrow$  all vertices  $v$  such that  $\ell_{\{\text{CurrentVertex}, v\}} > x$ 
  end while
  value(CurrentVertex)  $\leftarrow$   $i + 1$ 
   $i \leftarrow$   $i + 1$ 
end while

```

Example 2.5. We explain in detail how the algorithm runs on the pointed E-labelled tree on the left part of Fig. 2.6. The result is the middle picture in Fig. 2.6.

First the root is assigned value $\boxed{1}$. The edge adjacent to the root with smallest label is the left-most one, so that in this case $\ell_{e_1} = .4$ and j_1 is the left-most vertex at height 1. Continuing the process, j_1 has two adjacent edges of labels bigger than $\ell_{e_1} = .4$. We pick the one with smallest label e_2 , here $\ell_{e_2} = .7$, and call j_2 its other extremity (that is the second left-most vertex at height 2). Now, j_2 has no adjacent edge with label bigger than $\ell_{e_2} = .7$, so we assign value $\boxed{2}$ to j_2 . Starting now from $\boxed{2}$, its adjacent edge with smallest label is the edge e_3 with label .5 going to a leaf. This leaf has no other adjacent edges, so in particular none with label bigger than

.5. We therefore assign value $\boxed{3}$ to this leaf. Finally, in order to assign the last value, starting from $\boxed{3}$, we go through the edge e_3 , go through e_2 , then go through the edge with label .9 and arrive to a leaf which is given value $\boxed{4}$. The procedure Find_4 is over.

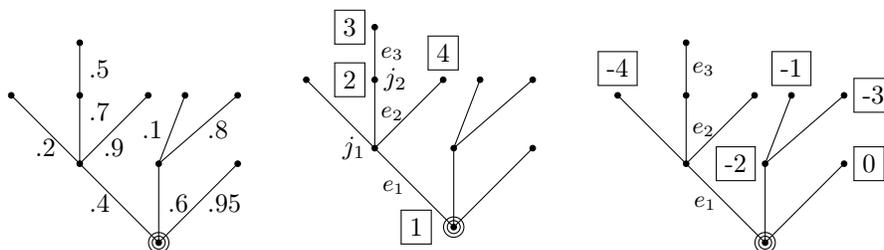


FIGURE 2.6. On the left: a plane tree τ with edge-labels ℓ . In the middle: $\text{Find}_4(\tau, \ell)$. On the right: $\overline{\text{Find}}_4(\tau, \ell)$.

Note that, by construction, this algorithm does not use the planar structure if τ is a plane tree. More precisely, if (τ, ℓ) is a plane E-labelled tree, then we have the commutation relation $\text{Find}_k(\text{Shape}(\tau), \ell) = \text{Shape}(\text{Find}_k(\tau, \ell))$.

Recall from Section 2.2 that with a minimal factorization F_n we have associated two different trees $\text{Tree}(F_n)$ (which is a non-plane EV-labelled tree) and $\mathcal{E}(F_n)$ (which is the pointed E-labelled non-plane tree, obtained from $\text{Tree}(F_n)$ by forgetting vertex labels). The following lemma explains how to go from $\mathcal{E}(F_n)$ to $\text{Tree}(F_n)$, using the above-defined algorithm.

Lemma 2.6. *Let F_n be a minimal factorization of $(1, 2, \dots, n)$. We have the identity $\text{Find}_n(\mathcal{E}(F_n)) = \text{Tree}(F_n)$.*

Proof: By construction $\text{Find}_n(\mathcal{E}(F_n))$ and $\text{Tree}(F_n)$ might only differ by their vertex labels. We prove by induction that for all $i \leq n$ the same vertex carries label i in both trees.

By definition, Find_n gives the value 1 to the root, and the root of $\mathcal{E}(F_n)$ is the vertex that used to have value 1 in $\text{Tree}(F_n)$. This proves the base case ($i = 1$) of the induction.

Fix $i \geq 1$ and assume that $\text{Find}_n(\mathcal{E}(F_n))$ gives value i to the vertex v_i of $\mathcal{E}(F_n)$ that used to have value i in $\text{Tree}(F_n)$. By construction, to assign value $i + 1$, $\text{Find}_n(\mathcal{E}(F_n))$ first considers the edge adjacent to v_i with minimum edge-label. Call a this label. This means that the transposition τ_a in F_n is of the form $(i j_1)$ for some j_1 . (This notation is consistent with the construction of Find_n .) By minimality of a , the transpositions $\tau_1, \dots, \tau_{a-1}$ fix i . Thus the partial product $\tau_1 \dots \tau_a$ maps i on j_1 . We then want to see where j_1 is mapped when we apply the next transpositions $\tau_{a+1}, \tau_{a+2}, \dots$. For this, we need to look for an edge adjacent to j_1 with a value bigger than a , which is exactly what Find_n does. If b is the smallest value of such an edge and j_2 the other extremity of this edge (again the notation is consistent with the one of the construction of Find_n), then $\tau_1 \dots \tau_b$ maps i to j_2 . The construction stops at j_r when there is no edge adjacent to j_r with a bigger

value than the previously considered edge, and then we know that $\tau_1 \dots \tau_{n-1}$ maps i to j_r . But, since $\tau_1 \dots \tau_{n-1} = (1, 2, \dots, n)$, we necessarily have $j_r = i + 1$. Thus Find_n precisely assigns $i + 1$ to j_r and this completes our induction step. \square

Remark 2.7. Note that if one replaces the labels (ℓ_e) of $\mathcal{E}(F_n)$ with labels (ℓ'_e) such that $\ell_e < \ell_f$ if and only if $\ell'_e < \ell'_f$ for every edges e, f (we say that the edge-labellings are *compatible*), one obtains the same vertex labels when running Find_n (since Find_n only uses the relative order of the labels).

Figs. 2.5 and 2.6 illustrate the previous lemma. Indeed the trees in the middle of Fig. 2.5 and in the left part of Fig. 2.6 are the same, with compatible edge-labellings. The procedure Find_4 indeed reassigns the labels 1, 2, 3 and 4 to some vertices of $\mathcal{E}(F_n)$, as they are in $\text{Tree}(F_n)$ (compare the left part of Fig. 2.5 and the middle picture in Fig. 2.6).

Lemma 2.6 explains how to reconstruct $\text{Tree}(F_n)$ from $\mathcal{E}(F_n)$. We are however interested in $\text{Tree}(\widetilde{F}_n)$, rather than $\text{Tree}(F_n)$. We therefore need to introduce a dual labelling procedure $\overline{\text{Find}}_k$, which assigns non positive labels. This procedure runs exactly as Find_k except that the order of label edges is taken as reversed. Namely we first look at the edge with *largest* label ℓ_1 incident to 1, call j_1 its extremity and then look for an edge of *largest* label $\ell_2 < \ell_1$, etc. Another difference is that $\overline{\text{Find}}_k$ now assigns labels $0, -1, \dots, -k$ to successively found vertices. An example of the outcome of this procedure is shown on Fig. 2.6.

The following lemma motivates the definition of this dual procedure.

Lemma 2.8. *Let F_n be a minimal factorization of $(1, 2, \dots, n)$. Then the tree $\overline{\text{Find}}_{n-1}(\mathcal{E}(F_n))$ is obtained from $\text{Tree}(F_n)$ by subtracting n to every vertex label.*

Proof: With a minimal factorization $F_n = (\tau_1, \dots, \tau_{n-1})$ of the full cycle $(1, 2, \dots, n)$, we can associate its reversed sequence $r(F_n) = (\tau_{n-1}, \dots, \tau_1)$, which is a factorization of the full cycle $(1, n, n-1, \dots, 2)$. Reversing the order of the edge labels in $\text{Tree}(F_n)$ yields $\text{Tree}(r(F_n))$. By Lemma 2.6 (which holds more generally for minimal factorizations of any cycle), the procedure $\overline{\text{Find}}$ thus assigns labels to $\mathcal{E}(F_n)$ in the order of the cycle $(1, n, n-1, \dots, 2)$, i.e. n gets the first label 0 (after the pointed vertex), $n-1$ gets the second label -1 , and so on. This proves the lemma. \square

Combining Lemmas 2.6 and 2.8, we see that the tree $\text{Tree}(\widetilde{F}_n)$, in which we are interested, is obtained by composing both procedures.

Corollary 2.9. *Let F_n be a minimal factorization of $(1, 2, \dots, n)$. Then*

$$\text{Tree}(\widetilde{F}_n) = \text{Find}_{\lfloor \frac{n}{2} \rfloor} \circ \overline{\text{Find}}_{\lfloor \frac{n-1}{2} \rfloor} (\mathcal{E}(F_n)).$$

2.4. Relabelling infinite trees. We now want to run the procedures Find and $\overline{\text{Find}}$ on infinite, yet locally finite, trees. Note that in general, Find_k (and $\overline{\text{Find}}_k$) may be ill-defined (since the inner “while” loop in Algorithm 1 may be infinite).

Let T be a locally finite tree (either plane, or non-plane pointed) and let $\ell = (\ell_e)$ be a family of distinct real numbers indexed by the edges of T . We say that $\mathbf{T} = (T, \ell)$ satisfies the property (\mathcal{I}) (resp. (\mathcal{D})) if there is no infinite increasing (resp. decreasing) path in T . If (T, ℓ) satisfies (\mathcal{I}) (resp. (\mathcal{D})), then it is clear that $\text{Find}_k(T, \ell)$ (resp. $\overline{\text{Find}}_k(T, \ell)$) is well defined for every $k \geq 1$ by construction.

In this case, we can also define a procedure $\text{Find}_\infty(T, \ell)$ (resp. $\overline{\text{Find}}_\infty(T, \ell)$) that assigns all labels in $\mathbb{Z}_{>0}$ (resp. $\mathbb{Z}_{<0}$) to the vertices of T . Under a simple assumption, combining both procedures labels *all* vertices of an infinite tree, as explained in the following lemma, where we say that T has *one end* if for every $r \geq 0$, $T \setminus \llbracket T \rrbracket_r$ has a unique infinite connected component.

Lemma 2.10. *Let T be an infinite locally finite E -labelled tree with one end (either plane, or non-plane pointed), satisfying both (\mathcal{I}) and (\mathcal{D}) . Then every vertex of the tree is either assigned a label by Find_∞ or by $\overline{\text{Find}}_\infty$, but not by both.*

Proof: (The reader may want to look at Fig. 2.7 to visualize the notation in this proof.) Let v be a vertex of T . Since T has one end, there exists a unique infinite injective path starting from the pointed vertex \emptyset . Denote by u the vertex of this path which is the closest to v (u could be the root vertex, or v itself). We first assume that $v \neq u$, i.e. v is not on the path from the root to infinity. Then u has at least two children, one of them, say u_1 , being an ancestor of v (possibly v itself) and one other, say u_2 , lying on the infinite path. We set $e_1 = \{u, u_1\}$ and $e_2 = \{u, u_2\}$, both being edges of T . To simplify the discussion, we also assume that u is not the root of the tree, and call e_0 the edge joining u to its parent. The labels of the edges e_0 , e_1 and e_2 are denoted by ℓ_0 , ℓ_1 and ℓ_2 , respectively. Whether v is assigned a label by Find_∞ or by $\overline{\text{Find}}_\infty$ depends on the relative order of ℓ_0 , ℓ_1 and ℓ_2 , as will be explained below.

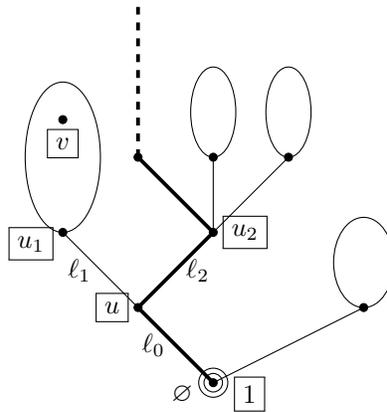


FIGURE 2.7. Notation of the proof of Lemma 2.10 (in bold, the infinite path).

Before going into a case distinction, let us make some remarks on the procedures Find_∞ and $\overline{\text{Find}}_\infty$, using the notion a *fringe subtrees* (a fringe subtree S of T is a subtree of T formed by one of its vertex and *all* its descendants). We claim that:

- when the algorithm Find_∞ (or $\overline{\text{Find}}_\infty$) enters a finite fringe subtree S (i.e. **CurrentVertex** is in S at some stage of Algorithm 1), it does not leave it before having assigned a label to every vertex in S .
- when the algorithm Find_∞ (or $\overline{\text{Find}}_\infty$) enters an infinite fringe subtree S , it never leaves S .

The first claim can be checked by induction, and the second one follows from the first one. Denote by respectively S_{u_1} and S_{u_2} the fringe subtrees rooted in u_1 and u_2 . By construction, S_{u_1} is finite and contains v , while S_{u_2} is infinite. Determining whether v is assigned a label by Find_∞ (or $\overline{\text{Find}}_\infty$) therefore boils down to determining whether Find_∞ (or $\overline{\text{Find}}_\infty$) enters S_{u_1} or S_{u_2} first.

From this reformulation it is now easy to see that:

- if $\ell_0 < \ell_1 < \ell_2$ or $\ell_1 < \ell_2 < \ell_0$ or $\ell_2 < \ell_0 < \ell_1$, then the vertex v is assigned a label by Find_∞ but not by $\overline{\text{Find}}_\infty$;
- if $\ell_1 < \ell_0 < \ell_2$ or $\ell_0 < \ell_2 < \ell_1$ or $\ell_2 < \ell_1 < \ell_0$, then the vertex v is assigned a label by $\overline{\text{Find}}_\infty$ but not by Find_∞ .

This proves the lemma in the case $v \neq u$ and $u \neq \emptyset$. If $v \neq u = \emptyset$, the above conclusion holds with the convention that $\ell_0 = \infty$. If $v = u \neq \emptyset$, the same holds with the convention that $\ell_1 = \infty$. The only remaining case is that of $v = \emptyset$, but it is clear that the root is assigned a label (namely the label 1) by Find_∞ and none by $\overline{\text{Find}}_\infty$. □

We now prove the following continuity lemma for the Find procedure, which is crucial to obtain our limit theorem for $\text{Tree}(\widetilde{\mathcal{F}}_n)$.

Lemma 2.11. *Consider a locally finite E-labelled tree \mathbf{T} with one end (either plane, or non-plane pointed), such that both (\mathcal{I}) and (\mathcal{D}) are satisfied. Let $(\mathbf{T}_n)_{n \geq 1}$ be a sequence of E-labelled trees, each with n vertices, such that \mathbf{T}_n converges to \mathbf{T} for the local topology on E-labelled trees. Then the convergence*

$$\text{Find}_{\lfloor \frac{n}{2} \rfloor} \circ \overline{\text{Find}}_{\lfloor \frac{n-1}{2} \rfloor}(\mathbf{T}_n) \xrightarrow{n \rightarrow \infty} \text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathbf{T})$$

holds for the local topology on EV-labelled trees.

Proof: For an EV-labelled tree τ , recall that we denote by $[[\tau]]'_h$ the V-labelled tree obtained from $[[\tau]]_h$ by forgetting the edge labels. Fix $k \geq 1$. We first prove that, for all $h \geq 1$,

$$[[\text{Find}_k(\mathbf{T}_n)]]_h \xrightarrow{n \rightarrow \infty} [[\text{Find}_k(\mathbf{T})]]_h. \tag{2.2}$$

It is enough to establish the result for every h sufficiently large. Since \mathbf{T} satisfies (\mathcal{I}) , by Lemma 2.10 and its proof, we may choose $h \geq 1$ such that Find_k does not visit a vertex with height greater than h in \mathbf{T} . Therefore Find_k only depends on $[[\mathbf{T}]]_{h+1}$. By assumption, we can take n sufficiently large so that $[[\mathbf{T}]]'_{h+1} = [[\mathbf{T}_n]]'_{h+1}$ and such that the edge labels of $[[\mathbf{T}]]_{h+1}$ and of $[[\mathbf{T}_n]]_{h+1}$ are compatible (in the sense of Remark 2.7). For such n , the execution of $\text{Find}_k(\mathbf{T}_n)$ is identical to that of $\text{Find}_k(\mathbf{T})$. As a consequence, $[[\text{Find}_k(\mathbf{T}_n)]]'_h = [[\text{Find}_k(\mathbf{T})]]'_h$ for n sufficiently large, and (2.2) follows since the edge-labels converge by assumption. The same holds replacing Find_k by $\overline{\text{Find}}_k$, and thus also by the composition $\text{Find}_k \circ \overline{\text{Find}}_k$ by using successively both statements.

We now use the fact that \mathbf{T} has one end. By Lemma 2.10, every vertex of \mathbf{T} is assigned a label by $\text{Find}_\infty \circ \overline{\text{Find}}_\infty$. In particular, for every fixed $h \geq 1$, there is an integer K_h such that all vertices at height at most h are assigned a label with absolute value smaller than K_h . Then it is clear that

$$[[\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathbf{T})]]_h = [[\text{Find}_{K_h} \circ \overline{\text{Find}}_{K_h}(\mathbf{T})]]_h.$$

From the first part of the proof, there exists an integer n_h such that for every $n \geq n_h$ we have

$$[[\text{Find}_{K_h} \circ \overline{\text{Find}}_{K_h}(\mathbf{T}_n)]]'_h = [[\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathbf{T})]'_h. \tag{2.3}$$

This implies that every vertex of height at most h in \mathbf{T}_n is assigned a label by $\text{Find}_k \circ \overline{\text{Find}}_k$ for every $k \geq K_h$ and $n \geq n_h$. Therefore, for $n > \max(2K_h, n_h)$, we have

$$[[\text{Find}_{\lfloor \frac{n}{2} \rfloor} \circ \overline{\text{Find}}_{\lfloor \frac{n-1}{2} \rfloor}(\mathbf{T}_n)]]'_h = [[\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathbf{T})]'_h.$$

This completes the proof since this holds for any $h \geq 1$ and since the edge-labels converge by assumption. \square

2.5. *Local convergence of the random minimal factorization tree.* We now have all the tools to prove the local convergence as $n \rightarrow \infty$ of the EV-labelled tree $\text{Tree}(\widetilde{\mathcal{F}}_n)$, which will in turn allow us to establish Theorem 1.1.

Theorem 2.12. *The convergence*

$$\frac{1}{n} \cdot \text{Tree}(\widetilde{\mathcal{F}}_n) \xrightarrow[n \rightarrow \infty]{(d)} \text{Shape}(\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathcal{T}_\infty^u))$$

holds in distribution in the set of all locally finite, non-plane, pointed, EV-labelled trees equipped with d_{loc} .

Proof of Theorem 2.12: In virtue of Skorokhod’s representation theorem (see e.g. Billingsley, 1999, Theorem 6.7), we may assume that the convergence of Proposition 2.4 holds almost surely, so that almost surely, for every $h \geq 1$,

$$[[\frac{1}{n} \cdot \mathcal{E}(\mathcal{F}_n)]]_h \xrightarrow[n \rightarrow \infty]{} [[\text{Shape}(\mathcal{T}_\infty^u)]]_h.$$

Since \mathcal{T}_∞^u has a.s. one end and satisfies a.s. conditions (\mathcal{I}) and (\mathcal{D}) , we can apply Lemma 2.11. We get that, almost surely, for every $h \geq 1$,

$$[[\text{Find}_{\lfloor \frac{n}{2} \rfloor} \circ \overline{\text{Find}}_{\lfloor \frac{n-1}{2} \rfloor}(\frac{1}{n} \cdot \mathcal{E}(\mathcal{F}_n))]]]_h \xrightarrow[n \rightarrow \infty]{} [[\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\text{Shape}(\mathcal{T}_\infty^u))]]]_h.$$

From Corollary 2.9, the left-hand side has the same distribution as $[[\frac{1}{n} \text{Tree}(\widetilde{\mathcal{F}}_n)]]_h$. Using the commutation between Find and Shape mentioned in Section 2.3, this completes the proof of the theorem. \square

We are finally in position to establish Theorem 1.1. Let us first define the limiting trajectories $(X_i)_{i \in \mathbb{Z}}$. We consider the limiting tree $\text{Shape}(\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathcal{T}_\infty^u))$ in Theorem 2.12. For a fixed $i \in \mathbb{Z}_{>0}$, we denote by $V_0^{(i)} = i, \dots, V_{K_i}^{(i)} = i + 1$ the labels of the successive vertices visited by Find_∞ when assigning the label $i + 1$; the number of such vertices $K_i + 1$ is random, note also that some of these labels might be bigger than i or negative, so that they are not assigned when we run Find_{i+1} on \mathcal{T}_∞^u , but are assigned later in the procedure $\text{Find}_\infty \circ \overline{\text{Find}}_\infty$. Finally, for $k \in \{1, \dots, K_i\}$, we denote $\ell_k^{(i)}$ the label of the edge between $V_{k-1}^{(i)}$ and $V_k^{(i)}$.

Then, setting $\ell_0^{(i)} = 0$ and $\ell_{K_i+1}^{(i)} = 2$, we define, for $t \in [0, 1]$,

$$X_i(t) = \sum_{k=0}^{K_i} V_k^{(i)} \mathbb{1}_{\ell_k^{(i)} \leq t < \ell_{k+1}^{(i)}} \tag{2.4}$$

(we take the convention $\ell_{K_i+1}^{(i)} = 2$ in order to have $X_i(1) = i + 1$). The construction is similar for $i \leq 0$, except that we consider the step where $\overline{\text{Find}}_\infty$ starts from the

vertex labelled $i + 1$ and assigns label i , and we denote by $V_{K_i}^{(i)} = i + 1, \dots, V_0^{(i)} = i$ the successive visited vertices (note that the order of indices is reversed), with the definition of $\ell_k^{(i)}$ being unchanged.

Proof of Theorem 1.1: We first introduce some notation. Fix $i > 0$. For $n \geq |i|$, run the procedure Find on $\frac{1}{n} \cdot \text{Tree}(\widetilde{\mathcal{F}}_n)$ starting from i until it assigns label $i + 1$. Denote by $V_0^{(n,i)} = i, \dots, V_{K_{n,i}}^{(n,i)} = i + 1$ the labels of the successively visited vertices (the number $K_{n,i} + 1$ of such vertices is a random variable depending on n) and by $\ell_k^{(n,i)}$ the label of the edge between $V_{k-1}^{(n,i)}$ and $V_k^{(n,i)}$ (for $k \in \{1, \dots, K_{n,i}\}$; since we consider $\frac{1}{n} \cdot \text{Tree}(\widetilde{\mathcal{F}}_n)$, these labels are in $[0, 1]$). Finally set $\ell_0^{(n,i)} = 0$ and $\ell_{K_{n,i}+1}^{(n,i)} = 2$. Recall from (1.1) the definition of trajectories $X_i^{(n)}$ of i in $\widetilde{\mathcal{F}}^{(n)}$. From the proof of Lemma 2.6, we have

$$\forall 0 \leq t \leq 1, \quad X_i^{(n)}(\lfloor nt \rfloor) = \sum_{k=0}^{K_{n,i}} V_k^{(n,i)} \mathbb{1}_{\ell_k^{(n,i)} \leq t < \ell_{k+1}^{(n,i)}}.$$

As above, we use a similar construction for $i < 0$ and the above relation holds as well in this case.

Fix $A \geq 1$ and observe that the set of all indices \mathcal{I}_A of all the trajectories $(X_i)_{i \in \mathbb{Z}}$ that enter the rectangle $[0, 1] \times [-A, A]$ satisfies the identity

$$\mathcal{I}_A = \{i \in \mathbb{Z} : \exists 0 \leq t \leq 1 : |X_i(t)| \leq A\}.$$

We note that an element i can be in \mathcal{I}_A if and only if there exists an increasing path in \mathcal{T}_∞^u from the vertex labelled i to some vertex label i' with $|i'| \leq A$. Since \mathcal{T}_∞^u is locally finite and contains a single path from the root to infinity, which a.s. contains infinitely many ascents and descents, for a given i' , the set of such i is a.s. finite. We conclude that $\#\mathcal{I}_A < \infty$ almost surely.

Now, by Skorokhod’s representation theorem we may assume that the convergence of Theorem 2.12 holds almost surely. Since $\#\mathcal{I}_A < \infty$ almost surely, we may fix (a random) $H > 1$ such that

- (1) for every $i \in \mathcal{I}_A$, all the vertices visited by the algorithm $\text{Find}_{|i|}$ and $\overline{\text{Find}}_{|i|}$ have height at most $H - 1$;
- (2) for every i' with $|i'| \leq A$, there is no decreasing path from i' which reaches height H or more.

We can find an integer $N > 1$ such that we have the identity $\llbracket \frac{1}{n} \cdot \text{Tree}(\widetilde{\mathcal{F}}_n) \rrbracket'_H = \llbracket \text{Shape}(\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathcal{T}_\infty^u)) \rrbracket'_H$ for every $n \geq N$; moreover, by possibly increasing N , we may assume that these V -labelled balls have compatible edge labellings. Condition (i) above implies that, for $n \geq N$, the procedures $\text{Find}_{|i|}$ and $\overline{\text{Find}}_{|i|}$ behave similarly on $\frac{1}{n} \cdot \text{Tree}(\widetilde{\mathcal{F}}_n)$ and \mathcal{T}_∞^u . Condition (ii) forces $\mathcal{I}_A^{(n)}$ to be constituted of labels i of vertices such that the trajectories of i “stay” at height at most H in the tree, so that $\mathcal{I}_A^{(n)} = \mathcal{I}_A$ for every $n \geq N$. As a consequence, for every $n \geq N$ and $i \in \mathcal{I}_A$, $K_{n,i} = K_i$ and $V_k^{(n,i)} = V_k^{(i)}$ for every $0 \leq k \leq K_n$. Also, for every $i \in \mathcal{I}_A$ and $0 \leq k \leq K_{n,i}$, $\ell_k^{(n,i)} \rightarrow \ell_k^{(i)}$ as $n \rightarrow \infty$. The desired result follows. \square

3. Combinatorial consequences

The goal of this section is to prove Corollary 1.2, using the local convergence of $\text{Tree}(\widetilde{\mathcal{F}}_n)$. We start in Section 3.1 by item (i), *i.e.* some results on the existence of limiting distributions for “local” statistics. In Section 3.2, we prove items (ii), (iii) and (iv) to illustrate how the explicit construction of the limit of $\text{Tree}(\widetilde{\mathcal{F}}_n)$, allows to compute limiting laws of such statistics. As we shall see, explicit computations quickly become quite cumbersome.

3.1. Existence of distribution limits for local statistics. We first need to introduce some notation. As in the Introduction, for a factorization $F = (\tau_1, \dots, \tau_{n-1}) \in \mathfrak{M}_n$ and an integer $1 \leq i \leq n$, let $\widetilde{T}_i^F = \{1 \leq k \leq n-1 : i \in \widetilde{\tau}_k\}$ be the set of indices of all transpositions moving $i \in \mathbb{Z}$ (transpositions are as before identified with two-element sets), and let $\widetilde{M}_i^F = \{1 \leq k \leq n-1 : \widetilde{\tau}_1 \cdots \widetilde{\tau}_{k-1}(i) \neq \widetilde{\tau}_1 \cdots \widetilde{\tau}_k(i)\}$ be the set of all indices of transpositions that affect the trajectory of $i \in \mathbb{Z}$.

These sets are easily read on the associated tree $\text{Tree}(F)$: in particular,

- the number $\#\widetilde{T}_i^{(n)}$ of transpositions moving i in F , is the degree of the node with label i in $\text{Tree}(F)$;
- the number $\#\widetilde{M}_i^{(n)}$ of transpositions that affect the trajectory of i in F is the distance between the vertices with labels i and $i+1$ in $\text{Tree}(\widetilde{\mathcal{F}}_n)$.

As before, taking a factorization \mathcal{F}_n uniformly at random among all minimal factorizations of size n , we use the following notation for the corresponding random sets:

$$\widetilde{T}_i^{(n)} := \widetilde{T}_i^{\mathcal{F}_n}, \quad \widetilde{M}_i^{(n)} := \widetilde{M}_i^{\mathcal{F}_n}.$$

The local convergence of EV-labelled trees implies the (joint) convergence of the degree of the vertex i and of the distance between the vertices i and $i+1$ (for every fixed i in \mathbb{Z}). Therefore the convergence in distribution in Corollary 1.2 (i) is an immediate consequence of Theorem 2.12. The statement of the marginals of the limiting distribution is proved below: in Corollary 3.2 for $\#\widetilde{T}_i^{(n)}$ and as a consequence of symmetry considerations in Section 4 for $\#\widetilde{M}_i^{(n)}$.

More generally, many other statistics converge jointly in distribution; here is another example.

Corollary 3.1. *Let $\widetilde{I}_1^{(n)}, \dots, \widetilde{I}_{M_n}^{(n)}$ be integers such that the transpositions of $\widetilde{\mathcal{F}}_n$ moving 1 are, in this order $(1, \widetilde{I}_1^{(n)}), \dots, (1, \widetilde{I}_{M_n}^{(n)})$. Then $\frac{1}{n}(\widetilde{I}_1^{(n)}, \widetilde{I}_2^{(n)}, \dots, \widetilde{I}_{M_n}^{(n)})$ converges in distribution.*

3.2. Some explicit computations. In this Section, we compute explicitly some limiting distribution related to the above convergence results. This is based on the explicit construction of the limiting tree in Theorem 2.12. We start by proving that $\#\widetilde{T}_i^{(n)}$ converges in distribution to a Poisson size-biased distribution (which is part of Corollary 1.2 (i)), for which only Proposition 2.4 is needed (that is the convergence of trees without vertex labels).

Corollary 3.2. *Fix $i \in \mathbb{Z}$. Then, for every $j \geq 1$,*

$$\mathbb{P}\left(\#\widetilde{T}_i^{(n)} = j\right) \xrightarrow{n \rightarrow \infty} \frac{e^{-1}}{(j-1)!}.$$

Proof: By using the action by conjugation of $(1, \dots, n)$ on minimal factorizations, we see that the distribution of the number of transpositions that act on i is independent from i . It is therefore enough to establish the result for $i = 1$.

By construction, $\#\tilde{\mathbb{T}}_1^{(n)}$ is the degree of the pointed vertex in $\mathcal{E}(\mathcal{F}_n)$. Local convergence of unlabelled or E-labelled tree implies the convergence of the root degree, so that, from Proposition 2.4, $\#\tilde{\mathbb{T}}_1^{(n)}$ converges to the root degree in $[[\text{Shape}(\mathcal{T}_\infty^u)]]_1$. By construction of \mathcal{T}_∞^u , the degree of its root vertex is a Poisson(1) size-biased distribution, thus giving the desired result. \square

The other parts of Corollary 1.2 need the full statement of Theorem 2.12 (that is with vertex labels). We first establish Corollary 1.2 (ii).

Proof of Corollary 1.2 (ii): By Theorem 1.1, as $n \rightarrow \infty$, we have

$$\mathbb{P}(X_1^{(n)}(k) \geq 1 \text{ for every } 0 \leq k \leq n) \longrightarrow \mathbb{P}(X_1(t) \geq 1 \text{ for every } t \in [0, 1]).$$

By definition of the limiting trajectories X_i (Eq. (2.4)), X_1 takes only positive values if and only if there are only positive labels between the path between 1 and 2 in the limiting tree $\text{Shape}(\text{Find}_\infty \circ \text{Find}_\infty(\mathcal{T}_\infty^u))$.

We will determine when this happens by distinguishing two cases:

- *Case 1: the edge e with smallest label adjacent to the root does not belong to the spine.* We call v its extremity which is not the root. Then the algorithm Find_∞ enters first the fringe subtree rooted at v . The vertex getting label 2 will therefore be in that subtree. Moreover, Find_∞ assigns a (positive) label to every vertex in that fringe subtree. We conclude that, in this case, the path between 1 and 2 indeed contains only positive labels.
- *Case 2: the edge e with smallest label adjacent to the root belongs to the spine.* Then we claim that the path between 1 and 2 contains only positive labels if and only if the second edge-label ℓ_2 of the spine is smaller than the first one (call it ℓ_1). Indeed, if $\ell_2 > \ell_1$, then the first nonroot vertex v on the spine gets a negative label (see the proof of Lemma 2.10) and lies on the path between 1 and 2. Conversely, if $\ell_2 < \ell_1$, this vertex v gets a positive label. Moreover, either this label is 2, or Find_∞ enters a finite fringe subtree, and will assign only positive labels, including 2, in this fringe subtree. In both cases, the path between 1 and 2 only contain positive labels.

By conditioning on the number of children of the root, we find that the probability of the first event is $\sum_{k=2}^\infty \frac{e^{-1}}{(k-1)!} \frac{k-1}{k} = 1/e$.

Let us now compute the probability that we are in the second case *and* that $\ell_2 < \ell_1$. Let us work conditionally given the number k of children of the root. The conditional probability that the edge with smallest label adjacent to the root is that on the spine is $1/k$. This smallest label has the distribution of the minimum of k independent uniform random variable in $[0, 1]$, that is density $k(1-x)^{k-1}$. Since ℓ_2 is uniform in $[0, 1]$, independently of the number of children of the root and the labels of the corresponding edges, conditionally on $\ell_1 = x$, the probability that $\ell_2 < \ell_1$ is simply x . Summing up, the probability that we are in the second case *and* $\ell_2 < \ell_1$ is

$$\sum_{k=1}^\infty \frac{e^{-1}}{(k-1)!} \frac{1}{k} \int_0^1 xk(1-x)^{k-1} dx = \int_0^1 xe^{-x} dx = 1 - 2/e,$$

where the first equality follows by exchanging sum and integral. The sum of the two probabilities is $1 - 1/e$, and this completes the proof. \square

Finally, we establish Corollary 1.2 (iii) and (iv), whose proofs are more involved.

Proof of Corollary 1.2 (iii) and (iv): We start with considering the limiting tree $\text{Shape}(\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathcal{T}_\infty^u))$ in Theorem 2.12. We denote u_1^∞ , u_2^∞ and $d_{1,2}^\infty$ the vertices with labels 1 and 2 in this tree, and their relative distance, respectively.

By Theorem 2.12 and the discussion in the beginning of Section 3.1 concerning the relation between $\#\tilde{\mathbb{T}}_i^{(n)}$, $\#\tilde{\mathbb{M}}_i^{(n)}$ and $\text{Tree}(\tilde{\mathcal{F}}_n)$, we have, for every $i, j \geq 1$,

$$\mathbb{P}\left(\#\tilde{\mathbb{T}}_1^{(n)} = i, \#\tilde{\mathbb{M}}_1^{(n)} = j\right) \xrightarrow{n \rightarrow \infty} \mathbb{P}\left(\deg(u_1^\infty) = i, d_{1,2}^\infty = j\right).$$

and

$$\mathbb{P}\left(\#\tilde{\mathbb{T}}_1^{(n)} = i, \#\tilde{\mathbb{T}}_2^{(n)} = j\right) \xrightarrow{n \rightarrow \infty} \mathbb{P}\left(\deg(u_1^\infty) = i, \deg(u_2^\infty) = j\right).$$

We also note that u_1^∞ , u_2^∞ and $d_{1,2}^\infty$ can be equivalently read on $\text{Find}_2(\mathcal{T}_\infty^u)$ instead of $\text{Shape}(\text{Find}_\infty \circ \overline{\text{Find}}_\infty(\mathcal{T}_\infty^u))$. Therefore, in order to compute the limiting probabilities, we only need to run Find_2 on \mathcal{T}_∞^u .

For integers $h \geq 1$, $0 \leq s \leq h$ and $L_h \subset [0, 1]$, we introduce the probability $P_h^s(k_0, \dots, k_{h-1}; k_h - 1)$ (resp. $P_{\geq h}^s(k_0, \dots, k_{h-1}; L_h)$) of the following conjunction of events, when running the algorithm Find_2 on \mathcal{T}_∞^u :

- the algorithm goes through exactly h edges (resp. at least h edges), i.e. u_2^∞ is at distance exactly h (resp. at least h) from the root;
- If, as in the description of the algorithm, we call j_1, j_2, \dots, j_h the vertices successively visited, then j_i has degree k_i for each $i < h$ (by convention, j_0 is the root of the tree);
- j_0, j_1, \dots, j_s are special vertices, while j_{s+1}, \dots, j_h are not;
- for $P_h^s(k_0, \dots, k_{h-1}; k_h - 1)$, we additionally require that j_h has $k_h - 1$ children (this shift makes formulas nicer).
- for $P_{\geq h}^s(k_0, \dots, k_{h-1}; L_h)$, we also require that the label ℓ_h of the edge from between j_{h-1} and j_h lies in L_h .

Recall that the degree of the root of \mathcal{T}_∞^u follows a size-biased Poisson distribution and that edges adjacent to the root are labeled by independent uniform variables in $[0, 1]$. If the root degree is k_0 , the minimum among labels of edges adjacent to the root has density $k_0(1 - \ell_1)^{k_0-1}$. Moreover, j_1 is uniformly distributed among the children of the roots, and so is the special vertex of height 1, so j_1 has a probability $1/k_0$ to be a special vertex. Therefore, the probability $P_{\geq 1}^0(k_0; L_1)$ and $P_{\geq 1}^1(k_0, L_1)$ are respectively given by

$$P_{\geq 1}^0(k_0; L_1) = \frac{e^{-1}}{(k_0 - 1)!} (k_0 - 1) \int_{\ell_1 \in L_1} (1 - \ell_1)^{k_0-1} d\ell_1;$$

$$P_{\geq 1}^1(k_0; L_1) = \frac{e^{-1}}{(k_0 - 1)!} \int_{\ell_1 \in L_1} (1 - \ell_1)^{k_0-1} d\ell_1.$$

Let us focus, e.g., on the case where j_1 is a special vertex. Then its offspring distribution is again a size-biased Poisson distribution. Conditionally on ℓ_1 and on

the fact that j_1 has k_1 children, the label ℓ_2 has density $\mathbb{1}_{\ell_2 > \ell_1} k_1 (\ell_1 + 1 - \ell_2)^{k_1 - 1}$. Again, the probability that j_2 is special is $1/k_1$. We therefore have

$$P_{\geq 2}^1(k_0, k_1; L_2) = \frac{e^{-1}}{(k_1 - 1)!} (k_1 - 1) \int_{\ell_2 \in L_2} \int_{\ell_1 < \ell_2} (\ell_1 + 1 - \ell_2)^{k_1 - 1} P_{\geq 1}^1(k_0; d\ell_1) d\ell_2;$$

$$P_{\geq 2}^2(k_0, k_1; L_2) = \frac{e^{-1}}{(k_1 - 1)!} \int_{\ell_2 \in L_2} \int_{\ell_1 < \ell_2} (\ell_1 + 1 - \ell_2)^{k_1 - 1} P_{\geq 1}^1(k_0; d\ell_1) d\ell_2.$$

Similarly, if j_1 is not a special vertex, we have

$$P_{\geq 2}^0(k_0, k_1; L_2) = \frac{e^{-1}}{k_1!} k_1 \int_{\ell_2 \in L_2} \int_{\ell_1 < \ell_2} (\ell_1 + 1 - \ell_2)^{k_1 - 1} P_{\geq 1}^0(k_0; d\ell_1) d\ell_2.$$

Continuing the reasoning, an easy induction proves that we have

$$P_{\geq h}^s(k_0, k_1, \dots, k_{h-1}; L_h) = \left(\prod_{i=0}^{h-1} \frac{e^{-1}}{(k_i - 1)!} \right) (k_s - 1)^*$$

$$\cdot \int_{\ell_h \in L_h} \int_{\ell_1 < \dots < \ell_h} (1 - \ell_1)^{k_0 - 1} (\ell_1 + 1 - \ell_2)^{k_1 - 1} \dots (\ell_{h-1} + 1 - \ell_h)^{k_{h-1} - 1} d\ell_1 \dots d\ell_h,$$

where $(k_s - 1)^* = 1$ if $s = h$ and $(k_s - 1)^* = k_s - 1$ otherwise.

Conditionally on the fact that the algorithm Find₂ goes through at least h edges, and conditionally on the label ℓ_h of the last visited edge, Find₂ will stop at height h if all labels of edges adjacent to j_h are smaller than ℓ_h , which happens with probability $\ell_h^{k_h - 1}$ (where $k_h - 1$ is the number of children of j_h). Again, the distribution of k_h depends on whether j_h is special or not, so that we should consider two cases separately: for $s < h$

$$P_h^s(k_0, \dots, k_{h-1}; k_h - 1) = \frac{e^{-1}}{(k_h - 1)!} \int_{[0,1]} \ell_h^{k_h - 1} P_{\geq h}^s(k_0, k_1, \dots, k_{h-1}; d\ell_h)$$

$$= \left(\prod_{i=0}^h \frac{e^{-1}}{(k_i - 1)!} \right) (k_s - 1) \int_{\ell_1 < \dots < \ell_h} \prod_{i=0}^h (\ell_i + 1 - \ell_{i+1})^{k_i - 1} d\ell_1 \dots d\ell_h, \quad (3.1)$$

with the convention $\ell_0 = 0$ and $\ell_{h+1} = 1$. Similarly, for $s = h$, we have

$$P_h^h(k_0, \dots, k_{h-1}; k_h - 1) = \frac{e^{-1}}{(k_h - 2)!} \int_{[0,1]} \ell_h^{k_h - 1} P_{\geq h}^s(k_0, k_1, \dots, k_{h-1}; d\ell_h)$$

$$= \left(\prod_{i=0}^{h-1} \frac{e^{-1}}{(k_i - 1)!} \right) \frac{e^{-1}}{(k_h - 2)!} \int_{\ell_1 < \dots < \ell_h} \prod_{i=0}^h (\ell_i + 1 - \ell_{i+1})^{k_i - 1} d\ell_1 \dots d\ell_h.$$

Note that this coincides with (3.1) for $s = h$, so that (3.1) is actually valid for every s in $\{0, 1, \dots, h\}$.

We now come back to the specific probabilities we want to evaluate. For item (i), we fix $h = j$ and $k_0 = i$ and sum over s and over k_1, \dots, k_h :

$$\mathbb{P}(\deg(u_1^\infty) = k_0, d_{1,2}^\infty = h) = \sum_{s=0}^h \sum_{k_1, \dots, k_h \geq 1} P_h^s(k_0, k_1, \dots, k_{h-1}; k_h - 1)$$

For $s = 0$, noting the sum over each k_i ($i \geq 1$) is the series expansion of an exponential, we have

$$\begin{aligned} \sum_{k_1, \dots, k_h \geq 1} P_h^0(k_0, k_1, \dots, k_{h-1}; k_h - 1) &= \frac{e^{-2}(k_0 - 1)}{(k_0 - 1)!} \int_{\ell_1 < \dots < \ell_h} (1 - \ell_1)^{k_0 - 1} e^{\ell_1} d\ell_1 \dots d\ell_h. \end{aligned}$$

A similar computation for $s \geq 1$ gives

$$\begin{aligned} \sum_{k_1, \dots, k_h \geq 1} P_h^s(k_0, k_1, \dots, k_{h-1}; k_h - 1) &= \frac{e^{-2}}{(k_0 - 1)!} \int_{\ell_1 < \dots < \ell_h} (1 - \ell_1)^{k_0 - 1} e^{\ell_1} (\ell_s + 1 - \ell_{s+1}) d\ell_1 \dots d\ell_h. \end{aligned}$$

Summing over s in $\{0, 1, \dots, h\}$, we find that

$$\begin{aligned} \mathbb{P}(\deg(u_1^\infty) = k_0, d_{1,2}^\infty = h) &= \frac{e^{-2}}{(k_0 - 1)!} \int_{\ell_1 < \dots < \ell_h} (1 - \ell_1)^{k_0 - 1} e^{\ell_1} (\ell_1 + h + k_0 - 2) d\ell_1 \dots d\ell_h. \end{aligned}$$

The integrand only depends on ℓ_1 . Besides, $\int_{\ell_1 < \dots < \ell_h} d\ell_2 \dots d\ell_h = \frac{(1 - \ell_1)^{h-1}}{(h-1)!}$ for any fixed ℓ_1 . Thus, we can rewrite the above integral as

$$\begin{aligned} \mathbb{P}(\deg(u_1^\infty) = k_0, d_{1,2}^\infty = h) &= \frac{e^{-2}}{(k_0 - 1)!(h - 1)!} \int_0^1 (1 - \ell_1)^{k_0 + h - 2} e^{\ell_1} (\ell_1 + h + k_0 - 2) d\ell_1 \\ &= \frac{e^{-2}}{(k_0 - 1)!(h - 1)!}, \end{aligned}$$

where the computation of the last integral is an easy calculus exercise. This shows Corollary 1.2 (iii).

To establish (iv), we fix $k_0 = i$ and $k_h = j$ (for the non-root vertex $j_h = u_2^\infty$, having $k_h - 1$ children means having degree k_h) and sum over h, s and k_1, \dots, k_{h-1} . Namely, we have

$$\mathbb{P}(\deg(u_1^\infty) = i, \deg(u_2^\infty) = j) = \sum_{h \geq 1} S_h,$$

where

$$S_h = \sum_{s=0}^h \sum_{k_1, \dots, k_{h-1} \geq 1} P_h^s(i, k_1, \dots, k_{h-1}; j - 1).$$

The case $h = 1$ is somewhat special since the last sum has only one summand corresponding to the empty list. In this case, we may have $s = 0$ and $s = 1$ giving

$$\begin{aligned} S_1 &= P_1^0(i; j - 1) + P_1^1(i; j - 1) \\ &= \frac{e^{-2}(i + j - 2)}{(i - 1)!(j - 1)!} \int_0^1 (1 - \ell_1)^{i-1} \ell_1^{j-1} d\ell_1 = \frac{e^{-2}(i + j - 2)}{(i + j - 1)!} \end{aligned} \tag{3.2}$$

Consider now the summands corresponding to $h \geq 2$. A similar computation as above, starting from (3.1) and separating the cases $s = 0$ and $s = h$, yields:

$$S_h = \frac{e^{-2}}{(i-1)!(j-1)!} \cdot \int_{\ell_1 < \dots < \ell_h} (1-\ell_1)^{i-1} \ell_h^{j-1} e^{\ell_1-\ell_h} (\ell_1-\ell_h+i+j+h-3) d\ell_1 \cdots d\ell_h.$$

The integrand only involves ℓ_1 and ℓ_h . Using the equality $\int_{\ell_1 < \dots < \ell_h} d\ell_2 \cdots d\ell_{h-1} = \frac{(\ell_h-\ell_1)^{h-2}}{(h-2)!}$ and using variables $x := \ell_1$ and $y := \ell_h$, we get

$$S_h = \frac{e^{-2}}{(i-1)!(j-1)!} \cdot \int_{x < y} \frac{(y-x)^{h-2}}{(h-2)!} (1-x)^{i-1} y^{j-1} e^{x-y} (x-y+i+j+h-3) dx dy.$$

Summing this over $h \geq 2$ and exchanging sum and integral (the terms are nonnegative), we obtain

$$\begin{aligned} \sum_{h \geq 2} S_h &= \frac{e^{-2}(i+j-1)}{(i-1)!(j-1)!} \cdot \int_{x < y} (1-x)^{i-1} y^{j-1} dx dy \\ &= \frac{e^{-2}(i+j-1)}{(i-1)!(j-1)!} \left(\frac{1}{i \cdot j} - \frac{(i-1)!(j-1)!}{(i+j)!} \right). \end{aligned}$$

Adding S_1 , which was computed in (3.2), we get

$$\begin{aligned} \mathbb{P}(\deg(u_1^\infty) = i, \deg(u_2^\infty) = j) &= \sum_{h \geq 1} S_h \\ &= e^{-2} \left(\frac{i+j-2}{(i+j-1)!} + \frac{i+j-1}{i!j!} - \frac{i+j-1}{(i+j)!} \right). \end{aligned}$$

This completes the proof of Corollary 1.2 (iv). □

4. A bijection and duality

In this section, we construct a bijection \mathcal{B} for minimal factorizations with the following property: if $F' = \mathcal{B}(F)$ with $F \in \mathfrak{M}_n$, then, for every $1 \leq i \leq n$, the number of transpositions that affect the trajectory of i in F is equal to the number of transpositions in F' containing i . Combinatorial consequences of this bijection are then discussed.

Our bijection is based on Goulden–Yong’s duality bijection (Goulden and Yong, 2002), which we now present (see Fig. 4.8 for an example; note that here we multiply from left to right, while in Goulden and Yong (2002) the multiplication is done from right to left). See Apostolakis (2018a,b) for extensions in a more general context. Let $F \in \mathfrak{M}_n$ be a minimal factorization. Recall from Section 2.2 its associated pointed non-plane EV-labelled tree $\mathcal{E}(F)$.

First draw $\mathcal{E}(F)$ inside the complex unit disk \mathbb{D} by identifying vertex j (for $1 \leq j \leq n$) with the complex number $e^{-\frac{2i\pi(j-1)}{n}}$. A *face* is a connected component of $\mathbb{D} \setminus \mathcal{E}(F)$. Then, by Goulden and Yong (2002), edges do not cross, and every face contains exactly one arc of \mathbb{S} of the form $\widehat{j, j+1}$ for a certain $1 \leq j \leq n$ (with the convention $n+1 = 1$ and by identifying $e^{-\frac{2i\pi(j-1)}{n}}$ with j , see Fig. 4.8). Conversely, every arc $\widehat{j, j+1}$ with $1 \leq j \leq n$ is contained in a face.

Some combinatorial information is easily read on $\mathcal{E}(F)$. Indeed, the number $\#T_i^F$ of transpositions in F containing i is simply the degree of the vertex labelled

i in $\mathcal{E}(F)$. Similarly, the number $\#M_j^F$ of transpositions that affect the trajectory of j in F is the number of edges lying around the face of $\mathcal{E}(F)$ containing the arc $\widehat{j, j+1}$. In particular, by applying a horizontal symmetry to $\mathcal{E}(F)$ we obtain the following identity on generating functions:

$$\sum_{F \in \mathfrak{M}_n} x^{\#T_1^F} y^{\#M_j^F} = \sum_{F \in \mathfrak{M}_n} x^{\#T_1^F} y^{\#M_{n+1-j}^F}. \quad (4.1)$$

Then, still following [Goulden and Yong \(2002\)](#), define the “dual” EV-labelled tree $\mathcal{E}^\dagger(F)$ as follows: the vertices are $e^{-\frac{2i\pi(j-1)}{n} - \frac{i\pi}{2n}}$ (which is given label j) for $1 \leq j \leq n$, and two vertices $e^{-\frac{2i\pi(j-1)}{n} - \frac{i\pi}{2n}}$ and $e^{-\frac{2i\pi(k-1)}{n} - \frac{i\pi}{2n}}$ are connected if the two faces containing $\widehat{j, j+1}$ and $\widehat{k, k+1}$ are adjacent in $\mathcal{E}(F)$. Moreover, the corresponding edge gets the label of the edge of $\mathcal{E}(F)$ separating these two faces.

Finally, define $\overline{\mathcal{E}}^\dagger(F)$ by changing the edge labels of $\mathcal{E}^\dagger(F)$ by “symmetrization”: exchange labels i and $n-i$ for every $1 \leq i \leq n/2$. It turns out that $\overline{\mathcal{E}}^\dagger(F)$ codes a minimal factorization, which allows to define $\mathcal{B}(F)$:

Lemma 4.1. *For every minimal factorization F , there exists a unique minimal factorisation $\mathcal{B}(F)$ such that $\overline{\mathcal{E}}^\dagger(F) = \mathcal{E}(\mathcal{B}(F))$.*

Proof: We recall [Goulden and Yong \(2002, Theorem 2.2\)](#) (adapted to the fact that here we multiply from left to right, while in [Goulden and Yong, 2002](#) the multiplication is done from right to left): in $\mathcal{E}(F)$, when turning along a face in clockwise order starting from the arc of \mathbb{S} in its boundary, the edge labels are decreasing. Therefore, in $\overline{\mathcal{E}}^\dagger(F)$, the edge labels are also decreasing in clockwise order in every face (starting each time from the circle arc contained in the face boundary). From [Goulden and Yong \(2002, Lemma 2.5\)](#), this condition implies the existence of a unique minimal factorisation whose associated drawing is $\overline{\mathcal{E}}^\dagger(F)$. This completes the proof. \square

The fact that \mathcal{B} is a bijection follows by definition of \mathcal{B} , since \mathcal{E} is a bijection.

Theorem 4.2. *Let F be a minimal factorization. For every $1 \leq i \leq n$:*

- (i) *the number of transpositions in F that affect the trajectory of i is equal to the number of transpositions in $\mathcal{B}(F)$ containing i , i.e. $\#M_i^F = \#T_i^{\mathcal{B}(F)}$;*
- (ii) *the number of transpositions in F containing i is equal to the number of transpositions in $\mathcal{B}(F)$ that affect the trajectory of $i-1$, i.e. $\#T_i^F = \#M_{i-1}^{\mathcal{B}(F)}$. (We use the convention $i-1 = n$ for $i = 1$.)*

When \mathcal{F}_n denotes a minimal factorizations of size n chosen uniformly at random, as in the Introduction we use the following notation for $1 \leq i \leq n$: $\mathbb{T}_i^{(n)} := T_i^{\mathcal{F}_n}$ and $\mathbb{M}_i^{(n)} := M_i^{\mathcal{F}_n}$. Theorem 4.2 implies that $\#\mathbb{T}_i^{(n)}$ and $\#\mathbb{M}_i^{(n)}$ have the same law. Combining with Corollary 3.2, we conclude that the limiting distribution of $\#\mathbb{M}_i^{(n)}$ is a size biased Poisson law, as claimed in Corollary 1.2.

Proof: Fix $1 \leq i \leq n$. For (i), observe that, by construction, the following numbers are all equal:

- the number of transpositions that affects the trajectory of i in F ;
- the number of edges adjacent to the face of $\mathcal{E}(F)$ containing the arc $\widehat{i, i+1}$;
- the degree of i in $\overline{\mathcal{E}}^\dagger(F) = \mathcal{E}(\mathcal{B}(F))$;

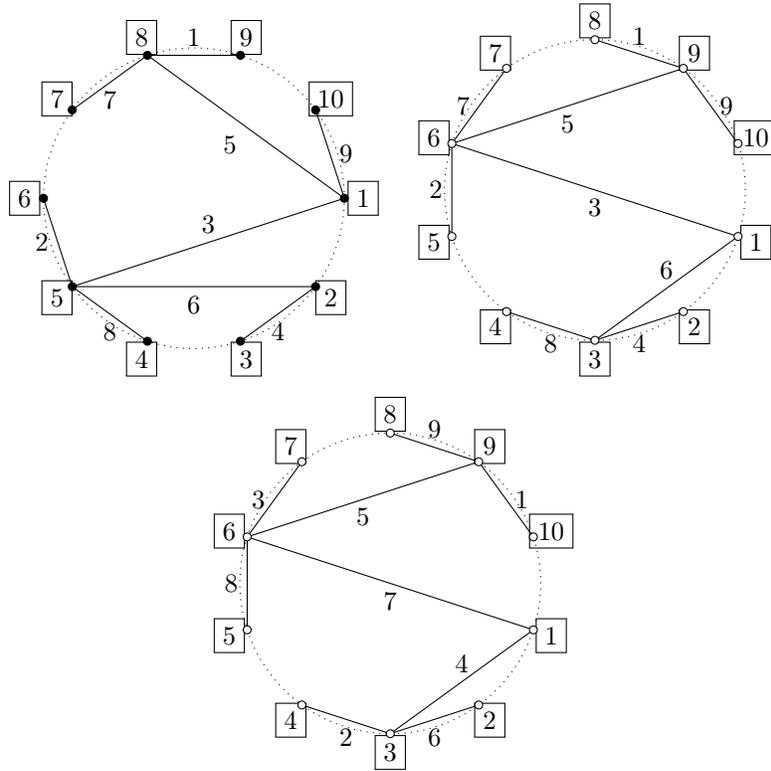


FIGURE 4.8. Several objects associated with the same minimal factorisation $F = ((8, 9) (5, 6) (1, 5) (2, 3) (1, 8) (2, 5) (7, 8) (4, 5) (1, 10))$ of $(1, 2, \dots, 10)$. From left to right: the EV-labelled tree $\mathcal{E}(F)$, its dual tree $\mathcal{E}^\dagger(F)$ and its symmetrized version $\bar{\mathcal{E}}^\dagger(F)$ which codes a minimal factorization $\mathcal{B}(F)$.

- the number of transpositions in $\mathcal{B}(F)$ containing i .

The proof of the second assertion is similar and is left to the reader. □

We can now prove the distributional identity stated in Theorem 1.3.

Proof of Theorem 1.3: The first equality in distribution is a probabilistic translation of the properties of the bijection \mathcal{B} (Theorem 4.2). The second is the result of applying to $\mathcal{E}(F)$ an axial symmetry around the diameter containing $e^{-\frac{i\pi(k-1)}{n}}$. □

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