



Order of fluctuations of the free energy in the SK model at critical temperature

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Abstract. We present an elementary approach to the order of fluctuations for the free energy in the Sherrington-Kirkpatrick mean field spin glass model at and near the critical temperature. It is proved that at the critical temperature the variance of the free energy is of $O((\log N)^2)$. In addition, we show that if one approaches the critical temperature from the low temperature regime at the rate $O(N^{-\alpha})$ for some $\alpha > 0$, then the variance is of $O((\log N)^2 + N^{1-\alpha})$.

1. Introduction

The Sherrington-Kirkpatrick (SK) model was initially introduced in [Sherrington and Kirkpatrick \(1975\)](#) in order to explain some strange magnetic properties of certain alloys. Over the past decades, it has received a great attention in the physics and mathematics communities. See [Mézard et al. \(1987\)](#) for physics treatments and [Panchenko \(2013\)](#); [Talagrand \(2003, 2011a,b\)](#) for recent mathematical progress.

The aim of this short note is to study the order of fluctuations of the free energy in the SK model at the critical temperature. For any $N \geq 1$, the Hamiltonian of the SK model is defined as

$$H_N(\sigma) = \frac{1}{\sqrt{N}} \sum_{1 \leq i, j \leq N} g_{ij} \sigma_i \sigma_j$$

for any $\sigma \in \Sigma_N := \{-1, +1\}^N$, where $(g_{ij})_{1 \leq i, j \leq N}$ is a family of independent standard Gaussian random variables. The covariance of H_N is described by

$$\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = NR(\sigma^1, \sigma^2)^2,$$

where $R(\sigma^1, \sigma^2) := N^{-1} \sum_{i=1}^N \sigma_i^1 \sigma_i^2$ is called the overlap between spin configurations $\sigma^1, \sigma^2 \in \Sigma_N$. The free energy and the Gibbs measure at (inverse) temperature

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$\beta > 0$ is defined as

$$F_N(\beta) = \log \sum_{\sigma \in \Sigma_N} \exp(\beta H_N(\sigma))$$

and

$$G_N(\sigma) = \frac{\exp(\beta H_N(\sigma))}{Z_N(\beta)}, \quad \forall \sigma \in \Sigma_N,$$

where $Z_N(\beta) := \sum_{\sigma \in \Sigma_N} \exp(\beta H_N(\sigma))$. Denote by σ^1, σ^2 i.i.d. samplings from G_N and by $\langle \cdot \rangle$ the expectation with respect to these random variables.

The SK model is known to exhibit a phase transition at the critical temperature $\beta_c := 1/\sqrt{2}$. In the high temperature regime $\beta < \beta_c$, the limiting free energy is equal to the anneal free energy, that is,

$$\lim_{N \rightarrow \infty} \frac{F_N(\beta)}{N} = \lim_{N \rightarrow \infty} \frac{\log \mathbb{E} Z_N}{N}$$

and if we sample two independent σ^1, σ^2 from the Gibbs measure, then they are essentially orthogonal to each other in the sense that $\lim_{N \rightarrow \infty} \mathbb{E} \langle R(\sigma^1, \sigma^2)^2 \rangle = 0$. In contrast, the model exhibits different behaviors in the lower temperature regime $\beta > \beta_c$, where we see that

$$\lim_{N \rightarrow \infty} \frac{F_N(\beta)}{N} < \lim_{N \rightarrow \infty} \frac{\log \mathbb{E} Z_N}{N}$$

and the two independent samplings σ^1, σ^2 have nonzero overlap, that is,

$$\lim_{N \rightarrow \infty} \mathbb{E} \langle R(\sigma^1, \sigma^2)^2 \rangle > 0.$$

Indeed, it was conjectured that the limiting distribution of $R(\sigma^1, \sigma^2)$ should be described by a probability measure whose support is an interval $[0, q]$ for some $q \in (0, 1)$. See [Mézard et al. \(1987\)](#); [Talagrand \(2003, 2011b\)](#) for more details.

Previous rigorous results on the order of fluctuations of the free energy is summarized as follows:

- **High temperature:** Aizenman, Lebowitz, and Ruelle ([Aizenman et al., 1987](#)) proved that the free energy has Gaussian fluctuations in the high temperature regime. Their result implies that the limit of $\text{Var}(F_N(\beta))$ exists and is finite.
- **Near critical temperature:** The problem of understanding the transition near the critical temperature was intensively studied in Talagrand's books [Talagrand \(2003, Section 2.14\)](#) and [Talagrand \(2011b, Section 11.7\)](#), where he showed that when $\beta = \beta_N$ approaches the critical temperature from the high temperature regime in the rate $\lim_{N \rightarrow \infty} N^{1/3}(\beta_c^2 - \beta_N^2) = c$, then the overlap undergoes a phase transition depending on whether c is finite or infinite. Exactly at the criticality β_c , he also proved that the overlap is controlled by $\mathbb{E} \langle R(\sigma^1, \sigma^2)^2 \rangle \leq C/\sqrt{N}$ for some universal constant C , from which it can be derived that $\text{Var}(F_N(\beta_c)) \leq C\sqrt{N}$, see [Remark 2.2](#) below.
- **Low temperature:** [Chatterjee \(2009\)](#) showed that

$$\text{Var}(F_N(\beta)) \leq \frac{C(\beta)N}{\log N},$$

where $C(\beta)$ is a constant independent of N . Incidentally, it was proved by [Chatterjee \(2019+, Theorem 2.4\)](#) that fluctuations of the free energy are at least of order 1 at any β .

The order of fluctuations of the free energy is also studied in the physics literature, for example [Aspelmeier \(2008\)](#); [Aspelmeier et al. \(2008\)](#); [Parisi and Rizzo \(2009\)](#). At the critical temperature, it is expected that

$$\text{Var}(F_N(\beta_c)) = \frac{1}{6} \log N + O(1).$$

In the low temperature regime, it is believed that $\text{Var}(F_N(\beta)) \asymp N^{2\mu}$ for some $\mu \geq 0$ ¹. However, physicists' simulations have not reached an agreement: while some suggested $\mu = 1/4$, it was also proposed that $\mu = 1/6$. See [Aspelmeier et al. \(2008, Section 7\)](#) for a detailed discussion. Notably, [Aspelmeier \(2008\)](#) argued that $\mu \leq 1/4$.

This note is focused on the order of fluctuation of F_N at the critical temperature. Our main result below contains two parts. First, at the critical temperature, we obtain an upper bound of order $(\log N)^2$ for the variance of the free energy. Second, we show that if one approaches the critical temperature from the low temperature regime in the rate $\beta_N = \sqrt{\beta_c^2 + N^{-\alpha}}$ for $\alpha \in (0, 1)$, then a polynomial bound $N^{1-\alpha}$ is obtained. This improves Chatterjee's bound $N/\log N$ in the near critical case.

Theorem 1.1. *The following statements hold:*

- (1) *There exists a constant $C > 0$ such that*

$$\text{Var}(F_N(\beta_c)) \leq C((\log N)^2 + 1), \quad \forall N \geq 1.$$

- (2) *For any fixed $\alpha > 0$ and $d > 0$, there exists a constant C depending only on α and d such that*

$$\text{Var}\left(F_N\left(\sqrt{\beta_c^2 + dN^{-\alpha}}\right)\right) \leq C((\log N)^2 + N^{1-\alpha}), \quad \forall N \geq 1.$$

Our approach is motivated by a work of Guerra and Toninelli ([Guerra and Toninelli, 2002](#)), where they derived the limit of the SK free energy and provided a rate of convergence. Their idea was to consider a coupled free energy with Hamiltonian of the form $\sqrt{s}(H_N(\sigma^1) + H_N(\sigma^2)) + \lambda NR(\sigma^1, \sigma^2)^2$, where \sqrt{s} is a varying temperature and λ is an auxiliary parameter. From this, they derived an ordinary differential inequality for this coupled free energy in the variable $s \geq 0$ and by solving this inequality, they obtained the rate of convergence of the free energy. Our argument adopts a different route by considering a coupled Hamiltonian that is related to the problem of chaos in disorder in the SK model considered in [Chatterjee \(2009, 2014\)](#).

2. Proof of Theorem 1.1

Let H'_N, H''_N be two independent copies of H_N . For $0 \leq t \leq 1$ and $\sigma, \rho \in \Sigma_N$, set

$$\begin{aligned} H_{N,t}^1(\sigma) &= \sqrt{t}H_N(\sigma) + \sqrt{1-t}H'_N(\sigma), \\ H_{N,t}^2(\rho) &= \sqrt{t}H_N(\rho) + \sqrt{1-t}H''_N(\rho). \end{aligned}$$

Note that

$$\mathbb{E}H_{N,t}^1(\sigma)H_{N,t}^2(\rho) = tNR(\sigma, \rho)^2.$$

¹ $a_N \asymp b_N$ means that there exist $c, C > 0$ independent of N such that $cb_N \leq a_N \leq Cb_N$.

For any $t \in [0, 1]$ and $\lambda \in \mathbb{R}$, define

$$\phi_N(t, \lambda) = \frac{1}{N} \mathbb{E} \log \sum_{\sigma, \rho \in \Sigma_N} \exp(\beta(H_{N,t}^1(\sigma) + H_{N,t}^2(\rho)) + \lambda\beta^2 NR(\sigma, \rho)^2).$$

Denote by $\langle \cdot \rangle_{t, \lambda}$ the Gibbs expectation associated to this free energy. That is, the expectation with respect to the measure

$$\frac{\sum_{\sigma, \rho} I((\sigma, \rho) \in \cdot) \exp(\beta(H_{N,t}^1(\sigma) + H_{N,t}^2(\rho)) + \lambda\beta^2 NR(\sigma, \rho)^2)}{\sum_{\sigma, \rho} \exp(\beta(H_{N,t}^1(\sigma) + H_{N,t}^2(\rho)) + \lambda\beta^2 NR(\sigma, \rho)^2)}.$$

Lemma 2.1. *For any $\beta > 0$ and $t \in [0, 1]$ satisfying $2\beta^2 t < 1$, we have that for any $N \geq 1$,*

$$\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t, 0} \leq \frac{2}{N(1 - 2\beta^2 t)} \log \frac{2}{1 - 2\beta^2 t}.$$

Proof: Note that for $t \in (0, 1)$ and any $\sigma, \sigma', \rho, \rho' \in \Sigma_N$,

$$\begin{aligned} \mathbb{E} \left(\frac{H_N(\sigma)}{\sqrt{t}} - \frac{H'_N(\sigma)}{\sqrt{1-t}} \right) H_{N,t}^1(\sigma') &= 0, \\ \mathbb{E} \left(\frac{H_N(\rho)}{\sqrt{t}} - \frac{H''_N(\rho)}{\sqrt{1-t}} \right) H_{N,t}^2(\rho') &= 0, \\ \mathbb{E} \left(\frac{H_N(\sigma)}{\sqrt{t}} - \frac{H'_N(\sigma)}{\sqrt{1-t}} \right) H_{N,t}^2(\rho') &= NR(\sigma, \rho')^2, \\ \mathbb{E} \left(\frac{H_N(\rho)}{\sqrt{t}} - \frac{H''_N(\rho)}{\sqrt{1-t}} \right) H_{N,t}^1(\sigma') &= NR(\sigma', \rho)^2. \end{aligned}$$

Using Gaussian integration by parts gives

$$\begin{aligned} \partial_t \phi_N(t, \lambda) &= \frac{1}{2N} \beta \mathbb{E} \left\langle \left(\frac{H_N(\sigma)}{\sqrt{t}} - \frac{H'_N(\sigma)}{\sqrt{1-t}} \right) + \left(\frac{H_N(\rho)}{\sqrt{t}} - \frac{H''_N(\rho)}{\sqrt{1-t}} \right) \right\rangle_{t, \lambda} \\ &= \frac{\beta^2}{2} (2\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t, \lambda} - \mathbb{E}\langle R(\sigma^1, \rho^2)^2 \rangle_{t, \lambda} - \mathbb{E}\langle R(\sigma^2, \rho^1)^2 \rangle_{t, \lambda}) \\ &= \beta^2 (\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t, \lambda} - \mathbb{E}\langle R(\sigma^1, \rho^2)^2 \rangle_{t, \lambda}), \end{aligned}$$

where the pairs (σ, ρ) , (σ^1, ρ^1) , and (σ^2, ρ^2) are i.i.d. copies from the Gibbs measure $\langle \cdot \rangle_{t, \lambda}$ and the second equality used symmetry between the distributions corresponding to the pairs (σ^1, ρ^2) and (σ^2, ρ^1) . Set

$$\Phi_N(t, \lambda) = \phi_N(t, \lambda - t).$$

Then from the above equation,

$$\begin{aligned} \partial_t \Phi_N(t, \lambda) &= \partial_t \phi_N(t, \lambda - t) - \partial_\lambda \phi_N(t, \lambda - t) \\ &= \beta^2 (\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t, \lambda-t} - \mathbb{E}\langle R(\sigma^1, \rho^2)^2 \rangle_{t, \lambda-t}) - \beta^2 \mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t, \lambda-t} \\ &= -\beta^2 \mathbb{E}\langle R(\sigma^1, \rho^2)^2 \rangle_{t, \lambda-t}. \end{aligned}$$

Note that $\partial_t \Phi_N(s, \lambda + t) = -\beta^2 \mathbb{E} \langle R(\sigma^1, \rho^2)^2 \rangle_{s, \lambda+t-s}$. It follows that

$$\begin{aligned} \phi_N(t, \lambda) &= \Phi_N(t, \lambda + t) \\ &= \int_0^t \partial_t \Phi_N(s, \lambda + t) ds + \Phi_N(0, \lambda + t) \\ &\leq \Phi_N(0, \lambda + t) \\ &= \phi_N(0, \lambda + t). \end{aligned}$$

Now using the convexity of ϕ_N in λ gives

$$\begin{aligned} \beta^2 \lambda \mathbb{E} \langle R(\sigma, \rho)^2 \rangle_{t,0} &= \lambda \partial_\lambda \phi_N(t, 0) \\ &\leq \phi_N(t, \lambda) - \phi_N(t, 0) \\ &\leq \phi_N(0, \lambda + t) - \phi_N(t, 0) \\ &= \phi_N(0, \lambda + t) - \phi_N(0, 0), \end{aligned} \tag{2.1}$$

where the last equation used that $\phi_N(t, 0) = \phi_N(0, 0)$. Note that under the measure $\mathbb{E} \langle \cdot \rangle_{0,0}$, σ and ρ are independent uniform random variables on Σ_N and hence, $NR(\sigma, \rho)$ is equal to the sum of N i.i.d. Rademacher random variables X_1, \dots, X_N in distribution. It is well-known (see, e.g., Talagrand, 2011b, Eq. (A.19)) that

$$\mathbb{E} \exp \left[x \left(\frac{X_1 + \dots + X_N}{\sqrt{N}} \right)^2 \right] \leq \frac{1}{\sqrt{1-2x}}, \quad \forall x \in [0, 1/2).$$

Consequently, using (2.1) and Jensen’s inequality, we have

$$\begin{aligned} \beta^2 \lambda \mathbb{E} \langle R(\sigma, \rho)^2 \rangle_{t,0} &\leq \phi_N(0, \lambda + t) - \phi_N(0, 0) \\ &= \frac{1}{N} \mathbb{E} \log \left\langle \exp(\beta^2(\lambda + t)NR(\sigma, \rho)^2) \right\rangle_{0,0} \\ &\leq \frac{1}{N} \log \mathbb{E} \left\langle \exp(\beta^2(\lambda + t)NR(\sigma, \rho)^2) \right\rangle_{0,0} \\ &\leq \frac{1}{N} \log \frac{1}{\sqrt{1-2\beta^2(\lambda + t)}} \end{aligned}$$

whenever $2\beta^2(\lambda + t) < 1$. In particular, plugging

$$\lambda = \frac{1}{2} \left(\frac{1}{2\beta^2} - t \right)$$

into the above inequality completes our proof. □

Proof of Theorem 1.1: Recall from Chatterjee (2009) that the variance of the free energy can be expressed as

$$\text{Var}(F_N(\beta)) = \beta^2 N \int_0^1 \mathbb{E} \langle R(\sigma, \rho)^2 \rangle_{t,0} dt. \tag{2.2}$$

For any $0 < \delta < 1/(2\beta^2) \leq 1$,

$$\begin{aligned} \int_0^{\frac{1}{2\beta^2}-\delta} \mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t,0} dt &\leq \frac{1}{N} \int_0^{\frac{1}{2\beta^2}-\delta} \frac{2}{1-2\beta^2 t} \log \frac{2}{1-2\beta^2 t} dt \\ &= \frac{1}{2N\beta^2} \left(\log \frac{1-2\beta^2 t}{2} \right)^2 \Big|_0^{\frac{1}{2\beta^2}-\delta} \\ &= \frac{1}{2N\beta^2} \left((\log(\beta^2 \delta))^2 - (\log 2)^2 \right) \\ &\leq \frac{1}{2N\beta^2} (\log(\beta^2 \delta))^2 \\ &\leq \frac{1}{N\beta^2} ((\log \delta)^2 + 4(\log \beta)^2), \end{aligned}$$

where we have used the inequality $(\log(ab))^2 \leq 2(\log a)^2 + 2(\log b)^2$ for any $a, b > 0$. On the other hand, noting that $|R(\sigma, \rho)| \leq 1$ implies

$$\int_{\frac{1}{2\beta^2}-\delta}^1 \mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t,0} dt \leq 1 - \frac{1}{2\beta^2} + \delta.$$

From this,

$$\text{Var}(F_N(\beta)) \leq \beta^2 \left(\frac{1}{\beta^2} ((\log \delta)^2 + 4(\log \beta)^2) + \left(1 - \frac{1}{2\beta^2} + \delta \right) N \right).$$

If $\beta = \beta_c$, we take $\delta = 1/N$ so that

$$\text{Var}(F_N(\beta)) \leq ((\log N)^2 + 4(\log 2)^2) + \frac{1}{2}$$

and this gives the first assertion. If $\beta^2 = \beta_c^2 + dN^{-\alpha}$ for $d > 0$, we take $\delta = dN^{-\alpha}$ and note that

$$1 - \frac{1}{2\beta^2} + \delta = \frac{2dN^{-\alpha}}{1 + 2dN^{-\alpha}} + dN^{-\alpha} \leq 3dN^{-\alpha},$$

which implies that as long as N is large enough,

$$\begin{aligned} \text{Var}(F_N(\beta)) &\leq \beta^2 \left(\frac{1}{\beta^2} ((-\alpha \log N + \log d)^2 + 4(\log \beta)^2) + 3dN^{1-\alpha} \right) \\ &= (-\alpha \log N + \log d)^2 + 4(\log \beta)^2 + 3d\beta^2 N^{1-\alpha} \end{aligned}$$

and the second assertion follows. \square

Remark 2.2. Consider the SK model at the critical temperature β_c . Recall from [Talagrand \(2011b, Chapter 11\)](#) that there exists a constant $C > 0$ such that $\mathbb{E}\langle R(\sigma^1, \sigma^2)^2 \rangle \leq C/\sqrt{N}$ for all $N \geq 1$. Also, it is known (see [Chatterjee, 2009](#)) that $t \in [0, 1] \mapsto \mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t,0}$ is a nondecreasing function with $\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{1,0} = \mathbb{E}\langle R(\sigma^1, \sigma^2)^2 \rangle$. These imply that $\mathbb{E}\langle R(\sigma, \rho)^2 \rangle_{t,0} \leq C/\sqrt{N}$ for all $t \in [0, 1]$. Consequently, from (2.2), $\text{Var}(F_N(\beta_c)) \leq C\beta_c^2\sqrt{N}$.

Remark 2.3. Another interesting example to consider is the pure p -spin model with $p \geq 3$, i.e., $\mathbb{E}H_N(\sigma^1)H_N(\sigma^2) = NR(\sigma^1, \sigma^2)^p$. In this case, the determination of the critical temperature $\beta_{p,c}$ for the corresponding free energy was studied in [Chen \(2019+\)](#) and it was established there that $\text{Var}(F_N(\beta)) \asymp N^{1-p/2}$ for all $\beta < \beta_{p,c}$. The current approach however does not seem to work at the criticality.

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