ALEA, Lat. Am. J. Probab. Math. Stat. 18, 1507–1515 (2021)
DOI: 10.30757/ALEA.v18-54

Random walk on the self-avoiding tree

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Abstract. We consider a modified version of the biased random walk on a tree constructed from the set of finite self-avoiding walks on the hexagonal lattice, and use it to construct probability measures on infinite self-avoiding walk. Under theses probability measures, we prove that the infinite self-avoiding walks have the Russo-Seymour-Welsh property of the exploration curve of the critical Bernoulli percolation.

1. Introduction

An n-step self-avoiding walk (SAW) (or a self-avoiding walk of length n) in a regular lattice $\mathbb{L}$ (such as the integer lattice $\mathbb{Z}^2$, triangular lattice $\mathbb{T}$, hexagonal lattice, etc) is a nearest neighbor path $\gamma = (\gamma_0, \gamma_1, \dots, \gamma_n)$ that visits no vertex more than once. Self-avoiding walks were first introduced as a lattice model for polymer chains (see Flory, 1953); while they are very easy to define, they are extremely difficult to analyze rigorously and there are still many basic open questions about them (see Madras and Slade, 1993, Chapter 1).

Let c_n be the number of SAWs of length n starting at the origin. The *connective constant* of $\mathbb{L}$, which we will denote by μ, is defined by

$$c_n = \mu^{n+o(n)} \quad \text{when} \quad n \to \infty.$$

The existence of the connective constant is easy to establish from the sub-multiplicativity relation $c_{n+m} \leq c_n c_m$, from which one can also deduce that $c_n \geq \mu^n$ for all n; the existence of μ was first observed by Hammersley and Morton (1954). Nienhuis (1982) gave a prediction that for all regular planar lattices, $c_n = \mu^n n^{\alpha+o(1)}$ where $\alpha = \frac{11}{32}$, and this prediction is known to hold under the assumption of the existence of a conformally invariant scaling limit, see *e.g.* Lawler et al. (2004).

We are interested in defining a natural family of probability measures on the set SAW_∞ of *infinite* self-avoiding walks. Such a family was constructed in Beffara and Huynh (2017) by using the biased random walk with one parameter on a particular tree which is called the *self-avoiding tree* (see Section 1.1 for the definition). In Beffara and Huynh (2017), the authors proved that under these

Received by the editors April 4h, 2020; accepted March 3rd, 2021.
2010 *Mathematics Subject Classification.* 82C24 (and 82C22, 82C41, 82C43, 60K35).
Key words and phrases. Self-avoiding walk, effective conductance, random walk on tree.

measures, the infinite self-avoiding walks almost surely visit the line $\mathbb{Z} \times \{0\}$ infinitely many times. However we don't know whether the infinite self-avoiding walks visit the interval $[n, 2n] \times \{0\}$ with a probability larger than a constant which do not depend on n.

In this paper, we construct a family of probability measures on the set SAW_∞ by using a biased random walk with a reinforcement, which we call the biased random walk with two parameters. We prove that under these measures, the infinite self-avoiding walks visit the interval $[n, 2n] \times \{0\}$ with a probability larger than a constant which do not depend on n.

1.1. *Background.* In this paper, we will focus on the case of hexagonal lattice $\mathbb{T}^*_+ := \mathbb{T}^* \cap \{y \geq 0\}$ (see Figure 2.4). Let $\mathcal{T}_{\mathbb{T}^*_+}$ be the tree whose vertices are the finite self-avoiding walks in $\mathbb{T}^*_+$ starting at the origin $o := (0,0)$, where two such vertices are adjacent when one walk is a one-step extension of the other. We will call this tree the *self-avoiding tree* on $\mathbb{T}^*_+$. Formally, denote by Ω_n the set of self-avoiding walks of length n starting at the origin and $V := \bigcup_{n=0}^{+\infty} \Omega_n$. Two elements $x, y \in V$ are adjacent if one path is an extension by one step of the other. We then define $\mathcal{T}_{\mathbb{T}^*_+} = (V, E)$. Denote by o its root.

Remark 1.1. Note that each infinite branch of $\mathcal{T}_{\mathbb{T}^*_+}$ is an infinite self-avoiding walk in the lattice $\mathbb{T}^*_+$.

Let $\mathcal{T}$ be an infinite, locally finite and rooted tree and denote by o its root. For any vertex ν of $\mathcal{T}$, denote by ν^{-1} its *parent* (we also say that ν is a *child* of ν^{-1}), *i.e.* the neighbor of ν with shortest distance from the root o. Denote by $\partial(\nu)$ the number of children of ν. In the case $\partial(\nu) \neq 0$, denote by $\nu_1, \cdots, \nu_{\partial(\nu)}$ its children. If a vertex has no child, it is called a *leaf.*
We define an order on $V(\mathcal{T})$ as follows: if $\nu, \mu \in V(\mathcal{T})$, we say that $\nu \leq \mu$ if the simple path joining the root o to μ passes through ν. For each $\nu \in V(\mathcal{T})$, we define the *subtree* of $\mathcal{T}$ rooted at ν, denoted by $\mathcal{T}^\nu$, where $V(\mathcal{T}^\nu) := \{\mu \in V(\mathcal{T}) : \nu \leq \mu\}$ and $E(\mathcal{T}^\nu) = E(\mathcal{T})|_{V(\mathcal{T}^\nu) \times V(\mathcal{T}^\nu)}$. Note that ν is the root of $\mathcal{T}^\nu$ and $\mathcal{T}^o = \mathcal{T}$.

1.2. *Random walk on trees.* Given an infinite, locally finite and rooted tree $\mathcal{T}$ with conductances (i.e positive numbers) assigned to the edges, we consider the random walk starting at the root that can go from a vertex to its parent or children and whose transition probabilities from a vertex are proportional to the conductances along the edges to be traversed.

Let $\lambda > 0$ and we consider conductances λ^n on edges at distance n from the root. In this cas, the random walk is called *biased random walk with one parameter* λ and denoted by RW_λ. Note that the conductances increase by a factor of λ as the distance increase 1, then the relative weights at a vertex are as shown in Figure 1.1.

Let $\lambda, \eta > 0$ and we define a modified version of RW_λ: if the relative weights at a vertex are as shown in Figure 1.2, then the random walk is called *biased random walk with two parameter* (λ, η), and denoted by $RW_{\lambda,\eta}$.

Let $\lambda, \eta > 0$ and consider the biased random walk $RW_{\lambda,\eta}$ on $\mathcal{T}$. For (λ, η) such that the biased random walk $RW_{\lambda,\eta}$ on $\mathcal{T}$ is transient, then almost surely, the random walk does not visit $\mathcal{T}_k$[1] anymore after a sufficiently large time. We can then define the *limit walk*, as denoted by $\omega^\infty_{\lambda,\eta}$ in the following way:

$$\omega^\infty_{\lambda,\eta}(i) = x_i \iff \left\{ \begin{array}{c} x_i \in \mathcal{T}_i \\ \exists n_0, \forall n > n_0 : X_n \in \mathcal{T}^{x_i} \end{array} \right\}.$$

Remark 1.2. Fix $\eta > 0$ and let $\lambda > 0$ such that the biased random walk $RW_{\lambda,\eta}$ on $\mathcal{T}$ is transient and then $\omega^\infty_{\lambda,\eta}$ is well defined. By letting λ goes to infinity, the law of first k-steps (for any k) of

[1]Denote by $\mathcal{T}_k$ the set of vertices of $\mathcal{T}$ at distance k from the root.

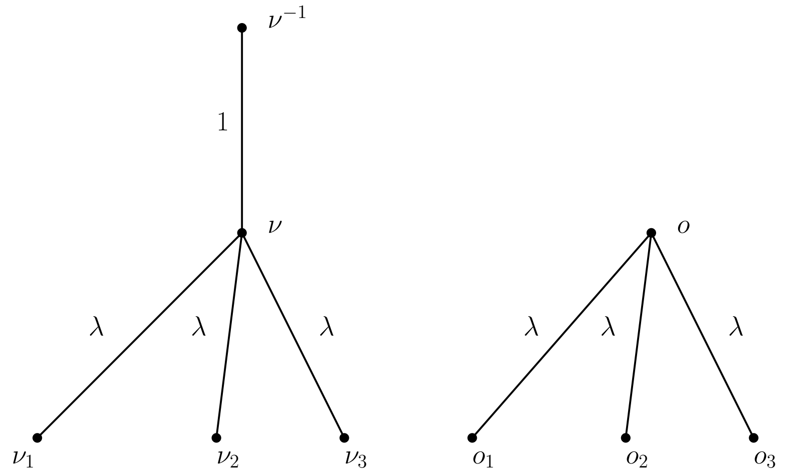

FIGURE 1.1. On the left: the relative weights at a vertex ν other than the root for the biased random walk with one parameter λ. On the right: the relative weights at the root for the biased random walk with one parameter λ.

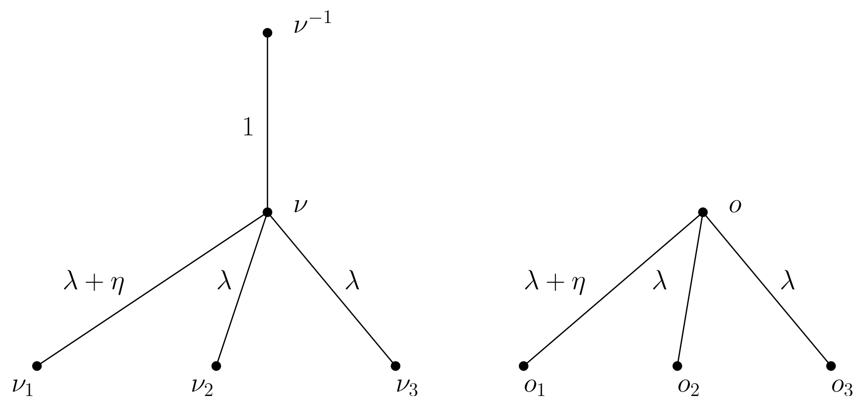

FIGURE 1.2. On the left: the relative weights at a vertex ν other than the root for the biased random walk with parameter (λ, η). On the right: the relative weights at the root for the biased random walk with parameter (λ, η).

$\omega^{\infty}_{\lambda,\eta}$ converges towards that of a stochastic process $\mathbf{X}$: at each step, $\mathbf{X}$ uniformly chooses one of its children (which can extend to infinity) and never returns to its parent. This stochatic process is called *biased random walk* with parameters ∞, and denoted by RW_∞.

Denote by $\mathbb{P}_{\lambda,\eta}$ the law of $\omega^{\infty}_{\lambda,\eta}$ and we write ω^{∞}_{∞} for the limit walk of RW_∞.

Recall that if $\mathcal{T}$ is a tree, we denote by $\widetilde{\mathcal{T}}$ the subtree obtained from $\mathcal{T}$ by recursively erasing all its leaves; in terms of our dynamical self-avoiding walk model, this corresponds to preventing the path from entering traps. The reader can easily check that the limit walk is the same on these trees without leaves as in the original ones, it is sufficient to prove the results in the case of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$. We consider the limit walk on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ instead of $\mathcal{T}_{\mathbb{T}^*_+}$.

1.3. *Main results.* Consider the self-avoiding tree $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$. We define an order on the children of a vertex of $\mathcal{T}_{\mathbb{T}^*_+}$ as in the following convention.

Convention: Let ν be a vertex of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ at distance n from the root (i.e $|v| = n$) and assume that $\partial v = 2$. By the construction of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$, ν is a self-avoiding walk of length n starting at the origin. Let α (resp. β) be the extension by one step of ν by choosing the left (resp. right) neighbor of $\nu(n)$ (see Figure 1.3). We then define two children of the vertex ν in the tree $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ by letting: $\nu_1 = \alpha$ and $\nu_2 = \beta$.

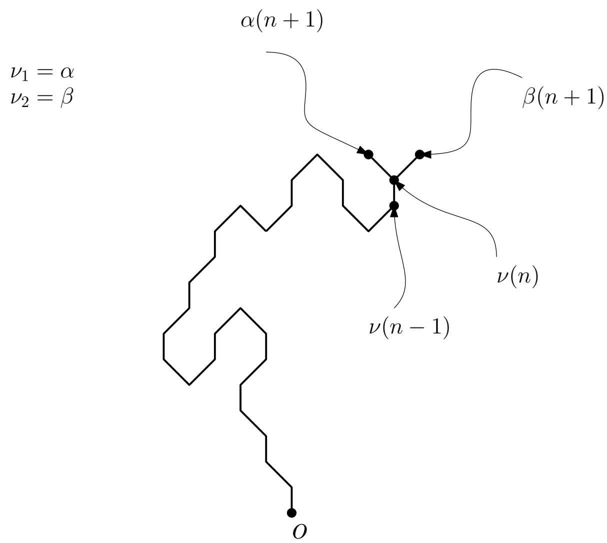

FIGURE 1.3. Two extensions α and β of ν

Let $\lambda, \eta > 0$ and consider the biased random walk with parameter (λ, η) on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$. Note that the limit walk $\omega^\infty_{\lambda,\eta}$ is a (random) infinite self-avoiding walk and $\mathbb{P}_{\lambda,\eta}$ is a probability measure on the set of infinite self-avoiding walks starting at the origin (denoted by SAW_∞) in the lattice $\mathbb{T}^*_+$. By the same argument used in (Beffara and Huynh, 2017, Section 6.2) and Remark 1.2, we can see that ω^∞_∞ (i.e $\lambda = \infty$) can be interpreted as the exploration curve $\gamma^{1/2}$ of the critical Bernoulli percolation on the hexagonal lattice – see Section 2 for a formal definition of $\gamma^{1/2}$. This is very useful because every feature of the curve $\gamma^{1/2}$ is also one for ω^∞_∞ and can therefore be restated in terms of the biased walk on the self-avoiding tree. One of these properties is that $\gamma^{1/2}$ reaches the interval $[n, 2n] \times \{0\}$ with a probability larger than a positive constant c_1 and smaller than another constant $c_2 < 1$ which do not depend on n. This property is called *RSW-property* (see Seymour and Welsh, 1978 and Russo, 1981 for more details) and the constant c_1 (resp. c_2) is called a *lower bound* (resp. *upper bound*) of the RSW-property.

In this paper, we prove that if $\eta > 2$ and λ is large enough, then the limit walk $\omega^\infty_{\lambda,\eta}$ has a lower bound of the RSW-property:

Theorem 1.3. *For all $\eta > 2$ and for all $\lambda > \frac{2\eta}{\eta-2}$: $\exists c \in]0,1[, \forall n \geq 1$, we have:*

$$\mathbb{P}_{\lambda,\eta}(\omega^\infty_{\lambda,\eta} \cap ([n, 2n] \times \{0\}) \neq \varnothing) \geq c.$$

1.4. *Open question.* The main idea of the proof of Theorem 1.3 is a coupling between the limit walk $\omega^\infty_{\lambda,\eta}$ $(\eta > 2)$ and the exploration curve of the *critical* Bernoulli percolation. In the case of $\eta \in [0, 2]$, we hope that there is a coupling between the limit walk $\omega^\infty_{\lambda,\eta}$ and the exploration curve $\gamma^{1/2}$ of the *critical* Bernoulli percolation on the hexagonal lattice. If this coupling exists, we have the following result:

Conjecture 1.4. *For all $\eta \geq 0$, there exists $\lambda_0 > 0$, for all $\lambda > \lambda_0$: $\exists c \in]0,1[, \forall n \geq 1$, we have:*

$$\mathbb{P}_{\lambda,\eta}(\omega^{\infty}_{\lambda,\eta} \cap ([n, 2n] \times \{0\}) \neq \varnothing) \geq c.$$

The upper bound's existence of RSW has not been studied in this paper, which suggests a direction for future research.

2. Exploration curve of Bernoulli percolation on the hexagonal lattice

Percolation theory was introduced by Broadbent and Hammersley (1957). For $p \in [0, 1]$, a face of $\mathbb{T}^*_+$ is open with probability p or closed with probability $1 - p$, independently of the others.

Let $p \in [0, 1]$ and we define the *exploration curve* as follows. We divide the hexagonal faces of the boundary $\partial\mathbb{T}^*_+$ into two parts: $\partial^-(\mathbb{T}^*_+)$ involves in the group on the left side of o and $\partial^+(\mathbb{T}^*_+)$ involves in the group on the right side of o (see Figure 2.4). We color the hexagons of $\partial^-(\mathbb{T}^*_+)$ in black and those of $\partial^+(\mathbb{T}^*_+)$ in white. Moreover, the colors of the hexagones in $\mathbb{T}^*_+$ is chosen at random: black with probability p and white with probability $1 - p$, independently of the others. We define the *exploration curve* γ^p starting at o which separates the black component containing $\partial^-(\mathbb{T}^*_+)$ from the white component containing $\partial^+(\mathbb{T}^*_+)$. Then the exploration curve γ^p is a self-avoiding walk using the vertices and edges of hexagonal lattice $\mathbb{T}^*_+$. See Figure 2.4.

We can define this interface γ^p in an equivalent, dynamical way, informally described as follows. At each step, γ^p looks at its three neighbors on the hexagonal lattice, one of which is occupied by the previous step of γ^p. For the next step, γ^p randomly chooses one of these neighbors that has not yet occupied by γ^p. If there is just one neighbor that has not yet been occupied, then we choose this neighbor and if there are two neighbors, then we choose the right neighbor with probability p and the left neighbor with probability $1 - p$.

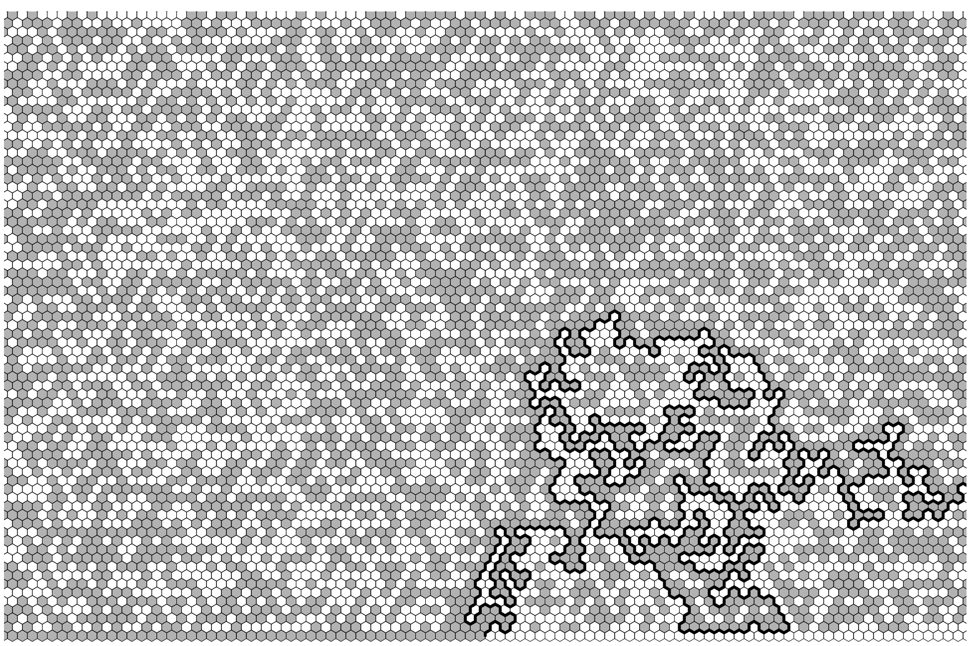

FIGURE 2.4. The hexagons on the right side of origin (i.e $\partial^+(\mathbb{T}^*_+)$) are colored in white and the hexagons on the left side of origin (i.e $\partial^-(\mathbb{T}^*_+)$) are colored in black.

We know that there exists $p_c = 1/2$ such that for $p < p_c$ there is almost surely no infinite cluster, while for $p > p_c$ there is almost surely an infinite cluster (Werner, 2009, Theorem 4.9).

Lemma 2.1 (Seymour and Welsh, 1978, Russo, 1981)**.** *Let $p = 1/2$, there exists a constant $c \in]0,1[$ such that for any $n \geq 1$:*

$$\mathbb{P}_p(\gamma^{1/2} \cap ([n, 2n] \times \{0\}) \neq \varnothing) \geq c. \tag{2.1}$$

3. Proof of Theorem 1.3

Recall that the limit walk on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ and $\mathcal{T}_{\mathbb{T}^*_+}$ have the same law. Then we investigate the limit walk on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ instead of $\mathcal{T}_{\mathbb{T}^*_+}$.

3.1. *The law of first steps of the limit walk.* We consider the biased random walk $RW_{\lambda,\eta}$ on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$. Recall that $\omega^{\infty}_{\lambda,\eta}$ is the associated limit walk and $\mathbb{P}_{\lambda,\eta}$ denotes its law.

Let $k \in \mathbb{N}^*$ and $y_1, y_2, \dots, y_k$ be k elements of $V(\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})$ such that the path $(o, y_1, y_2, \dots, y_k)$ in $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ is simple. For each λ such that $RW_{\lambda,\eta}$ on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ is transient, the law of first k steps of $\omega^{\infty}_{\lambda,\eta}$ is defined by:

$$\varphi^{\lambda,\eta,k}(y_1, y_2, \dots, y_k) = \mathbb{P}_{\lambda,\eta}\left(\omega^{\infty}_{\lambda,\eta}(0) = o, \omega^{\infty}_{\lambda,\eta}(1) = y_1, \omega^{\infty}_{\lambda,\eta}(2) = y_2, \dots, \omega^{\infty}_{\lambda,\eta}(k) = y_k\right). \tag{3.1}$$

Notation. Let ν be a vertex of the tree $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ and let μ be a vertex of $(\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\nu}$. Denote by $\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\nu}, \mu)$ for the probability of the event that the random walk $RW_{\lambda,\eta}$ on $(\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\nu}$, started at the root (i.e $X_0 = \nu$), visits μ at its first step (i.e $X_1 = \mu$) and never returns to the root. Finally, denote by $\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\nu})$ for the probability of the event that the random walk $RW_{\lambda,\eta}$ on $(\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\nu}$, started at the root (i.e $X_0 = \nu$) and never returns to the root.

Lemma 3.1 (Beffara and Huynh, 2017, Lemma 64)**.** *Let $k \in \mathbb{N}^*$ and $y_1, y_2, \dots, y_k$ be k elements of $V(\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})$ such that $(o, y_1, y_2, \dots, y_k)$ is a simple path starting at o of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$. We then have*

$$\varphi^{\lambda,\eta,k}(y_1, y_2, \dots, y_k) = \frac{\widetilde{\mathcal{C}}(\lambda, \eta, \widetilde{\mathcal{T}}_{\mathbb{T}^*_+}, y_1)}{\widetilde{\mathcal{C}}(\lambda, \eta, \widetilde{\mathcal{T}}_{\mathbb{T}^*_+})} \times \frac{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{y_1}, y_2)}{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{y_1})} \times \dots \times \frac{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{y_{k-1}}, y_k)}{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{y_{k-1}})}.$$

Fix $\eta > 2$ and $\lambda > 0$ such that $RW_{\lambda,\eta}$ on $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ is transient. For each finite path ω of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ starting at o, such that $\omega_{|\omega|}$ has two children, we define:

$$\alpha_\omega := \mathbb{P}\left(\omega^{\infty}_{\lambda,\eta}(|\omega| + 1) = (\omega_{|\omega|})_2 \middle| (\omega^{\infty}_{\lambda,\eta})_{|[0,|\omega|]} = \omega\right). \tag{3.2}$$

By using Lemma 3.1, we obtain:

$$\alpha_\omega = \frac{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\omega(|\omega|)}, (\omega_{|\omega|})_2)}{\widetilde{\mathcal{C}}(\lambda, \eta, (\widetilde{\mathcal{T}}_{\mathbb{T}^*_+})^{\omega(|\omega|)})}. \tag{3.3}$$

Denote by $\mathcal{A}$ the set of finite paths ω of $\widetilde{\mathcal{T}}_{\mathbb{T}^*_+}$ such that α_ω is well defined.

Lemma 3.2. *For all $\eta > 2$ and for all $\lambda > \frac{2\eta}{\eta - 2}$, we have:*

$$\min_{\omega \in \mathcal{A}} \alpha_\omega \geq 1/2. \tag{3.4}$$

Proof: Fix $\eta > 2$ and $\lambda > \frac{2\eta}{\eta-2}$. Let $\omega \in \mathcal{A}$ and consider $(X_n)_{n\geq 0}$ be the random walk $RW_{\lambda,\eta}$ on $(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}$ started at its root $\omega(|\omega|)$. We divide $(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}$ into two sub-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ presented in Figure 3.5. We then have:

$$\begin{aligned}\widetilde{\mathcal{C}}(\lambda,\eta,(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)},(\omega_{|\omega|})_2) &\geq \mathbb{P}\left(X_0=\omega_{|\omega|}; X_1=(\omega_{|\omega|})_2 \text{ and } \forall n\geq 1: X_n\neq \omega_{|\omega|}\right)\\ &\geq \frac{\lambda+\eta}{2\lambda+\eta}\,\widetilde{\mathcal{C}}(\lambda,\eta,\mathcal{T}_1).\end{aligned} \tag{3.5}$$

Let $\mathbb{N}$ be the regular tree of degree 1 and denote by $\widetilde{\mathcal{C}}(\lambda,\mathbb{N})$ for the probability of the event that the random walk $RW_{\lambda,0}$ on $\mathbb{N}$, started at the root (i.e $X_0 = 0$) and never returns to the root. By Rayleigh's monotonicity principle (see Lyons and Peres, 2016, page 35), we have:

$$\widetilde{\mathcal{C}}(\lambda,\eta,\mathcal{T}_1) \geq \widetilde{\mathcal{C}}(\lambda,\mathbb{N}). \tag{3.6}$$

On the other hand, we have:

$$\widetilde{\mathcal{C}}(\lambda,\mathbb{N}) = \frac{\lambda-1}{\lambda}. \tag{3.7}$$

Hence by (3.5), (3.6) and (3.7), we obtain:

$$\widetilde{\mathcal{C}}(\lambda,\eta,(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)},(\omega_{|\omega|})_2) \geq \frac{\lambda-1}{\lambda}\times\frac{\lambda+\eta}{2\lambda+\eta}. \tag{3.8}$$

Since $\lambda > \frac{2\eta}{\eta-2}$, by an simple computation we obtain:

$$\frac{\lambda-1}{\lambda}\times\frac{\lambda+\eta}{2\lambda+\eta} \geq 1/2. \tag{3.9}$$

By (3.8) and (3.9), we obtain:

$$\widetilde{\mathcal{C}}(\lambda,\eta,(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)},(\omega_{|\omega|})_2) \geq 1/2. \tag{3.10}$$

It is clear that $\widetilde{\mathcal{C}}(\lambda,\eta,(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)}) \leq 1$, hence we obtain:

$$\alpha_\omega \geq \widetilde{\mathcal{C}}(\lambda,\eta,(\widetilde{\mathcal{T}}_{\mathbb{T}_+^*})^{\omega(|\omega|)},(\omega_{|\omega|})_2) \geq 1/2,$$

this completes the proof of lemma. ☐

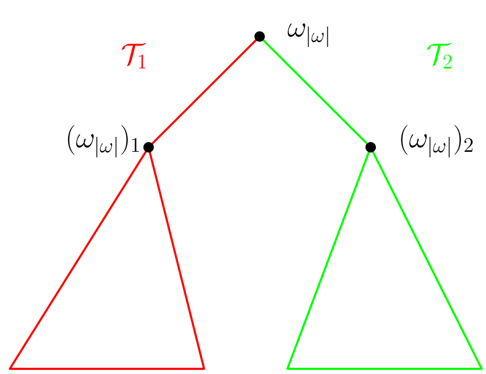

Figure 3.5.

3.2. *Proof of Theorem 1.3.* Consider the critical Bernoulli percolation on $\mathbb{T}_+^*$ with parameter 1/2. Given a configuration of percolation, we construct a random path $\gamma_{\lambda,\eta}^\infty$ starting at $o = (0,0)$ by the following way. At step n, $\gamma_{\lambda,\eta}^\infty$ looks at its three neighbors on the hexagonal lattice, one of which is occupied by the previous step of $\gamma_{\lambda,\eta}^\infty$. For the next step, $\gamma_{\lambda,\eta}^\infty$ randomly chooses one of these neighbors that has not yet occupied by $\gamma_{\lambda,\eta}^\infty$. If there is just one neighbor that has not yet been occupied, then we choose this neighbor. If there are two neighbors, then we choose the right neighbor and the left neighbor by the following rule. Let h be the hexagon which contains these neighbors and let γ be such that $(\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma$:

- If h is black, we choose the right neighbor;
- If h is white, we have two possibilities:
  (1) we choose the right neighbor with probability $\frac{\alpha_\gamma - 1/2}{1-1/2} \geq 0$ (by Lemma 3.2);
  (2) we choose the left neighbor with probability $\frac{1-\alpha_\gamma}{1-1/2}$.

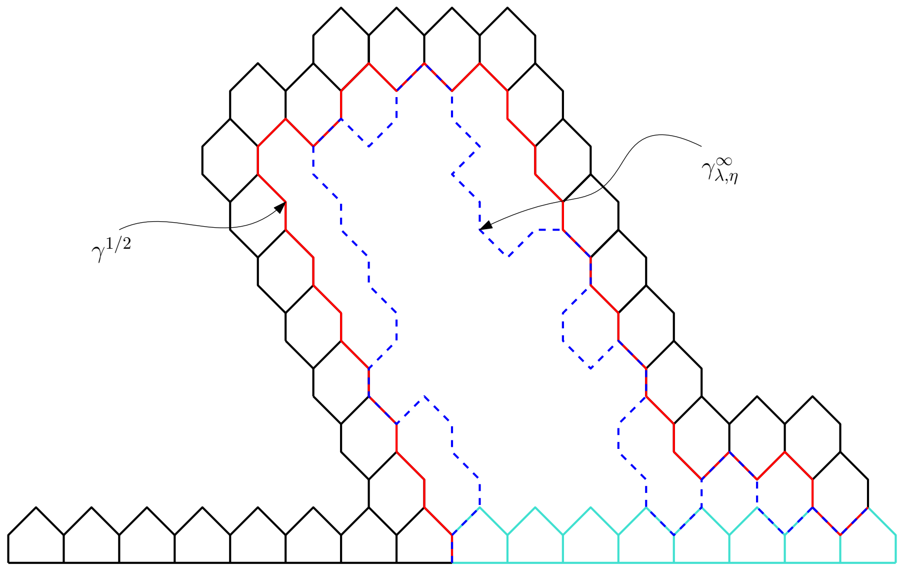

FIGURE 3.6. The exploration curve $\gamma^{1/2}$ is the red path and $\gamma_{\lambda,\eta}^\infty$ is the blue path. At each step, if $\gamma_{\lambda,\eta}^\infty$ visits a black hexagon: it always chooses the right neighbor.

Lemma 3.3. *$(\gamma_{\lambda,\eta}^\infty)$ has the same law as $(\omega_{\lambda,\eta}^\infty)$.*

Proof: First, by the construction of $\gamma_{\lambda,\eta}^\infty$ and the definition of limit walk, we have:

$$\mathbb{P}(\gamma_{\lambda,\eta}^\infty(0) = o) = \mathbb{P}(\omega_{\lambda,\eta}^\infty(0) = o) = 1.$$

Let $n > 0$ and denote by $\mathcal{A}$ the set of self-avoiding walk of length n starting at o which can extend to infinity (i.e the set of vertices of $\widetilde{\mathcal{T}}_{\mathbb{T}_+^*}$ at distance n from the root). Assume that for all $\gamma \in \mathcal{A}$, we have:

$$\mathbb{P}\left((\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma\right) = \mathbb{P}\left((\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma\right). \tag{3.11}$$

Let γ be an element of $\mathcal{A}$. We have two possibilities:

- If there is only one way to extend γ to a self-avoiding walk γ_1 of length $n+1$ (i.e the vertex γ of $\widetilde{\mathcal{T}}_{\mathbb{T}_+^*}$ has only one child γ_1), we then have:

$$\mathbb{P}\left(\gamma_{\lambda,\eta}^\infty(n+1) = \gamma_1 \middle| (\gamma_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma\right) = \mathbb{P}\left(\omega_{\lambda,\eta}^\infty(n+1) = \gamma_1 \middle| (\omega_{\lambda,\eta}^\infty)_{|[0,n]} = \gamma\right) = 1.$$

- If γ has two children γ_1 and γ_2, by the construction of $\gamma^{\infty}_{\lambda,\eta}$, we have:

$$\begin{aligned}\mathbb{P}\left(\gamma^{\infty}_{\lambda,\eta}(n+1)=\gamma_2\middle|(\gamma^{\infty}_{\lambda,\eta})_{|[0,n]}=\gamma\right)&=1/2+(1-1/2)\frac{\alpha_\gamma-1/2}{1-1/2}\\&=\alpha_\gamma.\end{aligned}\tag{3.12}$$

Hence by (3.3) and (3.12) we obtain:

$$\mathbb{P}\left(\gamma^{\infty}_{\lambda,\eta}(n+1)=\gamma_2\middle|(\gamma^{\infty}_{\lambda,\eta})_{|[0,n]}=\gamma\right)=\mathbb{P}\left(\omega^{\infty}_{\lambda,\eta}(n+1)=\gamma_2\middle|(\omega^{\infty}_{\lambda,\eta})_{|[0,n]}=\gamma\right)=\alpha_\gamma,$$

and

$$\mathbb{P}\left(\gamma^{\infty}_{\lambda,\eta}(n+1)=\gamma_1\middle|(\gamma^{\infty}_{\lambda,\eta})_{|[0,n]}=\gamma\right)=\mathbb{P}\left(\omega^{\infty}_{\lambda,\eta}(n+1)=\gamma_1\middle|(\omega^{\infty}_{\lambda,\eta})_{|[0,n]}=\gamma\right)=1-\alpha_\gamma.$$

□

Lemma 3.4. *We have the following inequality:*

$$\mathbb{P}\left(\gamma^{\infty}_{\lambda,\eta}\cap([n,2n]\times\{0\})\neq\varnothing\right)\geq\mathbb{P}\left(\gamma^{1/2}\cap([n,2n]\times\{0\})\neq\varnothing\right).$$

Proof: This is intuitively clear: informally, by the construction of $\gamma^{\infty}_{\lambda,\eta}$, the path $\gamma^{\infty}_{\lambda,\eta}$ always stays on the right of $\gamma^{1/2}$ (see Figure 3.6). A formal proof is easy but tedious to write, and is therefore omitted here. □

Theorem 1.3 is a straightforward consequence of Lemma 2.1, Lemma 3.3 and Lemma 3.4.

Acknowledgments. I am grateful to Vincent Beffara for previous discussions and the reviewer for his future research suggestion of the upper bound's existence.

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