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Speed of extinction for continuous state branching processes in subcritical Lévy environments: the strongly and intermediate regimes

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Abstract. In this paper, we study the speed of extinction of continuous state branching processes in subcritical Lévy environments. More precisely, when the associated Lévy process to the environment drifts to $-\infty$ and, under a suitable exponential change of measure (Esscher transform), the environment either drifts to $-\infty$ or oscillates. We extend recent results of Palau et al. (2016) and Li and Xu (2018), where the branching term is associated to a spectrally positive stable Lévy process and complement the recent article of Bansaye et al. (2021) where the critical case was studied. Our methodology combines a path analysis of the branching process together with its Lévy environment, fluctuation theory for Lévy processes and the asymptotic behaviour of exponential functionals of Lévy processes. As an application of the aforementioned results, we characterise the process conditioned to survival also known as the Q-process.

1. Introduction and main results

Continuous state branching processes in random environments (or CBPREs for short) are the continuous analogue, in time and space, of Galton-Watson processes in random environments (or GWREs for short). Roughly speaking, a process in this class is a time-inhomogeneous Markov process taking values in $[0, \infty]$, where 0 and ∞ are absorbing states, satisfying the quenched branching property; that is conditionally on the environment, the process started from $x + y$ is distributed as the sum of two independent copies of the same process but issued from x and y, respectively.

CBPREs provide a richer class of branching models which take into account the effect of the environment on demographic parameters and letting new phenomena appear. In particular, the

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classification of the asymptotic behaviour of rare events, such as the survival and explosion probabilities, is much more complex than the case when the environment is fixed since it may combine environmental and demographical stochasticities. Another interesting feature of CBPREs is that they also appear as scaling limits of GWREs, a very rich family of population models; see for instance Kurtz (1978) where the continuous path setting is considered and Bansaye and Simatos (2015) and Bansaye et al. (2021) where different classes of processes in random environment are studied including CBPREs.

An interesting family of CBPREs arises when we consider discrete population models in i.i.d. environments (see for instance Bansaye et al. (2021); Bansaye and Simatos (2015); Böinghoff and Hutzenthaler (2012)). The scaling limit of such population models in i.i.d. environments can be characterised by a stochastic differential equation whose linear term is driven by a Lévy process which captures the effect of the environment. This family of processes is known as *continuous state branching processes in Lévy environment* (or CBLEs for short) and their construction have been given by He et al. (2018) and by Palau and Pardo (2018), independently, as the unique non-negative strong solution of a stochastic differential equation which will be specified below.

The study of the long-term behaviour of CBLEs has attracted considerable attention in the last decade due to the interesting properties exhibited by these processes, such as an extra phase transition for the extinction probability in the subcritical regime. A list of key papers includes Bansaye et al. (2013), Böinghoff and Hutzenthaler (2012), He et al. (2018), Palau and Pardo (2017, 2018), Palau et al. (2016) and Xu (2021). In all these manuscripts, the speed of extinction has been computed for the case where the associated branching mechanism is either stable or Gaussian, since the survival probability can be computed explicitly in terms of exponential functionals of Lévy processes. More precisely, Böinghoff and Hutzenthaler (2012) and Palau and Pardo (2017) have studied the case when the random environment is driven by a Brownian motion with drift and when the branching term is given by a Feller diffusion and a stable continuous state branching process, respectively. Both studies exploited the explicit knowledge of the density of the exponential functional of a Brownian motion with drift. Bansaye et al. (2013) determined the long-term behaviour for stable branching mechanisms where the random environment is driven by a Lévy process with bounded variation paths. The case when the environment is driven by a general Lévy process satisfying some exponential moments and the branching mechanism is stable was treated, independently, by Li and Xu (2018) and Palau et al. (2016). Moreover, the results for the critical regime in the aforementioned two articles can be extended to the case when the Lévy environment has not finite second moment and satisfies the so-called Spitzer's condition (see Theorem 2.20 and Remark 2.21 in Patie and Savov (2018)). More recently, Xu (2021) gave an exact description for the speed of the extinction probability for CBLEs with stable branching mechanism and where the Lévy environment is heavy-tailed.

Little is known about the long-term behaviour of CBLEs when the associated branching mechanism is neither stable nor Gaussian. Up to our knowledge, the only study in this direction is the recent paper by Bansaye et al. (2021), where the speed of extinction of critical CBLEs for more general branching mechanisms was studied. More precisely, the authors in Bansaye et al. (2021) considered the case when the associated Lévy process in the environmental term satisfies the so-called Spitzer's condition and relax the assumption that the branching mechanism is stable. The strategy of their proof relies on the description of the extinction event under favorable environments, or in other words that the running infimum of the environment is positive, and the explicit behaviour of the exponential functional of Lévy processes under Spitzer's condition given in Patie and Savov (2018).

In this article, we are interested in understanding the asymptotic behaviour of the survival probability for CBLEs in the subcritical regime for a more general class of branching mechanisms rather than the stable case. Recall that in the subcritical regime, the underlying Lévy process drifts to $-\infty$. Moreover, as it was observed in Li and Xu (2018) and Palau et al. (2016) and in the discrete case by Afanasyev (1980) and Dekking (1987), there is another phase transition in the subcritical regime. These sub-regimes are known in the literature as: *strongly*, *intermediate* and *weakly subcritical regimes*, respectively (see e.g. Theorem 5.1 in Li and Xu (2018) or Proposition 2.2 in Palau et al. (2016)). The main contribution of this paper is to provide the precise asymptotic behaviour of the survival probability in the intermediate and strongly subcritical regimes, under some general assumptions on the Lévy process associated to the environment and the branching mechanism. Furthermore, we apply our main results to describe CBLEs conditioned on survival, also known as Q-processes, and we identify them as CBLEs with immigration (see for instance Theorem 5.3 in He et al. (2018), Theorem 1 in Palau and Pardo (2017) or below for a proper definition of the aforementioned class of processes).

In the strongly subcritical regime, we deduce that the survival probability decays exponentially with the same exponential rate as the expected population size (up to a multiplicative constant which is proportional to the initial population size). The key point in our arguments is to rewrite the probability of survival under a suitable change of measure which is associated to an exponential martingale of the Lévy environment. In order to do so, the existence of some exponential moments for the Lévy environment is required. Under this exponential change of measure, the Lévy environment remains in the subcritical regime, however the probability of survival now can be related to the Laplace transform of a CBLRE with immigration. In order to characterise the limit of the survival probability, we require the so-called Grey's condition which guarantees that the process is absorbed at 0 a.s.(see Corollary 4.4 in He et al. (2018)) and the characterisation of the Laplace transform of the aforementioned CBLRE with immigration in terms of an extension of the original environment to an homogeneous Lévy process indexed in the real line (see equation 5.6 in He et al. (2018)). The latter characterisation was used in He et al. (2018) to study the stationary distribution of CBLREs with immigration and requires a classical $x \log x$-moment condition on the Lévy measure associated to the branching mechanism. Thus, in order to guarantee the positivity of the limiting coefficient in our result, the $x \log x$-moment condition on the Lévy measure associated to the branching mechanism is required together with a $1/x$-moment condition on the stationary distribution of the CBLRE with immigration that appears in the probability of survival.

For the intermediate subcritical regime, we obtain that the speed of the survival probability decays exponentially with a polynomial factor of order 1/2 (up to a multiplicative constant which is proportional to the initial population size). In order to deduce our second main result, we combine the approach developed in Afanasyev et al. (2014); Geiger et al. (2003), for the discrete time setting, with fluctuation theory of Lévy processes. Similarly as in the strongly subcritical regime, we use an exponential change of measure under which the CBLE now oscillates. In other words, the latter observation allows us to follow a similar strategy developed in Bansaye et al. (2021) to study the extinction rate for CBLEs in the critical regime. More precisely, under this new measure, we split the event of survival in two parts, that is when the running supremum is either negative or positive; and then we show that only paths of the Lévy process with a very low running supremum give substantial contribution to the speed of survival. In this regime, we impose an $x \log x$-moment condition on the Lévy measure associated to the branching mechanism and a lower bound for the branching mechanism, which allows us to control the event of survival under favorable environments. In addition, our arguments require another technical condition which involves the branching mechanism and the Lévy process that we will specified below.

1.1. *Preliminaries.* Let $(\Omega^{(b)}, \mathcal{F}^{(b)}, (\mathcal{F}_t^{(b)})_{t\geq 0}, \mathbb{P}^{(b)})$ be a filtered probability space satisfying the usual hypothesis on which we may construct the demographic or branching term of the model that we are interested in. We suppose that $(B_t^{(b)}, t \geq 0)$ is a $(\mathcal{F}_t^{(b)})_{t\geq 0}$-adapted standard Brownian motion and $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ is a $(\mathcal{F}_t^{(b)})_{t\geq 0}$-adapted Poisson random measure on $\mathbb{R}_+^3$ with intensity $\mathrm{d}s\mu(\mathrm{d}z)\mathrm{d}u$

where μ satisfies

$$\int_{(0,\infty)} (z \wedge z^2)\mu(\mathrm{d}z) < \infty. \tag{1.1}$$

We denote by $\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ the compensated version of $N^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$. Further, we also introduce the so-called branching mechanism ψ, a convex function with the following Lévy-Khintchine representation

$$\psi(\lambda) = \psi'(0+)\lambda + \varrho^2\lambda^2 + \int_{(0,\infty)} \left(e^{-\lambda x} - 1 + \lambda x\right)\mu(\mathrm{d}x), \qquad \lambda \geq 0, \tag{1.2}$$

where $\varrho \geq 0$. Observe that the term $\psi'(0+)$ is well defined (finite) since condition (1.1) holds. Moreover, the function ψ describes the stochastic dynamics of the population.

On the other hand, for the environmental term, we consider another filtered probability space $(\Omega^{(e)}, \mathcal{F}^{(e)}, (\mathcal{F}^{(e)}_t)_{t\geq 0}, \mathbb{P}^{(e)})$ satisfying the usual hypotheses. Let us consider $\sigma \geq 0$ and α real constants; and π a measure concentrated on $\mathbb{R} \setminus \{0\}$ such that

$$\int_{\mathbb{R}} (1 \wedge z^2)\pi(\mathrm{d}z) < \infty.$$

Suppose that $(B^{(e)}_t, t \geq 0)$ is a $(\mathcal{F}^{(e)}_t)_{t\geq 0}$ - adapted standard Brownian motion, $N^{(e)}(\mathrm{d}s, \mathrm{d}z)$ is a $(\mathcal{F}^{(e)}_t)_{t\geq 0}$ - Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\mathrm{d}s\pi(\mathrm{d}z)$, and $\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z)$ its compensated version. We denote by $S = (S_t, t \geq 0)$ a Lévy process, that is a process with stationary and independent increments and càdlàg paths, with the following Lévy-Itô decomposition

$$S_t = \alpha t + \sigma B^{(e)}_t + \int_0^t \int_{(-1,1)} (e^z - 1)\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} (e^z - 1)N^{(e)}(\mathrm{d}s, \mathrm{d}z).$$

Note that S is a Lévy process with no jumps smaller than -1.

In our setting, the population size has no impact on the evolution of the environment or in other words we are considering independent processes for the demographic and environmental terms. More precisely, we work now on the space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ the direct product of the two probability spaces defined above, that is to say, $\Omega := \Omega^{(e)} \times \Omega^{(b)}, \mathcal{F} := \mathcal{F}^{(e)} \otimes \mathcal{F}^{(b)}, \mathcal{F}_t := \mathcal{F}^{(e)}_t \otimes \mathcal{F}^{(b)}_t$ for $t \geq 0$, and $\mathbb{P} := \mathbb{P}^{(e)} \otimes \mathbb{P}^{(b)}$. Therefore $(Z_t, t \geq 0)$, the *continuous state branching process in the Lévy environment* $(S_t, t \geq 0)$ is defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ as the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} Z_t =& Z_0 - \psi'(0+)\int_0^t Z_s \mathrm{d}s + \int_0^t \sqrt{2\varrho^2 Z_s}\mathrm{d}B^{(b)}_s \\ &+ \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z\widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t Z_{s-}\mathrm{d}S_s. \end{aligned} \tag{1.3}$$

According to Theorem 3.1 in He et al. (2018) or Theorem 1 in Palau and Pardo (2018), the equation has pathwise uniqueness and strong solution when $|\psi'(0+)| < \infty$. Furthermore, when conditioned on the environment, the process Z inherits the branching property of the underlying CSBP previously defined. Let us denote by $\mathbb{P}_z$, for its law starting from $z \geq 0$.

The analysis of the process Z is deeply related to the behaviour and fluctuations of the Lévy process $\xi = (\xi_t, t \geq 0)$, defined as follows

$$\xi_t = \overline{\alpha}t + \sigma B^{(e)}_t + \int_0^t \int_{(-1,1)} z\widetilde{N}^{(e)}(\mathrm{d}s, \mathrm{d}z) + \int_0^t \int_{(-1,1)^c} zN^{(e)}(\mathrm{d}s, \mathrm{d}z), \tag{1.4}$$

where

$$\overline{\alpha} := \alpha - \psi'(0+) - \frac{\sigma^2}{2} - \int_{(-1,1)} (e^z - 1 - z)\pi(\mathrm{d}z).$$

Note that, both processes S and ξ generate the same filtration. Actually, the process ξ is obtained from S, changing only the drift and jump sizes. We denote by $\mathbb{P}^{(e)}_x$, for the law of the process ξ starting from $x \in \mathbb{R}$ and when $x = 0$, we use the notation $\mathbb{P}^{(e)}$ for $\mathbb{P}^{(e)}_0$.

Further, under condition (1.1), the process $\big(Z_t e^{-\xi_t}, t \geq 0\big)$ is a quenched martingale implying that for any $t \geq 0$ and $z \geq 0$,

$$\mathbb{E}_z[Z_t \mid S] = z e^{\xi_t}, \qquad \mathbb{P}_z \text{ -a.s,} \tag{1.5}$$

see Bansaye et al. (2021). In other words, the process ξ plays an analogous role as the random walk associated to the logarithm of the offspring means in the discrete time framework and leads to the usual classification for the long-term behaviour of branching processes. More precisely, we say that the process Z is subcritical, critical or supercritical accordingly as ξ drifts to $-\infty$, oscillates or drifts to $+\infty$. In this manuscript, we focus on the subcritical regime.

In addition, under condition (1.1), there is another quenched martingale associated to $(Z_t e^{-\xi_t}, t \geq 0)$ which allows us to compute its Laplace transform, see for instance Proposition 2 in Palau and Pardo (2018) or Theorem 3.4 in He et al. (2018). In order to compute the Laplace transform of $Z_t e^{-\xi_t}$, we first introduce the unique positive solution $(v_t(s, \lambda, \xi), s \in [0, t])$ of the following backward differential equation

$$\frac{\partial}{\partial s} v_t(s, \lambda, \xi) = e^{\xi_s} \psi_0(v_t(s, \lambda, \xi) e^{-\xi_s}), \qquad v_t(t, \lambda, \xi) = \lambda, \tag{1.6}$$

where

$$\psi_0(\lambda) = \psi(\lambda) - \lambda \psi'(0+) = \varrho^2 \lambda^2 + \int_{(0,\infty)} \big(e^{-\lambda x} - 1 + \lambda x\big) \mu(\mathrm{d}x). \tag{1.7}$$

Then the process $\big(\exp\{-v_t(s, \lambda, \xi) Z_s e^{-\xi_s}\}, 0 \leq s \leq t\big)$ is a quenched martingale implying that for any $\lambda \geq 0$ and $t \geq s \geq 0$,

$$\mathbb{E}_z\Big[\exp\{-\lambda Z_t e^{-\xi_t}\} \;\Big|\; S, \mathcal{F}^{(b)}_s\Big] = \exp\{-Z_s e^{-\xi_s} v_t(s, \lambda, \xi)\}. \tag{1.8}$$

Moreover, we also consider the random semigroup $h_{s,t}(\lambda) = e^{-\xi_s} v_t(s, \lambda e^{\xi_t}, \xi)$ which is well defined for all $\lambda \geq 0$ and $s \in [0, t]$; and satisfies

$$\mathbb{E}_z\Big[e^{-\lambda Z_t} \Big| S, \mathcal{F}^{(b)}_s\Big] = \exp\{-Z_s h_{s,t}(\lambda)\}, \tag{1.9}$$

see Theorem 3.4 in He et al. (2018). According to Section 2 in He et al. (2018), the mapping $s \mapsto h_{s,t}(\lambda)$ is the unique positive pathwise solution to the integral differential equation

$$h_{s,t}(\lambda) = e^{\xi_t - \xi_s} \lambda - \int_s^t e^{\xi_r - \xi_s} \psi_0\big(h_{r,t}(\lambda)\big) \mathrm{d}r, \qquad 0 \leq s \leq t. \tag{1.10}$$

We close this subsection by introducing CBLEs with immigration which will play a fundamental role in our arguments. Let $b \geq 0$ be a positive constant and ν a Lévy measure concentrated on $(0, \infty)$ such that

$$\int_{(0,\infty)} (1 \wedge z) \nu(\mathrm{d}z) < \infty.$$

We say that $X = (X_t, t \geq 0)$ is a *continuous state branching process with immigration in the Lévy environment* S if it is the unique non-negative strong solution of the following stochastic differential equation

$$\begin{aligned} X_t = X_0 - (\psi'(0+) - b) \int_0^t X_s \mathrm{d}s + \int_0^t \sqrt{2\varrho^2 X_s} \mathrm{d}B^{(b)}_s + \int_0^t \int_{(0,\infty)} z N^{(i)}(\mathrm{d}s, \mathrm{d}z) \\ + \int_0^t \int_{(0,\infty)} \int_0^{Z_{s-}} z \widetilde{N}^{(b)}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u) + \int_0^t X_{s-} \mathrm{d}S_s, \end{aligned} \tag{1.11}$$

where $N^{(i)}(\mathrm{d}s, \mathrm{d}z)$ is a Poisson random measure with intensity $\mathrm{d}s\nu(\mathrm{d}z)$ (see Theorem 1 in Palau and Pardo (2017) or Theorem 5.1 in He et al. (2018)). The process X is characterised by the branching mechanism ψ and the immigration mechanism η which is given by the Laplace exponent of a subordinator, i.e.

$$\eta(\lambda) = b\lambda + \int_0^\infty (1 - e^{-\lambda x})\nu(\mathrm{d}x).$$

Let us denote its law starting from $z \geq 0$, by $\mathbb{Q}_z$.

According to Theorem 5.3 in He et al. (2018), the law of X is characterised as follows: for any $\lambda \geq 0$ and $t \geq 0$,

$$\mathbb{Q}_z\Big[e^{-\lambda X_t}\Big] = \mathbb{E}^{(e)}\left[\exp\left\{-zh_{0,t}(\lambda) - \int_0^t \eta(h_{s,t}(\lambda))\mathrm{d}s\right\}\right]. \tag{1.12}$$

For our purposes, we are interested in the limiting distribution of $\mathbb{Q}_z(X_t \in \mathrm{d}y)$, as t goes to ∞. The limiting distribution was derived by He et al. (2018) under general assumptions and can be characterised as we explain below. First, we require an extension of the functional $(v_t(s, \lambda, \xi), s \in [0, t])$ to negative times. In order to do so, let us consider an independent copy $(\xi'_t, t \geq 0)$ of the Lévy process ξ and construct the time homogeneous Lévy process $\Xi = (\Xi_t, -\infty < t < \infty)$, indexed by $\mathbb{R}$, as follows

$$\Xi_t := \begin{cases} -\lim_{s\downarrow -t}\xi'_s & \text{for } \; t < 0, \\ 0 & \text{for } \; t = 0, \\ \xi_t & \text{for } \; t > 0. \end{cases} \tag{1.13}$$

Note that the latter definition ensures that the Lévy process Ξ has càdlàg paths on $(-\infty, \infty)$ and, in particular, if ξ_t drifts to $-\infty$, as $t \to \infty$, a.s., then Ξ_t drifts to ∞, as $t \to -\infty$, a.s. We denote by $\mathbf{P}_x^{(e)}$ for the law of the process Ξ such that $\Xi_0 = x \in \mathbb{R}$ and, when $x = 0$, we use the notation $\mathbf{P}^{(e)}$ for $\mathbf{P}_0^{(e)}$.

With the definition of Ξ in hand, we introduce the map $s \in (-\infty, 0] \mapsto v_0(s, \lambda, \Xi)$ as the unique positive pathwise solution of

$$v_0(s, \lambda, \Xi) = \lambda - \int_s^0 e^{\Xi_r}\psi_0\big(e^{-\Xi_r}v_0(r, \lambda, \Xi)\big)\mathrm{d}r, \qquad s \leq 0. \tag{1.14}$$

Implicitly, it also follows that for $s \leq 0$ the map $s \mapsto h^{\Xi}_{s,0}(\lambda) = e^{-\Xi_s}v_0(s, \lambda e^{\Xi_0}, \Xi)$ is the unique positive pathwise solution to the equation

$$h^{\Xi}_{s,0}(\lambda) = e^{-\Xi_s}\lambda - \int_s^0 e^{\Xi_r - \Xi_s}\psi_0\big(h^{\Xi}_{r,0}(\lambda)\big)\mathrm{d}r, \qquad s \leq 0. \tag{1.15}$$

Hence by time homogeneity of the process Ξ, we have, for any $\lambda \geq 0$ and $t \geq 0$, that

$$\mathbb{Q}_z\Big[e^{-\lambda X_t}\Big] = \mathbf{E}^{(e)}\left[\exp\left\{-zh^{\Xi}_{-t,0}(\lambda) - \int_{-t}^0 \eta(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right], \tag{1.16}$$

see Section 5 in He et al. (2018) for further details. Finally, since ξ drifts to $-\infty$ and under the following $\log x$-moment condition for the Lévy measure ν,

$$\int_1^\infty \log u\, \nu(\mathrm{d}u) < \infty, \tag{1.17}$$

according to Theorem 5.6 in He et al. (2018), we have that there exists a probability measure Π on $[0, \infty)$ such that $\mathbb{Q}_z(X_t \in \cdot)$ converges weakly towards Π, as t goes to ∞, for every $z \geq 0$. Moreover,

$$\int_{[0,\infty)} e^{-\lambda y}\Pi(\mathrm{d}y) = \mathbf{E}^{(e)}\left[\exp\left\{-\int_{-\infty}^0 \eta(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]. \tag{1.18}$$

It is important to note that ψ' is the Laplace exponent of a subordinator. Indeed

$$\psi_0'(\lambda) = 2\varrho^2\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x}) x\mu(\mathrm{d}x),$$

so when we take $\eta \equiv \psi_0'$ in the previous discussion, condition (1.17) is reduced to the following assumption that will be used below.

Assumption 1.1. The Lévy measure μ satisfies

$$\int_1^\infty u \log u \,\mu(\mathrm{d}u) < \infty.$$

For our purposes, we require the following exponential moment condition on the Lévy environment ξ.

Assumption 1.2. there exists $\vartheta > 0$ such that

$$\int_{\{|x|>1\}} e^{\theta x}\pi(\mathrm{d}x) < \infty, \qquad \text{for all} \qquad \theta \in [0, \vartheta].$$

This condition is equivalent to the existence of the Laplace transform of ξ, that is $\mathbb{E}^{(e)}[e^{\theta\xi_1}]$ is well defined for $\theta \in [0, \vartheta]$ (see for instance Lemma 26.4 in Sato (2013)). The latter implies that we can introduce the Laplace exponent of ξ as follows $\Phi_\xi(\theta) := \log \mathbb{E}^{(e)}[e^{\theta\xi_1}]$, for $\theta \in [0, \vartheta]$. Again from Lemma 26.4 in Sato (2013), we also have $\Phi_\xi(\theta) \in C^\infty$ and $\Phi_\xi''(\theta) > 0$, for $\theta \in (0, \vartheta)$.

Another object which will be relevant for our analysis is the so-called exponential martingale associated to ξ, i.e.

$$M_t^{(\theta)} = \exp\Big\{\xi_t - t\Phi_\xi(\theta)\Big\}, \qquad t \geq 0,$$

which is well-defined for $\theta \in [0, \vartheta]$ under assumption (1.2). It is well-known that $(M_t^{(\theta)}, t \geq 0)$ is a $(\mathcal{F}_t^{(e)})_{t\geq 0}$-martingale and that it induces a change of measure which is known as the Esscher transform, that is to say

$$\mathbb{P}^{(e,\theta)}(\Lambda) := \mathbb{E}^{(e)}\Big[M_t^{(\theta)}\mathbf{1}_\Lambda\Big], \qquad \text{for} \quad \Lambda \in \mathcal{F}_t^{(e)}. \tag{1.19}$$

Similarly to the critical case, which was studied by Bansaye et al. (2021), the asymptotic analysis of the intermediate subcritical regime requires the notion of the renewal functions $U^{(\theta)}$ and $\widehat{U}^{(\theta)}$ under $\mathbb{P}^{(e,\theta)}$, which are associated to the supremum and infimum of ξ, respectively. More precisely, recall that the running infimum and supremum of ξ are defined by

$$\underline{\xi}_t = \inf_{0\leq s\leq t} \xi_s \qquad \text{and} \qquad \overline{\xi}_t = \sup_{0\leq s\leq t} \xi_s, \qquad \text{for} \qquad t \geq 0. \tag{1.20}$$

We also recall that the reflected processes $\xi - \underline{\xi}$ and $\overline{\xi} - \xi$ are Markov processes with respect to the filtration $(\mathcal{F}_t^{(e)})_{t\geq 0}$ and whose semigroups satisfy the Feller property (see for instance Proposition VI.1 in the monograph of Bertoin (1996)). The latter allows us to introduce the notion of local times for the reflected processes at 0. Let us denote by $L = (L_t, t \geq 0)$ and $\widehat{L} = (\widehat{L}_t, t \geq 0)$ the local times of $\overline{\xi} - \xi$ and $\xi - \underline{\xi}$ at 0, respectively, in the sense of Chapter IV in Bertoin (1996). Thus the renewal functions $U^{(\theta)}$ and $\widehat{U}^{(\theta)}$ are defined as follows

$$U^{(\theta)}(x) := \mathbb{E}^{(e,\theta)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\overline{\xi}_t \leq x\}} \mathrm{d}L_t\right] \quad \text{and} \quad \widehat{U}^{(\theta)}(x) := \mathbb{E}^{(e,\theta)}\left[\int_{[0,\infty)} \mathbf{1}_{\{\underline{\xi}_t \geq -x\}} \mathrm{d}\widehat{L}_t\right]. \tag{1.21}$$

The renewal functions are identically 0 on $(-\infty, 0]$, strictly positive on $(0, \infty)$ and satisfy

$$U^{(\theta)}(x) \leq C_1 x \qquad \text{and} \qquad \widehat{U}^{(\theta)}(x) \leq C_2 x \quad \text{for any} \quad x \geq 0, \tag{1.22}$$

where C_1, C_2 are finite constants (see for instance Lemma 6.4 and Section 8.2 in the monograph of Doney (2007)). For simplicity on exposition, when $\theta = 0$ we use the following notation $U^{(0)} = U$ and $\widehat{U}^{(0)} = \widehat{U}$. We refer to Section 2.1 for a proper definition or Section VI.4 in Bertoin (1996).

1.2. *Main results.* We are now ready to state our main results. In order to introduce our main results we require some technical assumptions on the branching mechanism and on the environment which will control the event of survival. Let us start with the strongly subcritical regime where some assumptions on the branching mechanism and on the environment are required. For the environment, we assume Assumption 1.2 with $\vartheta = 1$, which guarantees the existence of exponential positive moments on $[0, 1]$, together with $\Phi'_\xi(0) < 0$ and $\Phi'_\xi(1) < 0$. The latter guarantees the use of the Esscher transform at $\theta = 1$ in (1.19), and that the Lévy environment is strongly subcritical.

For the branching mechanism, we require two conditions: Assumption 1.1 for the Lévy measure μ and the so-called Grey's condition, i.e.

Assumption 1.3. the function ψ_0 satisfies

$$\int_1^\infty \frac{1}{\psi_0(\lambda)} \mathrm{d}\lambda < \infty.$$

The latter assumption guarantees that the process Z is absorbed at 0, $\mathbb{P}_z$-a.s., for $z > 0$, see Corollary 4.4 in He et al. (2018). It is important to note that Grey's condition is a necessary and sufficient condition for absorption, with positive probability, both for CSBPs (see Grey (1974)) and for CBLEs (see Theorem 4.1 in He et al. (2018)). On the other hand, the $x \log x$-moment condition for the Lévy measure μ is a necessary and sufficient condition for the ergodicity of the CBLE with immigration which is implicit under the Esscher transform of Z. The $x \log x$-moment condition also appears in the discrete setting to study the long-term behaviour of branching processes in a strongly subcritical random environment (see e.g. Theorem 1.1 in Afanasyev et al. (2005)).

It turns out that, in this regime, the survival probability decays with the same rate as the expected generation size, i.e. as $\mathbb{E}_z[Z_t]$ for t large enough, up to a multiplicative constant. A similar behaviour appears for subcritical Galton-Watson processes as well as for discrete branching processes in random environments in the strongly subcritical regime (see for instance Theorem 1.1 in Afanasyev et al. (2005)). Moreover from (1.5), we have

$$\mathbb{E}_z[Z_t] = z\mathbb{E}[e^{\xi_t}] = ze^{\Phi_\xi(1)t}.$$

In other words, the survival probability decays exponentially up to a multiplicative constant which is proportional to the initial state of the population as is stated below.

Theorem 1.4 (Strongly subcritical regime)**.** *Suppose that Assumptions 1.1, 1.2, with $\vartheta = 1$, $\Phi'_\xi(0) < 0$ and $\Phi'_\xi(1) < 0$ and 1.3 are fulfilled. We also assume that*

$$\int_0^\infty \mathbf{E}^{(e,1)} \left[\exp\left\{ -\int_{-1}^0 \psi'_0\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s \right\} \right] \mathrm{d}\lambda < \infty. \tag{1.23}$$

Then for every $z > 0$, we have

$$\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z\mathfrak{B}_1,$$

where

$$\mathfrak{B}_1 = \int_0^\infty \mathbf{E}^{(e,1)} \left[\exp\left\{ -\int_{-\infty}^0 \psi'_0(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s \right\} \right] \mathrm{d}\lambda \in (0, \infty).$$

It is important to note that condition (1.23) can be interpreted in terms of a CBLE with immigration. In other words, let $X = (X_t, t \geq 0)$ be a CBLE with immigration starting from 0 with branching and immigration mechanisms (ψ_0, ψ_0'); and with auxiliary Lévy process (to the environment) given by $(\xi, \mathbb{P}^{(e,1)})$. Let us denote by $\mathbb{Q}_0$ for its law. Hence, condition (1.23) is equivalent to

$$\mathbb{Q}_0\left[\frac{1}{X_1}\right] < \infty$$

and

$$\mathfrak{B}_1 = \int_{[0,\infty)} \frac{1}{y}\,\Pi_{\psi_0}(\mathrm{d}y),$$

where $\Pi_{\psi_0}(\mathrm{d}y)$ denotes the limiting probability distribution of $\mathbb{Q}_0(X_t \in \mathrm{d}y)$.

In general, it seems difficult to compute explicitly the constant $\mathfrak{B}_1$ except for the stable case, that is when $\psi_0(\lambda) = C\lambda^{1+\beta}$ with $\beta \in (0,1]$ and $C > 0$ (observe that $C = 2\varrho^2$ when $\beta = 1$), where $\mathfrak{B}_1$ can be computed explicitly and coincides with the constant that appears in Theorem 5.1 in Li and Xu (2018). Note that in this case ψ_0 satisfies Assumptions 1.1 and 1.3 and also we observe that the integral in condition (1.23) can be rewritten as follows

$$\int_0^\infty \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda = \mathbb{E}^{(e,1)}\left[\left(\int_0^1 e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right].$$

Now, using the exponential moment Assumption 1.2 with $\vartheta = 1$ and Lemma 2.2 in Li and Xu (2018), we deduce that the expectation in the right-hand side is finite. Therefore, for any CBLE with stable branching mechanism and associated Lévy environment satisfying Assumption 1.2 with $\vartheta = 1$, $\Phi_\xi'(0) < 0$ and $\Phi_\xi'(1) < 0$, we get from Theorem 1.4 that

$$\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z(\beta C)^{-1/\beta}\mathbb{E}^{(e,1)}\left[\left(\int_0^\infty e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right].$$

We refer to Subsection 2.2.1 for further details about the computation of this constant.

Our next main result deals with the intermediate subcritical regime. This regime is governed by the exponential moment condition on the environment (1.2) with $\vartheta > 1$, together with $\Phi_\xi'(0) < 0$ and $\Phi_\xi'(1) = 0$. In other words, the Lévy process ξ drifts to $-\infty$ under $\mathbb{P}^{(e)}$ and oscillates under the probability measure $\mathbb{P}^{(e,1)}$, induced by the Esscher transform (1.19). In order to state our result, we require to introduce the notion of Lévy processes conditioned to stay positive. According to Lemma 1 in Chaumont and Doney (2005), the process $(\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}_t^{(e)})_{t\geq 0}$. With the help of this martingale, we define a new measure which corresponds to the law of ξ *conditioned to stay positive*, as follows: for $\Lambda \in \mathcal{F}_t^{(e)}$ and $x > 0$,

$$\mathbb{P}_x^{\uparrow}(\Lambda) := \frac{1}{\widehat{U}(x)}\mathbb{E}_x^{(e)}\left[\widehat{U}(\xi_t)\mathbf{1}_{\{\underline{\xi}_t>0\}}\mathbf{1}_\Lambda\right]. \tag{1.24}$$

Similarly, we can define the process conditioned to stay positive under $\mathbb{P}^{(e,1)}$ but with the martingale associated with the renewal measure $\widehat{U}^{(1)}$ instead of $\widehat{U}$. For a complete overview on this theory, the reader is referred to the monographs of Bertoin (1996) and Doney (2007), see also Chaumont (1996) and Chaumont and Doney (2005) and references therein.

Similarly as in the critical regime studied by Bansaye et al. (2021), we require the following assumption on the branching mechanism

Assumption 1.5. there exist $\beta \in (0,1]$ and $C > 0$ such that

$$\psi_0(\lambda) \geq C\lambda^{1+\beta} \qquad \text{for} \qquad \lambda \geq 0.$$

This condition will help us to control the probability of survival under environments with large extrema, that is when the supremum of the environment is very large. We also note that it implies Grey's condition (1.3) and thus we ensure that Z gets extinct in finite time with positive probability.

Furthermore, we also assume that the Lévy measure μ, which is associated to the branching mechanism, satisfies the $x\log x$ moment condition (1.1).

Similarly as in the strongly subcritical case, the survival probability decays exponentially but now with a polynomial factor of order $1/2$, up to a multiplicative constant which is proportional to the initial state of the population. In other words, we have the following result.

Theorem 1.6 (Intermediate subcritical regime)**.** *Suppose that Assumptions 1.1, 1.2, with $\vartheta > 1$, $\Phi'_\xi(0) < 0$, $\Phi'_\xi(1) = 0$, and 1.5 hold. We also suppose that, for $x < 0$,*

$$\int_0^\infty \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty. \tag{1.25}$$

Then, for every $z > 0$, we have

$$\lim_{t\to\infty} t^{1/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = z\mathbb{E}^{(e,1)}\left[H_1\right]\sqrt{\frac{2}{\pi\Phi''_\xi(1)}}\mathfrak{B}_2,$$

where

$$\mathfrak{B}_2 = \lim_{x\to-\infty} U^{(1)}(-x)\mathbf{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathrm{d}\lambda\right].$$

We conclude this section with our last main result which is devoted to the study of the CBLE process conditioned to survival or Q-process. As we will see below and in an analogous sense to what has been proved for the classical Galton-Watson process (see for instance Section 12.3 in Kyprianou (2014)) and CSBP (see Theorem 4.1 in Lambert (2007)), the conditioned process has the same law as a CBLE with immigration but with a different random environment to the original. In particular, we extend recent results in Lambert (2007), for the case of CSBPs; and in Palau and Pardo (2018), for the continuous state branching processes in a Brownian environment with a stable branching mechanism. Let

$$T_0 = \{t \geq 0 : Z_t = 0\},$$

be the extinction time of the process Z.

Theorem 1.7 (Q-process)**.** *Let $(Z_t, t \geq 0)$ be a CBLE in a strongly subcritical regime, i.e. satisfying the conditions in Theorem 1.4, or in an intermediate subcritical regime, i.e. satisfying the conditions in Theorem 1.6. Then for all $z, t > 0$,*

(i) *The conditional laws $\mathbb{P}_z(\,\cdot \mid T_0 > t)$ converge, as $t\to\infty$, towards a limit law $\mathbb{P}^\natural$, in the sense that for any $t \geq 0$ and $\Lambda \in \mathcal{F}_t$,*

$$\lim_{s\to\infty}\mathbb{P}_z(\Lambda \mid T_0 > s) = \mathbb{P}^\natural_z(\Lambda).$$

(ii) *The probability measure $\mathbb{P}^\natural_z$ can be expressed as Doob h-transform of $\mathbb{P}_z$ based on the martingale*

$$D_t = Z_t e^{-\Phi_\xi(1)t},$$

that is, for $\Lambda \in \mathcal{F}_t$,

$$\mathbb{P}^\natural_z(\Lambda) := \mathbb{E}_z\left[\frac{Z_t}{z}e^{-\Phi_\xi(1)t}\mathbf{1}_\Lambda\right].$$

(iii) *The process Z, under $\mathbb{P}^\natural_z$, is a CBLE with immigration initiated at z, with immigration mechanism given by $\psi_0'(\lambda)$ and the auxiliary Lévy process (to the environment) is given by $(\xi, \mathbb{P}^{(e,1)})$. Moreover, for any $\lambda \geq 0$ and $t > 0$, we have*

$$\mathbb{E}^\natural_z\left[e^{-\lambda Z_t}\right] = \mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{0,t}(\lambda) - \int_0^t \psi_0'(h_{s,t}(\lambda))\mathrm{d}s\right\}\right].$$

The remainder of this paper is devoted to the proofs of the main results.

1.3. *Comments about our results.* Conditions (1.23) and (1.25) seem to be optimal. For instance in the strongly subcritical regime, it turns out that the probability of survival $e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0)$ is exactly $\mathbb{Q}_z[1/X_t]$, where $(X, \mathbb{Q}_z)$ is a CBLE with immigration starting from z with branching and immigration mechanisms (ψ_0, ψ_0'); and with auxiliary Lévy process (to the environment) given by $(\xi, \mathbb{P}^{(e,1)})$. In order to obtain the limit, some control is required in the Laplace transform of X_t and it is here where condition (1.23) appears by eliminating the term which depends on the starting population z, since it goes to 0 as t increases. Moreover, we also remark that condition (1.23) is not required in the discrete setting due to some monotonicity properties associated to the random walk and the sequence of probability generating functions (see Lemma 2.1 and 2.3 in Geiger et al. (2003)), properties that are lost in the continuous setting.

Regarding Assumption 1.5, it seems quite difficult to get rid of it since some explicit knowledge or properties of the functional $h_{0,t}(\infty)$ are needed to control the behaviour of

$$e^{-\xi_t}\mathbb{P}_z\big(Z_t > 0 \;\big|\; \xi\big), \qquad \text{for} \quad z > 0,$$

under favourable environments, as t increases. In other words, we can rewrite the probability of survival in a favourable environment as follows

$$e^{-t\Phi_\xi(1)}\mathbb{P}_{(z,x)}\left(Z_t > 0, \sup_{0\le s\le t} \xi_s \ge y\right) \;=\; \mathbb{E}_x^{(e,1)}\Big[e^{-\xi_t}\left(1 - \exp\left\{-zh_{0,t}(\infty)\right\}\right)\mathbf{1}_{\{\overline{\xi}_t \ge y\}}\Big],$$

where we recall that $\overline{\xi}_t$ denotes the running supremum up to time t, see (1.20).

In other words to control the right-hand side of the previous identity is somehow quite involved, contrary to the discrete setting where the quenched probability of survival can be upper bounded using a first moment estimate since the event of survival is equal to the event of the current population being bigger or equal to one, an strategy which cannot be used in our setting.

Finally, we believe that it is possible to obtain Theorem 1.6 with Assumption 1.2 but with the less restrictive condition that $\vartheta = 1$ with $\Phi_\xi'(0) < 0$ and $\Phi_\xi'(1) = 0$. It is important to note that even in the case when the branching mechanism is stable, a deeper analysis is required to deduce such result. More precisely, when $\psi_0(\lambda) = C\lambda^{1+\beta}$ with $\beta \in (0, 1)$ and $C > 0$, from Section 2.1.2 in Palau et al. (2016), we have

$$e^{-\xi_t}\mathbb{P}_z\big(Z_t > 0 \;\big|\; \xi\big) = e^{-\xi_t}\left(1 - \exp\left\{-z\left(\beta C\int_0^t e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right\}\right) \qquad \text{a.s.}$$

Even though the exponential functional of a Lévy process is a well studied object, it seems difficult to control the previous random variable, under $\mathbb{P}^{(e,1)}$, with the restriction that $\vartheta = 1$ due to the nature of the exponential functional together with $e^{-\xi_t}$.

We conjecture that, under the so-called Spitzer's condition

$$\frac{1}{t}\int_0^t \mathbb{P}^{(e,1)}(\xi_s \ge 0)\mathrm{d}s \to \rho \in (0, 1), \qquad \text{as} \quad t \to \infty,$$

together with Assumptions 1.1, 1.3 and (1.25), the survival probability must behave as follows: for $z > 0$,

$$e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) \sim z\mathfrak{B}_3 t^{-\rho}\ell(t), \quad \text{as} \quad t \to \infty,$$

where $\mathfrak{B}_3$ is a positive constant and ℓ is a slowly varying function at ∞.

2. Proofs

2.1. *Preliminaries on Lévy processes.* In this section, we briefly recall some important facts of Lévy processes and its fluctuation theory that we will require in what follows. Recall that $\mathbb{P}_x^{(e)}$ denotes the law of the Lévy process ξ starting from $x \in \mathbb{R}$ and when $x = 0$, we use the notation $\mathbb{P}^{(e)}$ for

$\mathbb{P}^{(e)}_0$. The dual process $\widehat{\xi}=-\xi$ is also a Lévy process satisfying that for any fixed time $t>0$, the processes

$$(\xi_{(t-s)^-}-\xi_t, 0\le s\le t) \qquad \text{and} \qquad (\widehat{\xi}_s, 0\le s\le t), \tag{2.1}$$

have the same law, with the convention that $\xi_{0-}=\xi_0$ (see for instance Lemma 3.4 in Kyprianou (2014)). For every $x\in\mathbb{R}$, let $\widehat{\mathbb{P}}^{(e)}_x$ be the law of $x+\xi$ under $\widehat{\mathbb{P}}^{(e)}$, that is the law of $\widehat{\xi}$ under $\mathbb{P}^{(e)}_{-x}$. In the sequel, we assume that ξ is not a compound Poisson process to avoid the possibility that in this case the process visits the same maxima or minima at distinct times which can make our analysis more involved.

Let us recall the definitions of the running infimum $\underline{\xi}$ and supremum $\overline{\xi}$ of ξ in (1.20) and that $L=(L_t,t\ge 0)$ and $\widehat{L}=(\widehat{L}_t,t\ge 0)$ are the local times of $\overline{\xi}-\xi$ and $\xi-\underline{\xi}$ at 0, respectively. If 0 is regular for $(-\infty,0)$ or regular downwards, i.e.

$$\mathbb{P}^{(e)}(\tau_0^-=0)=1,$$

where $\tau_0^-=\inf\{s>0:\xi_s\le 0\}$, then 0 is regular for the reflected process $\xi-\underline{\xi}$ and then, up to a multiplicative constant, $\widehat{L}$ is the unique additive functional of the reflected process whose set of increasing points is $\{t:\xi_t=\underline{\xi}_t\}$. If 0 is not regular downwards then the set $\{t:\xi_t=\underline{\xi}_t\}$ is discrete and we define the local time $\widehat{L}$ as the counting process of this set. The same properties holds for L by duality, i.e. if 0 is regular upwards then, up to a multiplicative constant, L is the unique additive functional whose set of increasing points is $\{t:\xi_t=\overline{\xi}_t\}$, otherwise L is the counting process of this set.

Let us denote by L^{-1} and $\widehat{L}^{-1}$ the right continuous inverse of the local times L and $\widehat{L}$, respectively. The range of the inverse local times L^{-1} and $\widehat{L}^{-1}$, correspond to the sets of real times at which new maxima and new minima occur, respectively. Next, we introduce the so called increasing ladder height process by

$$H_t=\overline{\xi}_{L^{-1}_t}, \qquad t\ge 0. \tag{2.2}$$

The pair (L^{-1},H) is a bivariate subordinator, as is the case of the pair $(\widehat{L}^{-1},\widehat{H})$ with

$$\widehat{H}_t=-\underline{\xi}_{\widehat{L}^{-1}_t}, \qquad t\ge 0.$$

Furthermore, it is important to note that by a simple change of variables, we can rewrite the renewal functions $U^{(\theta)}$ and $\widehat{U}^{(\theta)}$ in terms of the ascending and descending ladder height processes. Indeed, the measures induced by $U^{(\theta)}$ and $\widehat{U}^{(\theta)}$ can be rewritten as follows,

$$U^{(\theta)}(x)=\mathbb{E}^{(e,\theta)}\left[\int_0^\infty \mathbf{1}_{\{H_t\le x\}}\mathrm{d}t\right] \qquad \text{and} \qquad \widehat{U}^{(\theta)}(x)=\mathbb{E}^{(e,\theta)}\left[\int_0^\infty \mathbf{1}_{\{\widehat{H}_t\le x\}}\mathrm{d}t\right].$$

Roughly speaking, the renewal function $U^{(\theta)}(x)$ (resp. $\widehat{U}^{(\theta)}(x)$) "measures" the amount of time that the ascending (resp. descending) ladder height process spends on the interval $[0,x]$ and in particular induces a measure on $[0,\infty)$ which is known as the renewal measure. Finally, we mention that $U^{(\theta)}(0)=0$ if 0 is regular upwards and $U^{(\theta)}(0)=1$ otherwise, similarly $\widehat{U}^{(\theta)}(0)=0$ if 0 is regular upwards and $\widehat{U}^{(\theta)}(0)=1$ otherwise.

2.2. *Strongly subcritical regime.*

Proof of Theorem 1.4: Let $z>0$ and $x\in\mathbb{R}$; and denote by $\mathbb{P}_{(z,x)}$ for the law of the couple (Z,ξ) starting from z and x, respectively. We begin by noting that, conditioning on the environment and then using the exponential change of measure given in (1.19) with $\theta=1$, allow us to deduce

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t>0) &= e^{-\Phi_\xi(1)t}\mathbb{E}^{(e)}\Big[e^{-\xi_t}e^{\xi_t}\mathbb{P}_{(z,0)}\big(Z_t>0 \mid \xi\big)\Big] \\ &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\mathbb{P}_{(z,0)}\big(Z_t>0 \mid \xi\big)\Big]. \end{aligned}$$

Recall from (1.9), that for $\lambda \geq 0$ and $t \geq 0$, the random cumulant $h_{0,t}(\lambda) = e^{-\xi_0} v_t(0, \lambda e^{\xi_t}, \xi)$ satisfies

$$\mathbb{E}_{(z,0)}\big[e^{-\lambda Z_t} \bigm| \xi\big] = \exp\{-z h_{0,t}(\lambda)\}.$$

Now, we denote

$$G_t(\lambda) := e^{-\xi_t}\Big(1 - \exp\{-z h_{0,t}(\lambda)\}\Big), \qquad \text{for} \quad \lambda, t \geq 0,$$

and observe that the quenched survival probability of Z is given by

$$\mathbb{P}_{(z,0)}\big(Z_t > 0 \bigm| \xi\big) = 1 - \exp\{-z h_{0,t}(\infty)\}.$$

In other words,

$$G_t(0) = 0 \qquad \text{and} \qquad G_t(\infty) = e^{-\xi_t}\mathbb{P}_{(z,0)}\big(Z_t > 0 \bigm| \xi\big).$$

Since the map $\lambda \mapsto h_{0,t}(\lambda)$ is differentiable, then so does $G_t(\cdot)$. In view of the above arguments, we deduce

$$e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = \mathbb{E}^{(e,1)}\big[G_t(\infty)\big] = \mathbb{E}^{(e,1)}\left[\int_0^\infty G'_t(\lambda)\mathrm{d}\lambda\right]. \tag{2.3}$$

Hence in order to deduce our result, we would like to take the limit, as $t \to \infty$, in the above equality and then make use of the Dominated Convergence Theorem in order to interchange the limit with the integral on the right-hand side. With this purpose in mind, we need to find a function $g(\lambda)$ such that $\mathbb{E}^{(e,1)}\big[|G'_t(\lambda)|\big] \leq g(\lambda)$, for all $t \geq 1$, and

$$\int_0^\infty g(\lambda)\mathrm{d}\lambda < \infty. \tag{2.4}$$

First, we analyse $\mathbb{E}^{(e,1)}\big[|G'_t(\lambda)|\big]$. Note from the definition of $G_t(\lambda)$ that

$$G'_t(\lambda) = z e^{-\xi_t}\exp\big\{-z h_{0,t}(\lambda)\big\} h'_{0,t}(\lambda) = z\exp\big\{-z h_{0,t}(\lambda)\big\}\frac{\mathrm{d}}{\mathrm{d}u} v_t(0, u, \xi)\Big|_{u=\lambda e^{\xi_t}}, \tag{2.5}$$

where in the last equality we recall that $h_{0,t}(\lambda) = e^{-\xi_0} v_t(0, \lambda e^{\xi_t}, \xi)$. Moreover, by differentiating with respect to λ on both sides of the backward differential equation (1.6), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} v_t(0, \lambda, \xi) = 1 - \int_0^t \psi'_0\big(e^{-\xi_s} v_t(s, \lambda, \xi)\big)\frac{\mathrm{d}}{\mathrm{d}\lambda} v_t(s, \lambda, \xi)\mathrm{d}s.$$

Thus solving the above equation, we get

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} v_t(0, \lambda, \xi) = \exp\left\{-\int_0^t \psi'_0\big(e^{-\xi_s} v_t(s, \lambda, \xi)\big)\mathrm{d}s\right\}. \tag{2.6}$$

Then, it follows

$$\mathbb{E}^{(e,1)}\big[|G'_t(\lambda)|\big] = \mathbb{E}^{(e,1)}\big[G'_t(\lambda)\big] = z\mathbb{E}^{(e,1)}\left[\exp\left\{-z h_{0,t}(\lambda) - \int_0^t \psi'_0\big(h_{s,t}(\lambda)\big)\mathrm{d}s\right\}\right].$$

In other words, according to identity (1.12), $G'_t(\lambda)$ is the Laplace transform of a CBLE with immigration and whose immigration mechanism is given by ψ'_0.

In order to find the integrable function $g(\lambda)$ which dominates $\mathbb{E}^{(e,1)}[|G'_t(\lambda)|]$, for $t \geq 1$, we use another useful characterisation of $\mathbb{E}^{(e,1)}[G'_t(\lambda)]$. Recall that the homogeneous Lévy process Ξ defined in (1.13), allows to extend the definition of the map $s \mapsto h^{\Xi}_{s,0}(\lambda)$ for $s \leq 0$. The latter is the unique positive pathwise solution to (1.15). We write, for $\lambda > 0$ and $t \geq 0$,

$$\mathbb{E}^{(e,1)}\big[G'_t(\lambda)\big] = z\mathbf{E}^{(e,1)}\left[\exp\left\{-z h^{\Xi}_{-t,0}(\lambda) - \int_{-t}^0 \psi'_0\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right],$$

where $\mathbf{P}^{(e,1)}$ denotes the law of the homogeneous Lévy process Ξ constructed in (1.13) but with $(\xi, \mathbb{P}^{(e,1)})$. Now, we introduce the function

$$g(\lambda) := \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right].$$

Using the latter characterisation of $\mathbb{E}^{(e,1)}[G_t'(\lambda)]$ together with the non-negative property of ψ_0' and $h^{\Xi}_{-t,0}(\lambda)$, we deduce the following inequality, for $t \geq 1$,

$$\mathbb{E}^{(e,1)}\big[|G_t'(\lambda)|\big] \leq z\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-t}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right] \leq zg(\lambda).$$

Furthermore, observe from assumption (1.23), that the function $g(\cdot)$ is integrable. This allows us to apply Fubini's Theorem in (2.3), i.e.

$$e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) = \mathbb{E}^{(e,1)}\left[\int_0^\infty G_t'(\lambda)\mathrm{d}\lambda\right] = \int_0^\infty \mathbb{E}^{(e,1)}\big[G_t'(\lambda)\big]\mathrm{d}\lambda.$$

Thus, we appeal to the Dominated Convergence Theorem in (2.3) and get

$$\begin{aligned}\lim_{t\to\infty} e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t > 0) &= \lim_{t\to\infty}\int_0^\infty \mathbb{E}^{(e,1)}\big[G_t'(\lambda)\big]\mathrm{d}\lambda\\ &= \int_0^\infty \lim_{t\to\infty}\mathbb{E}^{(e,1)}\big[G_t'(\lambda)\big]\mathrm{d}\lambda = \int_0^\infty \mathbb{E}^{(e,1)}\left[\lim_{t\to\infty}G_t'(\lambda)\right]\mathrm{d}\lambda,\end{aligned}$$

where we have used again the Dominated Convergence Theorem in the last equality since the inequality $|G_t'(\lambda)| \leq z$ holds for all $t \geq 1$.

On the other hand, we also note that assumption $\Phi_\xi'(1) < 0$ implies $\xi_t \to -\infty$ as $t \to \infty$, $\mathbb{P}^{(e,1)}$-a.s. The latter implies that $\Xi_t \to \infty$ as $t \to -\infty$, $\mathbf{P}^{(e,1)}$-a.s. Next, thanks to the monotonicity property (see Proposition 2.3 in He et al. (2018)) of the map $-t \mapsto v_0(-t, \lambda, \Xi)$, we have

$$h^{\Xi}_{-t,0}(\lambda) = e^{-\Xi_{-t}}v_0(-t,\lambda,\Xi) \leq e^{-\Xi_{-t}}v_0(0,\lambda,\Xi) = e^{-\Xi_{-t}}\lambda.$$

It then follows that $\lim_{t\to\infty} h^{\Xi}_{-t,0}(\lambda) = 0$, $\mathbf{P}^{(e,1)}$-a.s., and thus

$$\begin{aligned}\mathbb{E}^{(e,1)}\left[\lim_{t\to\infty}G_t'(\lambda)\right] &= z\mathbf{E}^{(e,1)}\left[\lim_{t\to\infty}\exp\left\{-zh^{\Xi}_{-t,0}(\lambda) - \int_{-t}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right]\\ &= z\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right].\end{aligned}$$

The proof is complete once we have shown that

$$0 < \mathfrak{B}_1 := \int_0^\infty \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda < \infty.$$

From (1.18) (see also Corollary 5.7 in He et al. (2018) or the proof of Theorem 5.6 in the same reference), we see that under Assumption 1.1, we have

$$\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right] > 0.$$

Therefore

$$\begin{aligned}\int_0^\infty \mathbb{E}^{(e,1)}\left[\lim_{t\to\infty}G_t'(\lambda)\right]\mathrm{d}\lambda &= z\int_0^\infty \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^{0}\psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda\\ &= z\mathfrak{B}_1 > 0.\end{aligned}$$

Finally, since ψ_0' is non-negative and the condition in (1.23), we obtain the finiteness of $\mathfrak{B}_1$, that is

$$\mathfrak{B}_1 \le \int_0^\infty \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda = \int_0^\infty g(\lambda)\mathrm{d}\lambda < \infty.$$

This completes the proof.

□

2.2.1. *The stable case.* Now, let us compute the constant $\mathfrak{B}_1$ in the stable case and check that it coincides with the constant that appears in Theorem 5.1 in Li and Xu (2018). To this end, we recall the following identity of the previous proof

$$z\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right] = \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[G_t'(\lambda)\right],$$

where $G_t'(\lambda)$ is given in (2.5). We also recall that in this case $\psi_0(\lambda) = C\lambda^{1+\beta}$ with $\beta\in(0,1)$ and $C>0$, from which we observe that the $x\log x$-moment condition 1.1 and Grey's condition 1.3 are clearly satisfied. Moreover the backward differential equation in (1.6) can be solved explicitly (see e.g. Section 5 in Li and Xu (2018)), that is

$$v_t(s,\lambda,\xi) = \left(\lambda^{-\beta} + \beta C \mathtt{I}_{s,t}(\beta\xi)\right)^{-1/\beta},$$

where $\mathtt{I}_{s,t}(\beta\xi)$ denotes the exponential functional of the Lévy process $\beta\xi$, i.e.

$$\mathtt{I}_{s,t}(\beta\xi) := \int_s^t e^{-\beta\xi_u}\mathrm{d}u, \qquad 0\le s\le t. \tag{2.7}$$

In other words, we obtain

$$\begin{aligned}\mathbf{E}^{(e,1)}&\left[\exp\left\{-\int_{-\infty}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]\\ &= \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[\exp\{-zv_t(0,\lambda e^{\xi_t},\xi)\}\left(1+(\lambda e^{\xi_t})^\beta \beta C\mathtt{I}_{0,t}(\beta\xi)\right)^{-\frac{1}{\beta}-1}\right].\end{aligned}$$

Now appealing to duality, see (2.1), we get

$$e^{\beta\xi_t}\mathtt{I}_{0,t}(\beta\xi) = \int_0^t e^{-\beta(\xi_u-\xi_t)}\mathrm{d}u \overset{(d)}{=} \int_0^t e^{\beta\xi_u}\mathrm{d}u = \mathtt{I}_{0,t}(-\beta\xi), \tag{2.8}$$

and

$$v_t(0,\lambda e^{\xi_t},\xi) = e^{\xi_t}\left(\lambda^{-\beta}+\beta C e^{\beta\xi_t}\mathtt{I}_{0,t}(\beta\xi)\right)^{-1/\beta} \overset{(d)}{=} e^{\xi_t}\left(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\right)^{-1/\beta}.$$

Hence

$$\begin{aligned}\mathbf{E}^{(e,1)}&\left[\exp\left\{-\int_{-\infty}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]\\ &= \lim_{t\to\infty}\mathbb{E}^{(e,1)}\left[\exp\left\{-ze^{\xi_t}\left(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\right)^{-1/\beta}\right\}\left(1+\lambda^\beta\beta C\mathtt{I}_{0,t}(-\beta\xi)\right)^{-\frac{1}{\beta}-1}\right].\end{aligned} \tag{2.9}$$

Furthermore since $\xi_t\to-\infty$, as $t\to\infty$, $\mathbb{P}^{(e,1)}$-a.s., then $\mathtt{I}_{0,\infty}(-\beta\xi)$ is finite $\mathbb{P}^{(e,1)}$-a.s. Thus, it follows that

$$\lim_{t\to\infty}\exp\left\{-ze^{\xi_t}\left(\lambda^{-\beta}+\beta C\mathtt{I}_{0,t}(-\beta\xi)\right)^{-1/\beta}\right\} = 1, \qquad \mathbb{P}^{(e,1)} - \text{a.s.},$$

which yields,

$$\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-\infty}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right] = \mathbb{E}^{(e,1)}\left[\left(1+\beta C\lambda^\beta\mathtt{I}_{0,\infty}(-\beta\xi)\right)^{-\frac{1}{\beta}-1}\right]. \tag{2.10}$$

Next, we claim that condition (1.23) is satisfied under Assumption 1.2 with $\vartheta = 1$. We prove this claim below. Hence, using Fubini's Theorem we deduce

$$\begin{aligned}\mathfrak{B}_1 &= \int_0^\infty \mathbb{E}^{(e,1)}\left[\Big(1+\beta C\lambda^\beta \mathtt{I}_{0,\infty}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right]\mathrm{d}\lambda \\ &= (\beta C)^{-1/\beta}\mathbb{E}^{(e,1)}\left[\left(\int_0^\infty e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right],\end{aligned}$$

where in the last equality we have solved the integral with respect to λ.

Finally, we prove the claim that condition (1.23) is satisfied under Assumption 1.2 with $\vartheta = 1$. Recalling that the homogeneous Lévy process Ξ defined in (1.13), allows to extend the definition of the map $s \mapsto h^{\Xi}_{s,0}(\lambda)$ for $s \le 0$, and using identity (2.6) (with λe^{ξ_1} instead of λ), we get

$$\begin{aligned}\mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right] &= \mathbb{E}^{(e,1)}\left[\exp\left\{-\int_0^1 \psi_0'\big(h_{s,1}(\lambda)\big)\mathrm{d}s\right\}\right] \\ &= \mathbb{E}^{(e,1)}\left[\big(\lambda e^{\xi_1}\big)^{-\beta-1}\Big(\big(\lambda e^{\xi_1}\big)^{-\beta}+\beta C\mathtt{I}_{0,1}(\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right] \\ &= \mathbb{E}^{(e,1)}\left[\Big(1+\lambda^\beta\beta C e^{\beta\xi_1}\mathtt{I}_{0,1}(\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right] \\ &= \mathbb{E}^{(e,1)}\left[\Big(1+\lambda^\beta\beta C\mathtt{I}_{0,1}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right],\end{aligned}$$

where the last identity follows from (2.8). Then the integral in condition (1.23) satisfies

$$\begin{aligned}\int_0^\infty \mathbf{E}^{(e,1)}\left[\exp\left\{-\int_{-1}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\right]\mathrm{d}\lambda &= \int_0^\infty \mathbb{E}^{(e,1)}\left[\Big(1+\lambda^\beta\beta C\mathtt{I}_{0,1}(-\beta\xi)\Big)^{-\frac{1}{\beta}-1}\right]\mathrm{d}\lambda, \\ &= (\beta C)^{-1/\beta}\mathbb{E}^{(e,1)}\left[\left(\int_0^1 e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right],\end{aligned}$$

where in the last equality we have used again Fubini's Theorem and solve the integral with respect to λ. In particular, appealing once again to duality (2.1), we get

$$\mathbb{E}^{(e,1)}\left[\left(\int_0^1 e^{\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right] = e^{-\Phi_\xi(1)}\mathbb{E}^{(e)}\left[\left(\int_0^1 e^{-\beta\xi_u}\mathrm{d}u\right)^{-1/\beta}\right] = e^{-\Phi_\xi(1)}\mathbb{E}^{(e)}\big[\mathtt{I}_{0,1}(\beta\xi)^{-1/\beta}\big].$$

Moreover, under the exponential moment Assumption 1.2 with $\vartheta = 1$, from Lemma 2.2 in Li and Xu (2018), we deduce that

$$\mathbb{E}^{(e)}\Big[\mathtt{I}_{0,1}(\beta\xi)^{-1/\beta}\Big] \le e^{2\Phi_\xi'(0)}\mathbb{E}^{(e)}\big[e^{\xi_1}\big],$$

where the right-hand side is finite from our assumption. This prove our claim.

2.3. *Intermediate subcritical regime.* The aim of this section is to show Theorem 1.6. Throughout this section, we assume that the underlying Lévy process ξ fulfils conditions $\Phi_\xi'(0) < 0$ and $\Phi_\xi'(1) = 0$. In other words, ξ drifts to $-\infty$ under $\mathbb{P}^{(e)}$ and oscillates under the probability measure $\mathbb{P}^{(e,1)}$ defined by the Esscher transform (1.21).

Before moving to the proof of Theorem 1.6, we recall that, under the assumption that the Lévy process ξ possesses exponential moments of order $\vartheta > 1$, the probability that the supremum of ξ stays below 0 under $\mathbb{P}^{(e,1)}_x$, for $x < 0$, satisfies

$$\mathbb{P}^{(e,1)}_x\big(\overline{\xi}_t < 0\big) \sim \sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\big[H_1\big]U^{(1)}(-x)t^{-1/2}, \qquad \text{as} \quad t \to \infty, \tag{2.11}$$

where we recall that $U^{(1)}$ denotes the renewal function, under $\mathbb{P}^{(e,1)}$, and $(H_t, t \geq 0)$ the ascending ladder process, (see Lemma 11 in Hirano (2001)).

The proof of Theorem 1.6 uses the same notation as in the proof of Theorem 1.4 and is based on the following two lemmas. The first of which tells us, under our general assumptions (1.5) and the exponential moments condition (1.2) with $\vartheta > 1$, that only paths of Lévy processes with a low supremum contribute to the probability of survival.

Lemma 2.1. *Suppose that condition (1.2) holds with $\vartheta > 1$. We also assume that condition (1.5) is satisfied. Then for any $z > 0$, $x < 0$ and $0 < \delta < 1$, we have*

$$\lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) = 0.$$

Proof: Let $z > 0$, $x < 0$ and $0 < \delta < 1$. We begin by noting that conditioning on ξ and then using the Esscher transform allow us to deduce that

$$e^{-t\Phi_\xi(1)} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) \quad = \quad \mathbb{E}_x^{(e,1)}\Big[e^{-\xi_t} \mathbb{P}_{(z,x)}\big(Z_t > 0 \bigm| \xi\big) \mathbf{1}_{\{\overline{\xi}_{t-\delta} \geq y\}}\Big].$$

Further note from (1.8), that the survival probability conditioned on the environment is bounded from above by the functional $v_t(0, \infty, \xi - \xi_0)$, i.e.,

$$\mathbb{P}_{(z,x)}\big(Z_t > 0 \bigm| \xi\big) \quad = \quad 1 - \exp\big\{ - z v_t(0, \infty, \xi - \xi_0)\big\} \leq z v_t(0, \infty, \xi - \xi_0).$$

On the other hand, condition (1.5) allows us to find a lower bound for $v_t(0, \infty, \xi - \xi_0)$ in terms of the exponential functional of ξ. Indeed, we observe from the backward differential equation given in (1.6) that

$$\frac{\partial}{\partial s} v_t(s, \lambda e^{-\xi_0}, \xi - \xi_0) \geq C v_t^{1+\beta}(s, \lambda e^{-\xi_0}, \xi - \xi_0) e^{-\beta(\xi_s - \xi_0)}, \quad v_t(t, \lambda e^{-\xi_0}, \xi - \xi_0) = \lambda e^{-\xi_0}.$$

Integrating between 0 and t, we get

$$\frac{1}{v_t^\beta(0, \lambda e^{-\xi_0}, \xi - \xi_0)} - \frac{1}{(\lambda e^{-\xi_0})^\beta} \geq C\beta \int_0^t e^{-\beta(\xi_s - \xi_0)} \mathrm{d}s \quad \text{with} \quad C\beta > 0.$$

Now, letting $\lambda \uparrow \infty$ and taking into account that $\beta \in (0, 1)$ and $C > 0$, we deduce the following inequality for all $t \geq 0$,

$$v_t(0, \infty, \xi - \xi_0) \leq \big(C\beta \mathtt{I}_{0,t}(\beta(\xi - \xi_0))\big)^{-1/\beta}, \tag{2.12}$$

where $\mathtt{I}_{0,t}(\beta(\xi - \xi_0))$ is the exponential functional of the Lévy process $\beta(\xi - \xi_0)$, see (2.7). The latter implies that

$$\begin{aligned} e^{-t\Phi_\xi(1)} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) &\leq z(\beta C)^{-1/\beta} \mathbb{E}_x^{(e,1)}\Big[e^{-\xi_t} \mathtt{I}_{0,t}(\beta(\xi - \xi_0))^{-1/\beta} \mathbf{1}_{\{\overline{\xi}_{t-\delta} \geq y\}}\Big] \\ &= z(\beta C)^{-1/\beta} \mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta} \mathbf{1}_{\{\underline{\xi}_{t-\delta} \leq -y-x\}}\Big], \end{aligned}$$

where in the last equality we have appealed to the Duality Lemma given in (2.1) to see that

$$e^{-\xi_t} \mathtt{I}_{0,t}(\beta\xi)^{-1/\beta} \overset{(d)}{=} \left(\int_0^t e^{\beta\xi_s} \mathrm{d}s\right)^{-1/\beta} = \mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta}.$$

Finally, according to Li and Xu (2018, Lemma 3.5), we have

$$\lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} \mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta} \mathbf{1}_{\{\underline{\xi}_{t-\delta} \leq -y-x\}}\Big] = 0.$$

Therefore,

$$\begin{aligned} &\lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) \\ &\qquad \leq z(\beta C)^{-1/\beta} \lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} \mathbb{E}^{(e,1)}\Big[\mathtt{I}_{0,t}(-\beta\xi)^{-1/\beta} \mathbf{1}_{\{\underline{\xi}_{t-\delta} \leq -y-x\}}\Big] = 0, \end{aligned}$$

which concludes the proof. □

The following lemma studies the survival probability under environments with low extrema. More precisely, it confirms the statement that only paths of the Lévy process with a very low running supremum give a substantial contribution to the speed of the survival probability.

Lemma 2.2. *Suppose that condition (1.1) holds together with the exponential moment condition (1.2) with $\vartheta > 1$. We also assume that the integral condition in (1.25) holds. Then for every $z > 0$ and $x < 0$, we have*

$$\lim_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) = z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}}\mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(x),$$

where

$$\mathfrak{b}_2(x) = U^{(1)}(-x)\mathbf{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathrm{d}\lambda\right] \in (0,\infty). \tag{2.13}$$

Proof: Let $z > 0$ and assume that $\xi_0 = x < 0$. We begin by recalling that, under $\mathbb{P}^{(e,1)}$, the Lévy process ξ oscillates. In addition from the Esscher transform, we have the following identity

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,0)}\Big(Z_t > 0,\ \overline{\xi}_t < -x\Big) \\ &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\mathbb{P}_{(z,0)}\big(Z_t > 0 \mid \xi\big)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big] \end{aligned}$$

Recall from (1.9), that for any $\lambda \geq 0$ and $s \leq t$, the random cumulant $h_{s,t}(\lambda)$ satisfies

$$\begin{aligned} \mathbb{E}_{(z,x)}\big[e^{-\lambda Z_t} \mid \xi, \mathcal{F}^{(b)}_s\big] &= \mathbb{E}_{(z,0)}\Big[e^{-\lambda Z_t e^{\xi_t} e^{-\xi_t}} \mid \xi, \mathcal{F}^{(b)}_s\Big] \\ &= \exp\{-Z_s h_{s,t}(\lambda)\}. \end{aligned}$$

From the previous identity, we observe that the initial condition of the Lévy process ξ is irrelevant for the functional $h_{s,t}(\lambda)$. Further, recall that the quenched survival probability of the process $(Z_t, t \geq 0)$ is given by $\mathbb{P}_{(z,0)}\big(Z_t > 0 \mid \xi\big) = 1 - \exp\{-zh_{0,t}(\infty)\}$. Thus,

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\mathbb{P}_{(z,0)}\big(Z_t > 0 \mid \xi\big)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big] \\ &= \mathbb{E}^{(e,1)}\Big[e^{-\xi_t}\big(1 - \exp\big\{-zh_{0,t}(\infty)\big\}\big)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big]. \end{aligned}$$

Now, we use the same notation as in the proof of Theorem 1.4. Namely, we denote for each fixed $t \geq 0$, the function

$$G_t(\lambda) = e^{-\xi_t}\Big(1 - \exp\big\{-zh_{0,t}(\lambda)\big\}\Big), \qquad \text{for} \quad \lambda \geq 0.$$

Then,

$$G_t(0) = 0 \qquad \text{and} \qquad G_t(\infty) = e^{-\xi_t}\mathbb{P}_{(z,0)}\big(Z_t > 0 \mid \xi\big).$$

Since the map $\lambda \mapsto h_{0,t}(\lambda)$ is differentiable, then so does $G_t(\cdot)$. In view of the above arguments, we deduce

$$\begin{aligned} e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) &= \mathbb{E}^{(e,1)}\left[\mathbf{1}_{\{\overline{\xi}_t<-x\}}\int_0^\infty G_t'(\lambda)\mathrm{d}\lambda\right] \\ &= \int_0^\infty \mathbb{E}^{(e,1)}\Big[G_t'(\lambda)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big]\mathrm{d}\lambda, \end{aligned}$$

where in the last equality, the expectation and the integral may be exchanged using Fubini's Theorem. Recall the definition of the homogeneous Lévy process Ξ given in (1.13). Now, using the same strategy as in the proof of Theorem 1.4, that is extending the map $s \mapsto h^{\Xi}_{s,0}(\lambda)$, for $s \le 0$, and taking the derivate of $G_t(\cdot)$ computed in (2.5), we have

$$\begin{aligned}\mathbb{E}^{(e,1)}\Big[G_t'(\lambda)\mathbf{1}_{\{\overline{\xi}_t<-x\}}\Big] &= z\mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{0,t}(\lambda)-\int_0^t \psi_0'\big(h_{s,t}(\lambda)\big)\mathrm{d}s\right\}\mathbf{1}_{\{\overline{\xi}_t<-x\}}\right]\\ &= z\mathbf{E}^{(e,1)}\left[\exp\left\{-zh^{\Xi}_{-t,0}(\lambda)-\int_{-t}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathbf{1}_{\{\Xi_{-t}>x\}}\right],\end{aligned}$$

where we recall that $(\Xi, \mathbf{P}^{(e,1)})$ is the homogeneous Lévy process indexed in $\mathbb{R}$ associated to $(\xi, \mathbb{P}^{(e,1)})$. Next, we simplify the notation by introducing, for $t \ge 0$,

$$F_t(\lambda) := \exp\left\{-zh^{\Xi}_{-t,0}(\lambda)-\int_{-t}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}.$$

Hence, making use of the above observations, we deduce

$$e^{-\Phi_\xi(1)t}\mathbb{P}_{(z,x)}\Big(Z_t>0,\ \overline{\xi}_t<0\Big) = z\mathbf{P}^{(e,1)}\big(\Xi_{-t}>x\big)\int_0^\infty \mathbf{E}^{(e,1)}\Big[F_t(\lambda)\ \big|\ \Xi_{-t}>x\Big]\mathrm{d}\lambda.$$

Now, taking into account (2.11), we obtain that

$$\lim_{t\to\infty} t^{1/2}\mathbf{P}^{(e,1)}\big(\Xi_{-t}>x\big) = \lim_{t\to\infty} t^{1/2}\mathbb{P}^{(e,1)}\left(\overline{\xi}_t<-x\right) = \sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\big[H_1\big]U^{(1)}(-x), \tag{2.14}$$

and thus the proof of this lemma will be completed once we have shown

$$\begin{aligned}\lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}\Big[F_t(\lambda)\ \big|\ \Xi_{-t}>x\Big]\mathrm{d}\lambda &= \lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\ \big|\ \Xi_{-t}>0\Big]\mathrm{d}\lambda\\ &= \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathrm{d}\lambda\right]\\ &=: b(x).\end{aligned}$$

The arguments used to deduce the preceding limit are quite involved, for that reason we split its proof in three steps.

Step 1. Let us first introduce the following functions, for $r, \lambda \ge 0$ and $t \ge 0$,

$$f_r(t,\lambda) := \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\ \big|\ \Xi_{-(t+r)}>0\Big],$$

and

$$g_r(t,\lambda) := \mathbf{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-t}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\ \Big|\ \Xi_{-(t+r)}>0\right].$$

Since $F_t(\lambda)$ and $\exp\{-\int_{-t}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}\lambda\}$ are bounded random variables, we may deduce

$$f_r(t,\lambda) \to \mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_t(\lambda)\right] \quad \text{and} \quad g_r(t,\lambda) \to \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-t}^0 \psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right], \tag{2.15}$$

as $r \to \infty$. We first prove the convergence for $F_t(\lambda)$, since the same arguments will lead to the other convergence. Appealing to the Markov property, we have,

$$\mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\ \big|\ \Xi_{-(t+r)}>0\Big] = \mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda)\frac{\mathbf{P}^{(e,1)}_{-\Xi_{-t}}\big(\Xi_{-r}>0\big)}{\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-(t+r)}>0\big)}\mathbf{1}_{\{\Xi_{-t}>0\}}\right]. \tag{2.16}$$

Now since (2.14) holds, we have for $\epsilon > 0$ that there exists a constant $N_1 > 0$ (which depends on ϵ) such that the following inequality is satisfied for all $r \geq N_1$,

$$\frac{\mathbf{P}^{(e,1)}_{-\Xi_{-t}}\left(\Xi_{-r} > 0\right)}{\mathbf{P}^{(e,1)}_{-x}\left(\Xi_{-(t+r)} > 0\right)} \leq \frac{(1+\epsilon)}{(1-\epsilon)} \left(\frac{r}{t+r}\right)^{-1/2} \frac{U^{(1)}(\Xi_{-t})}{U^{(1)}(-x)},$$

Therefore, we deduce that for $r \geq N_1$,

$$\frac{\mathbf{P}^{(e,1)}_{-\Xi_{-t}}\left(\Xi_{-r} > 0\right)}{\mathbf{P}^{(e,1)}_{-x}\left(\Xi_{-(t+r)} > 0\right)} \leq \frac{(1+\epsilon)}{(1-\epsilon)} \left(1+\frac{t}{N_1}\right)^{1/2} \frac{U^{(1)}(\Xi_{-t})}{U^{(1)}(-x)}. \tag{2.17}$$

By definition of the process Ξ given in (1.13), we have that Ξ_{-t} and $-\overline{\xi}_t$ are equal in distribution and thus

$$\mathbf{E}^{(e,1)}_{-x}\left[U^{(1)}(\Xi_{-t})\mathbf{1}_{\{\Xi_{-t}>0\}}\right] = \mathbb{E}^{(e,1)}_{x}\left[U^{(1)}(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t<0\}}\right] = U^{(1)}(-x),$$

where in the second equality we have used that $(U^{(1)}(-\xi_t)\mathbf{1}_{\{\overline{\xi}_t<0\}}, t \geq 0)$ is a martingale with respect to $(\mathcal{F}^{(e)}_t)_{t\geq 0}$. Since $F_t(\lambda)$ is a bounded random variable, then we can apply the Dominated Convergence Theorem in (2.16) to obtain the first convergence in (2.15), i.e.

$$\begin{aligned}\lim_{r\to\infty} f_r(t,\lambda) &= \mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda)\lim_{r\to\infty}\frac{\mathbf{P}^{(e,1)}_{-\Xi_{-t}}\left(\Xi_{-r} > 0\right)}{\mathbf{P}^{(e,1)}_{-x}\left(\Xi_{-(t+r)} > 0\right)}\mathbf{1}_{\{\Xi_{-t}>0\}}\right] \\ &= \frac{1}{U^{(1)}(-x)}\mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda)U^{(1)}(\Xi_{-t})\mathbf{1}_{\{\Xi_{-t}>0\}}\right] =: \mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_t(\lambda)\right].\end{aligned}$$

Similarly, inequality (2.17) implies that the following upper bound also holds

$$g_r(t,\lambda) \leq C_1(t)\mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-t}^{0}\psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right],$$

where $C_1(t)$ is a positive constant which depends on t. We may now appeal to the Dominated Convergence Theorem together with our hypothesis (1.25), to deduce that for $t \geq 1$

$$\int_0^\infty g_r(t,\lambda)\mathrm{d}\lambda \to \int_0^\infty \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-t}^{0}\psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right]\mathrm{d}\lambda, \quad \text{as} \quad r\to\infty.$$

Furthermore, since $f_r(t,\lambda) \leq g_r(t,\lambda)$, an application of the *generalised* Dominated Convergence Theorem (see for instance Folland (1984, Exercise 2.20)) gives us

$$\lim_{r\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda) \mid \Xi_{-(t+r)} > 0\right]\mathrm{d}\lambda = \int_0^\infty \mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_t(\lambda)\right]\mathrm{d}\lambda. \tag{2.18}$$

Step 2. Let $1 \leq s \leq t$, $\lambda \geq 0$ and $\gamma \in (1,2]$. From the proof of Lemma 3.2 in Bansaye et al. (2021), we can deduce

$$\left|\mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda) - F_s(\lambda) \mid \Xi_{-\gamma t} > 0\right]\right| \leq C_2\mathbf{E}^{(e,1),\uparrow}_{-x}\left[|F_t(\lambda) - F_s(\lambda)|\right],$$

where C_2 is a positive constant. Hence

$$\left|\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\left[F_t(\lambda) - F_s(\lambda) \mid \Xi_{-\gamma t} > 0\right]\mathrm{d}\lambda\right| \leq C_2\int_0^\infty \mathbf{E}^{(e,1),\uparrow}_{-x}\left[|F_t(\lambda) - F_s(\lambda)|\right]\mathrm{d}\lambda.$$

In addition, we observe that by definition of $F_t(s)$ and taking into account that $s \le t$,

$$\begin{aligned}|F_t(\lambda) - F_s(\lambda)| &= \exp\left\{-\int_{-s}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\}\\ &\qquad \left|F_t(\lambda)\exp\left\{\int_{-s}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\} - \exp\{-zh^{\Xi}_{-s,0}(\lambda)\}\right|\\ &= \exp\left\{-\int_{-s}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\}\\ &\qquad \left|\exp\left\{-zh^{\Xi}_{-t,0}(\lambda) - \int_{-t}^{-s} \psi_0'\big(h^{\Xi}_{u,0}(\lambda)\big)\mathrm{d}u\right\} - \exp\{-zh^{\Xi}_{-s,0}(\lambda)\}\right|.\end{aligned}$$

Now, since $h^{\Xi}_{-s,0}(\lambda), \psi_0'$ are positive functions and $z > 0$, we have

$$\left|\exp\left\{-\left(zh^{\Xi}_{-t,0}(\lambda) + \int_{-t}^{-s} \psi_0'\big(h^{\Xi}_{u,0}(\lambda)\big)\mathrm{d}u\right)\right\} - \exp\{-zh^{\Xi}_{-s,0}(\lambda)\}\right| \le 2,$$

which yields for $s \ge 1$

$$|F_t(\lambda) - F_s(\lambda)| \le 2\exp\left\{-\int_{-1}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\}.$$

It then follows, from the previous calculations and our assumption (1.25) together with the Dominated Convergence Theorem, that

$$\lim_{s\to\infty}\lim_{t\to\infty}\left|\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda) - F_s(\lambda) \mid \Xi_{-\gamma t} > 0\Big]\mathrm{d}\lambda\right| = 0,$$

which in particular yields

$$\lim_{s\to\infty}\lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda) - F_s(\lambda) \mid \Xi_{-\gamma t} > 0\Big]\mathrm{d}\lambda = 0.$$

Thus, appealing to (2.18) in Step 1, we get

$$\begin{aligned}\lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda) \mid \Xi_{-\gamma t} > 0\Big]\mathrm{d}\lambda &= \lim_{s\to\infty}\lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_s(\lambda) \mid \Xi_{-\gamma t} > 0\Big]\mathrm{d}\lambda\\ &= \lim_{s\to\infty}\int_0^\infty \mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_s(\lambda)\right]\mathrm{d}\lambda.\end{aligned}$$

In order to deal with the above limit in the right-hand side, first note that $h^{\Xi}_{-s,0}(\lambda) \le \lambda e^{-\Xi_{-s}} \to 0$, as $s \to \infty$, $\mathbf{P}^{(e,1)}_{-x}$-a.s. Moreover, we have

$$\mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_s(\lambda)\right] \to \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-\infty}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\}\right], \quad \text{as} \quad s \to \infty,$$

and for $s \ge 1$,

$$\mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_s(\lambda)\right] \le \mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-1}^{0} \psi_0'(h^{\Xi}_{u,0}(\lambda))\mathrm{d}u\right\}\right].$$

Hence, we may now apply once again the Dominated Convergence Theorem to deduce that

$$\int_0^\infty \lim_{s\to\infty}\mathbf{E}^{(e,1),\uparrow}_{-x}\left[F_s(\lambda)\right]\mathrm{d}\lambda = b(x) < \infty.$$

In other words, we have

$$\lim_{t\to\infty}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda) \mid \Xi_{-\gamma t} > 0\Big]\mathrm{d}\lambda = b(x).$$

Next, from (2.14) we obtain

$$\begin{aligned}\lim_{t\to\infty}&\frac{1}{\mathbf{P}^{(e,1)}_{-x}(\Xi_{-t}>0)}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda\\ &=\lim_{t\to\infty}\frac{\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-\gamma t}>0\big)}{\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big)}\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mid \Xi_{-\gamma t}>0\Big]\mathrm{d}\lambda\\ &=\gamma^{-1/2}b(x).\end{aligned}$$

Since γ may be chosen arbitrarily close to 1, we have

$$\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda-b(x)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big)=o(1)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big).$$

Step 3. Let $\lambda\geq 0$, $t\geq 1$ and $\gamma\in(1,2]$ and denote

$$J_t(\lambda):=\frac{1}{\mathbf{P}^{(e,1)}_{-x}(\Xi_{-t}>0)}\mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\Big(\mathbf{1}_{\{\Xi_{-t}>0\}}-\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big)\Big].$$

Observe from (2.14) and the fact that $F_t(\lambda)\leq 1$, that the following holds

$$0\leq J_t(\lambda)\leq 1-\frac{\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-\gamma t}>0\big)}{\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big)}\to 1-\gamma^{-1/2},\quad \text{as}\quad t\to\infty.$$

Since γ may be taken arbitrary close to 1, we deduce that $J_t(\lambda)\to 0$ as $t\to\infty$. In addition,

$$\begin{aligned}J_t(\lambda)&\leq \mathbf{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-t}^0\psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\;\Big|\;\Xi_{-t}>0\right]\\ &\leq \mathbf{E}^{(e,1)}_{-x}\left[\exp\left\{-\int_{-1}^0\psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\;\Big|\;\Xi_{-t}>0\right]\\ &\leq C_3\mathbf{E}^{(e,1),\uparrow}_{-x}\left[\exp\left\{-\int_{-1}^0\psi_0'(h^{\Xi}_{s,0}(\lambda))\mathrm{d}s\right\}\right],\end{aligned}$$

where C_3 is a positive constant and the right-hand side is an integrable function in λ thanks to the assumption (1.25). Hence, appealing again to the Dominate Convergence Theorem, we see

$$\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\Big(\mathbf{1}_{\{\Xi_{-t}>0\}}-\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big)\Big]\mathrm{d}\lambda=o(1)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big).$$

We combine the previous limit with the conclusion of Steps 2 to deduce, as promised earlier, that

$$\begin{aligned}\int_0^\infty &\mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-t}>0\}}\Big]\mathrm{d}\lambda-b(x)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big)\\ &=\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-t}>0\}}\Big]\mathrm{d}\lambda-\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda\\ &\qquad+\int_0^\infty \mathbf{E}^{(e,1)}_{-x}\Big[F_t(\lambda)\mathbf{1}_{\{\Xi_{-\gamma t}>0\}}\Big]\mathrm{d}\lambda-b(x)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big)\\ &=o(1)\mathbf{P}^{(e,1)}_{-x}\big(\Xi_{-t}>0\big).\end{aligned}$$

Finally, similarly as in the proof of Theorem 1.4, we see that the Assumption 1.1 guarantees that $b(x)>0$. This concludes the proof. □

With Lemmas 2.1 and 2.2 in hand, we may now proceed to the proof of Theorem 1.6 following similar ideas as those used in Theorem 1.2 in Bansaye et al. (2021).

Proof of Theorem 1.6: Let $z, \epsilon > 0$ and $x < 0$. From Lemma 2.1, we have for every $\delta \in (0,1)$,

$$\lim_{y\to\infty} \limsup_{t\to\infty} t^{1/2} e^{-t\Phi_\xi(\gamma)} \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) = 0.$$

Then it follows that, we may choose $y > 0$ such that for t sufficiently large

$$\mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) \leq \epsilon \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} < y\Big).$$

Further, since $\{Z_t > 0\} \subset \{Z_{t-\delta} > 0\}$ for t large, we deduce that

$$\begin{aligned} \mathbb{P}_z(Z_t > 0) &= \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} \geq y\Big) + \mathbb{P}_{(z,x)}\Big(Z_t > 0,\ \overline{\xi}_{t-\delta} < y\Big) \\ &\leq (1+\epsilon)\mathbb{P}_{(z,x-y)}\Big(Z_{t-\delta} > 0,\ \overline{\xi}_{t-\delta} < 0\Big). \end{aligned}$$

In other words, for every $\epsilon > 0$ there exists $y' < 0$ such that

$$\begin{aligned} &(1-\epsilon)t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,y')}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) \leq t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_z(Z_t > 0) \\ &\leq (1+\epsilon)(t-\delta)^{1/2} e^{-\Phi_\xi(1)(t-\delta)} \mathbb{P}_{(z,y')}\Big(Z_{t-\delta} > 0,\ \overline{\xi}_{t-\delta} < 0\Big) \frac{t^{1/2} e^{-\Phi_\xi(1)t}}{(t-\delta)^{1/2} e^{-\Phi_\xi(1)(t-\delta)}}. \end{aligned}$$

Now, appealing to Lemma 2.2, we have

$$\lim_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_{(z,y')}\Big(Z_t > 0,\ \overline{\xi}_t < 0\Big) = z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y'),$$

where

$$\mathfrak{b}_2(y') = U^{(1)}(-y')\mathbf{E}^{(1),\uparrow}_{-y'}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathrm{d}\lambda\right]. \tag{2.19}$$

Hence, we obtain

$$\begin{aligned} (1-\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y') &\leq \lim_{t\to\infty} t^{1/2} e^{-t\Phi_\xi(1)} \mathbb{P}_z(Z_t > 0) \\ &\leq (1+\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y')e^{-\Phi_\xi(1)\delta}. \end{aligned}$$

On the other hand, we observe that y' is a sequence which may depend on ϵ. Further, this sequence y' goes to minus infinity as ϵ goes to 0. Then, for any sequence $y' = y_\epsilon$, we deduce that

$$\begin{aligned} 0 < (1-\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y_\epsilon) &\leq \lim_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_z(Z_t > 0) \\ &\leq (1+\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y_\epsilon) < \infty. \end{aligned}$$

Therefore, by letting $\epsilon \to 0$, we get

$$\begin{aligned} 0 < \liminf_{\epsilon\to 0}(1-\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y_\epsilon) &\leq \lim_{t\to\infty} t^{1/2} e^{-\Phi_\xi(1)t} \mathbb{P}_z(Z_t > 0) \\ &\leq \limsup_{\epsilon\to 0}(1+\epsilon)z\sqrt{\frac{2}{\pi\Phi''_\xi(1)}} \mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{b}_2(y_\epsilon)e^{-\Phi_\xi(1)\delta} < \infty. \end{aligned}$$

Since δ can be taken arbitrary close to 0, we deduce

$$\lim_{t\to\infty} t^{1/2}e^{-\Phi_\xi(1)t}\mathbb{P}_z(Z_t>0) = z\sqrt{\frac{2}{\pi\Phi_\xi''(1)}}\mathbb{E}^{(e,1)}\big[H_1\big]\mathfrak{B}_2,$$

where

$$\mathfrak{B}_2 := \lim_{\epsilon\to 0}\mathfrak{b}_2(y_\epsilon) = \lim_{\epsilon\to 0} U^{(1)}(-y_\epsilon)\mathbf{E}^{(1),\uparrow}_{-y_\epsilon}\left[\int_0^\infty \exp\left\{-\int_{-\infty}^0 \psi_0'\big(h^{\Xi}_{s,0}(\lambda)\big)\mathrm{d}s\right\}\mathrm{d}\lambda\right].$$

The proof is now complete.

□

2.4. *The Q process.*

Proof of Theorem 1.7: We first prove part (i). We only deduce it for the strongly subcritical regime, for the intermediate subcritical regime the arguments are basically the same. Let $z,t>0$ and $\Lambda\in\mathcal{F}_t$, from the Markov property, we obtain

$$\mathbb{P}_z\big(\Lambda \mid T_0>t+s\big) = \mathbb{E}_z\left[\mathbf{1}_{\{\Lambda,T_0>t\}}\frac{\mathbb{P}_{Z_t}(T_0>s)}{\mathbb{P}_z(T_0>t+s)}\right].$$

From Theorem 1.4, for any $\epsilon>0$ and t large enough, we deduce that

$$\frac{\mathbb{P}_{Z_t}(T_0>s)}{\mathbb{P}_z(T_0>t+s)} = \frac{e^{-\Phi_\xi(1)s}\mathbb{P}_{Z_t}(Z_s>0)e^{-\Phi_\xi(1)t}}{e^{-\Phi_\xi(1)(t+s)}\mathbb{P}_z(Z_{t+s}>0)} \le e^{-\Phi_\xi(1)t}\left(\frac{\epsilon+Z_t\mathfrak{B}_1}{-\epsilon+z\mathfrak{B}_1}\right).$$

Further, from (1.5), we have

$$e^{-\Phi_\xi(1)t}\mathbb{E}_z[Z_t \mid S] = ze^{\xi_t-\Phi_\xi(1)t},$$

where the random variable in the right-hand side above is integrable thanks to our exponential moment condition (1.2) with $\vartheta=1$. Hence, the Dominated Convergence Theorem implies that

$$\begin{aligned}\lim_{s\to\infty}\mathbb{P}_z\big(\Lambda \mid T_0>t+s\big) &= \mathbb{E}_z\left[\mathbf{1}_{\{\Lambda,T_0>t\}}\lim_{s\to\infty}\frac{\mathbb{P}_{Z_t}(T_0>s)}{\mathbb{P}_z(T_0>t+s)}\right]\\ &= \mathbb{E}_z\left[\mathbf{1}_{\{\Lambda,T_0>t\}}\frac{Z_t}{z}e^{-\Phi_\xi(1)t}\right] = \mathbb{E}_z\left[\frac{Z_t}{z}e^{-\Phi_\xi(1)t}\mathbf{1}_\Lambda\right].\end{aligned}$$

We now prove part (ii). The fact that the process $(e^{-\Phi_\xi(1)t}Z_t,\ t\ge 0)$ is a martingale follows directly from (1.5) by applying the Markov property as follows: for $0\le s\le t$,

$$\mathbb{E}_z\big[e^{-\Phi_\xi(1)(t+s)}Z_{t+s} \mid \mathcal{F}_s\big] = e^{-\Phi_\xi(1)(t+s)}\mathbb{E}_{Z_s}[Z_t] = e^{-\Phi_\xi(1)(t+s)}Z_s\mathbb{E}[e^{\xi_t}] = e^{-\Phi_\xi(1)s}Z_s,$$

which establishes the martingale property.

To deduce part (iii), we compute the Laplace transform of Z, under $\mathbb{P}^\natural_z$. Fix $z,t>0$ and $\lambda\ge 0$, using (1.9) and part (ii), we get

$$\begin{aligned}\mathbb{E}^\natural_z\Big[e^{-\lambda Z_t}\Big] &= \mathbb{E}_z\left[\frac{Z_t}{z}e^{-\Phi_\xi(1)t}e^{-\lambda Z_t}\right] = -\frac{e^{-\Phi_\xi(1)t}}{z}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}_z\Big[e^{-\lambda Z_t}\Big]\\ &= -\frac{e^{-\Phi_\xi(1)t}}{z}\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}^{(e)}\Big[\exp\{-zh_{0,t}(\lambda)\}\Big].\end{aligned}$$

Now, note that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\mathbb{E}^{(e)}\Big[\exp\{-zh_{0,t}(\lambda)\}\Big] = -z\mathbb{E}^{(e)}\left[\exp\{-zh_{0,t}(\lambda)\}e^{\xi_t}\frac{\mathrm{d}}{\mathrm{d}u}v_t(0,u,\xi)\Big|_{u=\lambda e^{\xi_t}}\right]. \tag{2.20}$$

Thus, from (2.6), we deduce

$$\begin{aligned}\mathbb{E}_z^{\natural}\left[e^{-\lambda Z_t}\right] &= e^{-\Phi_\xi(1)t}\mathbb{E}^{(e)}\left[\exp\{-zh_{0,t}(\lambda)\}e^{\xi_t}\frac{\mathrm{d}}{\mathrm{d}u}v_t(0,u,\xi)\Big|_{u=\lambda e^{\xi_t}}\right] \\ &= e^{-\Phi_\xi(1)t}\mathbb{E}^{(e)}\left[e^{\xi_t}\exp\left\{-zh_{0,t}(\lambda)-\int_0^t \psi_0'(h_{s,t}(\lambda))\mathrm{d}s\right\}\right] \\ &= \mathbb{E}^{(e,1)}\left[\exp\left\{-zh_{0,t}(\lambda)-\int_0^t \psi_0'(h_{s,t}(\lambda))\mathrm{d}s\right\}\right],\end{aligned}$$

where in the last equality we have used the definition of the Esscher transform given in (1.19). This completes the proof. □

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