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Transformations of infinitely divisible distributions via improper stochastic integrals

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Abstract. Let $X^{(\mu)}(ds)$ be an $\mathbb{R}^d$-valued homogeneous independently scattered random measure over $\mathbb{R}$ having μ as the distribution of $X^{(\mu)}((t,t+1])$. Let $f(s)$ be a nonrandom measurable function on an open interval (a,b) where $-\infty \leqslant a < b \leqslant \infty$. The improper stochastic integral $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$ is studied. Its distribution $\Phi_f(\mu)$ defines a mapping from μ to an infinitely divisible distribution on $\mathbb{R}^d$. Three modifications (compensated, essential, and symmetrized) and absolute definability are considered. After their domains are characterized, necessary and sufficient conditions for the domains to be very large (or very small) in various senses are given. The concept of the dual in the class of purely non-Gaussian infinitely divisible distributions on $\mathbb{R}^d$ is introduced and employed in studying some examples. The τ-measure τ of function f is introduced and whether τ determines Φ_f is discussed. Related transformations of Lévy measures are also studied.

1. Introduction

In Sato (2006a,b,c) is studied the improper stochastic integral $\int_0^{\infty-} f(s)dX_s^{(\mu)}$ or $\int_0^{\infty-} f(s)X^{(\mu)}(ds)$. Here $X_s^{(\mu)}$ is a Lévy process on $\mathbb{R}^d$ with distribution μ at time 1 and $X^{(\mu)}(ds)$ is the homogeneous independently scattered random measure associated with $X_s^{(\mu)}$, $f(s)$ is a nonrandom function locally $X^{(\mu)}$-integrable on the closed half line $[0,\infty)$, and $\int_0^{\infty-}$ is the limit in probability of $\int_0^t$ as $t \to \infty$. Let $\mathcal{L}(Y)$ denote the distribution of a random element Y. Let $ID(\mathbb{R}^d)$ denote the class of infinitely divisible distributions on $\mathbb{R}^d$. Given a function f, we define a mapping Φ_f from a subclass of $ID(\mathbb{R}^d)$ into $ID(\mathbb{R}^d)$ by $\Phi_f(\mu) = \mathcal{L}\left(\int_0^{\infty-} f(s)X^{(\mu)}(ds)\right)$. Such a mapping appears in many papers. With $f(s) = e^{-s}$, it appears in the representation of selfdecomposable distributions (see Rocha-Arteaga and Sato (2003) for

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references); with $f(s) = \log(1/s)$ for $s \in (0,1)$, it appears in the representation of the Goldie–Steutel–Bondesson class on $\mathbb{R}^d$ (see Barndorff-Nielsen and Thorbjørnsen (2002, 2006b) and Barndorff-Nielsen et al. (2006)). In Sato (2006b) a family of functions $f_\alpha(s)$ with $\alpha \geqslant 0$ such that, as $s \downarrow 0$, $f_\alpha(s) \sim \log(1/s)$ for all α and, as $s \to \infty$, $f_\alpha(s) \sim ce^{-s}$ with a constant $c > 0$ for $\alpha = 0$ and $f_\alpha(s) \sim (\alpha s)^{1/\alpha}$ for $\alpha > 0$, is studied. In Aoyama and Maejima (2007) the function $f(s)$ for $0 < s < 1$ defined by $s = \int_{f(s)}^{\infty} (2\pi)^{-1/2} e^{-v^2/2} dv$ (hence $f(s) \to \infty$ as $s \downarrow 0$ and $f(s) \to -\infty$ as $s \uparrow 1$) is utilized in the representation of the class of type G distributions on $\mathbb{R}^d$. In the case of the last example and in the case $f(s) = \log(1/s)$ for $s \in (0,1)$, we define $f(s)$ to be 0 for $s \in \{0\} \cup [1,\infty)$. Then the functions in all these examples are locally $X^{(\mu)}$-integrable on the closed half line $[0,\infty)$ for all $\mu \in ID(\mathbb{R}^d)$, so that they are in the framework of Sato (2006a,b,c). However, it would be more natural to consider an open interval (a,b) with $-\infty \leqslant a < b \leqslant \infty$ and a function $f(s)$ locally $X^{(\mu)}$-integrable on (a,b) and study improper stochastic integrals $\int_{a+}^{b-} f(s) X^{(\mu)}(ds)$, the limit in probability of $\int_p^q f(s) X^{(\mu)}(ds)$ as $p \downarrow a$ and $q \uparrow b$. In this paper we carry out the study of improper stochastic integrals of this type. Examples such as $\int_{0+}^{\pi-} (\sin s)^{-1} X^{(\mu)}(ds)$ and $\int_{0+}^{\pi-} (\cot s) X^{(\mu)}(ds)$ are in our mind; for some μ these integrands extended to $[0,\infty)$ with the value 0 outside of $(0,\pi)$ are not locally $X^{(\mu)}$-integrable on $[0,\infty)$. The improper integrals $\int_{-\infty}^{t} e^{s-t} X^{(\mu)}(ds)$, $t \in \mathbb{R}$, for stationary Ornstein–Uhlenbeck type processes in Maejima and Sato (2003) and $\int_{-\infty}^{\infty} ((t-s)_+{}^{\alpha} - (-s)_+{}^{\alpha}) X^{(\mu)}(ds)$ $(0 < \alpha < 1/2,\ u_+ = u \vee 0)$, $t \in \mathbb{R}$, for fractional Lévy processes in Marquardt (2006) are also included in our framework.

In Sections 2–4 we study improper integrals $\int_{a+}^{b-} f(s) X^{(\mu)}(ds)$ in general. It is largely parallel to the treatment of $\int_0^{\infty-} f(s) X^{(\mu)}(ds)$. Three modifications, that is, compensated, essential, and symmetrized improper integrals, are defined in addition to the usual improper integrals. They induce transformations of infinitely divisible distributions defined by

$$\Phi_f(\mu) = \mathcal{L}\left(\int_{a+}^{b-} f(s) X^{(\mu)}(ds)\right),$$

$$\Phi_{f,\mathrm{c}}(\mu) = \text{the set of distributions of the compensated limits of } \int_p^q f(s) X^{(\mu)}(ds) \text{ as } p \downarrow a \text{ and } q \uparrow b,$$

$$\Phi_{f,\mathrm{es}}(\mu) = \text{the set of distributions of the essential limits of } \int_p^q f(s) X^{(\mu)}(ds) \text{ as } p \downarrow a \text{ and } q \uparrow b,$$

$$\Phi_{f,\mathrm{sym}}(\mu) = \mathcal{L}\left(\text{symmetrized limit of } \int_p^q f(s) X^{(\mu)}(ds) \text{ as } p \downarrow a \text{ and } q \uparrow b\right).$$

We describe the Lévy–Khintchine triplets of these images and the domains $\mathfrak{D}(\Phi_f)$, $\mathfrak{D}(\Phi_{f,\mathrm{c}})$, $\mathfrak{D}(\Phi_{f,\mathrm{es}})$, and $\mathfrak{D}(\Phi_{f,\mathrm{sym}})$. Some distributions μ in $\mathfrak{D}(\Phi_f)$ satisfy a condition called absolute definability of $\int_{a+}^{b-} f(s) X^{(\mu)}(ds)$. Let

$$\mathfrak{D}^0(\Phi_f) = \left\{\mu\colon \int_{a+}^{b-} f(s) X^{(\mu)}(ds) \text{ is absolutely definable}\right\}.$$

In general

$$\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_f) \subset \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \subset \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) = \mathfrak{D}(\Phi_{f,\,\mathrm{sym}}).$$

Among these, $\mathfrak{D}^0(\Phi_f)$ and $\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$ play especially important roles in further study. The relation of these definitions to the previous ones in Sato (2006a,b,c) will be given in Remarks 3.16 and 4.10.

Sections 5–9 deal with special problems. In Section 5 we introduce the concept of the dual in the class of purely non-Gaussian infinitely divisible distributions on $\mathbb{R}^d$, and study some improper stochastic integrals on a finite interval by transforming them to those on an infinite interval with respect to the Lévy process with the dual distribution at time 1. We call $\mu' \in ID(\mathbb{R}^d)$ with Lévy–Khintchine triplet $(0, \nu', \gamma')$ the dual of $\mu \in ID(\mathbb{R}^d)$ with $(0, \nu, \gamma)$ if

$$\nu'(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(\iota(x)) |x|^2 \nu(dx)$$

and $\gamma' = -\gamma$, where $\iota(x) = |x|^{-2}x$, the inversion of x.

In Section 6 we seek conditions in order that the domains are very large. Specifically, the condition for the domains being the whole class $ID(\mathbb{R}^d)$ is given. This amplifies some results in Barndorff-Nielsen and Pérez-Abreu (2005). Other conditions such as for $ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$ and for $ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}^0(\Phi_f)$ are given and the relations of those conditions are discussed (see below for the definition of $ID_{\mathrm{AB}}(\mathbb{R}^d)$). Conditions in order that the domains are very small are also considered.

For a real-valued measurable function f on (a, b) we introduce the measure τ given by

$$\tau(B) = \int_a^b 1_{\{f(s) \in B\}} ds,$$

and call it the τ-measure of f. In Section 7 we study whether τ determines $\mathfrak{D}(\Phi_f)$ and its variants and $\Phi_f(\mu)$ and its variants. Roughly speaking, the answer is yes for $\mathfrak{D}^0(\Phi_f)$ and $\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$, but no for $\mathfrak{D}(\Phi_f)$. Further, under some conditions including decrease of f, we address the problem whether τ determines f. The τ-measure is a development of ideas in Aoyama and Maejima (2007) and Barndorff-Nielsen and Thorbjørnsen (2006a,b).

In one dimension the class of infinitely divisible distributions concentrated on $[0, \infty)$ is important in theory and applications. Its multivariate analogue is given with $[0, \infty)$ replaced by a proper cone K in $\mathbb{R}^d$. Some results in Section 6 paraphrased for such distributions are given in Section 8.

In the transformation from $\mu \in ID(\mathbb{R}^d)$ to $\widetilde{\mu} = \Phi_f(\mu) \in ID(\mathbb{R}^d)$ their Lévy measures ν and $\widetilde{\nu}$ are related as

$$\widetilde{\nu}(B) = \int_a^b ds \int_{\mathbb{R}^d} 1_B(f(s)x)\, \nu(dx) = \int_{\mathbb{R} \setminus \{0\}} \nu\left(\frac{1}{u}B\right) \tau(du) \tag{1.1}$$

for Borel sets B in $\mathbb{R}^d \setminus \{0\}$, where τ is the τ-measure of f. Defining $\Psi_f(\nu)$ by $\Psi_f(\nu) = \widetilde{\nu}$, we study in Section 9 the relation between Ψ_f and $\Phi_{f,\,\mathrm{es}}$, and give counterparts of some results in Sections 6 and 8. The transformation of the form of the right extreme of (1.1) is introduced by Maejima and Rosiński (2002) for the standard Gaussian distribution τ and by Barndorff-Nielsen and Pérez-Abreu (2005, 2007) and Barndorff-Nielsen and Thorbjørnsen (2006a,b) for measures τ on $(0, \infty)$.

Let us prepare some general concepts used in this paper. Let $\widehat{\mu}(z)$, $z \in \mathbb{R}^d$, be the characteristic function of μ. Let $C_\mu(z)$ be the cumulant function of $\mu \in ID(\mathbb{R}^d)$. That is, $C_\mu(z)$ is the unique complex-valued continuous function on $\mathbb{R}^d$ satisfying $C_\mu(0) = 0$ and $\widehat{\mu}(z) = e^{C_\mu(z)}$. Sometimes we write $C_X(z) = C_\mu(z)$, using an $\mathbb{R}^d$-valued random variable X with $\mathcal{L}(X) = \mu$. We use the Lévy–Khintchine triplet (A, ν, γ) of μ in the form

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2} \right) \nu(dx) + i\langle \gamma, z\rangle, \tag{1.2}$$

where A is a $d \times d$ symmetric nonnegative-definite matrix, ν is a measure on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$, and γ is an element of $\mathbb{R}^d$. We have one-to-one correspondence between μ and (A, ν, γ). Let $\mu = \mu_{(A,\nu,\gamma)}$ denote the distribution corresponding to (A, ν, γ). The measure ν is called the Lévy measure of μ. The distribution $\mu \in ID(\mathbb{R}^d)$ with $A = 0$ is called purely non-Gaussian. Let $ID_0(\mathbb{R}^d)$ denote the class of purely non-Gaussian infinitely divisible distributions on $\mathbb{R}^d$. Following Sato (1999), we call $\mu = \mu_{(A,\nu,\gamma)}$

of type A if $A = 0$ and $\nu(\mathbb{R}^d) < \infty$,

of type B if $A = 0$, $\nu(\mathbb{R}^d) = \infty$, and $\displaystyle\int_{|x|\leqslant 1} |x|\nu(dx) < \infty$,

of type C if $A \neq 0$ or $\displaystyle\int_{|x|\leqslant 1} |x|\nu(dx) = \infty$.

The class of $\mu \in ID(\mathbb{R}^d)$ of type A or B is denoted by $ID_{\mathrm{AB}}(\mathbb{R}^d)$. Sometimes we omit $\mathbb{R}^d$ in the notation $ID(\mathbb{R}^d)$, $ID_0(\mathbb{R}^d)$, and $ID_{\mathrm{AB}}(\mathbb{R}^d)$.

If $\mu = \mu_{(A,\mu,\gamma)}$ in $ID(\mathbb{R}^d)$ satisfies $\int_{|x|\leqslant 1} |x|\nu(dx) < \infty$, then there is a unique $\gamma^0 \in \mathbb{R}^d$ such that

$$C_\mu(z) = -\frac{1}{2}\langle z, Az\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1)\, \nu(dx) + i\langle \gamma^0, a\rangle, \qquad z \in \mathbb{R}^d. \tag{1.3}$$

We express this fact by saying that μ has triplet $(A, \nu, \gamma^0)_0$ (see Remark 8.4 of Sato (1999)). This γ^0 is called the drift of μ.

Let $\mathrm{Lvm}(ID(\mathbb{R}^d))$ denote the class of Lévy measures of distributions in $ID(\mathbb{R}^d)$, that is, $\mathrm{Lvm}(ID(\mathbb{R}^d))$ is the class of measures ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty$. Similarly, let $\mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d))$ denote the class of Lévy measures of distributions in $ID_{\mathrm{AB}}(\mathbb{R}^d)$, that is, $\mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d))$ is the class of measures ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (|x| \wedge 1)\nu(dx) < \infty$.

Equalities among random variables are always understood to be almost surely. We use the words *decrease* and *increase* in the wide sense allowing flatness. When we say that a function is *real-valued* or $\mathbb{R}$-*valued*, the values ∞ and $-\infty$ are not allowed. The class of Borel sets in $\mathbb{R}^d$ is denoted by $\mathcal{B}(\mathbb{R}^d)$. The class of bounded Borel sets in $\mathbb{R}$ is denoted by $\mathcal{B}^0_{\mathbb{R}}$. The class of Borel sets B in (a, b) such that $\inf_{x\in B} x > a$ and $\sup_{x\in B} x < b$ is denoted by $\mathcal{B}^0_{(a,b)}$. We simply write (2006a), (2006b), and (2006c), indicating Sato's respective papers.

Inspired by some results in Barndorff-Nielsen and Pérez-Abreu (2005), the author sent three memos to a small circle in January 2005. Some theorems in Section 6 are developments of those memos. This paper has grown up from that part. The author thanks Makoto Maejima, Víctor Pérez-Abreu, and Ole Barndorff-Nielsen for having constant interest in this study and for informing him of their related results. He also thanks Jan Rosiński for giving him valuable remarks to the manuscript.

2. Stochastic integrals of nonrandom functions

First we give definition and existence of homogeneous independently scattered random measures.

Definition 2.1. A class $X = \{X(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ of $\mathbb{R}^d$-valued random variables is called an *independently scattered random measure* on $\mathbb{R}$ if (1) for any sequence $B_1, B_2, \dots$ of disjoint sets in $\mathcal{B}^0_{\mathbb{R}}$ with $\bigcup_{n=1}^{\infty} B_n \in \mathcal{B}^0_{\mathbb{R}}$, $\sum_{n=1}^{\infty} X(B_n)$ converges a. s. and equals $X(\bigcup_{n=1}^{\infty} B_n)$, (2) for any finite sequence $B_1, \dots, B_n$ of disjoint sets in $\mathcal{B}^0_{\mathbb{R}}$, $X(B_1), \dots, X(B_n)$ are independent, (3) $X(\{a\}) = 0$ for every $a \in \mathbb{R}$. It is called *homogeneous* if, in addition, $\mathcal{L}(X(B)) = \mathcal{L}(X(B + t))$ for all $B \in \mathcal{B}^0_{\mathbb{R}}$ and $t \in \mathbb{R}$. (It follows from (1) that $X(\emptyset) = 0$.)

If $X = \{X(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ is an $\mathbb{R}^d$-valued independently scattered random measure, then $\mathcal{L}(X(B)) \in ID(\mathbb{R}^d)$ for all $B \in \mathcal{B}^0_{\mathbb{R}}$.

Proposition 2.2. *For any $\mu \in ID(\mathbb{R}^d)$ there exists a unique (in law) $\mathbb{R}^d$-valued homogeneous independently scattered random measure $X = \{X(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ on $\mathbb{R}$ such that $\mathcal{L}(X((t, t + 1])) = \mu$ for all $t \in \mathbb{R}$.*

See Rajput and Rosinski (1989); Sato (2004) or Maejima and Sato (2003) for the proof.

Fix $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$ and let $X^{(\mu)} = \{X^{(\mu)}(B) \colon B \in \mathcal{B}^0_{\mathbb{R}}\}$ be an $\mathbb{R}^d$-valued homogeneous independently scattered random measure such that $\mathcal{L}(X((t, t+1])) = \mu$. Let (a, b) be an open interval with $-\infty \leqslant a < b \leqslant \infty$. The following definition of integrals with respect to $X^{(\mu)}$ is similar to that in Urbanik and Woyczyński (1967); Rajput and Rosinski (1989); Kwapień and Woyczyński (1992) and Sato (2004, 2006a).

Definition 2.3. Call $f(s)$ a *simple function on* (a, b), if $f(s) = \sum_{j=1}^{n} r_j 1_{B_j}(s)$ for some n, where $B_1, \dots, B_n$ are disjoint Borel sets in (a, b) and $r_1, \dots, r_n \in \mathbb{R}$. For a simple function $f(s)$ of this form, define

$$\int_B f(s) X^{(\mu)}(ds) = \sum_{j=1}^{n} r_j X^{(\mu)}(B \cap B_j)$$

for $B \in \mathcal{B}^0_{(a,b)}$. An $\mathbb{R}$-valued measurable function $f(s)$ on (a, b) is called *locally $X^{(\mu)}$-integrable on* (a, b), if there is a sequence of simple functions f_n, $n = 1, 2, \dots$, on (a, b) such that $f_n(s) \to f(s)$ Lebesgue almost everywhere on (a, b) as $n \to \infty$ and that, for every $B \in \mathcal{B}^0_{(a,b)}$, the sequence $\int_B f_n(s) X^{(\mu)}(ds)$ converges in probability as $n \to \infty$. The limit is denoted by $\int_B f(s) X^{(\mu)}(ds)$.

Using the Nikodým theorem, we can prove that if $f(s)$ is locally $X^{(\mu)}$-integrable on (a, b), then, for every $B \in \mathcal{B}^0_{(a,b)}$, $\int_B f(s) X^{(\mu)}(ds)$ does not depend on the choice of the sequence of simple functions satisfying the conditions above. If $f(s)$ is locally $X^{(\mu)}$-integrable on (a, b), then, for p and q satisfying $a < p < q < b$, $\int_{(p,q]} f(s) X^{(\mu)}(ds)$, $\int_{[p,q)} f(s) X^{(\mu)}(ds)$, $\int_{(p,q)} f(s) X^{(\mu)}(ds)$, and $\int_{[p,q]} f(s) X^{(\mu)}(ds)$ are identical almost surely; they are denoted by $\int_p^q f(s) X^{(\mu)}(ds)$.

Definition 2.4. Let $\mathbf{L}_{(a,b)}(X^{(\mu)})$ denote the class of locally $X^{(\mu)}$-integrable functions on (a,b).

Proposition 2.5. *If $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$, then*

$$\int_q^p |C_\mu(f(s)z)|ds < \infty, \quad z \in \mathbb{R}^d \qquad \text{for all } p,q \text{ with } a<p<q<b \tag{2.1}$$

and

$$C_{\int_B f(s)X^{(\mu)}(ds)}(z) = \int_B C_\mu(f(s)z)ds, \quad z \in \mathbb{R}^d \qquad \text{for } B \in \mathcal{B}^0_{(a,b)}. \tag{2.2}$$

See Proposition 2.17 of Sato (2004).

Theorem 2.6. *Let $f(s)$ be an $\mathbb{R}$-valued measurable function on (a,b).*

(i) *Suppose that $A \neq 0$. Then $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ if and only if*

$$\int_p^q f(s)^2 ds < \infty \qquad \text{for all } p,q \text{ with } a<p<q<b. \tag{2.3}$$

(ii) *Suppose that $A = 0$. Then $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ if and only if*

$$\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\,\nu(dx) < \infty \quad \text{for all } p,q \text{ with } a<p<q<b, \tag{2.4}$$

$$\begin{gathered}\int_p^q \left| f(s)\gamma + \int_{\mathbb{R}^d} f(s)x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| ds < \infty \\ \text{for all } p,q \text{ with } a<p<q<b. \end{gathered} \tag{2.5}$$

Proof. Let

$$\begin{aligned}\varphi(s,u) = u^2 \operatorname{tr} A &+ \int_{\mathbb{R}^d} (|ux|^2 \wedge 1)\nu(dx) \\ &+ \left| u\gamma + u\int_{\mathbb{R}^d} x \left(\frac{1}{1+|ux|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right|. \end{aligned} \tag{2.6}$$

Then $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ if and only if

$$\int_p^q \varphi(s, f(s))ds < \infty \qquad \text{for all } p,q \text{ with } a<p<q<b. \tag{2.7}$$

See (2006a) for the proof. Property (2.7) is equivalent to saying that

$$\int_p^q f(s)^2 \operatorname{tr} A\, ds < \infty \qquad \text{for all } p,q \text{ with } a<p<q<b \tag{2.8}$$

together with (2.4) and (2.5).

(i) Suppose that $A \neq 0$. Since $\operatorname{tr} A > 0$, (2.8) is equivalent to (2.3). Since

$$\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\,\nu(dx) \leqslant \int_p^q (f(s)^2+1)ds \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\,\nu(dx),$$

(2.4) follows from (2.3). Further,

$$\int_p^q |f(s)|ds \leqslant \left((q-p)\int_p^q |f(s)|^2 ds \right)^{1/2} < \infty.$$

Thus, (2.5) also follows from (2.3), since we have

$$\begin{aligned}\int_p^q ds \left| \int_{\mathbb{R}^d} f(s)x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| \\ \leqslant \int_p^q ds \left| \int_{\mathbb{R}^d} \frac{f(s)x(|x|^2+|f(s)x|^2)\nu(dx)}{(1+|f(s)x|^2)(1+|x|^2)} \right| \leqslant \int_p^q \frac{1+f(s)^2}{2} ds \int_{\mathbb{R}^d} \frac{|x|^2\nu(dx)}{1+|x|^2} \\ < \infty \quad \text{for all } p,q \text{ with } a<p<q<b,\end{aligned}$$

using $r/(1+r^2) \leqslant 1/2$ for all $r \geqslant 0$.

(ii) Suppose that $A = 0$. If $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$, then (2.4) and (2.5) hold, as we have seen above. Conversely, assume that (2.4) and (2.5) hold. Then (2.8) is satisfied since $\operatorname{tr} A = 0$. Hence $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. □

Remark 2.7. Suppose that $A = 0$. Then $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ if and only if (2.4) holds and

$$\int_p^q |f(s)| \left| \gamma - \int_{\mathbb{R}^d} \frac{x|f(s)x|^2\nu(dx)}{(1+|f(s)x|^2)(1+|x|^2)} \right| ds < \infty \qquad \text{for all } p,q \text{ with } a<p<q<b. \tag{2.9}$$

Indeed, since

$$\int_p^q ds \left| \int_{\mathbb{R}^d} \frac{f(s)x\,|x|^2}{(1+|f(s)x|^2)(1+|x|^2)} \nu(dx) \right| \leqslant \frac{1}{2} \int_p^q ds \int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2} \nu(dx) < \infty,$$

(2.5) is rewritten into (2.9).

Remark 2.8. In all cases (irrespective of whether $A = 0$ or not) condition (2.3), that is, local square-integrability on (a,b), is sufficient for $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. See the proof of (i) of Theorem 2.6.

Proposition 2.9. *Suppose that $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. Let $(A_p^q, \nu_p^q, \gamma_p^q)$ be the triplet of $\int_p^q f(s)X^{(\mu)}(ds)$ for $a<p<q<b$. Then*

$$A_p^q = \int_p^q f(s)^2 A ds, \tag{2.10}$$

$$\nu_p^q(B) = \int_p^q ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d) \text{ with } 0 \notin B \tag{2.11}$$

$$\gamma_p^q = \int_p^q ds \left(f(s)\gamma + \int_{\mathbb{R}^d} f(s)x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right). \tag{2.12}$$

See Corollary 2.19 of (2006a).

Let us consider the case of ID_{AB}.

Theorem 2.10. *Suppose that $\mu \in ID_{\mathrm{AB}}(\mathbb{R}^d)$. Let $f(s)$ be an $\mathbb{R}$-valued measurable function on (a,b). Then the following statements are equivalent.*

(a) *$f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ and*

$$\mathcal{L}\left(\int_p^q f(s)X^{(\mu)}(ds) \right) \in ID_{\mathrm{AB}}(\mathbb{R}^d) \quad \text{for all } p,q \text{ with } a<p<q<b. \tag{2.13}$$

(b) *The Lévy measure ν and the drift γ^0 of μ satisfy*

$$\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x| \wedge 1)\,\nu(dx) < \infty \quad \text{for all } p,q \text{ with } a<p<q<b, \tag{2.14}$$

$$\int_p^q |f(s)\gamma^0| ds < \infty \quad \textit{for all } p,q \textit{ with } a<p<q<b. \tag{2.15}$$

Proof. Assume (a). We use Propositions 2.5 and 2.9. The triplet $(A_p^q, \nu_p^q, \gamma_p^q)$ of $\int_p^q f(s)X^{(\mu)}(ds)$ satisfies $A_p^q = 0$ and $\int_{\mathbb{R}^d}(|x|\wedge 1)\nu_p^q(dx) < \infty$. We have (2.14) since

$$\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|\wedge 1)\,\nu(dx) = \int_{\mathbb{R}^d} (|x|\wedge 1)\,\nu_p^q(dx)$$

from (2.11). We have

$$\infty > \int_p^q |C_\mu(f(s)z)| ds = \int_p^q ds \left| \int_{\mathbb{R}^d} (e^{i\langle f(s)z,x\rangle} - 1)\,\nu(dx) + \langle f(s)z, \gamma^0\rangle \right|$$

and

$$\int_p^q ds \int_{\mathbb{R}^d} |e^{i\langle f(s)z,x\rangle} - 1|\,\nu(dx) \leqslant I_1 + I_2,$$

where

$$\begin{aligned} I_1 &= \int_p^q ds \int_{\mathbb{R}^d} |\langle f(s)x, z\rangle| 1_{\{|f(s)x|\leqslant 1\}}\nu(dx) \\ &\leqslant |z| \int_p^q ds \int_{\mathbb{R}^d} |f(s)x| 1_{\{|f(s)x|\leqslant 1\}}\nu(dx) \leqslant |z| \int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|\wedge 1)\,\nu(dx) < \infty \end{aligned}$$

and

$$I_2 = 2\int_p^q ds \int_{\mathbb{R}^d} 1_{\{|f(s)x|>1\}}\nu(dx) \leqslant 2\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|\wedge 1)\,\nu(dx) < \infty.$$

Therefore

$$\int_p^q |\langle f(s)\gamma^0, z\rangle| ds < \infty \quad \text{for all } z.$$

Choosing $z = (\delta_{jk})_{1\leqslant k\leqslant d}$, we see that $\gamma^0 = (\gamma_j^0)_{1\leqslant j\leqslant d}$ satisfies $\int_p^q |f(s)\gamma_j^0| ds < \infty$. Hence (2.15) is satisfied. Thus (b) is obtained. Note that $\int_p^q f(s)X^{(\mu)}(ds)$ has drift

$$(\gamma^0)_p^q = \int_p^q f(s)\gamma^0 ds. \tag{2.16}$$

Conversely assume (b). We have

$$\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|^2\wedge 1)\,\nu(dx) \leqslant \int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|\wedge 1)\,\nu(dx) < \infty.$$

Since $\gamma^0 = \gamma - \int_{\mathbb{R}^d} x(1+|x|^2)^{-1}\nu(dx)$, we have

$$\begin{aligned} &\int_p^q ds \left| f(s)\gamma + \int_{\mathbb{R}^d} f(s)x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| \\ &= \int_p^q ds \left| f(s)\gamma^0 + \int_{\mathbb{R}^d} \frac{f(s)x}{1+|f(s)x|^2}\nu(dx) \right|, \end{aligned}$$

which is finite. Indeed, we have (2.15) and, using $(1+r^2)^{-1} \leqslant 2(1+r)^{-1}$ for $r \geqslant 0$, we have

$$\begin{aligned} \int_p^q ds \int_{\mathbb{R}^d} \frac{|f(s)x|}{1+|f(s)x|^2}\nu(dx) &\leqslant 2\int_p^q ds \int_{\mathbb{R}^d} \frac{|f(s)x|}{1+|f(s)x|}\nu(dx) \\ &\leqslant 2\int_p^q ds \int_{\mathbb{R}^d} (|f(s)x|\wedge 1)\nu(dx) < \infty. \end{aligned}$$

It follows from Theorem 2.6 that $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. The triplet $(A_p^q, \nu_p^q, \gamma_p^q)$ of $\int_p^q f(s)X^{(\mu)}(ds)$ satisfies $A_p^q = 0$ and (2.11). Thus

$$\int_{\mathbb{R}^d} (|x| \wedge 1)\, \nu_p^q(dx) = \int_p^q ds \int_{\mathbb{R}^d} (|f(s)x| \wedge 1)\, \nu(dx) < \infty.$$

Hence we obtain (2.13). □

3. Improper stochastic integrals on (a, b)

Fix $-\infty \leqslant a < b \leqslant \infty$. Let us define improper stochastic integrals on (a, b) with nonrandom integrands and their modifications. For $(a, b) = (0, \infty)$ Cherny and Shiryaev (2005) study stochastic integrals up to infinity (with random integrands in general) in semi-martingale approach, but we are treating a simpler situation without using semi-martingales. Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$ and let $X^{(\mu)}$ be as in Section 2. *In this section throughout, we assume that* $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$.

Definition 3.1. We say that *the improper stochastic integral on* (a, b) *of* f *with respect to* $X^{(\mu)}$ *is definable* if $\int_p^q f(s)X^{(\mu)}(ds)$ is convergent in probability in $\mathbb{R}^d$ as $p \downarrow a$ and $q \uparrow b$. The limit is written as $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$ and its distribution is written as $\Phi_f(\mu)$.

Definition 3.2. We say that *the essential improper integral on* (a, b) *of* f *with respect to* $X^{(\mu)}$ *is definable* if there is a nonrandom $\mathbb{R}^d$-valued function $g(p, q)$, $a < p < q < b$ such that $\int_p^q f(s)X^{(\mu)}(ds) - g(p, q)$ is convergent in probability in $\mathbb{R}^d$ as $p \downarrow a$ and $q \uparrow b$. Notice that there is a freedom of choice of $g(p, q)$. Let $\Phi_{f,\,\mathrm{es}}(\mu)$ denote the class of the distributions of all such limits.

Definition 3.3. We say that *the compensated improper integral on* (a, b) *of* f *with respect to* $X^{(\mu)}$ *is definable* if there is $\theta \in \mathbb{R}^d$ such that $\int_{a+}^{b-} f(s)X^{(\mu*\delta_{-\theta})}(ds)$ is definable. Here $\delta_{-\theta}$ is the distribution concentrated at $-\theta$. As there may be a freedom of choice of θ, let $\Phi_{f,\,\mathrm{c}}(\mu)$ denote the class of the distributions of all such limits.

Definition 3.4. Let $X^{(\mu)\sharp}$ be an independent copy of $X^{(\mu)}$. We say that *the symmetrized improper integral on* (a, b) *of* f *with respect to* $X^{(\mu)}$ *is definable* if $\int_{a+}^{b-} f(s)(X^{(\mu)}(ds) - X^{(\mu)\sharp}(ds))$ is definable. Let $\Phi_{f,\,\mathrm{sym}}(\mu)$ denote the distribution of the limit.

Note that $\Phi_f(\mu)$ and $\Phi_{f,\,\mathrm{sym}}(\mu)$ are elements of $ID(\mathbb{R}^d)$, while $\Phi_{f,\,\mathrm{es}}(\mu)$ and $\Phi_{f,\,\mathrm{c}}(\mu)$ are subsets of $ID(\mathbb{R}^d)$. *Thus we consider* Φ_f *and* $\Phi_{f,\,\mathrm{sym}}$ *as transformations of* $\mu \in ID(\mathbb{R}^d)$ *into* $ID(\mathbb{R}^d)$, *and* $\Phi_{f,\,\mathrm{es}}$ *and* $\Phi_{f,\,\mathrm{c}}$ *as transformations of* $\mu \in ID(\mathbb{R}^d)$ *with values being subsets of* $ID(\mathbb{R}^d)$.

Sometimes we say that $\Phi_f(\mu)$ is definable if $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$ is definable. We say that $\Phi_{f,\,\mathrm{es}}(\mu)$ [resp. $\Phi_{f,\,\mathrm{c}}(\mu)$, $\Phi_{f,\,\mathrm{sym}}(\mu)$] is definable if the essential [resp. compensated, symmetrized] improper integral on (a, b) of f with respect to $X^{(\mu)}$ is definable.

Theorem 3.5. *The following three statements are equivalent.*

(a) $\Phi_f(\mu)$ *is definable.*

(b) *For each* $z \in \mathbb{R}^d$, $\int_p^q C_\mu(f(s)z)ds$ *is convergent in* $\mathbb{C}$ *as* $p \downarrow a$ *and* $q \uparrow b$.

(c) *The triplet* (A,ν,γ) *satisfies the following:*

$$\int_a^b f(s)^2 \mathrm{tr}\, A\, ds < \infty, \tag{3.1}$$

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty, \tag{3.2}$$

$$\gamma_p^q \text{ is convergent in } \mathbb{R}^d \text{ as } p \downarrow a \text{ and } q \uparrow b. \tag{3.3}$$

Proof. Similar to Proposition 5.5 of (2006a) and Propositions 2.2 and 2.6 of (2006c). Here the equivalence of statement (c) is based on an analogue of Lemma 5.4 of (2006a). □

Theorem 3.6. $\Phi_{f,\mathrm{es}}(\mu)$ *is definable if and only if* (3.1) *and* (3.2) *hold.*

Proof. Similar to Proposition 5.6 of (2006a). □

Theorem 3.7. $\Phi_{f,\mathrm{c}}(\mu)$ *is definable if and only if* (3.1), (3.2), *and*

$$\begin{gathered}\int_p^q f(s)\left(\gamma^\sharp + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right) ds \\ \text{is convergent in } \mathbb{R}^d \text{ with some } \gamma^\sharp \in \mathbb{R}^d \text{ as } p \downarrow a \text{ and } q \uparrow b.\end{gathered} \tag{3.4}$$

Proof. This theorem follows from Definition 3.3 and Theorem 3.5. □

Corollary 3.8. *Suppose that* $\int_p^q f(s)ds$ *is convergent as* $p \downarrow a$ *and* $q \uparrow b$. *Then,* $\Phi_{f,\mathrm{c}}(\mu)$ *is definable if and only if* $\Phi_f(\mu)$ *is definable.*

This follows from Theorems 3.5 and 3.7.

Theorem 3.9. $\Phi_{f,\mathrm{sym}}(\mu)$ *is definable if and only if* (3.1) *and* (3.2) *hold.*

Proof. The law of $\int_p^q f(s)(X^{(\mu)}(ds) - X^{(\mu)\sharp}(ds))$ has triplet $(2A_p^q, (\nu_p^q)_{\mathrm{sym}}, 0)$, where

$$(\nu_p^q)_{\mathrm{sym}}\,(B) = \nu_p^q(B) + \nu_p^q(-B). \tag{3.5}$$

Hence the condition for definability of $\Phi_{f,\mathrm{sym}}(\mu)$ is the same as (3.1) and (3.2). □

Theorem 3.10. *If* $\Phi_f(\mu)$ *is definable, then* $\Phi_f(\mu)$ *has triplet* $(A_a^b, \nu_a^b, \gamma_{a+}^{b-})$ *given by*

$$A_a^b = \int_a^b f(s)^2 A ds, \tag{3.6}$$

$$\nu_a^b(B) = \int_a^b ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d) \text{ with } 0 \notin B \tag{3.7}$$

$$\gamma_{a+}^{b-} = \lim_{p\downarrow a,\, q\uparrow b} \int_p^q ds \left(f(s)\gamma + \int_{\mathbb{R}^d} f(s)x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right). \tag{3.8}$$

Proof. Similar to Proposition 5.5 of (2006a) and Proposition 2.6 of (2006c). □

Theorem 3.11. *If* $\Phi_{f,\mathrm{es}}(\mu)$ *is definable, then* $\Phi_{f,\mathrm{es}}(\mu)$ *is the class of all infinitely divisible distributions* $\mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})}$ *on* $\mathbb{R}^d$ *such that* $\widetilde{A}$ *is* A_a^b *of* (3.6), $\widetilde{\nu}$ *is* ν_a^b *of* (3.7), *and* $\widetilde{\gamma} \in \mathbb{R}^d$.

Proof. Obvious from Definition 3.2 and Theorem 3.10. □

We write the limit of $\int_p^q f(s)ds$ as $p \downarrow a$ and $q \uparrow b$ as $\int_{a+}^{b-} f(s)ds$.

Theorem 3.12. *Suppose that $\Phi_{f,\mathrm{c}}(\mu)$ is definable and that $f(s)$ is locally integrable on (a,b).*

(i) *If $\int_p^q f(s)ds$ converges to a nonzero real number as $p \downarrow a$ and $q \uparrow b$, then $\Phi_{f,\mathrm{c}}(\mu)$ is not a singleton, and $\Phi_{f,\mathrm{c}}(\mu) = \Phi_{f,\mathrm{es}}(\mu)$.*

(ii) *Assume that one of the following two conditions is satisfied:*

(a) *$\int_p^q f(s)ds$ converges to zero as $p \downarrow a$ and $q \uparrow b$,*

(b) *$\int_p^q f(s)ds$ is not convergent as $p \downarrow a$ and $q \uparrow b$.*

Then $\Phi_{f,\mathrm{c}}(\mu)$ consists of a single distribution $\mu_{(\widetilde{A},\widetilde{\nu},\widetilde{\gamma})} \in ID(\mathbb{R}^d)$, where $\widetilde{A}$ is A_a^b of (3.6) and $\widetilde{\nu}$ is ν_a^b of (3.7).

Proof. (i) Suppose that $\int_p^q f(s)ds$ is convergent as $p \downarrow a$ and $q \uparrow b$ and $\int_{a+}^{b-} f(s)ds \neq 0$. Since $\Phi_{f,\mathrm{c}}(\mu)$ is definable, it follows from Theorem 3.7 that (3.1), (3.2), and (3.4) hold. For any $\theta \in \mathbb{R}^d$,

$$\int_p^q f(s)\left(\gamma - \theta + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds$$

tends to

$$\int_{a+}^{b-} f(s)\left(\gamma^\sharp + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds + \int_{a+}^{b-} f(s)ds(\gamma - \theta - \gamma^\sharp),$$

where $\gamma^\sharp$ is that of (3.4). Hence $\Phi_{f,\mathrm{c}}(\mu) = \Phi_{f,\mathrm{es}}(\mu)$.

(ii) If (a) is satisfied, then it follows from (3.4) that

$$\int_p^q f(s)\left(\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds$$

is convergent as $p \downarrow a$ and $q \uparrow b$ and, for any $\theta \in \mathbb{R}^d$, $\int_{a+}^{b-} f(s)X^{(\mu*\delta_{-\theta})}(ds)$ is definable and does not depend on θ. If condition (b) is satisfied, there is only one $\theta \in \mathbb{R}^d$ such that $\int_{a+}^{b-} f(s)X^{(\mu*\delta_{-\theta})}(ds)$ is definable; indeed, if it is definable for θ and also for some $\theta' \neq \theta$ in place of θ, then

$$\begin{aligned}\int_p^q & f(s)ds(\theta' - \theta) \\ &= \int_p^q f(s)\left(\gamma - \theta + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds \\ &\quad - \int_p^q f(s)\left(\gamma - \theta' + \int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right)ds,\end{aligned}$$

which is convergent as $p \downarrow a$ and $q \uparrow b$, a contradiction. □

Theorem 3.13. *Suppose that $\Phi_{f,\mathrm{c}}(\mu)$ is definable and that $f(s)$ is locally integrable on (a,b). Suppose, further, that $\Phi_{f,\mathrm{c}}(\mu)$ is a singleton $\{\widetilde{\mu}\}$. If $\int_{\mathbb{R}^d} |x|\widetilde{\mu}(dx) < \infty$, then $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$.*

Proof. We may assume that $f(s)$ is not identically zero. Suppose $\int_{\mathbb{R}^d} |x|\widetilde{\mu}(dx) < \infty$. Then $\int_{|x|>1} |x|\nu_a^b(dx) < \infty$, and hence $\int_{|x|>1} |x|\nu_p^q(dx) < \infty$. We also have $\int_{|x|>1} |x|\nu(dx) < \infty$, since, for any $a > 0$,

$$\int_{|x|>1} |x|\nu_p^q(dx) = \int_p^q ds \int_{|f(s)x|>1} |f(s)x|\nu(dx)$$

$$\geqslant \int_{|f(s)|>a} |f(s)|ds \int_{|x|>1/a} |x|\nu(dx).$$

Using θ such that $\int_{a+}^{b-} f(s)X^{(\mu*\delta_{-\theta})}(ds)$ is definable, we have

$$\begin{aligned}\int_p^q & C_{\mu*\delta_{-\theta}}(f(s)z)ds \\ &= \int_p^q \left[-\frac{1}{2}\langle z, f(s)^2Az\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,f(s)x\rangle} - 1 - i\langle z, f(s)x\rangle)\nu(dx)\right] ds \\ &\quad + i\int_p^q f(s)ds\left[\int_{\mathbb{R}^d}\langle z,x\rangle(1-(1+|x|^2)^{-1})\nu(dx) + \langle \gamma-\theta, z\rangle\right].\end{aligned}$$

As $p \downarrow a$ and $q \uparrow b$, the left-hand side and the first term of the right-hand side are convergent. Hence the second term of the right-hand side is also convergent, but the limit must be zero, as condition (a) or (b) of Theorem 3.12 is satisfied. Therefore

$$C_{\widetilde{\mu}}(z) = -\frac{1}{2}\langle z, A_a^b z\rangle + \int_{\mathbb{R}^d}(e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_a^b(dx),$$

which shows that $\int_{\mathbb{R}^d} x\widetilde{\mu}(dx) = 0$. □

Even if we assume that $f(s)$ is locally integrable on (a,b), that $\Phi_{f,\,\mathrm{c}}(\mu)$ is a singleton $\{\widetilde{\mu}\}$, and that $\int_{\mathbb{R}^d}|x|\mu(dx)$ is finite, these assumptions do not imply finiteness of $\int_{\mathbb{R}^d}|x|\widetilde{\mu}(dx)$. Examples for this fact are given in pp. 36–37 of (2006b).

Theorem 3.14. *If $\Phi_{f,\,\mathrm{sym}}(\mu)$ is definable, then $\Phi_{f,\,\mathrm{sym}}(\mu)$ has triplet $(2A_{a+}^{b-}, (\nu_{a+}^{b-})_{\mathrm{sym}}, 0)$, where A_{a+}^{b-} is given by* (3.6) *and*

$$(\nu_{a+}^{b-})_{\mathrm{sym}}(B) = \nu_{a+}^{b-}(B) + \nu_{a+}^{b-}(-B) \tag{3.9}$$

with ν_{a+}^{b-} given by (3.7).

Proof. This follows from Theorem 3.9 and its proof. □

The following result will be useful later.

Theorem 3.15. *Suppose that $\mu \in ID_{\mathrm{AB}}(\mathbb{R}^d)$. Then the following two statements* (a) *and* (b) *are equivalent.*

(a) *$\Phi_f(\mu)$ is definable and $\Phi_f(\mu) \in ID_{\mathrm{AB}}(\mathbb{R}^d)$.*

(b) *The triplet $(0,\nu,\gamma^0)_0$ of μ satisfies*

$$\int_a^b ds \int_{\mathbb{R}^d}(|f(s)x| \wedge 1)\,\nu(dx) < \infty \tag{3.10}$$

and

$$\int_p^q f(s)\gamma^0 ds \text{ is convergent as } p \downarrow a \text{ and } q \uparrow b. \tag{3.11}$$

If statements (a) *and* (b) *are true, then $\widetilde{\mu} = \Phi_f(\mu)$ has triplet $(0,\widetilde{\nu},\widetilde{\gamma}^0)_0$, where $\widetilde{\nu}$ is ν_a^b of* (3.7) *and*

$$\widetilde{\gamma}^0 = \int_{a+}^{b-} f(s)\gamma^0 ds. \tag{3.12}$$

Proof. Assume (b). Let us show (a). Condition (3.11) means that either $\gamma^0 = 0$ or $\int_p^q f(s)ds$ is convergent as $p \downarrow a$ and $q \uparrow b$. We make an argument similar to that in the proof that (b) implies (a) in Theorem 2.10. Thus we have (3.2) and (3.3), observing that

$$\gamma_p^q = \int_p^q ds \left(f(s)\gamma^0 + \int_{\mathbb{R}^d} \frac{f(s)x}{1+|f(s)x|^2}\nu(dx) \right)$$

with

$$\int_a^b ds \int_{\mathbb{R}^d} \frac{|f(s)x|}{1+|f(s)x|^2}\nu(dx) < \infty.$$

It follows from Theorem 3.5 that $\Phi_f(\mu)$ is definable. Let $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$ be the triplet of $\widetilde{\mu} = \Phi_f(\mu)$. Then $\widetilde{A} = 0$ and $\widetilde{\nu}$ is ν_a^b of (3.7). Hence we have $\int_{\mathbb{R}^d} (|x| \wedge 1)\widetilde{\nu}(dx) < \infty$ from (3.10), that is, $\widetilde{\mu} \in ID_{\mathrm{AB}}$. We have

$$\begin{aligned} C_{\widetilde{\mu}}(z) &= \lim_{p\downarrow a,\, q\uparrow b} C_{\int_p^q f(s)X^{(\mu)}(ds)}(z) = \lim_{p\downarrow a,\, q\uparrow b} \int_p^q C_\mu(f(s)z)ds \\ &= \lim_{p\downarrow a,\, q\uparrow b} \int_p^q ds \left(\int_{\mathbb{R}^d} (e^{i\langle f(s)z, x\rangle} - 1)\, \nu(dx) + i\langle \gamma^0, f(s)z\rangle \right). \end{aligned} \tag{3.13}$$

Thus

$$C_{\widetilde{\mu}}(z) = \int_a^b ds \int_{\mathbb{R}^d} (e^{i\langle z, f(s)x\rangle} - 1)\, \nu(dx) + i \left\langle \int_{a+}^{b-} f(s)\gamma^0 ds, z \right\rangle,$$

since condition (3.10) allows us to use the dominated convergence theorem. Hence we obtain (3.12).

Assume (a). Let us show (b). Let $\widetilde{\mu} = \Phi_f(\mu)$ with triplet $(\widetilde{A}, \widetilde{\nu}, \widetilde{\gamma})$. Since $\widetilde{\nu}$ is ν_a^b of (3.7), we obtain (3.10) from $\int_{\mathbb{R}^d} (|x| \wedge 1)\widetilde{\nu}(dx) < \infty$. We have (3.13) and

$$\lim_{p\downarrow a,\, q\uparrow b} \int_p^q ds \int_{\mathbb{R}^d} (e^{i\langle f(s)z, x\rangle} - 1)\, \nu(dx)$$

exists in $\mathbb{R}^d$. Hence $\lim_{p\downarrow a,\, q\uparrow b} \int_p^q \langle f(s)\gamma^0, z\rangle ds$ exists in $\mathbb{R}^d$. It follows that $\lim_{p\downarrow a,\, q\uparrow b} \int_p^q f(s)\gamma_j^0 ds$ exists in $\mathbb{R}$ for $j = 1, \ldots, d$. Hence we get (3.11). The last part of the theorem is obtained in the course of our discussion. □

Remark 3.16. We are considering Φ_f, $\Phi_{f,\mathrm{c}}$, $\Phi_{f,\mathrm{es}}$, and $\Phi_{f,\mathrm{sym}}$ as two-sided improper integrals (that is, $p \downarrow a$ and $q \uparrow b$). We can reduce Φ_f, $\Phi_{f,\mathrm{es}}$, and $\Phi_{f,\mathrm{sym}}$ to one-sided improper integrals, but we cannot always reduce $\Phi_{f,\mathrm{c}}$ to one-sided improper integrals.

Fix $c \in (a,b)$. Let $\mu \in ID(\mathbb{R}^d)$ and $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. Then it is not hard to show the following (i)–(iii).

(i) $\Phi_f(\mu)$ is definable if and only if $\int_{a+}^c f(s)X^{(\mu)}(ds)$ and $\int_c^{b-} f(s)X^{(\mu)}(ds)$ are definable, that is, $\int_p^c f(s)X^{(\mu)}(ds)$ and $\int_c^q f(s)X^{(\mu)}(ds)$ are convergent in probability as $p \downarrow a$ and $q \uparrow b$, respectively.

(ii) $\Phi_{f,\mathrm{es}}(\mu)$ is definable if and only if there are nonrandom $\mathbb{R}^d$-valued functions $g(p)$, $p \in (a,c)$, and $h(q)$, $q \in (c,b)$, such that $\int_p^c f(s)X^{(\mu)}(ds) - g(p)$ and $\int_c^q f(s)X^{(\mu)}(ds) - h(q)$ are convergent in probability as $p \downarrow a$ and $q \uparrow b$, respectively.

(iii) Let $X^{(\mu)\sharp}(\mu)$ be an independent copy of $X^{(\mu)}$. $\Phi_{f,\,\mathrm{sym}}(\mu)$ is definable if and only if $\int_{a+}^{c} f(s)(X^{(\mu)}(ds)-X^{(\mu)\sharp}(ds))$ and $\int_{c}^{b-} f(s)(X^{(\mu)}(ds)-X^{(\mu)\sharp}(ds))$ are definable.

(iv) Is it true that $\Phi_{f,\,\mathrm{c}}(\mu)$ is definable if and only if there are θ and θ' in $\mathbb{R}^d$ such that $\int_{a+}^{c} f(s)X^{(\mu*\delta_{-\theta})}(ds)$ and $\int_{c}^{b-} f(s)X^{(\mu*\delta_{-\theta'})}(ds)$ are definable? The answer is affirmative if $\int_p^c f(s)ds$ is convergent as $p\downarrow a$ or if $\int_c^q f(s)ds$ is convergent as $q\uparrow b$. However, the answer is negative in general. For a counter-example, let $(a,b)=(0,\infty)$, $f(s)=s^{-1}$, and $\mu=\mu_{(0,\nu,\gamma)}$ such that $\int_{|x|<\varepsilon}\nu(dx)=\int_{|x|>1/\varepsilon}\nu(dx)=0$ for some $\varepsilon\in(0,1)$ and $\int_{\mathbb{R}^d}x\nu(dx)\neq 0$; use Example 4.5 and Proposition 5.3 in the later sections; notice that then $\Phi_{f,\,\mathrm{c}}(\mu)$ is not definable since $\int x(1+|x|^2)^{-1}\nu(dx)\neq -\int x|x|^2(1+|x|^2)^{-1}\nu(dx)$.

4. Domains of Φ_f, $\Phi_{f,\,\mathrm{es}}$, $\Phi_{f,\,\mathrm{c}}$, and $\Phi_{f,\,\mathrm{sym}}$ and domain of absolute definability

We continue to fix $-\infty\leqslant a<b\leqslant\infty$ and the dimension d. Let f be an $\mathbb{R}$-valued measurable function on (a,b). Let $\mathfrak{D}(\Phi_f)$ [resp. $\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$, $\mathfrak{D}(\Phi_{f,\,\mathrm{c}})$, $\mathfrak{D}(\Phi_{f,\,\mathrm{sym}})$] denote the class of $\mu\in ID(\mathbb{R}^d)$ such that $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$ and $\Phi_f(\mu)$ [resp. $\Phi_{f,\,\mathrm{es}}(\mu)$, $\Phi_{f,\,\mathrm{c}}(\mu)$, $\Phi_{f,\,\mathrm{sym}}(\mu)$] is definable. Further we introduce the following notion.

Definition 4.1. Let $\mu\in ID(\mathbb{R}^d)$ and $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$. We say that the improper integral on (a,b) of f with respect to $X^{(\mu)}$, $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$, is *absolutely definable* if

$$\int_a^b |C_\mu(f(s)z)|ds<\infty \quad \text{for all } z\in\mathbb{R}^d. \tag{4.1}$$

If (4.1) holds, then $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$ is definable, which follows from Theorem 3.5. Let $\mathfrak{D}^0(\Phi_f)$ denote the class of $\mu\in ID(\mathbb{R}^d)$ for which $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$ and $\int_{a+}^{b-} f(s)X^{(\mu)}(ds)$ is absolutely definable.

Theorem 4.2. *Let $\mu=\mu_{(A,\nu,\gamma)}\in ID(\mathbb{R}^d)$. Then $\mu\in\mathfrak{D}^0(\Phi_f)$ if and only if* (3.1) *and* (3.2) *hold and*

$$\int_a^b \left| f(s)\left(\gamma+\int_{\mathbb{R}^d} x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right)\right| ds<\infty. \tag{4.2}$$

Proof. See (2006b), Proposition 2.3. Recall that (3.1), (3.2), and (4.2) imply $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$. □

Theorem 4.2 shows that $\mu\in\mathfrak{D}^0(\Phi_f)$ if and only if f is $X^{(\mu)}$-integrable over (a,b) in the sense of Rajput and Rosinski (1989); see Theorem 2.7 of their paper. The author owes this remark to Jan Rosiński.

Theorem 4.3. *The following relations are true:*

$$\mathfrak{D}^0(\Phi_f)\subset\mathfrak{D}(\Phi_f)\subset\mathfrak{D}(\Phi_{f,\,\mathrm{c}})\subset\mathfrak{D}(\Phi_{f,\,\mathrm{es}})=\mathfrak{D}(\Phi_{f,\,\mathrm{sym}}). \tag{4.3}$$

Proof. This is a consequence of the descriptions of these domains obtained from Theorems 3.5–3.9 and 4.2. □

Remark 4.4. If we restrict ourselves to symmetric infinitely divisible distributions, then the five domains in (4.3) are identical. That is, if μ is symmetric and $\mu \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}})$, then $\mu \in \mathfrak{D}^0(\Phi_f)$. Indeed, then the location parameter γ of μ is zero and the integral in (4.2) is zero.

For two functions f and g, we write $f(s) \asymp g(s)$, $s \to \infty$, if there are positive constants c_1 and c_2 such that $0 < c_1 g(s) \leqslant f(s) \leqslant c_2 g(s)$ for all large s.

Example 4.5. Let $f(s)$ be a measurable function on (a, ∞) with a finite such that $\int_a^t f(s)^2 ds < \infty$ for all $t \in (a, \infty)$ and

$$f(s) \asymp s^{-1/\alpha} \quad \text{as } s \to \infty$$

with $0 < \alpha < 2$. Since we will have Remark 4.10, the results of (2006c) say the following. Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$. Let γ^1 be mean of μ if it exists.

(i) Consider the condition

$$\int_{|x|>1} |x|^\alpha \nu(dx) < \infty. \tag{4.4}$$

Then, for $0 < \alpha < 2$,

$$\mu \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad \Leftrightarrow \quad (4.4).$$

(ii) If $0 < \alpha < 1$, then

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\,\mathrm{es}}). \tag{4.5}$$

(iii) If $1 < \alpha < 2$, then

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\,\mathrm{es}}), \tag{4.6}$$

$$\mathfrak{D}(\Phi_f) \cap ID_0 \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \cap ID_0, \tag{4.7}$$

and

$$\mu \in \mathfrak{D}(\Phi_f) \quad \Leftrightarrow \quad (4.4) \text{ and } \gamma^1 = 0.$$

(iv) Let $\alpha = 1$ and suppose, in addition, that, for some $s_0 > a \vee 0$ and $c > 0$,

$$\int_{s_0}^\infty |f(s) - cs^{-1}| ds < \infty.$$

Consider the conditions

$$\lim_{t\to\infty} \int_{s_0}^t s^{-1} ds \int_{|x|>s} x\nu(dx) \text{ exists in } \mathbb{R}^d, \tag{4.8}$$

$$\int_{s_0}^\infty s^{-1} ds \left| \int_{|x|>s} x\nu(dx) \right| < \infty. \tag{4.9}$$

Then

$$\mathfrak{D}^0(\Phi_f) \cap ID_0 \subsetneqq \mathfrak{D}(\Phi_f) \cap ID_0 \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \cap ID_0 \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \cap ID_0 \tag{4.10}$$

and, letting (4.4) mean (4.4) with $\alpha = 1$,

$$\mu \in \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \quad \Leftrightarrow \quad (4.4) \text{ and } (4.8),$$

$$\mu \in \mathfrak{D}(\Phi_f) \quad \Leftrightarrow \quad (4.4),\ (4.8), \text{ and } \gamma^1 = 0,$$

$$\mu \in \mathfrak{D}^0(\Phi_f) \quad \Leftrightarrow \quad (4.4),\ (4.9), \text{ and } \gamma^1 = 0.$$

Example 4.6. Let $f(s)$ be a measurable function on (a,∞) with a finite such that $\int_a^t f(s)^2 ds < \infty$ for all $t \in (a,\infty)$ and there are positive constants α, c_1, and c_2 satisfying

$$e^{-c_2 s^\alpha} \leqslant f(s) \leqslant e^{-c_1 s^\alpha} \quad \text{for all large } s.$$

Then

$$\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad \Leftrightarrow \quad \int_{\mathbb{R}^d} (\log^+ |x|)^{1/\alpha} \nu(dx) < \infty,$$

and (4.5) holds. See Theorem 5.15 of (2006a) and Proposition 4.3 of (2006b). Using notation $\Phi = \Phi_f$ for $f(s) = e^{-s}$ on $(0,\infty)$, we know that the range of Φ is the class $L_0(\mathbb{R}^d)$ of all selfdecomposable distributions on $\mathbb{R}^d$ (see Rocha-Arteaga and Sato (2003) for references). Jurek (1983) shows that, for $m = 1,2,\ldots$, the subclass $L_m(\mathbb{R}^d)$ of $L_0(\mathbb{R}^d)$ is the range of Φ_f for $f(s) = \exp(-((m+1)!\, s)^{1/(m+1)})$ on $(0,\infty)$ (see also Rocha-Arteaga and Sato (2003)).

Example 4.7. Let $f(s)$ be a measurable function on (a,∞) with a finite such that $\int_a^t f(s)^2 ds < \infty$ for all $t \in (a,\infty)$ and

$$f(s) \asymp e^{-e^s} \quad \text{as } s \to \infty.$$

Then

$$\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad \Leftrightarrow \quad \int_{|x|>e} (\log\log |x|) \nu(dx) < \infty,$$

and (4.5) holds.

Proof is as follows. Let $h(s) = e^{-e^s}$. We have $c_1 h(s) \leqslant f(s) \leqslant c_2 h(s)$ for $s \geqslant s_0$ with $c_1, c_2, \ldots$ positive constants and $s_0 > 2e$. Clearly we have (3.1). Let $s = h^{-1}(u) = \log\log(1/u)$ be the inverse function of $u = h(s)$. Let $I = \int_{s_0}^\infty ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx)$. Then,

$$I = \int_{s_0}^\infty ds \int_{|f(s)x|\leqslant 1} |f(s)x|^2 \nu(dx) + \int_{s_0}^\infty ds \int_{|f(s)x|>1} \nu(dx) = I_1 + I_2 \quad \text{(say)},$$

$$I_2 \leqslant \int_{|x|>c_2^{-1}/h(s_0)} \nu(dx) \int_{s_0}^{h^{-1}(c_2^{-1}|x|^{-1})} ds \leqslant \int_{|x|>c_3} (\log\log|x|)\nu(dx) + c_4,$$

$$\begin{aligned} I_1 \leqslant c_2^2 &\int_{|x|>c_1^{-1}/h(s_0)} |x|^2 \nu(dx) \int_{h^{-1}(c_1^{-1}|x|^{-1})}^\infty h(s)^2 ds \\ &+ c_2^2 \int_{|x|\leqslant c_1^{-1}/h(s_0)} |x|^2 \nu(dx) \int_{s_0}^\infty h(s)^2 ds < \infty, \end{aligned}$$

since

$$\lim_{u\downarrow 0} \frac{1}{u^2} \int_{h^{-1}(u)}^\infty h(s)^2 ds = \lim_{u\downarrow 0} \frac{ue^{-s}}{2u} = 0.$$

The estimate of I from below is similar. Hence I is finite if and only if $\int_{|x|>e} (\log\log|x|)\nu(dx) < \infty$.

To see (4.5), it is enough to show (4.2) with a, b replaced by s_0, ∞. Let

$$J = \int_{s_0}^\infty f(s)ds \int_{\mathbb{R}^d} \frac{|x|^3 \nu(dx)}{(1+f(s)^2|x|^2)(1+|x|^2)}.$$

Since $\int_{s_0}^\infty f(s)ds < \infty$ and $f(s) \to 0$, it is enough to show that $J < \infty$. Now,

$$J \leqslant c_2 c_1^{-1} \int_{\mathbb{R}^d} \frac{|x|^2 \nu(dx)}{1+|x|^2} \int_{s_0}^\infty \frac{c_1 h(s)|x| ds}{1 + c_1^2 h(s)^2 |x|^2} < \infty,$$

since, for any $r > 0$,

$$\int_{s_0}^{\infty} \frac{rh(s)ds}{1+r^2h(s)^2} = \int_0^{h(s_0)} \frac{ru(-ds/du)du}{1+r^2u^2} = \int_0^{h(s_0)} \frac{rdu}{(1+r^2u^2)(\log(1/u))}$$
$$\leqslant c_5 \int_0^{h(s_0)} \frac{rdu}{1+r^2u^2} \leqslant c_5 \int_0^{\infty} \frac{dv}{1+v^2}.$$

Example 4.8. Let $f(s)$ be a measurable function on (a,∞) with a finite such that $\int_a^t f(s)^2 ds < \infty$ for all $t \in (a,\infty)$ and

$$f(s) \asymp s^{-1}(\log s)^{-\beta} \quad \text{as } s \to \infty$$

with $\beta \in \mathbb{R}$.

(i) Consider the condition

$$\int_{|x|>2} |x|(\log|x|)^{-\beta}\nu(dx) < \infty. \tag{4.11}$$

Then, for $\beta \in \mathbb{R}$,

$$\mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad \Leftrightarrow \quad (4.11).$$

(ii) If $0 < \beta \leqslant 1$, then

$$\mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{es}}). \tag{4.12}$$

(iii) If $1 < \beta \leqslant 2$, then

$$\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{es}}). \tag{4.13}$$

Proof of (i) is similar to the first half of the proof in Example 4.7. This time $h(s) = s^{-1}(\log s)^{-\beta}$ and $s_0 > a \vee 1$ is chosen so large that $h(s)$ is strictly decreasing for $s \geqslant s_0$. We define I, I_1, and I_2 in the same way. Then

$$I_1 \leqslant c_6 \int_{|x|>c_7} |x|(\log h^{-1}(c_1^{-1}|x|^{-1}))^{-\beta}\nu(dx) + c_8$$
$$\leqslant c_9 \int_{|x|>c_7} |x|(\log|x|)^{-\beta}\nu(dx) + c_8$$

since $\int_t^{\infty} h(s)^2 ds \sim t^{-1}(\log t)^{-2\beta} = h(t)(\log t)^{-\beta}$ as $t \to \infty$ and $\log h^{-1}(u) \sim \log(1/u)$ as $u \downarrow 0$, and

$$I_2 \leqslant c_{10} \int_{|x|>c_{11}} |x|(\log|x|)^{-\beta}\nu(dx)$$

since $h^{-1}(u) \sim u^{-1}(\log(1/u))^{-\beta}$ as $u \downarrow 0$. We can estimate I from below similarly. Hence $I < \infty$ if and only if (4.11) holds.

Prof of (ii). Let $0 < \beta \leqslant 1$. We have $\int_{s_0}^{\infty} f(s)ds = \infty$ for some s_0. Hence, to show $\mathfrak{D}(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{c}})$, choose $\mu_{(A,\nu,\gamma)}$ with ν symmetric and $\gamma \neq 0$. In order to show $\mathfrak{D}(\Phi_{f,\,\mathrm{c}}) \subsetneqq \mathfrak{D}(\Phi_{f,\,\mathrm{es}})$, consider $\mu_{(A,\nu,\gamma)}$ with ν concentrated on the x_1-axis. For simplicity of notation let $d = 1$. Assume that $\int_{(-\infty,2]} \nu(dx) = 0$, $\int_{(2,\infty)} x\nu(dx) = \infty$, and $\int_{(2,\infty)} x(\log x)^{-\beta}\nu(dx) < \infty$. Then

$$\int_{(2,\infty)} x\left(\frac{1}{1+(f(s)x)^2} - \frac{1}{1+x^2}\right)\nu(dx) \to \infty \quad \text{as } s \to \infty,$$

from which it follows that $\mu \notin \mathfrak{D}(\Phi_{f,\,\mathrm{c}})$.

Proof of (iii). Let $1 < \beta \leqslant 2$. Then $\int_a^\infty f(s)ds$ is finite. Thus $\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_{f,\mathrm{c}})$ as Corollary 3.8 says. To show $\mathfrak{D}(\Phi_{f,\mathrm{c}}) \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{es}})$, let $d = 1$, $\int_{(-\infty,2]} \nu(dx) = 0$, and $\nu(dx) = x^{-2}dx$ on $(2,\infty)$. We have

$$\int_{s_0}^q f(s)ds \int_2^\infty \left(\frac{1}{1+(f(s)x)^2} - \frac{1}{1+x^2} \right) x^{-1}dx \to \infty \quad \text{as } q \to \infty,$$

because, using $\int_r^\infty x^{-1}(1+x^2)^{-1}dx \sim \log(1/r)$ as $r \downarrow 0$, we have

$$\begin{aligned} \int_{s_0}^q f(s)ds \int_2^\infty \frac{x^{-1}dx}{1+(f(s)x)^2} &= \int_{s_0}^q f(s)ds \int_{2f(s)}^\infty \frac{x^{-1}dx}{1+x^2} \\ &\geqslant c_{11} \int_{s_0}^q h(s) \log \frac{1}{2c_2 h(s)} ds \\ &= c_{11} \int_{s_0}^q s^{-1}(\log s)^{-\beta} \log(2^{-1}c_2^{-1}s(\log s)^\beta)ds \geqslant c_{12} \int_{s_0}^q s^{-1}(\log s)^{1-\beta} ds \to \infty. \end{aligned}$$

Hence $\mu_{(A,\nu,\gamma)} \notin \mathfrak{D}(\Phi_{f,\mathrm{c}})$.

Remark 4.9. Let f be a locally square-integrable function on (a,b) with $-\infty \leqslant a < b \leqslant \infty$.

(i) If μ_1 and μ_2 are in $\mathfrak{D}(\Phi_f)$, then $\mu_1 * \mu_2 \in \mathfrak{D}(\Phi_f)$. That is, $\mathfrak{D}(\Phi_f)$ is closed under convolution. Also, $\mathfrak{D}^0(\Phi_f)$, $\mathfrak{D}(\Phi_{f,\mathrm{c}})$, and $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ are closed under convolution. Indeed, f is in $\mathbf{L}_{(a,b)}(X)$ for all X, and the conditions in Section 3 work, since convolution gives addition of triplets.

(ii) If μ_1 and μ_2 are in $ID(\mathbb{R}^d)$ and $\mu_1 * \mu_2 \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$, then μ_1 and μ_2 are in $\mathfrak{D}(\Phi_{f,\mathrm{es}})$. Use Theorem 3.6.

(iii) With some choice of f, there are μ_1 and μ_2 in $ID(\mathbb{R}^d)$ such that $\mu_1 * \mu_2 \in \mathfrak{D}(\Phi_f)$, $\mu_1 \notin \mathfrak{D}(\Phi_f)$, and $\mu_2 \notin \mathfrak{D}(\Phi_f)$. Use Example 4.5 with $f(s) = s^{-1/\alpha}$, $1 < \alpha < 2$, on $(1,\infty)$.

(iv) If $\mu_2 \in ID(\mathbb{R}^d)$ and if μ_1 and $\mu_1 * \mu_2$ are in $\mathfrak{D}(\Phi_f)$, then $\mu_2 \in \mathfrak{D}(\Phi_f)$. Use Theorem 3.5.

We give some comments on the relation with improper stochastic integrals studied in (2006a,b,c).

Remark 4.10. Let $\mathbf{L}_{[0,\infty)}(X^{(\mu)})$ be the class of locally $X^{(\mu)}$-integrable functions on $[0,\infty)$ in Definition 2.16 of (2006a). Then the following (a) and (b) are equivalent:

(a) $f \in \mathbf{L}_{[0,\infty)}(X^{(\mu)})$.

(b) $f \in \mathbf{L}_{(0,\infty)}(X^{(\mu)})$ and, for some t (equivalently, for any t) in $(0,\infty)$, $\int_{0+}^t f(s)X^{(\mu)}(ds)$ is absolutely definable.

In this case,

$$\int_0^t f(s)X^{(\mu)}(ds) = \int_{0+}^t f(s)X^{(\mu)}(ds), \tag{4.14}$$

the left-hand side being defined in (2006a) since $f \in \mathbf{L}_{[0,\infty)}(X^{(\mu)})$.

Proof is as follows. As Theorem 3.1 of (2006a) says, $f \in \mathbf{L}_{[0,\infty)}(X^{(\mu)})$ if and only if, for all $t \in (0,\infty)$,

$$\int_0^t f(s)^2 \mathrm{tr}\, A ds < \infty,$$

$$\int_0^t ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty,$$

$$\int_0^t \left| f(s)\gamma + \int_{\mathbb{R}^d} f(s)x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| ds < \infty.$$

Combine this with Theorem 2.6 and the one-sided version of Theorem 4.2. Then we see the equivalence of (a) and (b). To prove (4.14), it is enough to show that $\int_{1/n}^t f(s)X^{(\mu)}(ds)$ tends to $\int_0^t f(s)X^{(\mu)}(ds)$ in probability as $n \to \infty$. Let, for large n,

$$F_n = \int_0^t f(s)X^{(\mu)}(ds) - \int_{1/n}^t f(s)X^{(\mu)}(ds) = \int_0^{1/n} f(s)X^{(\mu)}(ds).$$

Then the triplet (A_n, ν_n, γ_n) of the $\mathcal{L}(F_n)$ satisfies $A_n \to 0$, $\int_{R^d}(|x|^2 \wedge 1)\, \nu_n(dx) \to 0$, and $\gamma_n \to 0$ as $n \to \infty$. Thus $F_n \to 0$ in probability.

In (2006a,b,c) $\int_0^{\infty-} f(s)X^{(\mu)}(ds)$ is said to be definable if $f \in \mathbf{L}_{[0,\infty)}(X^{(\mu)})$ and if $\int_0^t f(s)X^{(\mu)}(ds)$ is convergent in probability as $t \to \infty$. It follows from the results above that the following (c) and (d) are equivalent:

(c) $\int_0^{\infty-} f(s)X^{(\mu)}(ds)$ is definable.

(d) $f \in \mathbf{L}_{(0,\infty)}(X^{(\mu)})$, $\int_1^{\infty-} f(s)X^{(\mu)}(ds)$ is definable, and $\int_{0+}^1 f(s)X^{(\mu)}(ds)$ is absolutely definable.

In this case,

$$\int_0^{\infty-} f(s)X^{(\mu)}(ds) = \int_{0+}^{\infty-} f(s)X^{(\mu)}(ds).$$

The domain of those μ for which $\int_0^{\infty-} f(s)X^{(\mu)}(ds)$ is definable is denoted by $\mathfrak{D}[f, \mathbb{R}^d]$ in (2006a) and by $\mathfrak{D}(\Phi_f)$ in (2006b,c). The domains for essential and compensated improper integrals for $\int_0^{\infty-} f(s)X^{(\mu)}(ds)$ are respectively denoted by $\mathfrak{D}_{\mathrm{es}}[f, \mathbb{R}^d]$ and $\mathfrak{D}_{\mathrm{c}}[f, \mathbb{R}^d]$ in (2006a) and by $\mathfrak{D}_{\mathrm{e}}(\Phi_f)$ (or $\mathfrak{D}_{\mathrm{es}}(\Phi_f)$) and $\mathfrak{D}_{\mathrm{c}}(\Phi_f)$ in (2006b,c).

5. Duals of infinitely divisible distributions

We introduce the concept of duals of purely non-Gaussian infinitely divisible distributions on $\mathbb{R}^d$. Utilizing this concept, we relate some improper stochastic integrals on $(0, b)$ with $0 < b < \infty$ to those on (a, ∞) with a finite.

Definition 5.1. Let $\mu = \mu_{(0,\nu,\gamma)} \in ID_0(\mathbb{R}^d)$. We call the distribution $\mu' \in ID_0(\mathbb{R}^d)$ the *dual* of μ if the triplet $(0, \nu', \gamma')$ of μ' satisfies

$$\nu'(B) = \int_{\mathbb{R}^d \setminus \{0\}} 1_B(\iota(x))|x|^2 \nu(dx), \quad B \in \mathcal{B}(\mathbb{R}^d), \tag{5.1}$$

$$\gamma' = -\gamma, \tag{5.2}$$

where $\iota(x) = |x|^{-2}x$, the inversion of x, which maps $\mathbb{R}^d \setminus \{0\}$ onto itself.

The simple form (5.2) of the relation of location parameters of μ and μ' is due to the fact that we are using the centering function $x(1 + |x|^2)^{-1}$ in the Lévy–Khintchine representation in this paper.

The relation (5.1) implies

$$\int_{\mathbb{R}^d} h(x)\nu'(dx) = \int_{\mathbb{R}^d \setminus \{0\}} h(\iota(x))|x|^2 \nu(dx) \tag{5.3}$$

for all nonnegative measurable function $h(x)$ and thus

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu'(dx) = \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx). \tag{5.4}$$

Proposition 5.2. (i) *The dual of the dual of* μ *is* μ *itself. That is,* $\mu'' = \mu$ *for all* $\mu \in ID_0(\mathbb{R}^d)$.

(ii) $\int_{|x|\leqslant 1} |x|^{2-\alpha}\nu'(dx) = \int_{|x|\geqslant 1} |x|^{\alpha}\nu(dx)$ *for* $\alpha \in \mathbb{R}$.

(iii) *The dual* μ' *is of type* A *if and only if* μ *has finite second moment.*

(iv) *The dual* μ' *is of type* A *or* B *if and only if* μ *has finite mean.*

(v) *If* μ' *is of type* A *or* B, *then*

$$(\textit{drift of } \mu') = -(\textit{mean of } \mu).$$

(vi) *Let* $0 < \alpha < 2$. *Then* μ' *is* $(2-\alpha)$*-stable if and only if* μ *is* α*-stable.*

(vii) *Let* $0 < \alpha < 2$. *Then* μ' *is strictly* $(2-\alpha)$*-stable if and only if* μ *is strictly* α*-stable.*

Proof. (i) Using (5.3), we have

$$\begin{aligned}\nu''(B) &= \int_{\mathbb{R}^d\setminus\{0\}} 1_B(\iota(x))|x|^2\nu'(dx)\\ &= \int_{\mathbb{R}^d\setminus\{0\}} 1_B(\iota(\iota(x)))|\iota(x)|^2|x|^2\nu(dx) = \nu(B)\end{aligned}$$

for all B. Further, $\gamma'' = -\gamma' = \gamma$.

(ii) Using (5.3) again, we have

$$\int_{|x|\leqslant 1} |x|^{2-\alpha}\nu'(dx) = \int_{|\iota(x)|\leqslant 1} |\iota(x)|^{2-\alpha}|x|^2\nu(dx) = \int_{|x|\geqslant 1} |x|^{\alpha}\nu(dx).$$

(iii) and (iv) Use (ii) with $\alpha = 2$ and 1, respectively.

(v) Let $(\gamma^0)'$ and γ^1 be the drift of μ' and the mean of μ, respectively. Then

$$(\gamma^0)' = \gamma' - \int_{\mathbb{R}^d} \frac{x\nu'(dx)}{1+|x|^2}, \quad \gamma^1 = \gamma + \int_{\mathbb{R}^d} \frac{x|x|^2\nu(dx)}{1+|x|^2}.$$

Hence

$$(\gamma^0)' = -\gamma - \int_{\mathbb{R}^d} \frac{\iota(x)}{1+|\iota(x)|^2}|x|^2\nu(dx) = -\gamma - \int_{\mathbb{R}^d} \frac{x}{1+|x|^{-2}}\nu(dx) = -\gamma^1.$$

(vi) Let $0 < \alpha < 2$. Then a nontrivial distribution μ is α-stable if and only if $A = 0$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)r^{-\alpha-1}dr,$$

where S is the unit sphere and λ is a nonzero finite measure on S. In this case,

$$\int h(\iota(x))|x|^2\nu(dx) = \int_S \lambda(d\xi)\int_0^\infty h(r^{-1}\xi)r^2r^{-\alpha-1}dr$$

and thus

$$\nu'(B) = \int_S \lambda(d\xi)\int_0^\infty 1_B(r^{-1}\xi)r^{1-\alpha}dr = \int_S \lambda(d\xi)\int_0^\infty 1_B(u\xi)u^{-1-(2-\alpha)}du.$$

(vii) If $0 < \alpha < 2$ and μ is nontrivial, then μ is strictly α-stable if and only if μ is α-stable with additional condition $\gamma^0 = 0$ $(0 < \alpha < 1)$, $\gamma^1 = 0$ $(1 < \alpha < 2)$, or $\int_S \xi\lambda(d\xi) = 0$ $(\alpha = 1)$. Moreover, δ_γ with $\gamma \neq 0$ is strictly 1-stable. See Theorem

14.7 of Sato (1999). Now use (v) and the fact that ν and ν' has the identical λ-measure as seen in the proof of (vi). □

The following fact is in duality with the fact in Example 4.5.

Proposition 5.3. *Let $f(s)$ be a measurable function on $(0,b)$ with $0<b<\infty$ such that $\int_t^b f(s)^2 ds<\infty$ for all $t\in(0,b)$ and*

$$f(s)\asymp s^{-1/(2-\alpha)} \quad \text{as } s\downarrow 0$$

with $0<\alpha<2$. Let $\mu=\mu_{(A,\nu,\gamma)}\in ID(\mathbb{R}^d)$. Let γ^0 be drift of μ if it exists.

(i) *Consider the condition*

$$A=0 \text{ and } \int_{|x|<1}|x|^{2-\alpha}\nu(dx)<\infty. \tag{5.5}$$

Then, for $0<\alpha<2$,

$$\mu\in\mathfrak{D}(\Phi_{f,\mathrm{es}}) \quad\Leftrightarrow\quad (5.5).$$

(ii) *If $0<\alpha<1$, then*

$$\mathfrak{D}^0(\Phi_f)=\mathfrak{D}(\Phi_f)=\mathfrak{D}(\Phi_{f,\mathrm{c}})=\mathfrak{D}(\Phi_{f,\mathrm{es}}). \tag{5.6}$$

(iii) *If $1<\alpha<2$, then*

$$\mathfrak{D}^0(\Phi_f)=\mathfrak{D}(\Phi_f)\subsetneqq\mathfrak{D}(\Phi_{f,\mathrm{c}})=\mathfrak{D}(\Phi_{f,\mathrm{es}}) \tag{5.7}$$

and

$$\mu\in\mathfrak{D}(\Phi_f) \quad\Leftrightarrow\quad (5.5) \text{ and } \gamma^0=0.$$

(iv) *Let $\alpha=1$ and suppose, in addition, that, for some $s_0\in(0,b)$ and $c>0$,*

$$\int_0^{s_0}|f(s)-cs^{-1}|ds<\infty.$$

Consider the conditions

$$\lim_{t\downarrow 0}\int_t^{s_0}s^{-1}ds\int_{|x|<s}x\nu(dx) \text{ exists in } \mathbb{R}^d, \tag{5.8}$$

$$\int_0^{s_0}s^{-1}ds\left|\int_{|x|<s}x\nu(dx)\right|<\infty. \tag{5.9}$$

Then

$$\mathfrak{D}^0(\Phi_f)\subsetneqq\mathfrak{D}(\Phi_f)\subsetneqq\mathfrak{D}(\Phi_{f,\mathrm{c}})\subsetneqq\mathfrak{D}(\Phi_{f,\mathrm{es}}) \tag{5.10}$$

and, letting (5.5) mean (5.5) with $\alpha=1$,

$$\mu\in\mathfrak{D}(\Phi_{f,\mathrm{c}}) \quad\Leftrightarrow\quad (5.5) \text{ and } (5.8),$$
$$\mu\in\mathfrak{D}(\Phi_f) \quad\Leftrightarrow\quad (5.5),\ (5.8),\text{ and } \gamma^0=0,$$
$$\mu\in\mathfrak{D}^0(\Phi_f) \quad\Leftrightarrow\quad (5.5),\ (5.9),\text{ and } \gamma^0=0.$$

Proof. From our assumption there are $c_1,c_2>0$ and $s_0\in(0,b)$ such that

$$c_1 s^{-1/(2-\alpha)}\leqslant f(s)\leqslant c_2 s^{-1/(2-\alpha)} \quad \text{for } s\in(0,s_0]. \tag{5.11}$$

Let $u_0=s_0^{-\alpha/(2-\alpha)}$ and let $g(u)=1/f(u^{-(2-\alpha)/\alpha})$ for $u\geqslant u_0$. Then $g(u)\asymp u^{-1/\alpha}$ as $u\to\infty$, since

$$\frac{g(u)}{u^{-1/\alpha}}=\frac{u^{1/\alpha}}{f(u^{-(2-\alpha)/\alpha})}=\frac{s^{-1/(2-\alpha)}}{f(s)},$$

where $s = u^{-(2-\alpha)/\alpha}$. As this shows that $g(u)$ is locally bounded on $[u_0, \infty)$, we have $\int_{u_0}^{v} g(u)^2 du < \infty$ for all $v \in (u_0, \infty)$.

(i) If $\mu \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}})$, then $A = 0$, since $\int_0^{s_0} f(s)^2 ds \geqslant c_1 \int_0^{s_0} s^{-2/(2-\alpha)} ds = \infty$. We have

$$\begin{aligned}
\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad &\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \\
&\overset{(*)}{\Leftrightarrow} \quad A = 0 \text{ and } \mu_{(0,\nu',\gamma')} \in \mathfrak{D}(\Phi_{g,\,\mathrm{es}}) \\
&\overset{(**)}{\Leftrightarrow} \quad A = 0 \text{ and } \int_{|x|>1} |x|^{\alpha} \nu'(dx) < \infty \quad \overset{(***)}{\Leftrightarrow} \quad (5.5),
\end{aligned}$$

where $\mu_{(0,\nu',\gamma')}$ is the dual of $\mu_{(0,\nu,\gamma)}$. Write $\beta = (2-\alpha)/\alpha$. The equivalence $(*)$ comes from the equality that

$$\begin{aligned}
\int_0^{s_0} ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\nu(dx) &= \beta \int_{u_0}^{\infty} u^{-2/\alpha} du \int_{\mathbb{R}^d} \left(\left| \frac{x}{g(u)} \right|^2 \wedge 1 \right) \nu(dx) \\
&= \beta \int_{u_0}^{\infty} u^{-2/\alpha} du \int_{\mathbb{R}^d} (|g(u)x|^{-2} \wedge 1)|x|^2 \nu'(dx) \\
&= \beta \int_{u_0}^{\infty} \frac{u^{-2/\alpha}}{g(u)^2} du \int_{\mathbb{R}^d} (1 \wedge |g(u)x|^2) \nu'(dx).
\end{aligned}$$

The equivalence $(**)$ comes from Example 4.5 and $(***)$ from Proposition 5.2.

(ii) We have

$$\begin{aligned}
&\int_0^{s_0} f(s) ds \left| \gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| \\
&\quad = \beta \int_{u_0}^{\infty} \frac{u^{-2/\alpha}}{g(u)} du \left| \gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|x/g(u)|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right| \\
&\quad = \beta \int_{u_0}^{\infty} \frac{u^{-2/\alpha}}{g(u)^2} g(u) du \left| \gamma' + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|g(u)x|^2} - \frac{1}{1+|x|^2} \right) \nu'(dx) \right|.
\end{aligned}$$

Hence, for $0 < \alpha < 1$,

$$\begin{aligned}
\mu \in \mathfrak{D}^0(\Phi_f) \quad &\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu,\gamma)} \in \mathfrak{D}^0(\Phi_f) \\
&\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu',\gamma')} \in \mathfrak{D}^0(\Phi_g) \quad \Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu',\gamma')} \in \mathfrak{D}(\Phi_{g,\,\mathrm{es}}) \\
&\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}) \quad \Leftrightarrow \quad \mu \in \mathfrak{D}(\Phi_{f,\,\mathrm{es}}).
\end{aligned}$$

(iii) We have, for $v = t^{-\alpha/(2-\alpha)}$,

$$\begin{aligned}
&\int_t^{s_0} f(s) ds \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(s)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\
&\quad = \beta \int_{u_0}^{v} \frac{u^{-2/\alpha}}{g(u)} du \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|x/g(u)|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\
&\quad = -\beta \int_{u_0}^{v} \frac{u^{-2/\alpha}}{g(u)^2} g(u) du \left(\gamma' + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|g(u)x|^2} - \frac{1}{1+|x|^2} \right) \nu'(dx) \right).
\end{aligned}$$

Thus, for $1 < \alpha < 2$, using Example 4.5,

$$\begin{aligned}
\mu \in \mathfrak{D}(\Phi_f) \quad &\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Phi_f) \\
&\Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu',\gamma')} \in \mathfrak{D}(\Phi_g)
\end{aligned}$$

$$\Leftrightarrow \quad A=0, \int_{|x|>1}|x|^\alpha \nu'(dx)<\infty, \text{ and } (\text{mean of } \mu_{(0,\nu',\gamma')})=(\gamma^1)'=0$$
$$\Leftrightarrow \quad (5.5) \text{ and } (\text{drift of } \mu_{(0,\nu,\gamma)})=\gamma^0=0.$$

and, in a similar way, $\mu\in\mathfrak{D}^0(\Phi_f)$ if and only if (5.5) and $\gamma^0=0$. Now we see that $\mu\in\mathfrak{D}(\Phi_{f,\mathrm{c}})$ if and only if (5.5) holds. Hence $\mathfrak{D}(\Phi_f)=\mathfrak{D}^0(\Phi_f)$ and $\mathfrak{D}(\Phi_{f,\mathrm{c}})=\mathfrak{D}(\Phi_{f,\mathrm{es}})$. Since $\mathfrak{D}(\Phi_g)\cap ID_0 \subsetneqq \mathfrak{D}(\Phi_{g,\mathrm{es}})\cap ID_0$, we have $\mathfrak{D}(\Phi_f)\subsetneqq\mathfrak{D}(\Phi_{f,\mathrm{es}})$.

(iv) ($\alpha=1$) As in the proof of (ii),

$$\begin{aligned}\mu\in\mathfrak{D}^0(\Phi_f) \quad &\Leftrightarrow \quad A=0 \text{ and } \mu_{(0,\nu,\gamma)}\in\mathfrak{D}^0(\Phi_f)\\ &\Leftrightarrow \quad A=0 \text{ and } \mu_{(0,\nu',\gamma')}\in\mathfrak{D}^0(\Phi_g)\\ &\Leftrightarrow \quad A=0, \int_{|x|>1}|x|^\alpha\nu'(dx)<\infty, \int_{u_0}^\infty u^{-1}du\left|\int_{|x|>u}x\nu'(dx)\right|<\infty, (\gamma^1)'=0\\ &\Leftrightarrow \quad (5.5), (5.9), \text{ and } \gamma^0=0.\end{aligned}$$

For the last equivalence, note that

$$\int_{u_0}^\infty u^{-1}du\left|\int_{|x|>u}x\nu'(dx)\right|=\int_0^{s_0}s^{-1}ds\left|\int_{|x|<s}x\nu(dx)\right|.$$

Let

$$I(t)=\int_t^{s_0}f(s)ds\left(\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

Let us write $c_3, c_4, \ldots$ for positive constants. We have

$$\begin{aligned}I(t)=&\int_t^{s_0}(f(s)-cs^{-1})ds\left(\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right)\\ &+\int_t^{s_0}cs^{-1}ds\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|cs^{-1}x|^2}\right)\nu(dx)\\ &+\int_t^{s_0}cs^{-1}ds\left(\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|cs^{-1}x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right)\\ =&\,I_1(t)+I_2(t)+I_3(t) \qquad \text{(say)}.\end{aligned}$$

Under (5.5), $I_1(t)$ and $I_2(t)$ are convergent as $t\downarrow 0$, since

$$\begin{aligned}&\int_0^{s_0}|f(s)-cs^{-1}|ds\left|\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right|\\ &\quad\leqslant\int_0^{s_0}|f(s)-cs^{-1}|ds\left(|\gamma|+\int_{\mathbb{R}^d}\frac{|x|^3+|x||f(s)x|^2}{(1+|f(s)x|^2)(1+|x|^2)}\nu(dx)\right)\\ &\quad\leqslant\int_0^{s_0}|f(s)-cs^{-1}|ds\left(|\gamma|+2\int_{|x|<1}|x|\nu(dx)+c_3\int_{|x|\geqslant 1}\nu(dx)\right)\end{aligned}$$

(note that $f(s)\geqslant c_4>0$) and

$$\begin{aligned}&\int_0^{s_0}cs^{-1}ds\left|\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|cs^{-1}x|^2}\right)\nu(dx)\right|\\ &\quad\leqslant\int_0^{s_0}cs^{-1}ds\int_{\mathbb{R}^d}\frac{|x|\,||cs^{-1}x|^2-|f(s)x|^2|}{(1+|f(s)x|^2)(1+|cs^{-1}x|^2)}\nu(dx)\\ &\quad\leqslant c_5\int_0^{s_0}|cs^{-1}-f(s)|ds\int_{\mathbb{R}^d}\frac{|x|^3(cs^{-1})^2}{(1+|f(s)x|^2)(1+|cs^{-1}x|^2)}\nu(dx)\end{aligned}$$

$$\leqslant c_5 \int_0^{s_0} |cs^{-1} - f(s)| ds \left(\int_{|x|<1} |x| \nu(dx) + c_6 \int_{|x|\geqslant 1} \nu(dx) \right).$$

Hence, under (5.5), $I(t)$ is convergent if and only if $I_3(t)$ is convergent. We have, with $v = t^{-1}$,

$$\begin{aligned} I_3(t) &= \int_{u_0}^{v} cu^{-1} du \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|cux|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right) \\ &= -c^2 \int_{u_0}^{v} c^{-1} u^{-1} du \left(\gamma' + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|c^{-1}u^{-1}x|^2} - \frac{1}{1+|x|^2} \right) \nu'(dx) \right). \end{aligned}$$

Let $h(u) = c^{-1}u^{-1}$ for $u \geqslant u_0 = s_0^{-1}$. Then it follows that

$$\begin{aligned} &\mu \in \mathfrak{D}(\Phi_f) \quad \Leftrightarrow \quad A = 0 \text{ and } \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Phi_f) \Leftrightarrow A = 0 \text{ and } \mu_{(0,\nu',\gamma')} \in \mathfrak{D}(\Phi_h) \\ &\Leftrightarrow \quad A = 0, \int_{|x|>1} |x| \nu'(dx) < \infty, \lim_{v\to\infty} \int_{u_0}^{v} u^{-1} du \int_{|x|>u} x \nu'(dx) \text{ exists}, (\gamma^1)' = 0 \\ &\Leftrightarrow \quad (5.5), (5.8), \text{ and } \gamma^0 = 0. \end{aligned}$$

It follows from this that $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{c}})$ if and only if (5.5) and (5.8) hold. We get relation (5.10), using (4.10) with h in place of f. Thus, in showing that $\mathfrak{D}^0(\Phi_f) \subsetneqq \mathfrak{D}(\Phi_f)$, we make indirect use of the elaborate construction in (2006c) of a distribution in the difference of the two classes. □

Proposition 5.4. *Let $f(s)$ be a measurable function on $(0, b)$ with $0 < b < \infty$ such that $\int_t^b f(s)^2 ds < \infty$ for all $t \in (0, b)$ and*

$$f(s) \asymp s^{-1/2} \quad \text{as } s \downarrow 0.$$

Then

$$\mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\mathrm{es}}) \quad \Leftrightarrow \quad A = 0 \text{ and } \int_{|x|<1} |x|^2 \log(1/|x|) \nu(dx) < \infty,$$

and (5.6) *holds.*

Proof. We have (5.11) with $\alpha = 0$. Note that $\int_0^{s_0} f(s)^2 ds = \infty$ and $\int_0^{s_0} f(s) ds < \infty$. Let $g(u) = 1/f(e^{-2u})$ for $u \geqslant u_0 = s_0^{-1}$. Then $g(u) \asymp e^{-u}$ as $u \to \infty$, since

$$\frac{g(u)}{e^{-u}} = \frac{e^u}{f(e^{-2u})} = \frac{s^{-1/2}}{f(s)},$$

where $s = e^{-2u}$. We have

$$\begin{aligned} \int_0^{s_0} ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1) \nu(dx) &= \int_{u_0}^{\infty} 2e^{-2u} du \int_{\mathbb{R}^d} \left(\left| \frac{x}{g(u)} \right|^2 \wedge 1 \right) \nu(dx) \\ &= \int_{u_0}^{\infty} \frac{2e^{-2u}}{g(u)^2} du \int_{\mathbb{R}^d} (1 \wedge |g(u)x|^2) \nu'(dx). \end{aligned}$$

Hence, using Example 4.6, we have

$$\begin{aligned} &\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}}) \quad \Leftrightarrow \quad A = 0, \ \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\mathrm{es}}) \quad \Leftrightarrow \quad A = 0, \ \mu_{(0,\nu',\gamma')} \in \mathfrak{D}(\Phi_{g,\mathrm{es}}) \\ &\quad \Leftrightarrow \quad A = 0, \int_{|x|>1} \log|x| \nu'(dx) < \infty \end{aligned}$$

and the last integral equals $\int_{|x|<1}|x|^2\log(1/|x|)\nu(dx)$. We have

$$\begin{aligned}
&\int_0^{s_0} f(s)ds\left|\gamma+\int_{\mathbb{R}^d}x\left(\frac{1}{1+|f(s)x|^2}-\frac{1}{1+|x|^2}\right)\nu(dx)\right|\\
&\quad\leqslant\int_0^{s_0}ds\int_{|x|\leqslant 1}\frac{|f(s)x|^3\nu(dx)}{(1+|f(s)x|^2)(1+|x|^2)}+c_3\\
&\quad\leqslant c_4\int_0^{s_0}ds\int_{|x|\leqslant 1}\frac{|s^{-1/2}x|^3\nu(dx)}{(1+c_5|s^{-1/2}x|^2)(1+|x|^2)}+c_3\\
&\quad\leqslant c_4\int_0^{s_0}ds\int_{|x|\leqslant 1\wedge s^{1/2}}|s^{-1/2}x|^3\nu(dx)+c_5\int_0^{s_0}ds\int_{s^{1/2}<|x|\leqslant 1}s^{-1/2}|x|\nu(dx)+c_3\\
&\quad=c_4\int_{|x|\leqslant 1\wedge s_0^{1/2}}|x|^3\nu(dx)\int_{|x|^2}^{s_0}s^{-3/2}ds+c_5\int_{|x|\leqslant 1}|x|\nu(dx)\int_0^{s_0\wedge|x|^2}s^{-1/2}ds+c_3,
\end{aligned}$$

which is finite. Hence $\mathfrak{D}^0(\Phi_f)=\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$ and we obtain (5.6). □

Proposition 5.5. *Let $f(s)$ be a measurable function on $(0,b)$ with $0<b<\infty$ such that $\int_t^b f(s)^2ds<\infty$ for all $t\in(0,b)$ and*

$$f(s)\asymp s^{-1}(\log(1/s))^{-\beta}\quad \text{as } s\downarrow 0$$

with $\beta\in\mathbb{R}$. Then

$$\mu_{(A,\nu,\gamma)}\in\mathfrak{D}(\Phi_{f,\,\mathrm{es}})\quad\Leftrightarrow\quad A=0 \text{ and } \int_{|x|<1/2}|x|(\log(1/|x|))^{-\beta}\nu(dx)<\infty.$$

If $0<\beta\leqslant 1$, then (4.12) holds. If $1<\beta\leqslant 2$, then (4.13) holds.

Proof. Choose s_0 large enough and let u_0 be such that $s_0=u_0^{-1}(\log u_0)^{-2\beta}$. Let $g(u)=1/f(s)$ for $u\geqslant u_0$, where $s=u^{-1}(\log u)^{-2\beta}$. Then we can prove that $g(u)\asymp u^{-1}(\log u)^{-\beta}$ as $u\to\infty$,

$$\int_0^{s_0}ds\int_{\mathbb{R}^d}(|f(s)x|^2\wedge 1)\nu(dx)=\int_{u_0}^{\infty}(-ds/du)g(u)^{-2}du\int_{\mathbb{R}^d}(1\wedge|g(u)x|^2)\nu'(dx),$$

and $(-ds/du)g(u)^{-2}\asymp 1$. Using Example 4.8 (i), the rest of the proof is similar.

Proof of (4.12) or (4.13) for $0<\beta\leqslant 1$ or $1<\beta\leqslant 2$, respectively, is done by the same method as the proof of (ii) and (iii) of Example 4.8. □

6. Properties of f and largeness of various domains

Fix $-\infty\leqslant a<b\leqslant\infty$ and the dimension d. Let f be an $\mathbb{R}$-valued measurable function on (a,b). We use the subclasses $ID_0(\mathbb{R}^d)$ and $ID_{\mathrm{AB}}(\mathbb{R}^d)$ of $ID(\mathbb{R}^d)$ introduced in Section 1.

Theorem 6.1. *The following statements are equivalent.*

(a) $ID(\mathbb{R}^d)=\mathfrak{D}^0(\Phi_f)$.
(b) $ID(\mathbb{R}^d)=\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$.
(c) $ID_0(\mathbb{R}^d)\subset\mathfrak{D}^0(\Phi_f)$.
(d) $ID_0(\mathbb{R}^d)\subset\mathfrak{D}(\Phi_{f,\,\mathrm{es}})$.
(e) $\int_a^b 1_{\{f(s)\neq 0\}}ds<\infty$ *and* $\int_a^b f(s)^2ds<\infty$.

As we have the relation of the various domains in Theorem 4.3, statement (b) or (d) with $\Phi_{f,\,\mathrm{es}}$ replaced by any one of Φ_f, $\Phi_{f,\,\mathrm{c}}$, and $\Phi_{f,\,\mathrm{sym}}$ is also equivalent to statements (a)–(e).

Proof of Theorem 6.1. Clearly (a) $\Rightarrow$ (b) $\Rightarrow$ (d), and (a) $\Rightarrow$ (c) $\Rightarrow$ (d). Therefore it is enough to show that (e) $\Rightarrow$ (a), and (d) $\Rightarrow$ (e).

Assume (e). Then (3.1), (3.2), and (4.2) hold. Indeed, (3.1) is obvious, (3.2) follows as

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) \leqslant \int_a^b (f(s)^2+1) 1_{\{f(s)\neq 0\}} ds \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\, \nu(dx) < \infty,$$

and (4.2) follows as in the proof of Theorem 2.6 (i). Hence we obtain (a) by virtue of Theorem 4.2.

Assume (d). Then, for every $\mu \in ID_0$, f is in $\mathbf{L}_{(a,b)}(X^{(\mu)})$ and $\Phi_{f,\,\mathrm{es}}(\mu)$ is definable. Hence

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty \quad \text{for all } \nu \in \mathrm{Lvm}(ID(\mathbb{R}^d)) \tag{6.1}$$

by virtue of Theorem 3.6. Let us show (e) in two steps.

Step 1. Suppose that $\int_a^b f(s)^2 ds = \infty$. Let

$$k(r) = \int_a^b f(s)^2 1_{\{|f(s)|\leqslant 1/r\}} ds \quad \text{for } r>0.$$

For every $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$ we have

$$\begin{aligned} \int_{|x|\leqslant 1} |x|^2 k(|x|)\nu(dx) &= \int_a^b ds \int_{|x|\leqslant 1} f(s)^2 1_{\{|f(s)|\leqslant 1/|x|\}} |x|^2 \nu(dx) \\ &\leqslant \int_a^b ds \int_{|x|\leqslant 1} (|f(s)x|^2 \wedge 1)\nu(dx) < \infty \end{aligned} \tag{6.2}$$

by (6.1). Considering Lebesgue measure on $\{|x| \leqslant 1\}$ as ν, we see that $k(|x|) < \infty$ for almost every x with $|x| \leqslant 1$. Hence $k(r) < \infty$ for almost every r in $(0,1]$. Therefore $k(r)$ is finite for all $r>0$ and increases to ∞ as $r \downarrow 0$. Choose r_n, $n=1,2,\ldots$, such that $1 > r_n > r_{n+1} > 0$ and $k(r_n) \geqslant n$. Let $\rho = \sum_{n=1}^{\infty} n^{-2} \delta_{r_n}$ and let

$$\nu(B) = \int_S \lambda(d\xi) \int_{(0,1]} 1_B(r\xi)\, r^{-2} \rho(dr) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a finite nonzero measure on the unit sphere $S = \{\xi \in \mathbb{R}^d \colon |\xi| = 1\}$. Then $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$, since

$$\int_{|x|\leqslant 1} |x|^2 \nu(dx) = \int_S \lambda(d\xi) \int_{(0,1]} \rho(dr) = \lambda(S) \sum_{n=1}^{\infty} n^{-2} < \infty.$$

But

$$\begin{aligned} \int_{|x|\leqslant 1} |x|^2 k(|x|)\nu(dx) &= \int_S \lambda(d\xi) \int_{(0,1]} k(r)\rho(dr) \\ &= \lambda(S) \sum_{n=1}^{\infty} k(r_n)\rho(\{r_n\}) \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty, \end{aligned}$$

which contradicts (6.2). Therefore, $\int_a^b f(s)^2 ds < \infty$.

Step 2. Suppose that $\int_a^b 1_{\{f(s)\neq 0\}} ds = \infty$. Let

$$h(r) = \int_a^b 1_{\{|f(s)|>1/r\}} ds \quad \text{for } r > 0.$$

We have $h(r) \leqslant r^2 \int_a^b f(s)^2 ds < \infty$. For every $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$ we have

$$\begin{aligned} \int_{|x|>1} h(|x|)\nu(dx) &= \int_a^b ds \int_{|x|>1} 1_{\{|f(s)|>1/|x|\}}\nu(dx) \\ &\leqslant \int_a^b ds \int_{|x|>1} (|f(s)x|^2 \wedge 1)\nu(dx) < \infty \end{aligned} \tag{6.3}$$

by (6.1). As $r \uparrow \infty$, $h(r)$ increases to ∞. Choose r_n, $n = 1, 2, \ldots$, such that $1 < r_n < r_{n+1}$ and $h(r_n) \geqslant n$. Let $\rho = \sum_{n=1}^{\infty} n^{-2}\delta_{r_n}$ and let

$$\nu(B) = \int_S \lambda(d\xi) \int_{(1,\infty)} 1_B(r\xi)\,\rho(dr) \quad \text{for } B \in \mathcal{B}(\mathbb{R}^d),$$

where λ is a finite nonzero measure on S. Then $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$ and

$$\begin{aligned} \int_{|x|>1} h(|x|)\nu(dx) &= \int_S \lambda(d\xi) \int_{(1,\infty)} h(r)\rho(dr) \\ &= \lambda(S) \sum_{n=1}^{\infty} h(r_n)\rho(\{r_n\}) \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty, \end{aligned}$$

which contradicts (6.3). Therefore, $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$.

Steps 1 and 2 combined imply (e). □

Theorem 6.2. *The following statements are equivalent.*

(a) $ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}^0(\Phi_f)$ *and* $\Phi_f(ID_{\mathrm{AB}}(\mathbb{R}^d)) \subset ID_{\mathrm{AB}}(\mathbb{R}^d)$.

(b) $ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$ *and* $\Phi_{f,\mathrm{es}}(ID_{\mathrm{AB}}(\mathbb{R}^d)) \subset ID_{\mathrm{AB}}(\mathbb{R}^d)$.

(c) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* $\int_a^b |f(s)| ds < \infty$.

Statement (b) with $\Phi_{f,\mathrm{es}}$ replaced by any one of Φ_f, $\Phi_{f,\mathrm{c}}$, and $\Phi_{f,\mathrm{sym}}$ is equivalent to statements (a)–(c).

Proof of Theorem 6.2. Asssume (c). Let $\mu \in ID_{\mathrm{AB}}$. Then $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ from Theorem 2.10. As (3.11) is clear and (3.10) follows from

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x| \wedge 1)\,\nu(dx) \leqslant \int_a^b (|f(s)| + 1) 1_{\{f(s)\neq 0\}} ds \int_{\mathbb{R}^d} (|x| \wedge 1)\,\nu(dx),$$

Theorem 3.15 is applicable and (a) of Theorem 3.15 holds for all $\mu \in ID_{\mathrm{AB}}$. We have (4.2), recalling the proof of Theorem 2.10. Thus, using Theorem 4.2, we see that (a) is true.

Clearly (a) implies (b).

Assume (b). Then

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x| \wedge 1)\,\nu(dx) < \infty \quad \text{for all } \nu \in \mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d)).$$

Let us show (c) in two steps. The argument is similar to the corresponding part of the proof of Theorem 6.1, but we give a complete proof.

Step 1. Suppose that $\int_a^b |f(s)|ds = \infty$. Let

$$k(r) = \int_a^b |f(s)| 1_{\{|f(s)|\leqslant 1/r\}} ds \quad \text{for } r > 0.$$

For every $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$ we have

$$\begin{aligned}\int_{|x|\leqslant 1} |x|k(|x|)\nu(dx) &= \int_a^b ds \int_{|x|\leqslant 1} |f(s)x| 1_{\{|f(s)|\leqslant 1/|x|\}} \nu(dx) \\ &\leqslant \int_a^b ds \int_{|x|\leqslant 1} (|f(s)x| \wedge 1)\nu(dx) < \infty.\end{aligned}$$

It follows that $k(r)$ is finite for all $r > 0$ and increases to ∞ as $r \downarrow 0$. Choosing r_n, $n = 1, 2, \ldots$, such that $1 > r_n > r_{n+1} > 0$ and $k(r_n) \geqslant n$, let $\rho = \sum_{n=1}^{\infty} n^{-2}\delta_{r_n}$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_{(0,1]} 1_B(r\xi)\, r^{-1} \rho(dr),$$

where λ is a finite nonzero measure on the unit sphere S. Then $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$, since

$$\int_{|x|\leqslant 1} |x|\nu(dx) = \int_S \lambda(d\xi) \int_{(0,1]} \rho(dr) = \lambda(S) \sum_{n=1}^{\infty} n^{-2} < \infty.$$

But

$$\int_{|x|\leqslant 1} |x|k(|x|)\nu(dx) = \lambda(S) \sum_{n=1}^{\infty} k(r_n)\rho(\{r_n\}) \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty,$$

a contradiction. Hence $\int_a^b |f(s)|ds < \infty$.

Step 2. Suppose that $\int_a^b 1_{\{f(s)\neq 0\}} ds = \infty$. Let

$$h(r) = \int_a^b 1_{\{|f(s)|> 1/r\}} ds \quad \text{for } r > 0.$$

Then $h(r) \leqslant r \int_a^b |f(s)|ds < \infty$. As $r \uparrow \infty$, $h(r)$ increases to ∞. For every $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$

$$\begin{aligned}\int_{|x|>1} h(|x|)\nu(dx) &= \int_a^b ds \int_{|x|>1} 1_{\{|f(s)|>1/|x|\}} \nu(dx) \\ &\leqslant \int_a^b ds \int_{|x|>1} (|f(s)x| \wedge 1)\nu(dx) < \infty.\end{aligned}$$

Choosing r_n, $n = 1, 2, \ldots$, such that $1 < r_n < r_{n+1}$ and $h(r_n) \geqslant n$, let $\rho = \sum_{n=1}^{\infty} n^{-2}\delta_{r_n}$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_{(1,\infty)} 1_B(r\xi)\, \rho(dr),$$

with a finite nonzero measure λ on S. Then $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$ and

$$\int_{|x|>1} h(|x|)\nu(dx) = \lambda(S) \sum_{n=1}^{\infty} h(r_n)\rho(\{r_n\}) \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty,$$

a contradiction. Therefore, $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$. □

Example 6.3. Let $(a,b)=(0,b)$ with b finite. Suppose that $\int_t^b f(s)^2 ds<\infty$ for all $t\in(0,b)$ and that $f(s)\asymp s^{-\beta}$ as $s\downarrow 0$. If $0<\beta<1/2$, then f satisfies (e) of Theorem 6.1. If $0<\beta<1$, then f satisfies (c) of Theorem 6.2.

In relation to the two theorems above, it is interesting to consider the condition that $ID_{\mathrm{AB}}\subset\mathfrak{D}(\Phi_{f,\mathrm{es}})$ and the condition that $ID_{\mathrm{AB}}\subset\mathfrak{D}^0(\Phi_f)$. We have the following two theorems.

Theorem 6.4. *The following statements are equivalent.*

(a) $ID_{\mathrm{AB}}(\mathbb{R}^d)\subset\mathfrak{D}(\Phi_{f,\mathrm{es}})$.

(b) *The function $f(s)$ is locally integrable on (a,b), $\int_a^b 1_{\{f(s)\neq 0\}}ds<\infty$,*

$$\int_a^b f(s)^2 1_{\{|f(s)|\leqslant 1/r\}}ds=O(1/r)\quad \text{as } r\downarrow 0, \tag{6.4}$$

and

$$\int_a^b 1_{\{|f(s)|>1/r\}}ds=O(r)\quad \text{as } r\downarrow 0. \tag{6.5}$$

Proof. Let

$$k(r)=\int_a^b f(s)^2 1_{\{|f(s)|\leqslant 1/r\}}ds \quad\text{and}\quad h(r)=\int_a^b 1_{\{|f(s)|>1/r\}}ds.$$

Assume (b). By virtue of Theorem 2.10, $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$ for any $\mu\in ID_{\mathrm{AB}}$. Thus, it follows from Theorem 3.6 that statement (a) is true if

$$\int_a^b ds\int_{\mathbb{R}^d}(|f(s)x|^2\wedge 1)\,\nu(dx)<\infty \quad \text{for all } \nu\in\mathrm{Lvm}(ID_{\mathrm{AB}}). \tag{6.6}$$

Let $\nu\in\mathrm{Lvm}(ID_{\mathrm{AB}})$. Then

$$\begin{aligned}&\int_a^b ds\int_{\mathbb{R}^d}(|f(s)x|^2\wedge 1)\,\nu(dx)\\&\quad\leqslant\int_a^b ds\int_{|x|\leqslant 1}(|f(s)x|^2\wedge 1)\,\nu(dx)+\int_a^b 1_{\{f(s)\neq 0\}}ds\int_{|x|>1}\nu(dx)=I_1+I_2,\end{aligned}$$

say. Clearly I_2 is finite. Using (6.4), (6.5), and $\int_a^b 1_{\{f(s)\neq 0\}}ds<\infty$, we have

$$\begin{aligned}I_1&=\int_{|x|\leqslant 1}|x|^2\nu(dx)\int_a^b f(s)^2 1_{\{|f(s)|\leqslant 1/|x|\}}ds+\int_{|x|\leqslant 1}\nu(dx)\int_a^b 1_{\{|f(s)|>1/|x|\}}ds\\&=\int_{|x|\leqslant 1}|x|^2 k(|x|)\nu(dx)+\int_{|x|\leqslant 1}h(|x|)\nu(dx)\leqslant c\int_{|x|\leqslant 1}|x|\nu(dx)<\infty,\end{aligned}$$

where c is a constant. Hence (6.6) holds. Hence (a) is true.

Conversely, assume (a). Then, for every $\mu\in ID_{\mathrm{AB}}$ with triplet $(0,\nu,\gamma^0)_0$, $f\in\mathbf{L}_{(a,b)}(X^{(\mu)})$, and hence

$$\int_p^q |f(s)|\left|\gamma^0+\int_{\mathbb{R}^d}\frac{x\nu(dx)}{1+|f(s)x|^2}\right|ds<\infty$$

for all p,q with $a<p<q<b$ (recall Theorem 2.6). Considering the case $\nu=0$, we see that $\int_p^q |f(s)|ds<\infty$, that is, $f(s)$ is locally integrable on (a,b). Further (a) implies (6.6). Thus, for every $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$,

$$\begin{aligned}\int_{|x|\leqslant 1} |x|^2 k(|x|)\,\nu(dx) &= \int_a^b f(s)^2 ds \int_{|x|\leqslant 1} |x|^2 1_{\{|f(s)|\leqslant 1/|x|\}}\nu(dx)\\ &\leqslant \int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\,\nu(dx) < \infty\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^d} h(|x|)\,\nu(dx) &= \int_a^b ds \int_{\mathbb{R}^d} 1_{\{|f(s)|>1/|x|\}}\nu(dx)\\ &\leqslant \int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\,\nu(dx) < \infty.\end{aligned}$$

Using an appropriate ν, we see that $k(r)$ and $h(r)$ are finite almost everywhere.

Step 1. Suppose that $\int_a^b 1_{\{f(s)\neq 0\}} ds = \infty$. Then we have a contradiction exactly in the same way as in Step 2 in the proof of Theorem 6.2. Hence $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$.

Step 2. Suppose that $\limsup_{r\downarrow 0} r^{-1}h(r) = \infty$. Choose a sequence $r_n \leqslant 1$ decreasing to 0 such that ${r_n}^{-1}h(r_n) \geqslant n$. Let $\rho = \sum_{n=1}^{\infty} n^{-2}\delta_{r_n}$ and

$$\nu(B) = \int_S \lambda(d\xi) \int_{(0,1]} 1_B(r\xi) r^{-1}\rho(dr)$$

with a finite nonzero measure λ on S. Then $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$ but

$$\int_{|x|\leqslant 1} h(|x|)\,\nu(dx) = \lambda(S) \sum_{n=1}^{\infty} h(r_n){r_n}^{-1} n^{-2} \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty,$$

which is a contradiction. Hence we obtain (6.5).

Step 3. Suppose that $\limsup_{r\downarrow 0} r\,k(r) = \infty$. Choose a sequence $r_n \leqslant 1$ decreasing to 0 such that $r_n k(r_n) \geqslant n$. Define ρ and ν by the same formulas as in Step 2. Then $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$ but

$$\int_{|x|\leqslant 1} |x|^2 k(|x|)\,\nu(dx) = \lambda(S) \sum_{n=1}^{\infty} r_n k(r_n) n^{-2} \geqslant \lambda(S) \sum_{n=1}^{\infty} n^{-1} = \infty,$$

a contradiction. Thus (6.4) is true. □

Theorem 6.5. *The following statements are equivalent.*

(a) $ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}^0(\Phi_f)$.

(b) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* $\int_a^b |f(s)| ds < \infty$.

Proof. Assume (a). Let $\nu \in \mathrm{Lvm}(ID_{\mathrm{AB}})$. Then, for any γ^1 and γ^2 in $\mathbb{R}^d$, $\mu_{(0,\nu,\gamma^1)}$ and $\mu_{(0,\nu,\gamma^1)}$ are in $\mathfrak{D}^0(\Phi_f)$. Recall Theorem 4.2. We see that (4.2) is true for $\gamma = \gamma^1$ and $\gamma = \gamma^2$. Hence $\int_a^b |f(s)(\gamma^1 - \gamma^2)| ds < \infty$. It follows that $\int_a^b |f(s)| ds < \infty$. We can prove that $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ in the same way as Step 2 in the proof of Theorem 6.1.

Conversely, assume (b). Then, $ID_{\mathrm{AB}} \subset \mathfrak{D}^0(\Phi_f)$ and $\Phi_f(ID_{\mathrm{AB}}) \subset ID_{\mathrm{AB}}$ by virtue of Theorem 6.2. Thus, a fortiori, (a) holds. □

Remark 6.6. Consider the following conditions:

$$ID = \mathfrak{D}^0(\Phi_f), \tag{6.7}$$

$$ID = \mathfrak{D}(\Phi_{f,\mathrm{es}}), \tag{6.8}$$

$$ID_{\mathrm{AB}} \subset \mathfrak{D}^0(\Phi_f) \quad \text{and} \quad \Phi_f(ID_{\mathrm{AB}}) \subset ID_{\mathrm{AB}}, \tag{6.9}$$

$$ID_{\mathrm{AB}} \subset \mathfrak{D}(\Phi_{f,\mathrm{es}}) \quad \text{and} \quad \Phi_{f,\mathrm{es}}(ID_{\mathrm{AB}}) \subset ID_{\mathrm{AB}}, \tag{6.10}$$

$$ID_{\mathrm{AB}} \subset \mathfrak{D}^0(\Phi_f), \tag{6.11}$$

$$ID_{\mathrm{AB}} \subset \mathfrak{D}(\Phi_{f,\mathrm{es}}). \tag{6.12}$$

Then, it follows from the theorems in this section that

$$(6.7) \Leftrightarrow (6.8) \Rightarrow (6.9) \Leftrightarrow (6.10) \Leftrightarrow (6.11) \Rightarrow (6.12).$$

Further, we can show that condition (6.8) is strictly stronger than condition (6.10) and that condition (6.10) is strictly stronger than condition (6.12). They are proved by the use of the analytical expressions of the conditions. Indeed, it is obvious that (6.8) is strictly stronger than (6.10), since there is $f(s)$ on a finite open interval (a,b) such that $\int_a^b |f(s)|ds < \infty$ but $\int_a^b f(s)^2 ds = \infty$. To show that (6.10) is strictly stronger than (6.12), consider the function $f(s)$ in Example 6.7 or 6.8 below.

Example 6.7. Let $f(s)$ be as in Proposition 5.3 (iv). Thus $(a,b) = (0,b)$ with b finite and $f(s) \asymp s^{-1}$ as $s \downarrow 0$. Then, $f(s)$ satisfies (6.4) and (6.5) and $\int_0^1 f(s)ds = \infty$. Hence this $f(s)$ satisfies (6.12) but does not satisfy (6.10). We have shown $ID_{\mathrm{AB}} = \mathfrak{D}(\Phi_{f,\mathrm{es}})$ in Proposition 5.3. We have

$$\begin{aligned}&\{\mu \in ID_{\mathrm{AB}} \colon \Phi_{f,\mathrm{es}}(\mu) \subset ID_{\mathrm{AB}}\}\\ &\quad = \left\{\mu = \mu_{(0,\nu,\gamma)} \in ID_{\mathrm{AB}} \colon \int_{|x|\leqslant 1/2} |x| \log(1/|x|)\, \nu(dx) < \infty\right\}.\end{aligned} \tag{6.13}$$

This again shows that $\Phi_{f,\mathrm{es}}(ID_{\mathrm{AB}}) \not\subset ID_{\mathrm{AB}}$.

Let us prove (6.13). Let $\mu \in ID_{\mathrm{AB}}$. Let $\widetilde{\nu}$ denote the Lévy measure of $\Phi_{f,\mathrm{es}}(\mu)$. Then $\widetilde{\nu}(B) = \int_0^b ds \int_{\mathbb{R}^d} 1_B(f(s)x)\, \nu(dx)$. Hence

$$\begin{aligned}\int_{|x|\leqslant 1} |x| \widetilde{\nu}(dx) &\leqslant c_2 \int_0^{s_0} ds \int_{\mathbb{R}^d} |s^{-1}x| 1_{\{c_1|s^{-1}x|\leqslant 1\}} \nu(dx) + c_3\\ &= c_2 \int_{|x|\leqslant c_1^{-1}s_0} |x| \nu(dx) \int_{c_1|x|}^{s_0} s^{-1} ds + c_3 \leqslant c_2 \int_{|x|\leqslant c_1^{-1}s_0} |x| \log(1/|x|)\, \nu(dx) + c_4,\end{aligned}$$

and the converse estimate is similar.

Example 6.8. Let $(a,b) = (0,b)$ with b finite and $f(s) \asymp s^{-1}(\log(1/s))^{-1}$ as $s \downarrow 0$. We can prove that

$$\begin{aligned}\mathfrak{D}(\Phi_{f,\mathrm{es}}) = \{\mu = \mu_{(A,\nu,\gamma)} \in ID \colon A = 0 \text{ and}\\ \int_{|x|\leqslant 1/2} |x| (\log(1/|x|))^{-1} \nu(dx) < \infty\}\end{aligned} \tag{6.14}$$

and

$$\begin{aligned}&\{\mu \in ID_{\mathrm{AB}} \colon \Phi_{f,\mathrm{es}}(\mu) \subset ID_{\mathrm{AB}}\}\\ &\quad = \left\{\mu = \mu_{(0,\nu,\gamma)} \in ID_{\mathrm{AB}} \colon \int_{|x|\leqslant 1/(2e)} |x| \log\log(1/|x|)\, \nu(dx) < \infty\right\}.\end{aligned} \tag{6.15}$$

It follows from (6.14) that

$$ID_{\mathrm{AB}} \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{es}}). \tag{6.16}$$

The assertion (6.15) implies that $\Phi_{f,\mathrm{es}}(ID_{\mathrm{AB}}) \not\subset ID_{\mathrm{AB}}$, but this fact follows also from $\int_0^{1/2} f(s)ds = \infty$. Thus, like Example 6.7, this example satisfies (6.12) and does not satisfy (6.10). However, property (6.16) differs from property $ID_{\mathrm{AB}} = \mathfrak{D}(\Phi_{f,\mathrm{es}})$ of Example 6.7.

Let us prove (6.14). Let $\mu = \mu_{(A,\nu,\gamma)}$. Then $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ if and only if $A = 0$ and (3.2) holds. We have

$$c_1 s^{-1}(\log(1/s))^{-1} \leqslant f(s) \leqslant c_2 s^{-1}(\log(1/s))^{-1}$$

for $0 < s < s_0$ with $c_1, c_2 > 0$ and $s_0 \in (0, b \wedge (1/2))$. Now

$$\begin{aligned}\int_0^{s_0} ds \int_{|f(s)x|>1} \nu(dx) &= \int_{|x|>1} \nu(dx) \int_0^{s_0} 1_{\{|f(s)x|>1\}} ds \\ &\quad + \int_{|x|\leqslant 1} \nu(dx) \int_0^{s_0} 1_{\{|f(s)x|>1\}} ds = I_1 + I_2 \quad \text{(say)},\\ \int_0^{s_0} ds \int_{|f(s)x|\leqslant 1} |f(s)x|^2 \nu(dx) &= \int_{|x|>1} \nu(dx) \int_0^{s_0} |f(s)x|^2 1_{\{|f(s)x|\leqslant 1\}} ds \\ &\quad + \int_{|x|\leqslant 1} |x|^2 \nu(dx) \int_0^{s_0} f(s)^2 1_{\{|f(s)x|\leqslant 1\}} ds = J_1 + J_2 \quad \text{(say)}.\end{aligned}$$

Both I_1 and J_1 are bounded by $s_0 \int_{|x|>1} \nu(dx)$. We have

$$I_2 \leqslant \int_{|x|\leqslant 1} \nu(dx) \int_0^{s_0} 1_{\{s\log(1/s) < c_2|x|\}} ds,$$

$$J_2 \leqslant c_2 \int_{|x|\leqslant 1} |x|^2 \nu(dx) \int_0^{s_0} s^{-2}(\log(1/s))^{-2} 1_{\{s\log(1/s)\geqslant c_1|x|\}} ds,$$

and similar estimates from below. Then $I_2 + J_2 < \infty$ if and only if

$$\int_{|x|\leqslant 1/2} |x|(\log(1/|x|))^{-1} \nu(dx) < \infty,$$

since, letting $u = s\log(1/s)$, we have $du/ds = \log(1/s) - 1 \sim \log(1/u)$ as $s \downarrow 0$ (equivalently, as $u \downarrow 0$), and since, as $r \downarrow 0$,

$$\int_0^{s_0} 1_{\{s\log(1/s)<r\}} ds \sim \int_0^r \left(\frac{du}{ds}\right)^{-1} du \sim \int_0^r (\log(1/u))^{-1} du \sim r(\log(1/r))^{-1}$$

and

$$\begin{aligned}\int_0^{s_0} s^{-2}(\log(1/s))^{-2} 1_{\{s\log(1/s)\geqslant r\}} ds &\sim \int_r^{\varepsilon} u^{-2} \left(\frac{du}{ds}\right)^{-1} du \\ &\sim \int_r^{\varepsilon} u^{-2}(\log(1/u))^{-1} du \sim r^{-1}(\log(1/r))^{-1}\end{aligned}$$

with a small number $\varepsilon > 0$.

Proof of (6.15) is as follows. Let $\mu \in ID_{\mathrm{AB}}$. Then $\mu \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$ from (6.14). For the Lévy measure $\widetilde{\nu}$ of any distribution in $\Phi_{f,\mathrm{es}}(\mu)$, we have

$$\begin{aligned}\int_{|x|\leqslant 1} |x| \widetilde{\nu}(dx) &\leqslant \int_{\mathbb{R}^d} c_2 |x| \nu(dx) \int_0^{s_0} s^{-1}(\log(1/s))^{-1} 1_{\{s\log(1/s)\geqslant c_1|x|\}} ds + c_3 \\ &= I_1 + I_2 + c_3,\end{aligned}$$

where I_1 and I_2 are the repeated integrals with the integration over $\mathbb{R}^d$ replaced by that over $\{|x|>1\}$ and $\{|x|\leqslant 1\}$, respectively. Then

$$I_1 \leqslant c_1^{-1}c_2 \int_{|x|>1} \nu(dx) \int_0^{s_0} ds < \infty$$

and, with $u = s\log(1/s)$ and some $\varepsilon > 0$,

$$\int_0^{s_0} s^{-1}(\log(1/s))^{-1} 1_{\{s\log(1/s)\geqslant r\}} ds \sim \int_r^{\varepsilon} u^{-1}\left(\frac{du}{ds}\right)^{-1} du$$
$$\sim \int_r^{\varepsilon} u^{-1}(\log(1/u))^{-1} du \sim \log\log(1/r)$$

as $r \downarrow 0$. Hence, $I_2 < \infty$ if and only if

$$\int_{|x|\leqslant 1/(2e)} |x| \log\log(1/|x|)\, \nu(dx) < \infty.$$

The estimate of $\int_{|x|\leqslant 1} |x|\widetilde{\nu}(dx)$ from below is similar.

Example 6.9. Let $g_\alpha(u) = \int_u^\infty v^{-\alpha-1}e^{-v}dv$ for $u \in (0,\infty)$ with $\alpha \in \mathbb{R}$. Let $b_\alpha = g_\alpha(0+)$. Thus $b_\alpha = \Gamma(-\alpha)$ for $\alpha < 0$ and $b_\alpha = \infty$ for $\alpha \geqslant 0$. Consider $(0, b_\alpha)$ as (a,b). Let $u = f_\alpha(s)$, $s \in (0, b_\alpha)$, be the inverse function of $s = g_\alpha(u)$, $u \in (0,\infty)$. Then $f_\alpha(s)$ strictly decreases from ∞ to 0 as s goes from 0 to b_α. We have

$$f_\alpha(s) \sim \log(1/s) \quad \text{as } s \downarrow 0 \text{ for } \alpha \in \mathbb{R},$$

as Proposition 1.1 of (2006c) says. It follows that, for $f = f_\alpha$ with $\alpha < 0$, (6.7)–(6.12) hold. For $\alpha \geqslant 0$, Proposition 1.1 of (2006c) shows that, as $s \uparrow \infty$,

$$f_0(s) \sim ce^{-s} \quad \text{with a constant } c > 0,$$
$$f_\alpha(s) \sim (\alpha s)^{-1/\alpha} \quad \text{for } \alpha > 0,$$
$$f_1(s) = s^{-1} - s^{-2}\log s + o(s^{-2}\log s).$$

The five domains in Theorem 4.3 for $f = f_\alpha$ on $(0,\infty)$ with $\alpha \geqslant 0$ are described in (2006c), Theorems 2.4 and 2.8. The case $\alpha = 1$ is a special case of Example 4.5. We have $f_{-1}(s) = \log(1/s)$ for $s \in (0, b_{-1}) = (0,1)$. This case ($\alpha = -1$) was first introduced in Barndorff-Nielsen and Thorbjørnsen (2002) with the notation Υ for $\Phi_{f_{-1}}$ and explored by Barndorff-Nielsen et al. (2006) in connection with the Goldie–Steutel–Bondesson class $B(\mathbb{R}^d)$ and the Thorin class $T(\mathbb{R}^d)$.

Example 6.10. Let $g_{\beta,\alpha}(u) = (\Gamma(\alpha-\beta))^{-1}\int_u^1 (1-v)^{\alpha-\beta-1}v^{-\alpha-1}dv$ for $u \in (0,1)$ with $-\infty < \beta < \alpha < \infty$. Let $b_{\beta.\alpha} = g_{\beta,\alpha}(0+)$, which equals $\Gamma(-\alpha)/\Gamma(-\beta)$ for $\alpha < 0$ and ∞ for $\alpha \geqslant 0$. Let $u = f_{\beta,\alpha}(s)$, $s \in (0, b_{\beta,\alpha})$, be the inverse function of $s = g_{\beta,\alpha}(u)$, $u \in (0,1)$. Then $f_{\beta,\alpha}(s)$ strictly decreases from 1 to 0 as s goes from 0 to $b_{\beta,\alpha}$. Now Proposition 1.1 of (2006c) says that, as $s \uparrow \infty$,

$$f_{\beta,0}(s) \sim c_\beta e^{-\Gamma(-\beta)s} \quad \text{with a constant } c_\beta > 0 \text{ for } \beta < 0,$$
$$f_{\beta,\alpha}(s) \sim (\alpha\Gamma(\alpha-\beta)s)^{-1/\alpha} \quad \text{for } \alpha > 0 \text{ and } \beta < \alpha,$$
$$f_{\beta,1}(s) = (\Gamma(1-\beta))^{-1}s^{-1} + \beta(\Gamma(1-\beta))^{-2}s^{-2}\log s + o(s^{-2}\log s) \quad \text{for } \beta < 1.$$

From these behaviors Theorems 2.4 and 2.8 of (2006c) show that the five domains in Theorem 4.3 for $f = f_{\beta,\alpha}$ on $(0,\infty)$ with $\alpha \geqslant 0$ do not depend on β and are the same as those for $f = f_\alpha$ on $(0,\infty)$. We have $f_{-1,0}(s) = e^{-s}$ and thus $\Phi_{f_{-1,0}}$ equals

Φ of Example 4.6. The family $\{\Phi_{f_{\beta,\alpha}}\}$ has a close connection with the family $\{\Phi_{f_\alpha}\}$ in Example 6.9. Namely, Theorem 3.1 of (2006c) proves that

$$\Phi_{f_\alpha} = \Phi_{f_\beta}\Phi_{f_{\beta,\alpha}} = \Phi_{f_{\beta,\alpha}}\Phi_{f_\beta} \quad \text{for } -\infty < \beta < \alpha < \infty,$$

including the equality of the domains of both sides. A special case of this equality with $\alpha = 0$ and $\beta = -1$ is given in Barndorff-Nielsen et al. (2006).

At the end of this section, let us consider the case where $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ is very small.

Theorem 6.11. $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ *equals the class* $\{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$ *if and only if*

$$f(s) \text{ is locally integrable on } (a,b) \text{ and } \int_a^b (f(s)^2 \wedge 1)ds = \infty. \tag{6.17}$$

Notice that $\int_a^b (f(s)^2 \wedge 1)ds = \infty$ implies $\int_a^b f(s)^2 ds = \infty$ and $b - a = \infty$, but that the converse is not true. For example, consider $(a,b) = (0,\infty)$ and $f(s) = \sum_{n=1}^\infty n 1_{[n,n+n^{-2})}(s)$.

Proof of Theorem 6.11. We use Theorems 2.6 and 3.6.

The "only if" part: The function $f(s)$ is locally integrable on (a,b), since $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ for any $\mu = \delta_\gamma$ with $\gamma \in \mathbb{R}^d$. Further $\int_a^b (f(s)^2 \wedge 1)ds = \infty$, because otherwise $\mu = \mu_{(0,\nu,0)}$ with $\nu = \delta_{x_0}$, $|x_0| = 1$, belongs to $\mathfrak{D}(\Phi_{f,\mathrm{es}})$. Hence (6.17) holds.

The "if" part: Let $\mu = \mu_{(A,\nu,\gamma)} \in \mathfrak{D}(\Phi_{f,\mathrm{es}})$. Then $A = 0$, since $\int_a^b f(s)^2 ds = \infty$. If $\nu \neq 0$, then we have, using $0 < c \leqslant 1$ such that $\nu(\{|x| > c\}) > 0$,

$$\begin{aligned}\infty &> \int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\,\nu(dx) \geqslant \int_a^b ds \int_{|x|>c} ((f(s)^2c^2) \wedge 1)\,\nu(dx) \\ &\geqslant c^2 \int_{|x|>c} \nu(dx) \int_a^b (f(s)^2 \wedge 1)ds,\end{aligned}$$

which contradicts (6.17). Hence $\mu = \delta_\gamma$. Conversely, any $\mu = \delta_\gamma$ is in $\mathfrak{D}(\Phi_{f,\mathrm{es}})$. □

Remark 6.12. (i) $\mathfrak{D}(\Phi_{f,\mathrm{c}}) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$ if and only if (6.17) holds.

(ii) $\mathfrak{D}^0(\Phi_f) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$ if and only if (6.17) holds and $\int_a^b |f(s)|ds < \infty$.

(iii) $\mathfrak{D}(\Phi_f) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}$ if and only if (6.17) holds and $\int_{a+}^{b-} f(s)ds$ exists in $\mathbb{R}$.

(iv) $\mathfrak{D}^0(\Phi_f) = \{\delta_0\}$ if and only if $\int_a^b (f(s)^2 \wedge 1)ds = \infty$ and $\int_a^b |f(s)|ds = \infty$.

(v) Assume that $f(s)$ is locally integrable on (a,b). Then, $\mathfrak{D}(\Phi_f) = \{\delta_0\}$ if and only if $\int_a^b (f(s)^2 \wedge 1)ds = \infty$ and $\int_p^q f(s)ds$ is not convergent in $\mathbb{R}$ as $p \downarrow a$ and $q \uparrow b$.

These facts are proved similarly to Theorem 6.11. Use $\nu = \delta_{x_0} + \delta_{-x_0}$ with $|x_0| = 1$ instead of $\nu = \delta_{x_0}$.

Example 6.13. Let $(a,b) = (a,\infty)$ with a finite. Suppose that $\int_a^t f(s)^2 ds < \infty$ for all $t \in (a,\infty)$ and that $f(s) \asymp s^{-1/\alpha}$ as $s \to \infty$. If $\alpha \geqslant 2$, then

$$\mathfrak{D}^0(\Phi_f) = \mathfrak{D}(\Phi_f) = \{\delta_0\} \subsetneqq \mathfrak{D}(\Phi_{f,\mathrm{c}}) = \mathfrak{D}(\Phi_{f,\mathrm{es}}) = \{\delta_\gamma \colon \gamma \in \mathbb{R}^d\}.$$

7. The τ-measure of function f

Let us define the τ-measure of f. Functions $f(s)$ and $f_j(s)$ in the following are $\mathbb{R}$-valued. We fix the dimension d in $ID(\mathbb{R}^d)$.

Definition 7.1. Let $f(s)$ be a measurable function $f(s)$ on (a, b) with $-\infty \leqslant a < b \leqslant \infty$. Define a measure τ on $\mathbb{R}$ as

$$\tau(B) = \int_a^b 1_{\{f(s)\in B\}} ds, \qquad B \in \mathcal{B}_{\mathbb{R}}. \tag{7.1}$$

We call τ the τ-*measure* of function f.

It follows from (7.1) that

$$\int_{\mathbb{R}} h(u)\tau(du) = \int_a^b h(f(s))ds \tag{7.2}$$

for all nonnegative measurable functions h on $\mathbb{R}$.

We discuss two questions. The first is whether the τ-measure τ of f determines the domain $\mathfrak{D}(\Phi_f)$ and its variants. The second is under what conditions a given measure τ is the τ-measure of some f.

Theorem 7.2. *Suppose that $f_1(s)$ and $f_2(s)$ are measurable functions on (a_1, b_1) and (a_2, b_2), respectively, with identical τ-measure. Then,*

$$\mathfrak{D}^0(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_2}), \tag{7.3}$$

$$\Phi_{f_1}(\mu) = \Phi_{f_2}(\mu) \quad \textit{for all } \mu \in \mathfrak{D}^0(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_2}), \tag{7.4}$$

$$\mathfrak{D}(\Phi_{f_1,\mathrm{es}}) = \mathfrak{D}(\Phi_{f_2,\mathrm{es}}), \tag{7.5}$$

$$\Phi_{f_1,\mathrm{es}}(\mu) = \Phi_{f_2,\mathrm{es}}(\mu) \quad \textit{for all } \mu \in \mathfrak{D}(\Phi_{f_1,\mathrm{es}}) = \mathfrak{D}(\Phi_{f_2,\mathrm{es}}). \tag{7.6}$$

Proof. It follows from (7.2) that, for any $\mu \in ID(\mathbb{R}^d)$,

$$\int_{a_j}^{b_j} |C_\mu(f_j(s)z)|ds = \int_{\mathbb{R}} |C_\mu(uz)|\,\tau(du) \quad \text{for } j = 1, 2,\ z \in \mathbb{R}^d.$$

Hence we obtain (7.3) from Definition 4.1.

Proof of (7.4). Let $\mu \in \mathfrak{D}^0(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_2})$. We have $\int_{\mathbb{R}} |C_\mu(uz)|\,\tau(du) < \infty$. Hence $C_\mu(f_j(s)z)$ is integrable on (a_j, b_j) and it follows from (7.2) that

$$\int_{a_j}^{b_j} C_\mu(f_j(s)z)ds = \int_{\mathbb{R}} C_\mu(uz)\,\tau(du).$$

Denote $\widetilde{\mu}_j = \Phi_{f_j}(\mu)$. Then

$$C_{\widetilde{\mu}_j}(z) = \lim_{p\downarrow a_j, q\uparrow b_j} \int_p^q C_\mu(f_j(s)z)ds = \int_{a_j}^{b_j} C_\mu(f_j(s)z)ds.$$

Thus $\widetilde{\mu}_1 = \widetilde{\mu}_2$.

Proof of (7.5). Notice that

$$\int_{a_j}^{b_j} f_j(s)^2 \operatorname{tr} A\, ds = \int_{\mathbb{R}} u^2 \operatorname{tr} A\, \tau(du),$$

$$\int_{a_j}^{b_j} ds \int_{\mathbb{R}^d} (|f_j(s)x|^2 \wedge 1)\, \nu(dx) = \int_{\mathbb{R}} \tau(du) \int_{\mathbb{R}^d} (|ux|^2 \wedge 1)\, \nu(dx).$$

Use Theorem 3.6.

Proof of (7.6). Notice that

$$\int_{a_j}^{b_j} f_j(s)^2\,A\,ds = \int_{\mathbb{R}} u^2\,A\,\tau(du),$$

$$\int_{a_j}^{b_j} ds \int_{\mathbb{R}^d} 1_B(f_j(s)x)\,\nu(dx) = \int_{\mathbb{R}} \tau(du) \int_{\mathbb{R}^d} 1_B(ux)\,\nu(dx)$$

for $B \in \mathcal{B}(\mathbb{R}^d)$ with $0 \notin B$, and use Theorem 3.11. □

Proposition 7.3. (i) *There are measurable functions $f_1(s)$ and $f_2(s)$ on $(0,\infty)$ with identical τ-measure, such that $\mathfrak{D}(\Phi_{f_1}) \neq \mathfrak{D}(\Phi_{f_2})$.*

(ii) *There are measurable functions $f_1(s)$ and $f_2(s)$ on $(0,\infty)$ with identical τ-measure, such that $\Phi_{f_1}(\mu) \neq \Phi_{f_2}(\mu)$ for some $\mu \in \mathfrak{D}(\Phi_{f_1}) \cap \mathfrak{D}(\Phi_{f_2})$.*

Proof. We prove (i) and (ii) together. In the following f_3 serves as f_2 in (ii). Let $f_1(s) = s^{-1}\sin s$ for $s \in (0,\infty)$. Let $c_n = \int_{n\pi}^{(n+1)\pi} s^{-1}\sin s\,ds$. Then

$$\int_0^{\infty-} f_1(s)ds = \sum_{n=0}^{\infty} c_n = \frac{\pi}{2}.$$

Let $\sum_{m=0}^{\infty} d_m$ be a rearrangement of $\sum_{n=0}^{\infty} c_n$ (that is for each m there is a unique n such that $d_m = c_n$, and for each n there is a unique m such that $c_n = d_m$), such that $\sum_{m=0}^{\infty} d_m = \infty$. Let $\sum_{m=0}^{\infty} e_m$ be a rearrangement of $\sum_{n=0}^{\infty} c_n$ such that $\sum_{m=0}^{\infty} e_m = 0$. Those rearrangements exist, since $\sum_{n=0}^{\infty} c_n$ is convergent but not absolutely. Define $f_2(m\pi + s) = f_1(n\pi + s)$ for $0 < s \leqslant \pi$ if $d_m = c_n$. Similarly define $f_3(m\pi + s) = f_1(n\pi + s)$ for $0 < s \leqslant \pi$ if $e_m = c_n$. Then f_1, f_2, and f_3 have an identical τ-measure τ. We have

$$\int_0^{\infty-} f_2(s)ds = \sum_{m=0}^{\infty} d_m = \infty \quad \text{and} \quad \int_0^{\infty-} f_3(s)ds = \sum_{m=0}^{\infty} e_m = 0.$$

Notice that $\int_0^{\infty} f_j(s)^2 ds = \int_{\mathbb{R}} u^2\tau(du) < \infty$ for $j = 1,2,3$. Now consider $\mu = \mu_{(A,\nu,\gamma)}$ with ν symmetric and $\gamma \neq 0$. Use Theorems 3.5 and 3.10. Then we see that μ belongs to $\mathfrak{D}(\Phi_{f_1})$ and $\mathfrak{D}(\Phi_{f_3})$, but not to $\mathfrak{D}(\Phi_{f_2})$; $\Phi_{f_1}(\mu)$ and $\Phi_{f_3}(\mu)$ have a common Gaussian part and a common Lévy measure, but the location parameter of $\Phi_{f_1}(\mu)$ equals $(\pi/2)\gamma$ and that of $\Phi_{f_3}(\mu)$ equals 0. □

Proposition 7.4. *Suppose that $f_1(s)$ and $f_2(s)$ are measurable functions on (a_1,b_1) and (a_2,b_2), respectively, with identical τ-measure τ and that $\tau(\mathbb{R}\setminus\{0\}) < \infty$ and $\int_{\mathbb{R}} u^2\tau(du) < \infty$. Then,*

$$\mathfrak{D}^0(\Phi_{f_1}) = \mathfrak{D}^0(\Phi_{f_2}) = ID(\mathbb{R}^d) \tag{7.7}$$

and

$$\Phi_{f_1}(\mu) = \Phi_{f_2}(\mu) \quad \textit{for all } \mu \in ID(\mathbb{R}^d). \tag{7.8}$$

Proof. Since

$$\int_{a_j}^{b_j} 1_{\{f_j(s)\neq 0\}}ds = \tau(\mathbb{R}\setminus\{0\}) \quad \text{and} \quad \int_{a_j}^{b_j} f_j(s)^2 ds = \int_{\mathbb{R}} u^2\tau(du),$$

we can apply Theorem 6.1 to show (7.7). We obtain (7.8) from (7.4) of Theorem 7.2. □

Example 7.5. Let τ be a measure on $\mathbb{R}$ with $\tau(\mathbb{R}) = b < \infty$ and let $g(u) = \tau((u,\infty))$. Assume that $g(u)$ is continuous and strictly decreasing from b to 0 as u moves from $-\infty$ to ∞. Let $u = f(s)$, $s \in (0,b)$, be the inverse function of $s = g(u)$. Then $f(s)$ is continuous and strictly decreasing from ∞ to $-\infty$ as s moves from 0 to b. The measure τ is recovered as the τ-measure of f, since

$$\tau((u_1,u_2]) = g(u_1) - g(u_2) = \int_0^b 1_{[g(u_2),g(u_1))}(s)ds = \int_0^b 1_{(u_1,u_2]}(f(s))ds$$

for $u_1 < u_2$.

For example, let τ be standard Gaussian distribution on $\mathbb{R}$. Then $u = f(s)$, $s \in (0,1)$, is the inverse function of $s = g(u) = (2\pi)^{-1/2}\int_u^\infty e^{-v^2/2}dv$, $u \in \mathbb{R}$, and Proposition 7.4 applies. These τ and f are considered by Aoyama and Maejima (2007). They show that the range $\{\Phi_f(\mu)\colon \mu \in ID(\mathbb{R}^d)\}$ is the class of multivariate type G distributions introduced by Maejima and Rosiński (2002).

Example 7.6. Let τ be a measure on $(0,\infty)$ with total mass $b \leqslant \infty$ such that $g(u) = \tau((u,\infty))$, $u > 0$, is finite, continuous, and strictly decreasing from b to 0 as u moves from 0 to ∞. Let $u = f(s)$, $s \in (0,b)$, be the inverse function of $s = g(u)$. Then $f(s)$ is continuous and strictly decreasing from ∞ to 0 as s moves from 0 to b. The measure τ is the τ-measure of f. The pairs $g_\alpha(u)$ and $f_\alpha(s)$ with $\alpha \in \mathbb{R}$ in Example 6.9 are special cases; in particular, if $\alpha < 0$, then the τ-measure is Γ-distribution (with parameters $-\alpha$, 1) multiplied by $\Gamma(-\alpha)$ and Proposition 7.4 applies. For another example, if τ is Mittag-Leffler distribution with parameter $\alpha \in (0,1)$ (see Example 24.12 of Sato (1999)), then it satisfies the condition above and Proposition 7.4 again applies (τ has finite moments of all orders as is shown in p. 74 of Barndorff-Nielsen and Thorbjørnsen (2006b)), the corresponding Φ_f is studied by Barndorff-Nielsen and Thorbjørnsen (2006a) with notation Υ^α.

We introduce some conditions on f and τ.

Definition 7.7. We say that a function $f(s)$ on (a,b) satisfies *Condition* (A) if $f(s)$ is decreasing, left-continuous, not constant, and

$$\inf_{t\in(a,b)} f(t) < f(s) < \sup_{t\in(a,b)} f(t) \quad \text{for all } s \in (a,b). \tag{7.9}$$

The left-continuity in Condition (A) being inessential in the following sense. If $f(s)$ satisfies Condition (A) except the left-continuity requirement, then the left-continuous modification $f^-(s)$ defined by $f^-(s) = f(s-)$ satisfies Condition (A) and, for all $\mu \in ID(\mathbb{R}^d)$,

$$\int_p^q f(s)X^{(\mu)}(ds) = \int_p^q f^-(s)X^{(\mu)}(ds) \quad \text{whenever } a < p < q < b,$$

because $f(s) = f^-(s)$ except for at most countably many s.

Definition 7.8. We say that a measure τ on $\mathbb{R}$ satisfies *Condition* (B) if τ is not identically zero and if, for $a' = \inf \operatorname{Supp}(\tau)$ and $b' = \sup \operatorname{Supp}(\tau)$, τ has the following properties:

$$a' < b', \tag{7.10}$$

$$\tau((p,q)) < \infty \quad \text{whenever } a' < p < q < b', \tag{7.11}$$

$$\text{either } a' = -\infty, \text{ or } a' > -\infty \text{ and } \tau(\{a'\}) = 0, \tag{7.12}$$

$$\text{either } b' = \infty, \text{ or } b' < \infty \text{ and } \tau(\{b'\}) = 0, \tag{7.13}$$

Theorem 7.9. (i) *Let $-\infty \leqslant a < b \leqslant \infty$. If $f(s)$ is a function on (a, b) satisfying Condition (A), then the τ-measure τ of f satisfies Condition (B).*

(ii) *If τ is a measure on $\mathbb{R}$ satisfying Condition (B), then there are an interval (a, b) with $-\infty \leqslant a < b \leqslant \infty$ and a function $f(s)$ on (a, b) satisfying Condition (A) such that τ is the τ-measure of f.*

Assertion (i) of this theorem is straightforward from the definitions of Conditions (A) and (B). In order to show (ii), we prepare a lemma, which is an extension of Lemma 7.1 of Meyer (1962).

Lemma 7.10. *Let $-\infty \leqslant A' < B' \leqslant -\infty$ and let $G(u)$ be an $\mathbb{R}$-valued, increasing, right-continuous function on (A', B') which is not a constant function. Let*

$$A = \inf_{u \in (A', B')} G(u) \quad \text{and} \quad B = \sup_{u \in (A', B')} G(u). \tag{7.14}$$

Define

$$F(s) = \inf\{u \in (A', B') \colon G(u) > s\} \quad \text{for } s \in (A, B). \tag{7.15}$$

Then $F(s)$ takes values in (A', B') and is increasing and right-continuous and

$$G(u) = \inf\{s \in (A, B) \colon F(s) > u\} \ (\text{with } \inf \emptyset = B) \quad \text{for } u \in (A', B'). \tag{7.16}$$

Moreover, for any nonnegative measurable function $h(u)$ on (A', B'),

$$\int_{(A', B')} h(u) dG(u) = \int_{(A, B)} h(F(s)) ds. \tag{7.17}$$

Proof. It is clear that $F(s)$ is (A', B')-valued and increasing. For $s \in (A, B)$ we have $F(s) \leqslant F(s+)$. If $F(s) < F(s+)$, then there is r such that $F(s) < r < F(t)$ for all $t \in (s, B)$ and thus $G(r) > s$ and $G(r) \leqslant t$ for all $t \in (s, B)$, which is impossible. Hence $F(s)$ is right-continuous. Extend $F(s)$ to $\widetilde{F}(s)$ on $[A, B]$ by

$$\widetilde{F}(s) = \inf\{u \in (A', B') \colon G(u) > s\} \ (\text{with } \inf \emptyset = B').$$

Then $\widetilde{F}(s)$ is $[A', B']$-valued, increasing, and right-continuous. For any $u \in (A', B')$,

$$\widetilde{F}(G(u)) = \inf\{v \in (A', B') \colon G(v) > G(u)\} \geqslant u$$

and hence $\widetilde{F}(G(u + \varepsilon)) \geqslant u + \varepsilon$ for all small $\varepsilon > 0$, from which it follows that $\inf\{s \in [A, B] \colon \widetilde{F}(s) > u\}$ is $\leqslant G(u + \varepsilon)$, hence $\leqslant G(u)$. We now have

$$G(u) = \inf\{s \in [A, B] \colon \widetilde{F}(s) > u\} \quad \text{for } u \in (A', B') \tag{7.18}$$

because, if not, there is r such that $G(u) > r > \inf\{s \in [A, B] \colon \widetilde{F}(s) > u\}$ and thus $\widetilde{F}(r) > u$ showing that $G(u) \leqslant r$, which is absurd. The right-hand side of (7.16) equals the right-hand side of (7.18) for all $u \in (A', B')$ because, if not, then for some $u \in (A', B')$ and $t \in (A, B)$

$$\inf\{s \in (A, B) \colon F(s) > u\} > t > \inf\{s \in [A, B] \colon \widetilde{F}(s) > u\},$$

which implies $F(t) \leqslant u$ and $F(t) > u$, a contradiction. Therefore (7.16) is true. For any $v \in (A', B')$,

$$\int_{(A', B')} 1_{(A', v]}(u) dG(u) = \int_{(A', v]} dG(u) = G(v) - G(A'+) = G(v) - A$$

and

$$\begin{aligned}\int_{(A,B)} 1_{(A',v]}(F(s))ds &= \int_{(A,B)} 1_{\{F(s)\leqslant v\}}ds\\ &= \inf\{s\in(A,B)\colon F(s)>v\}-A = G(v)-A\end{aligned}$$

(note that if $F(s)>v$ for all $s\in(A,B)$, then $A=G(v)>-\infty$ by (7.16)). Hence (7.17) holds for $h=1_{(A',v]}$ with $v\in(A',B')$. Therefore (7.17) holds for general h. (In this lemma the roles of $G(u)$ and $F(s)$ are not symmetric, as we do not necessarily have $A'=\inf_{s\in(A,B)}F(s)$ and $B'=\sup_{s\in(A,B)}F(s)$.) □

Proof of Theorem 7.9 (ii). Suppose that τ satisfies Condition (B). Fix a point $c\in(a',b')$ and define $A'=a'$, $B'=b'$, and

$$G(u)=\begin{cases}\tau((c,u]) & \text{for } c<u<B',\\ 0 & \text{for } u=c,\\ -\tau((u,c]) & \text{for } A'<u<c.\end{cases}$$

Then $G(u)$ is finite, increasing, right-continuous, and not constant. Now we apply Lemma 7.10. Let A, B, and $F(s)$ on $s\in(A,B)$ be defined by (7.14) and (7.15). Then $A=-\tau((A',c])$, $B=\tau((c,B'))$, and $F(s)$ is (A',B')-valued, increasing, and right-continuous. Let $a=-B$, $b=-A$, and $f(s)=F(-s)$ for $s\in(a,b)$. Then $f(s)$ is (A',B')-valued, decreasing, and left-continuous. We claim that $f(s)$ satisfies Condition (A). If $f(s)$ is constant, then, for some $u_0\in(A',B')$, $F(s)=u_0$ for all $s\in(A,B)$, and hence (use (7.15)) $G(u_0)\geqslant B=\tau((c,B'))$, which contradicts Condition (B). Thus $f(s)$ is not constant. Let us show (7.9), that is,

$$\inf_{t\in(A,B)}F(t)<F(s)<\sup_{t\in(A,B)}F(t)\quad \text{for } s\in(A,B). \tag{7.19}$$

If $A'<u<B'$, then $A<G(u)<B$ by Condition (B). We have $G(A'+)=A$ and $G(B'-)=B$. Hence, using (7.15), we obtain $F(A+)=A'$ and $F(B-)=B'$. Thus (7.19) follows, since $A'<F(s)<B'$ for $s\in(A,B)$. Hence $f(s)$ satisfies Condition (A). For any nonnegative measurable h,

$$\begin{aligned}\int_{(a,b)}h(f(s))ds &= \int_{(-B,-A)}h(F(-s))ds=\int_{(A,B)}h(F(s))ds\\ &= \int_{(A',B')}h(u)dG(u)=\int_{(a',b')}h(u)\tau(du)=\int_{\mathbb{R}}h(u)\tau(du)\end{aligned}$$

by virtue of (7.17) and Condition (B). Therefore τ is the τ-measure of f. □

Example 7.11. The pairs of f and τ in Examples 7.5 and 7.6 satisfy Conditions (A) and (B). If τ is the probability measure with distribution function equal to Cantor function, then τ satisfies Condition (B) and the function f associated in Theorem 7.9 with Condition (A) increases only with jumps.

8. Transformations of infinitely divisible distributions on proper cone

A subset K of $\mathbb{R}^d$ is called a cone if it is a nonempty closed convex set such that (1) $x\in K$ and $\alpha\geqslant 0$ imply $\alpha x\in K$ and (2) $K\neq\{0\}$. If, moreover, $x\in K$ implies $-x\notin K$, then K is called a proper cone. For example, K is a proper cone in $\mathbb{R}$ if and only if K is either $[0,\infty)$ or $(-\infty,0]$. In $\mathbb{R}^2$, K is a proper cone which is nondegenerate (that is, not contained in any one-dimensional linear subspace), if and only if there are linearly independent $x^{(1)}$ and $x^{(2)}$ such that $K = \{\alpha_1 x^{(1)} + \alpha_2 x^{(2)} \colon \alpha_1 \geqslant 0 \text{ and } \alpha_2 \geqslant 0\}$. In $\mathbb{R}^3$, there are many proper cones such as triangular cones and circular cones. If K is the set of $(x_j)_{1\leqslant j\leqslant 3}$ such that

$$\begin{pmatrix} x_1 & x_3 \\ x_3 & x_2 \end{pmatrix}$$

is nonnegative-definite, then K is linearly isomorphic to a circular cone in $\mathbb{R}^3$; see Pedersen and Sato (2003).

In this section let K be a proper cone in $\mathbb{R}^d$. Let $ID(K)$ denote the class of infinitely divisible distributions supported on K (that is, $\mathrm{Supp}(\mu) \subset K$).

Proposition 8.1. *Let $\mu = \mu_{(A,\mu,\gamma)}$ be in $ID(\mathbb{R}^d)$. Then $\mu \in ID(K)$ if and only if μ is of type A or B,* Supp$(\nu) \subset K$, *and the drift γ^0 is in K.*

This proposition is given in Skorohod (1986) and also in E 22.11 of Sato (1999).

Let $\mathrm{Lvm}(ID(K))$ denote the class of Lévy measures of infinitely divisible distributions supported on K. That is, $\mathrm{Lvm}(ID(K))$ is the class of measures ν on $\mathbb{R}^d$ satisfying $\nu(\{0\}) = 0$, $\mathrm{Supp}(\nu) \subset K$, and $\int_K (|x| \wedge 1)\nu(dx) < \infty$.

Now let $-\infty \leqslant a < b \leqslant \infty$. Let f be a nonnegative measurable function on (a, b). The following propositions are the counterparts of Theorem 2.10 and Theorems 3.15, 6.2, 6.4, and 6.5.

Proposition 8.2. *Let $\mu \in ID(K)$. Let $f(s)$ be a nonnegative, $\mathbb{R}$-valued measurable function on (a, b). Then the following two statements are equivalent.*

(a) *$f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ and*

$$\mathcal{L}\left(\int_p^q f(s) X^{(\mu)}(ds)\right) \in ID(K) \quad \text{for all } p, q \text{ with } a < p < q < b. \tag{8.1}$$

(b) *The Lévy measure ν and the drift γ^0 of μ satisfy* (2.14) *and* (2.15).

Proof. Assume (a). Then we have (b), using Theorem 2.10.

Conversely assume (b). Theorem 2.10 says that $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ and that (2.13) is true. Let $(A, \nu, \gamma^0)_0$ and $(A_p^q, \nu_p^q, (\gamma^0)_p^q)_0$ be the triplets of μ and $\int_p^q f(s) X^{(\mu)}(ds)$, respectively. Then $A_p^q = 0$ from (2.13) and, using Proposition 8.1, we obtain $\mathrm{Supp}(\nu) \subset K$ from (2.11) and $(\gamma^0)_p^q \in K$ from (2.16) since f is nonnegative and K is a cone. Hence (8.1) is true. □

Proposition 8.3. *Let $\mu \in ID(K)$. Let $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ and $f \geqslant 0$. Then $\Phi_f(\mu)$ is definable and $\Phi_f(\mu) \in ID(K)$ if and only if statement* (b) *of Theorem 3.15 is true.*

Proof. Use Theorem 3.15. The "only if" part is because $ID(K) \subset ID_{\mathrm{AB}}$. For the "if" part, use Proposition 8.1 and assumption $f \geqslant 0$. □

Proposition 8.4. *Let $f(s)$ be a nonnegative, $\mathbb{R}$-valued measurable function on (a, b). Then the following statements are equivalent.*

(a) $ID(K) \subset \mathfrak{D}^0(\Phi_f)$ *and* $\Phi_f(ID(K)) \subset ID(K)$.

(b) $ID(K) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$ *and, for any* $\mu \in ID(K)$, $\Phi_{f,\mathrm{es}}(\mu) \cap ID(K) \neq \emptyset$.

(c) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* $\int_a^b f(s) ds < \infty$.

Proof. (c) $\Rightarrow$ (a): Using Theorem 6.2, we have $ID(K) \subset ID_{\mathrm{AB}}(\mathbb{R}^d) \subset \mathfrak{D}^0(\Phi_f)$ and $\Phi_f(ID(K)) \subset \Phi_f(ID_{\mathrm{AB}}(\mathbb{R}^d)) \subset ID_{\mathrm{AB}}(\mathbb{R}^d)$. For any $\mu \in ID(K)$, $\widetilde{\mu} = \Phi_f(\mu)$ is not only in ID_{AB} but also in $ID(K)$ because $f \geqslant 0$ (use (3.7) and (3.12)).

(a) $\Rightarrow$ (b): Obvious since $\mathfrak{D}^0(\Phi_f) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$.

(b) $\Rightarrow$ (c): We have, from Theorem 3.11,

$$\int_a^b ds \int_K (|f(s)x| \wedge 1)\, \nu(dx) < \infty \quad \text{for all } \nu \in \mathrm{Lvm}(ID(K)). \tag{8.2}$$

Hence the proof is similar to that of the corresponding part of Theorem 6.2 (replace the unit sphere S by $K \cap S$). □

Proposition 8.5. *Let $f(s)$ be a nonnegative, $\mathbb{R}$-valued measurable function on (a,b). Then $ID(K) \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$ if and only if statement* (b) *of Theorem 6.4 is true.*

Proof. The "if" part: Obvious from Theorem 6.4.

The "only if" part: It follows that

$$\int_a^b ds \int_K (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty \quad \text{for all } \nu \in \mathrm{Lvm}(ID(K)). \tag{8.3}$$

Hence we can make a discussion similar to the corresponding part of the proof of Theorem 6.4. □

Proposition 8.6. *Let $f(s)$ be a nonnegative, $\mathbb{R}$-valued measurable function on (a,b). Then $ID(K) \subset \mathfrak{D}^0(\Phi_f)$ if and only if statement* (b) *of Theorem 6.5 is true.*

Proof. The "if" part follows from Theorem 6.5. The "only if" part is similar to the corresponding part of the proof of Theorem 6.5. □

9. Transformations of Lévy measures

Fix $-\infty \leqslant a < b \leqslant \infty$ and the dimension d. Let $f(s)$ be an $\mathbb{R}$-valued measurable function on (a,b).

Definition 9.1. For a measure ν on $\mathbb{R}^d$, let $\nu^\sharp$ be the measure given by

$$\nu^\sharp(B) = \int_a^b ds \int_{\mathbb{R}^d} 1_B(f(s)x)\nu(dx), \qquad B \in \mathcal{B}(\mathbb{R}^d). \tag{9.1}$$

If $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$ and if $[\nu^\sharp]_{\mathbb{R}^d\setminus\{0\}}$, the restriction of $\nu^\sharp$ to $\mathbb{R}^d \setminus \{0\}$, is in $\mathrm{Lvm}(ID(\mathbb{R}^d))$, then define

$$\Psi_f(\nu) = [\nu^\sharp]_{\mathbb{R}^d\setminus\{0\}}. \tag{9.2}$$

The domain $\mathfrak{D}(\Psi_f)$ of Ψ_f is defined as

$$\mathfrak{D}(\Psi_f) = \{\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))\colon [\nu^\sharp]_{\mathbb{R}^d\setminus\{0\}} \in \mathrm{Lvm}(ID(\mathbb{R}^d))\}. \tag{9.3}$$

Since

$$\int_{\mathbb{R}^d\setminus\{0\}} h(x)\nu^\sharp(dx) = \int_a^b ds \int_{\mathbb{R}^d} h(f(s)x)1_{\mathbb{R}^d\setminus\{0\}}(f(s)x)\, \nu(dx) \tag{9.4}$$

for all nonnegative measurable functions h on $\mathbb{R}^d$, a measure ν on $\mathbb{R}^d$ belongs to $\mathfrak{D}(\Psi_f)$ if and only if $\nu(\{0\}) = 0$, $\int_{\mathbb{R}^d}(|x|^2 \wedge 1)\, \nu(dx) < \infty$, and

$$\int_a^b ds \int_{\mathbb{R}^d} (|f(s)x|^2 \wedge 1)\, \nu(dx) < \infty. \tag{9.5}$$

The following theorem shows a close relationship between the transformation Ψ_f and the essential improper integrals.

Theorem 9.2. *Assume that f is locally square-integrable on (a,b). Let ν and $\widetilde{\nu}$ be in* $\mathrm{Lvm}(ID(\mathbb{R}^d))$. *Then the following statements are equivalent.*

(a) $\nu \in \mathfrak{D}(\Psi_f)$ *and* $\widetilde{\nu} = \Psi_f(\nu)$.

(b) *If $\mu \in ID_0(\mathbb{R}^d)$ has Lévy measure ν, then μ belongs to $\mathfrak{D}(\Phi_{f,\mathrm{es}})$ and any distribution in $\Phi_{f,\mathrm{es}}(\mu)$ has Lévy measure $\widetilde{\nu}$.*

Proof. Use Remark 2.8 and Theorems 3.6 and 3.11. The assumption of local square-integrability of f on (a,b) is needed because the theorems in Section 3 presupposes that $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$. □

We give some results similar to those in Sections 6 and 8.

Proposition 9.3. *The following statements are equivalent.*

(a) $\mathrm{Lvm}(ID(\mathbb{R}^d)) = \mathfrak{D}(\Psi_f)$.

(b) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* $\int_a^b f(s)^2 ds < \infty$.

Proof. Statement (a) is equivalent to saying that any $\nu \in \mathrm{Lvm}(ID(\mathbb{R}^d))$ satisfies (9.5). If (b) is true, then (a) follows from Theorem 6.1. Conversely, if (a) is true, then we obtain (b) exactly in the same way as in Steps 1 and 2 in the proof of Theorem 6.1 (here we cannot directly apply this theorem, as we do not assume $f \in \mathbf{L}_{(a,b)}(X^{(\mu)})$ and we cannot use Theorem 3.6 in order to say that (a) implies $ID_0 \subset \mathfrak{D}(\Phi_{f,\mathrm{es}})$). □

Proposition 9.4. *The following statements are equivalent.*

(a) $\mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d)) \subset \mathfrak{D}(\Psi_f)$ *and* $\Psi_f(\mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d))) \subset \mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d))$.

(b) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* $\int_a^b |f(s)| ds < \infty$.

Proof. Theorem 6.2 tells that statement (b) is equivalent to condition (b) of Theorem 6.2. Hence, if statement (b) is true, then statement (a) follows. Conversely, if statement (a) is true, then we obtain statement (b) as Steps 1 and 2 in the proof of Theorem 6.2 work. □

Proposition 9.5. *The following statements are equivalent.*

(a) $\mathrm{Lvm}(ID_{\mathrm{AB}}(\mathbb{R}^d)) \subset \mathfrak{D}(\Psi_f)$.

(b) $\int_a^b 1_{\{f(s)\neq 0\}} ds < \infty$ *and* (6.4) *and* (6.5) *hold.*

Proof. If (b) is true, then (a) follows, as the proof of (6.6) in the first half of the proof of Theorem 6.4 works. If (a) is true, then (b) is true as in Steps 1, 2, and 3 of the second half of the proof of Theorem 6.4. □

Proposition 9.6. *The following statements are equivalent.*

(a) $\mathfrak{D}(\Psi_f)$ *consists only of the zero measure.*

(b) $\int_a^b (f(s)^2 \wedge 1) ds = \infty$.

Proof. Almost the same proof as that of Theorem 6.11 works. □

Proposition 9.7. *Let K be a proper cone in $\mathbb{R}^d$. Assume that $f(s)$ is nonnegative. Then*

$$\mathrm{Lvm}(ID(K)) \subset \mathfrak{D}(\Psi_f) \quad \textit{and} \quad \Psi_f(\mathrm{Lvm}(ID(K))) \subset \mathrm{Lvm}(ID(K)) \tag{9.6}$$

if and only if statement (c) *of Proposition 8.4 is true.*

Proof. If (c) of Proposition 8.4 is true, then (b) of Proposition 8.4 is true and (9.6) holds. Conversely, if (9.6) holds, then (c) of Proposition 8.4 is true by an argument similar to the corresponding part of the proof of Theorem 6.2. □

Proposition 9.8. *Let K be a proper cone in $\mathbb{R}^d$. Assume that $f(s) \geqslant 0$. Then*

$$\mathrm{Lvm}(ID(K)) \subset \mathfrak{D}(\Psi_f) \tag{9.7}$$

if and only if statement (b) *of Proposition 9.5 is true.*

Proof. The inclusion (9.7) is equivalent to (8.3). Hence the proof is similar to that of Proposition 9.5. □

Using the τ-measure τ of f introduced in Section 7, we can express the transformation Ψ_f as in the following proposition. If we restrict our attention to measures τ on $(0, \infty)$, the transformation Ψ_f in this form is identical with the transformation discussed by Barndorff-Nielsen and Pérez-Abreu (2005) under the name of Upsilon transformation Υ_τ; it is called a generalized Upsilon transformation in Barndorff-Nielsen and Thorbjørnsen (2006b). Their studies are being made in Barndorff-Nielsen and Maejima (2007) and Barndorff-Nielsen et al. (2007).

Proposition 9.9. *Let τ be the τ-measure of f. Then, $\nu \in \mathfrak{D}(\Psi_f)$ if and only if ν is in* $\mathrm{Lvm}(ID(\mathbb{R}^d))$ *and satisfies*

$$\int_{\mathbb{R}} \tau(du) \int_{\mathbb{R}^d} (|ux|^2 \wedge 1)\, \nu(dx) < \infty. \tag{9.8}$$

If $\nu \in \mathfrak{D}(\Psi_f)$, then, for $B \in \mathcal{B}_{\mathbb{R}^d \setminus \{0\}}$,

$$(\Psi_f(\nu))(B) = \int_{\mathbb{R}} \tau(du) \int_{\mathbb{R}^d} 1_B(ux)\, \nu(dx) = \int_{\mathbb{R}\setminus\{0\}} \nu\left(\frac{1}{u}B\right) \tau(du). \tag{9.9}$$

Proof. Immediate from (7.2), (9.5), and Definition 9.1. □

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