

Limit theorems for an epidemic model on the complete graph

Thomas G. Kurtz, Elcio Lebensztayn, Alexandre R. Leichsenring, Fábio P. Machado

Department of Mathematics, University of Wisconsin, Madison Wisconsin 53706, U.S.A.
E-mail address: kurtz@math.wisc.edu

Departamento de Estatística, IME – USP, CEP 05508-090, São Paulo, SP, Brasil.
E-mail address: elcio@ime.usp.br

E-mail address: aleleich@ime.usp.br

E-mail address: fmachado@ime.usp.br

Abstract. We study the following random walks system on the complete graph with n vertices. At time zero, there is a number of active and inactive particles living on the vertices. Active particles move as continuous-time, rate 1, random walks on the graph, and, any time a vertex with an inactive particle on it is visited, this particle turns into active and starts an independent random walk. However, for a fixed integer $L \geq 1$, each active particle dies at the instant it reaches a total of L jumps without activating any particle. We prove a Law of Large Numbers and a Central Limit Theorem for the proportion of visited vertices at the end of the process.

1. Description of the model

We study a continuous-time random walks system formulated as a model for information spreading through a network. The dynamics of this model is described as follows. For $n \geq 3$, let \mathcal{K}_n be the complete graph with n vertices and $L \geq 1$ be a fixed integer. At time zero, there is one particle at each vertex of \mathcal{K}_n ; one of them is active, the others are inactive. The active particle begins to move as a continuous-time, rate 1, random walk on \mathcal{K}_n ; and, as soon as any active particle visits an inactive one, the latter becomes active and starts an independent random walk. However each active particle dies at the instant it reaches a total of L jumps (consecutive or not) without activating any particle. We may think that each active particle starts with L lives and loses one life whenever it jumps on a vertex which

Received by the editors August 15, 2007; accepted March 26, 2008.

2000 Mathematics Subject Classification. 60F05, 60J27, 60K35, 92B05.

Key words and phrases. Epidemic model, random walk, complete graph, Markov chains, Law of Large Numbers, Central Limit Theorem.

has been already visited by the process. Observe that the process eventually finishes (when there are no active particles). Other initial configurations are considered in the sequel.

For a realization of this model on \mathcal{K}_n , let $V^{(n)}(t)$ be the number of visited vertices at time t , and $A_i^{(n)}(t)$ be the number of active particles with i lives at time t ($i = 1, \dots, L$). Notice that $\{(V^{(n)}(t), A_1^{(n)}(t), \dots, A_L^{(n)}(t))\}_{t \geq 0}$ is a continuous-time Markov chain in \mathbb{Z}^{L+1} with transitions and rates given by

transition	rate
$u_0 = (1, 0, \dots, 0, 1)$	$\left(\frac{n-V}{n-1}\right) \sum_{i=1}^L A_i$
$u_1 = (0, -1, 0, \dots, 0)$	$\left(\frac{V-1}{n-1}\right) A_1$
$u_j = (0, \dots, 0, \overset{j}{1}, \overset{j+1}{-1}, 0, \dots, 0)$	$\left(\frac{V-1}{n-1}\right) A_j \quad j = 2, \dots, L.$

In words, u_0 indicates the transition of the process in which an inactive particle is activated, u_1 represents the death of an active particle with one life, and, for $j = 2, \dots, L$, u_j indicates the event that an active particle with j lives loses one life.

Let us define a more general initial configuration. We suppose that there exist $\rho_0^{(n)} \in [1/n, 1]$ and $\rho_i^{(n)} \geq 0$, $i = 1, \dots, L$, such that $\sum_{i=1}^L \rho_i^{(n)} > 0$ and $n\rho_i^{(n)}$ is an integer for every $i = 0, \dots, L$. The process begins with

$$V^{(n)}(0) = n\rho_0^{(n)} \text{ and } A_i^{(n)}(0) = n\rho_i^{(n)}, \quad i = 1, \dots, L.$$

As already mentioned, this process eventually ends. Let

$$\gamma^{(n)} = \inf \left\{ t : \sum_{i=1}^L A_i^{(n)}(t) = 0 \right\}$$

be the absorption time of the process running on \mathcal{K}_n . Our main purpose is to establish limit theorems for the *coverage* of \mathcal{K}_n , that is, for the proportion $n^{-1} V^{(n)}(\gamma^{(n)})$ of visited vertices at the end of the process. Throughout the paper, we assume that

$$\rho_i = \lim_{n \rightarrow \infty} \rho_i^{(n)} \text{ exists for every } i = 0, \dots, L, \quad (1.1)$$

and define

$$\rho = \sum_{i=1}^L i \rho_i$$

the limiting proportion of lives at time zero. Note that $\rho_0 = \rho = 0$ for the one-particle-per-vertex initial configuration.

Seen as a random walks system, the model under study is a variant of the so-called frog model. In the discrete-time setup (simultaneous jumps), this model is often considered on an infinite connected graph, and the lifetime of an active particle is either infinite or a random variable which is independent of the trajectory. The main subjects are shape theorems on \mathbb{Z}^d and phase transition on \mathbb{Z}^d and homogeneous trees. See Alves et al. (2002a,b), Lebensztayn et al. (2005) and references therein. As far as we know, only Alves et al. (2006) deals with this model on complete graphs. The authors study a critical parameter related to the coverage

in the case of geometrically distributed lifetimes, and present a couple of results obtained from computational analysis, simulations and mean field approximation in the case $L = 1$. The present work was motivated by the results and open problems presented there, and provides rigorous and explicit answers for the continuous-time model in a more general setup. Regarding other continuous-time versions, we refer to Ramírez and Sidoravicius (2004) and Kesten and Sidoravicius (2005) for shape theorems on \mathbb{Z}^d .

The studied model can be thought as a model for the evolution of a disease in a population or the spreading of a virus in a computer network. To understand this interpretation, first consider the case $L = 1$ and think the active particles as diffusion agents (viruses) which move along the vertices (individuals/computers). During its life, at the time points of a homogeneous Poisson process with intensity 1, a virus chooses at random a new position in the graph. If the contacted individual is still susceptible, then it catches the disease and the virus duplicates. Once that happens, this individual is regarded as immunized, that is, it activates an anti-virus which will kill any virus that tries to infect it in the future. In the general case, the parameter L is interpreted as the initial resistance of the virus, which weakens each time the contacted individual is already immunized. The main problem we study in this paper refers to the distribution of the proportion of the population which is visited by the disease after all the viruses are dead.

Let us underline that in our model the infection is spread by the random walks, while the individuals are represented by the vertices of the graph. Nevertheless, in terms of rates, the case $L = 1$ has a certain similarity to the model known as general stochastic epidemic (Markovian SIR), which was originated with Bartlett (1949). Recalling this model, we denote by $X(t)$ and $Y(t)$ the number of susceptible and infective individuals, respectively, at time t . Then, (X, Y) is a Markov process with the following transition table:

from	to	at rate
(i, j)	$(i - 1, j + 1)$	$\lambda ij/n$
	$(i, j - 1)$	γj .

Here, n is the initial number of susceptibles in the population, λ is the rate at which a given infective makes contact with other individuals, and γ is the parameter of the exponential lifetimes of the infective individuals. The most important result in this context is known as the Threshold Theorem, which in brief identifies the ratio λ/γ as a threshold quantity to determine whether the epidemic builds up or not for n large. There are also limit theorems concerning the asymptotic distribution of the number of susceptible individuals that ultimately become infected. We observe that the principal difficulty in studying our model is that the classical coupling techniques cannot be applied, by virtue of the dependence between the progeny and the lifetime of the active particles. For more details on epidemic models, see Andersson and Britton (2000) and the references therein.

2. Main results

First we state the Weak Law of Large Numbers for the coverage of \mathcal{K}_n . Recall that we are assuming (1.1) and that $\rho = \sum_{i=1}^L i \rho_i$.

Definition 2.1. Consider the function $f : [\rho_0, 1) \rightarrow \mathbb{R}$ given by

$$f(v) = \rho + (L + 1)(v - \rho_0) + \log\left(\frac{1 - v}{1 - \rho_0}\right). \quad (2.1)$$

We define $v_\infty = v_\infty(L, \rho_0, \rho)$ as the unique solution of $f(v) = 0$ satisfying $f'(v) \leq 0$, and

$$\gamma_\infty = \gamma_\infty(L, \rho_0, \rho) = \rho + (L + 1)(v_\infty - \rho_0).$$

Figure 1 illustrates the four possible cases of the graph of f depending on the values of ρ_0 and ρ . Observe that v_∞ is the unique root of f , except in the case that $\rho = 0$ and $\rho_0 < L/(L + 1)$. Moreover,

$$v_\infty = \inf\{v \in [\rho_0, 1) : f(v) \leq 0\}. \quad (2.2)$$

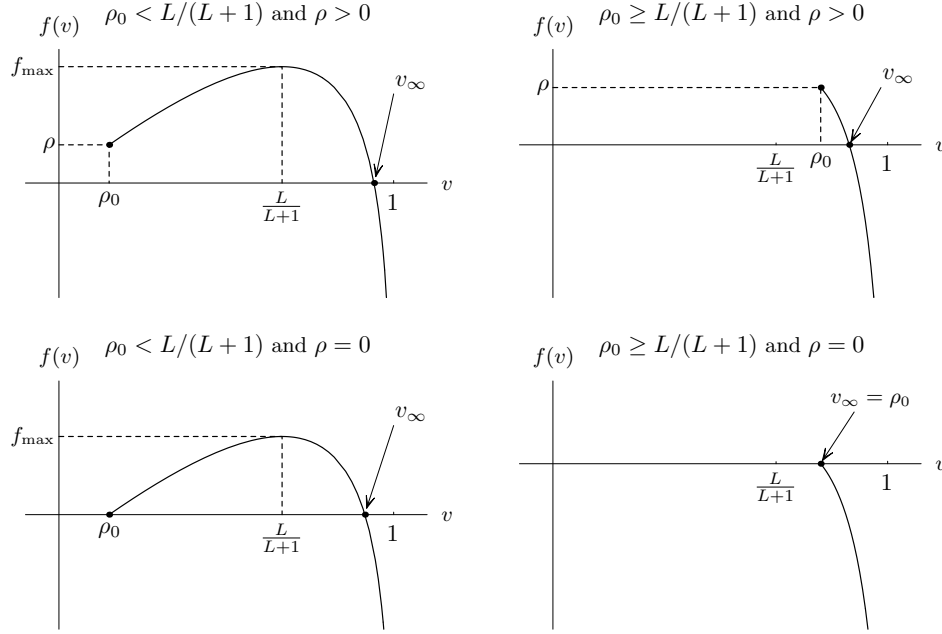


Figure 1: Behavior of f – The four possible cases in terms of ρ_0 and ρ .

Theorem 2.2.

$$\lim_{n \rightarrow \infty} \frac{V^{(n)}(\gamma^{(n)})}{n} = v_\infty \quad \text{in probability.}$$

It is worthwhile recalling that, by Skorohod's theorem, the convergence in Theorem 2.2 can be represented by versions defined on the same probability space, in such a way that the convergence is almost sure. In Section 3, we present such a construction, based on the theory of density dependent Markov chains.

Remark 2.3. One can express the limiting coverage v_∞ as

$$v_\infty = 1 + (L + 1)^{-1} W_0(-(1 - \rho_0)(L + 1)e^{-(1 - \rho_0)(L + 1) - \rho}),$$

where W_0 is the principal branch of the so-called Lambert W function (which is the inverse of the function $x \mapsto x e^x$). More details about this function can be found in Corless et al. (1996).

In particular, for the one-particle-per-vertex initial configuration,

$$v_\infty = 1 + W_0(-c e^{-c})/c, \quad (2.3)$$

where $c = L + 1$. Formulae similar to (2.3) appear in the study of the final size of epidemic models and the relative size of the giant component in certain random graphs. See respectively Andersson and Britton (2000) and Bollobás (1998, p. 241) for more details.

Next we present the Central Limit Theorem for the coverage.

Theorem 2.4. *Suppose that $\rho > 0$ or that $\rho = 0$ and $\rho_0 < L/(L + 1)$. Then,*

$$\sqrt{n} \left(\frac{V^{(n)}(\gamma^{(n)})}{n} - v_\infty \right) \xrightarrow{\mathcal{D}} N(0, \sigma^2) \text{ as } n \rightarrow \infty,$$

where $\xrightarrow{\mathcal{D}}$ denotes convergence in distribution, and $N(0, \sigma^2)$ is the Gaussian distribution with mean zero and variance given by

$$\sigma^2 = \sigma^2(L, \rho_0, \rho) = \frac{(1 - v_\infty)(v_\infty - \rho_0 - \gamma_\infty(1 - \rho_0)(1 - v_\infty))}{(1 - \rho_0)((L + 1)v_\infty - L)^2}. \quad (2.4)$$

Observe that, for the one-particle-per-vertex initial configuration, the asymptotic variance simplifies to

$$\sigma^2 = \frac{v_\infty(1 - v_\infty)}{(L + 1)v_\infty - L}.$$

Table 1 exhibits the approximate values of v_∞ , γ_∞ and σ^2 in this case for $L = 1, \dots, 6$; Table 2 does the same for $L = 1, 2$ and some arbitrarily chosen values of ρ_0 and ρ .

L	1	2	3	4	5	6
v_∞	0.7968	0.9405	0.9802	0.9930	0.9975	0.9991
γ_∞	1.5936	2.8214	3.9207	4.9651	5.9849	6.9936
σ^2	0.2727	0.06814	0.02111	0.007179	0.002549	0.0009228

Table 1: One-particle-per-vertex initial configuration.

	$\rho_0 = 0.05, \rho = 0.2$		$\rho_0 = 0.45, \rho = 0.05$		$\rho_0 = 0.7, \rho = 0.1$	
	$L = 1$	$L = 2$	$L = 1$	$L = 2$	$L = 1$	$L = 2$
v_∞	0.8397	0.9473	0.6477	0.8353	0.7585	0.7972
γ_∞	1.7794	2.8919	0.4453	1.2060	0.2171	0.3916
σ^2	0.1896	0.0589	0.8180	0.3229	0.1289	0.3234

Table 2: Values of v_∞ , γ_∞ and σ^2 .

Finally, we prove

Theorem 2.5. *Let $N^{(n)}$ be the number of jumps that the process makes until absorption. Then,*

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}(N^{(n)})}{n} = \gamma_\infty.$$

3. Proofs

The proofs of the results rely strongly on the theory of density dependent Markov chains, presented in Chapter 11 of Ethier and Kurtz (1986). To fit the notation, we define for $t \geq 0$,

$$\begin{aligned} Z^{(n)}(t) &= (V^{(n+1)}(t) - 1, A_1^{(n+1)}(t), A_2^{(n+1)}(t), \dots, A_L^{(n+1)}(t)), \\ z^{(n)}(t) &= \frac{Z^{(n)}(t)}{n} = (v^{(n)}(t), a_1^{(n)}(t), a_2^{(n)}(t), \dots, a_L^{(n)}(t)). \end{aligned}$$

In addition, let

$$\begin{aligned} a^{(n)}(t) &= \sum_{i=1}^L a_i^{(n)}(t), \\ y^{(n)}(t) &= \sum_{i=1}^L i a_i^{(n)}(t), \text{ and} \\ \tau^{(n)} &= \gamma^{(n+1)} = \inf\{t : a^{(n)}(t) = 0\}. \end{aligned}$$

Notice that we are dealing with the model on \mathcal{K}_{n+1} , and that $a^{(n)}(t)$ and $y^{(n)}(t)$ are respectively the ‘‘proportions’’ of active particles and lives at time t . We use the quotation marks due to the division by n .

We observe that $\{z^{(n)}(t)\}_{t \geq 0}$ is a density dependent Markov chain with

$$\begin{aligned} \beta_{u_0}(v, a_1, \dots, a_L) &= (1 - v) \sum_{i=1}^L a_i, \text{ and} \\ \beta_{u_j}(v, a_1, \dots, a_L) &= v a_j, \quad j = 1, \dots, L. \end{aligned}$$

Hence, applying Theorem 6.4.1 of Ethier and Kurtz (1986), we write

$$\begin{aligned} v^{(n)}(t) &= \frac{(n+1)\rho_0^{(n+1)} - 1}{n} + \frac{1}{n} Y_0 \left(n \int_0^t (1 - v^{(n)}(s)) a^{(n)}(s) ds \right), \\ a_i^{(n)}(t) &= \frac{(n+1)\rho_i^{(n+1)}}{n} + \frac{1}{n} Y_{i+1} \left(n \int_0^t v^{(n)}(s) a_{i+1}^{(n)}(s) ds \right) - \frac{1}{n} Y_i \left(n \int_0^t v^{(n)}(s) a_i^{(n)}(s) ds \right), \\ &\quad i = 1, \dots, L-1, \\ a_L^{(n)}(t) &= \frac{(n+1)\rho_L^{(n+1)}}{n} + \frac{1}{n} Y_0 \left(n \int_0^t (1 - v^{(n)}(s)) a^{(n)}(s) ds \right) \\ &\quad - \frac{1}{n} Y_L \left(n \int_0^t v^{(n)}(s) a_L^{(n)}(s) ds \right), \end{aligned}$$

where $\{Y_0, Y_1, \dots, Y_L\}$ is a set of independent standard Poisson processes. Thus, we have coupled all the processes with different values of n .

In brief, here is the main idea to prove Theorems 2.2 and 2.4. Using a random time change, we speed up the process without affecting where it is absorbed. That is, we define a coupled process $\{\tilde{z}^{(n)}(t)\}_{t \geq 0}$ having the same transitions as

$\{z^{(n)}(t)\}_{t \geq 0}$, though with time at a fast pace. Then, $\{(\tilde{v}^{(n)}(t), \tilde{y}^{(n)}(t))\}_{t \geq 0}$ turns out to be a Markov chain that converges to a deterministic limit as $n \rightarrow \infty$ and for which we can apply Theorem 11.4.1 of Ethier and Kurtz (1986).

Our task is done once we prove the following claims.

Claim 3.1. $\lim_{n \rightarrow \infty} v^{(n)}(\tau^{(n)}) = v_\infty$ almost surely.

Claim 3.2. Suppose that $\rho > 0$ or that $\rho = 0$ and $\rho_0 < L/(L+1)$. Then,

$$\sqrt{n} \left(v^{(n)}(\tau^{(n)}) - v_\infty \right) \xrightarrow{\mathcal{D}} N(0, \sigma^2) \text{ as } n \rightarrow \infty,$$

where σ^2 is given by (2.4).

Claim 3.3. $\lim_{n \rightarrow \infty} n^{-1} \mathbb{E} (N^{(n+1)}) = \gamma_\infty$.

3.1. *Random time change.* We define

$$\begin{aligned} \theta^{(n)}(t) &= \int_0^t a^{(n)}(s) ds, \quad 0 \leq t \leq \tau^{(n)}, \\ \alpha^{(n)}(s) &= \inf\{t : \theta^{(n)}(t) > s\}, \quad 0 \leq s \leq \int_0^\infty a^{(n)}(u) du, \end{aligned}$$

and let $\tilde{z}^{(n)}(t) = z^{(n)}(\alpha^{(n)}(t))$. Hence, this time-changed system is described by

$$\begin{aligned} \tilde{v}^{(n)}(t) &= \frac{(n+1)\rho_0^{(n+1)} - 1}{n} + \frac{1}{n} Y_0 \left(n \int_0^t (1 - \tilde{v}^{(n)}(s)) ds \right), \\ \tilde{a}_i^{(n)}(t) &= \frac{(n+1)\rho_i^{(n+1)}}{n} + \frac{1}{n} Y_{i+1} \left(n \int_0^t \tilde{v}^{(n)}(s) \frac{\tilde{a}_{i+1}^{(n)}(s)}{\tilde{a}^{(n)}(s)} ds \right) - \frac{1}{n} Y_i \left(n \int_0^t \tilde{v}^{(n)}(s) \frac{\tilde{a}_i^{(n)}(s)}{\tilde{a}^{(n)}(s)} ds \right), \\ &\quad i = 1, \dots, L-1, \\ \tilde{a}_L^{(n)}(t) &= \frac{(n+1)\rho_L^{(n+1)}}{n} + \frac{1}{n} Y_0 \left(n \int_0^t (1 - \tilde{v}^{(n)}(s)) ds \right) - \frac{1}{n} Y_L \left(n \int_0^t \tilde{v}^{(n)}(s) \frac{\tilde{a}_L^{(n)}(s)}{\tilde{a}^{(n)}(s)} ds \right), \end{aligned}$$

for $t \leq \int_0^\infty a^{(n)}(u) du$.

Figure 2 shows the evolution of $a^{(n)}$ before and after the random time change $\alpha^{(n)}$ in a simulation with $n = 10$, $L = 1$ and initial configuration of one particle per vertex. The next proposition discloses the effect of the random time change. In order to prove it, one uses that $\alpha^{(n)}$ (like its inverse $\theta^{(n)}$) is a strictly increasing, continuous, piecewise linear function.

Proposition 3.4. *The process $\{\tilde{z}^{(n)}(t)\}$ has the same transitions as $\{z^{(n)}(t)\}$, and the waiting times are identically distributed random variables following an exponential distribution with parameter n .*

Therefore, defining

$$\tilde{\tau}^{(n)} = \inf\{t : \tilde{a}^{(n)}(t) = 0\},$$

we conclude that $v^{(n)}(\tau^{(n)}) = \tilde{v}^{(n)}(\tilde{\tau}^{(n)})$.

3.2. *Dimension reduction.* Although the time-changed system has a nontrivial deterministic limit as $n \rightarrow \infty$ (which is defined only for $t \in [0, \gamma_\infty]$), one cannot use Theorem 11.4.1 of Ethier and Kurtz (1986) directly in order to prove the desired limit theorems in the case $L \geq 2$. To overcome this problem, we work with a reduced Markov chain.

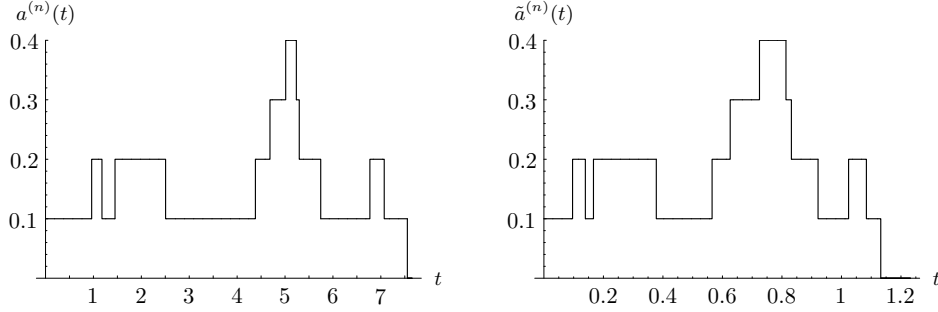


Figure 2: Evolution of $a^{(n)}$ and $\tilde{a}^{(n)}$. (Simulation with $n = 10$, $L = 1$ and initial configuration of one particle per vertex).

Let $\tilde{x}^{(n)}(t) = (\tilde{v}^{(n)}(t), \tilde{y}^{(n)}(t))$, $t \geq 0$. Note that $\{\tilde{x}^{(n)}(t)\}_{t \geq 0}$ is a density dependent Markov chain with transitions $(1, L)$ and $(0, -1)$, and

$$\beta_{(1,L)}(v, y) = 1 - v, \quad \beta_{(0,-1)}(v, y) = v.$$

So $\tilde{y}^{(n)}(t)$ can be written as

$$\tilde{y}^{(n)}(t) = \left(\frac{n+1}{n} \right) \sum_{i=1}^L i \rho_i^{(n+1)} + \frac{L}{n} Y_0 \left(n \int_0^t (1 - \tilde{v}^{(n)}(s)) ds \right) - \frac{1}{n} Y_1^* \left(n \int_0^t \tilde{v}^{(n)}(s) ds \right),$$

where Y_1^* is a standard Poisson process independent of Y_0 . In addition, notice that $\tilde{\tau}^{(n)} = \inf\{t : \tilde{y}^{(n)}(t) = 0\}$.

Using Theorem 11.2.1 of Ethier and Kurtz (1986), we conclude that the limiting deterministic system is governed by the following system of ordinary differential equations

$$\begin{cases} v'(t) = 1 - v(t), \\ y'(t) = L - (L+1)v(t), \end{cases}$$

with initial conditions $v(0) = \rho_0$ and $y(0) = \rho$. Summing up,

Lemma 3.5. *Let $x(t) = (v(t), y(t))$, where*

$$\begin{aligned} v(t) &= 1 - (1 - \rho_0) e^{-t}, \\ y(t) &= f(v(t)) = \rho + (L+1)(v(t) - \rho_0) - t, \end{aligned} \tag{3.1}$$

and f is given by (2.1). We have that $\tilde{x}^{(n)}(t)$ converges almost surely to $x(t)$, uniformly on bounded time intervals.

With respect to $\tilde{v}^{(n)}(t)$, we prove a little more.

Lemma 3.6. *$\tilde{v}^{(n)}(t)$ converges almost surely to $v(t)$, uniformly on \mathbb{R} .*

Proof. Given any $\varepsilon > 0$, we take $t_0 = t_0(\varepsilon)$ such that $v(t) \geq 1 - \varepsilon/2$ for all $t \geq t_0$. By the uniform convergence on the interval $[0, t_0]$, there exists $n_0 = n_0(\varepsilon)$ such that

$$|\tilde{v}^{(n)}(t) - v(t)| \leq \varepsilon/2 \text{ for all } n \geq n_0 \text{ and } t \in [0, t_0].$$

Therefore, for all $n \geq n_0$ and $t \geq t_0$,

$$\tilde{v}^{(n)}(t) \geq \tilde{v}^{(n)}(t_0) \geq v(t_0) - \varepsilon/2 \geq 1 - \varepsilon,$$

whence $|\tilde{v}^{(n)}(t) - v(t)| \leq \varepsilon$. \square

3.3. Proof of Claim 3.1. First notice that $v(\cdot)$ in (3.1) establishes a one-to-one correspondence between $[0, \infty)$ and $[\rho_0, 1)$. From (2.2) and (3.1), it follows that $\gamma_\infty = \inf\{t : y(t) \leq 0\}$ and $v_\infty = v(\gamma_\infty)$. Hence, recalling that $v^{(n)}(\tau^{(n)}) = \tilde{v}^{(n)}(\tilde{\tau}^{(n)})$, we obtain Claim 3.1 from Lemma 3.6 and the next result.

Lemma 3.7. $\lim_{n \rightarrow \infty} \tilde{\tau}^{(n)} = \gamma_\infty$ almost surely.

Proof. Assuming that $\rho > 0$, this is exactly the first main statement of the proof of Theorem 11.4.1 from Ethier and Kurtz (1986). Here the drift function is $F(v, y) = (1 - v, L - (L + 1)v)$, $\varphi(v, y) = y$, and

$$\nabla \varphi(x(\gamma_\infty)) \cdot F(x(\gamma_\infty)) = y'(\gamma_\infty) = L - (L + 1)v_\infty < 0. \quad (3.2)$$

For the sake of completeness, let us recall the original argument. The fact that $y(0) > 0$ and (3.2) imply that $y(\gamma_\infty - \varepsilon) > 0$ and $y(\gamma_\infty + \varepsilon) < 0$ for $0 < \varepsilon < \gamma_\infty$. The convergence stated in Lemma 3.7 follows as y_n converges to y almost surely uniformly on bounded time intervals.

In the case that $\rho = 0$ and $\rho_0 < L/(L + 1)$, we argue that the result is also valid because $y'(0) > 0$ and the fact that (3.2) still holds. Finally, we note that if $\rho = 0$ and $\rho_0 \geq L/(L + 1)$, then $y(t) < 0$ for all $t > 0$, and again the almost sure convergence of y_n to y uniformly on bounded intervals yields that $\lim_{n \rightarrow \infty} \tilde{\tau}^{(n)} = 0 = \gamma_\infty$ almost surely. \square

3.4. Proof of Claim 3.2. We keep on applying Theorem 11.4.1 from Ethier and Kurtz (1986). For this, we adopt the notations presented there, except by the Gaussian process V defined in p. 458, that we would rather denote by $U = (U_v, U_y)$. Observing that we can use the mentioned theorem in the case that $\rho > 0$ or that $\rho = 0$ and $\rho_0 < L/(L + 1)$, we obtain that

$$\sqrt{n} \left(\tilde{v}^{(n)}(\tilde{\tau}^{(n)}) - v_\infty \right) \xrightarrow{\mathcal{D}} U_v(\gamma_\infty) - \frac{1 - v_\infty}{L - (L + 1)v_\infty} U_y(\gamma_\infty) \text{ as } n \rightarrow \infty.$$

The resulting normal distribution clearly has mean zero, so to prove Claim 3.2 it remains to calculate the variance, an easy (though tedious) computation, whose main steps are as follows. The matrix of partial derivatives of the drift function F and the matrix G are

$$\partial F(v, y) = \begin{pmatrix} -1 & 0 \\ -(L + 1) & 0 \end{pmatrix} \quad \text{and} \quad G(v, y) = \begin{pmatrix} 1 - v & L(1 - v) \\ L(1 - v) & L^2(1 - v) + v \end{pmatrix}.$$

Moreover, the solution Φ of the matrix equation

$$\frac{\partial}{\partial t} \Phi(t, s) = \partial F(v(t), y(t)) \Phi(t, s), \quad \Phi(s, s) = I_2$$

is given by

$$\Phi(t, s) = \begin{pmatrix} e^{-(t-s)} & 0 \\ (L + 1)(e^{-(t-s)} - 1) & 1 \end{pmatrix}.$$

Hence, the covariance matrix of the Gaussian process U is

$$\text{Cov}(U(t), U(r)) = \int_0^{t \wedge r} \Phi(t, s) G(v(s), y(s)) [\Phi(r, s)]^T ds. \quad (3.3)$$

But we need to find out $\text{Cov}(U(\gamma_\infty), U(\gamma_\infty))$. Using (3.3), we obtain that

$$\text{Cov}(U(t), U(t)) = (1 - \rho_0) \times \begin{pmatrix} e^{-t}(1 - e^{-t}) & e^{-t}((L+1)(1 - e^{-t}) - t) \\ e^{-t}((L+1)(1 - e^{-t}) - t) & (L+1)^2 e^{-t}(1 - e^{-t}) + ((1 - \rho_0)^{-1} - 2(L+1)e^{-t})t \end{pmatrix}.$$

Therefore, since $(1 - \rho_0)e^{-\gamma_\infty} = 1 - v_\infty$ and $(1 - \rho_0)(1 - e^{-\gamma_\infty}) = v_\infty - \rho_0$,

$$\begin{aligned} \text{Var}(U_v(\gamma_\infty)) &= \frac{(1 - v_\infty)(v_\infty - \rho_0)}{1 - \rho_0}, \\ \text{Var}(U_y(\gamma_\infty)) &= \frac{(L+1)^2(1 - v_\infty)(v_\infty - \rho_0)}{1 - \rho_0} + (1 - 2(L+1)(1 - v_\infty))\gamma_\infty, \\ \text{Cov}(U_v(\gamma_\infty), U_y(\gamma_\infty)) &= \frac{(L+1)(1 - v_\infty)(v_\infty - \rho_0)}{1 - \rho_0} - (1 - v_\infty)\gamma_\infty. \end{aligned}$$

Using the well-known properties of the variance and simplifying properly, we get formula (2.4). \square

3.5. Proof of Claim 3.3. Claim 3.3 is a corollary of Lemma 3.7. Let us recall that $N^{(n+1)}$ is the number of transitions that $\tilde{x}^{(n)}$ makes until the time $\tilde{\tau}^{(n)}$. We can write $\tilde{\tau}^{(n)} = \sum_{i=1}^{N^{(n+1)}} R_i$, where $\{R_i\}_{i \geq 1}$ is a set of independent and identically distributed random variables having exponential distribution with parameter n , and $N^{(n+1)}$ is independent of $\{R_i\}_{i \geq 1}$. By Wald's equation,

$$\mathbb{E}(\tilde{\tau}^{(n)}) = \frac{\mathbb{E}(N^{(n+1)})}{n}.$$

Thus, in view of Lemma 3.7, the desired result is established once we show that the sequence $\{\tilde{\tau}^{(n)}\}$ is uniformly integrable. To prove this fact, we note that $N^{(n+1)} \leq \bar{N}^{(n+1)}$, where

$$\bar{N}^{(n+1)} = (n+1) \left((L+1)(1 - \rho_0^{(n+1)}) + \sum_{i=1}^L i \rho_i^{(n+1)} \right).$$

This upper bound is achieved when $\tilde{v}^{(n)}(\tilde{\tau}^{(n)}) = 1$ (or, equivalently, \mathcal{K}_{n+1} is fully visited), in which case there are $(n+1) \left(L(1 - \rho_0^{(n+1)}) + \sum_{i=1}^L i \rho_i^{(n+1)} \right)$ transitions $(0, -1)$ and $(n+1)(1 - \rho_0^{(n+1)})$ transitions $(1, L)$. Consequently,

$$\tilde{\tau}^{(n)} \leq \Gamma^{(n)} \sim \text{Gamma}(\bar{N}^{(n+1)}, n).$$

Since $\sup_n \mathbb{E}(\Gamma^{(n)})^2 < \infty$, we have that $\{\Gamma^{(n)}\}$ is uniformly integrable, and so is $\{\tilde{\tau}^{(n)}\}$. \square

Acknowledgments

The authors would like to thank the anonymous referee for helpful comments that led to improving the paper. Thomas Kurtz's research was supported in part by NSF grant DMS 05-03983, Elcio Lebensztayn was supported by FAPESP (05/04001-5), Alexandre Leichsenring by CNPq (141980/2006-0), and Fábio Machado by CNPq (306927/2007-1). Thanks are also due to CNPq Edital Universal 483967/2006-8.

References

- O. S. M. Alves, E. Lebensztayn, F. P. Machado and M. Z. Martinez. Random walks systems on complete graphs. *Bull. Braz. Math. Soc. (N. S.)* **37** (4), 571–580 (2006).
- O. S. M. Alves, F. P. Machado and S. Yu. Popov. Phase transition for the frog model. *Electron. J. Probab.* **7** (16), 21 pp. (2002a).
- O. S. M. Alves, F. P. Machado and S. Yu. Popov. The shape theorem for the frog model. *Ann. Appl. Probab.* **12** (2), 533–546 (2002b).
- H. Andersson and T. Britton. Stochastic epidemic models and their statistical analysis. In *Lecture Notes in Statistics*, volume 151. Springer-Verlag, New York (2000).
- M. S. Bartlett. Some evolutionary stochastic processes. *J. Roy. Statist. Soc. Ser. B* **11** (2), 211–229 (1949).
- B. Bollobás. Modern graph theory. In *Graduate Texts in Mathematics*, volume 184. Springer-Verlag, New York (1998).
- R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey and D. E. Knuth. On the lambert w function. *Adv. Comput. Math.* **5** (4), 329–359 (1996).
- S. N. Ethier and T. G. Kurtz. *Markov processes: Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986).
- H. Kesten and V. Sidoravicius. The spread of a rumor or infection in a moving population. *Ann. Probab.* **33** (6), 2402–2462 (2005).
- E. Lebensztayn, F. P. Machado and S. Popov. An improved upper bound for the critical probability of the frog model on homogeneous trees. *J. Stat. Phys.* **119** (1-2), 331–345 (2005).
- A. F. Ramírez and V. Sidoravicius. Asymptotic behavior of a stochastic combustion growth process. *J. Eur. Math. Soc.* **6** (3), 293–334 (2004).