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General Υ-transformations

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Abstract. In this paper we introduce a general class of transformations of (all or most of) the class $\mathfrak{M}_L(\mathbb{R}^d)$, of d-dimensional Lévy measures on $\mathbb{R}^d$, into itself. We refer to transformations of this type as Υ transformations (or Upsilon transformations). Closely associated to these are mappings of the set $\mathcal{ID}(\mathbb{R}^d)$ of all infinitely divisible laws on $\mathbb{R}^d$ into itself. In considerable generality, the mappings are one-to-one, regularising and bi-continuous. Furthermore, in many cases the transformations have a stochastic interpretation in terms of random integrals with respect to Lévy processes.

1. Introduction

In this paper we associate to any Lévy measure γ on $(0,\infty)$ certain transformations, which we refer to as Upsilon-transformations corresponding to γ. There are (at least) three natural ways of viewing the Upsilon transformations, namely, listed in decreasing order of generality,

(a) Transformations of Lévy measures: $\Upsilon_\gamma\colon D \to \mathfrak{M}_L(\mathbb{R}^d)$, where the domain $D \subseteq \mathfrak{M}_L(\mathbb{R}^d)$ depends on γ.

(b) Transformations of infinitely divisible probability measures: $\Upsilon^\gamma\colon D' \to \mathcal{ID}(\mathbb{R}^d)$, where the domain $D' \subseteq \mathcal{ID}(\mathbb{R}^d)$ depends on γ.

(c) Transformations of infinitely divisible probability measures given in terms of random integrals:

$$\mu \mapsto L\Big\{ \int f_\gamma(t)\,\mathrm{d}Z_t \Big\},$$

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where $L\{Y\}$ denotes the law of a random variable Y, f_γ is a fixed deterministic function and (Z_t) is a Lévy process such that $L\{Z_1\} = \mu$.

In the following we briefly describe the main features established in the paper of the above three points of view.

(a) Transformations of Lévy measures. For a σ-finite Borel measure ρ on $\mathbb{R}^d$, we define a new Borel measure $\Upsilon_\gamma(\rho)$ on $\mathbb{R}^d$ by the formula:

$$\big[\Upsilon_\gamma(\rho)\big](B) = \int_0^\infty \rho(x^{-1}B)\,\gamma(\mathrm{d}x), \tag{1.1}$$

for any Borel set B. If $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) < \infty$, then formula (1.1) produces a new Lévy measure $\Upsilon_\gamma(\rho)$ from any Lévy measure ρ, but if $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) = \infty$, this is only true for certain Lévy measures ρ, and we refer to the class of such ρ as the Lévy domain of Υ_γ, denoted by $\mathsf{dom}_L\Upsilon_\gamma$ (cf. Section 3). The mapping $\Upsilon_\gamma\colon \mathsf{dom}_L\Upsilon_\gamma \to \mathfrak{M}_L(\mathbb{R}^d)$ is termed the Upsilon transformation of Lévy measures associated to γ. Such transformations generally have a regularising effect, as we point out in Section 2, and they arise naturally in the study of random integrals and series representations of infinitely divisible laws (see e.g. Rosiński, 1984 and Rosiński, 1990). An application of Upsilon transformations to the construction of Lévy copulas with special properties is discussed in Barndorff-Nielsen and Lindner (2006). In the case where $d = 1$ and the Lévy measure ρ is concentrated on $(0,\infty)$, the measure $\Upsilon_\gamma(\rho)$ equals the multiplicative convolution $\rho \circledast \gamma$ of ρ and γ, and this reveals a commutativity of the roles of ρ and γ in the construction. In addition to domains we also study the ranges and continuity properties of the mappings Υ_γ. In many aspects the derived results turn out to be closely similar to those of unbounded operators on Banach spaces. Thus, we prove that Υ_γ is continuous on $\mathsf{dom}_L\Upsilon_\gamma$ if and only if it is Lévy bounded, that is if and only if $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) < \infty$, which, as mentioned above, is equivalent to having $\mathsf{dom}_L\Upsilon_\gamma = \mathfrak{M}_L(\mathbb{R}^d)$. In this case we also show that Υ_γ is a closed mapping in the sense that it takes closed subsets of $\mathfrak{M}_L(\mathbb{R}^d)$ to new closed subsets of $\mathfrak{M}_L(\mathbb{R}^d)$. This immediately implies that Υ_γ is a homeomorphism whenever it is injective. The topology on $\mathfrak{M}_L(\mathbb{R}^d)$, to which the above results refer, is that of Lévy weak convergence, as introduced in Section 5. The question of injectivity of Υ_γ is delicate. In Section 6 we give some partial results which may be used to establish injectivity for rather general classes of Upsilon transformations. A more detailed analysis will be given in a forthcoming paper.

(b) Transformations of infinitely divisible laws. If $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) < \infty$, then we associate to γ a mapping $\Upsilon^\gamma\colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$, which may be defined in terms of cumulant transforms by the equality

$$C_{\Upsilon^\gamma(\mu)}(z) = \int_0^\infty C_\mu(tz)\,\gamma(\mathrm{d}t), \qquad (z \in \mathbb{R}^d), \tag{1.2}$$

where C_μ denotes the cumulant transform of μ (for its definition, see (7.1) below). From equation (1.2) it it is easy to derive that Υ^γ preserves the affine structure of $\mathcal{ID}(\mathbb{R}^d)$, in the sense that

(i) $\Upsilon^\gamma(\mu_1 * \mu_2) = \Upsilon^\gamma(\mu_1) * \Upsilon^\gamma(\mu_2)$, $\qquad (\mu_1, \mu_2 \in \mathcal{ID}(\mathbb{R}^d))$,
(ii) $\Upsilon^\gamma(T_B\mu) = T_B\Upsilon^\gamma(\mu)$, $\qquad (B \in M_d(\mathbb{R}),\ \mu \in \mathcal{ID}(\mathbb{R}^d))$,
(iii) $\{\Upsilon^\gamma(\delta_c) \mid c \in \mathbb{R}^d\} \subseteq \{\delta_c \mid c \in \mathbb{R}^d\}$,

where $T_B\mu$ denotes the transformation of μ by the linear mapping T_B associated to the $d \times d$-matrix B, and δ_c denotes the Dirac measure at c. As a consequence of (i)–(iii), for any non-zero γ such that $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) < \infty$, the range of Υ^γ is a subset of $\mathit{ID}(\mathbb{R}^d)$, which contains all Dirac measures and is closed under convolution and linear transformations. We prove in addition that the range is closed in the topology of weak convergence. These properties of the ranges are shared by many important classes of infinitely divisible probability measures (e.g., for $d = 1$, the selfdecomposable laws and the Goldie-Steutel-Bondesson class), and, as we shall indicate, a significant number of such classes are in fact realised as ranges of Upsilon transformations. If γ is a σ-finite Borel measure on $(0,\infty)$ such that $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) = \infty$, then the integral in the right hand side of (1.2) is generally not well-defined for all measures μ from $\mathit{ID}(\mathbb{R}^d)$, and (1.2) only gives rise to a mapping on a restricted class of measures μ. Nonetheless, interesting examples of such mappings with restricted domains have already appeared in the literature. For instance if γ is the measure with Lebesgue-density $t^{-1}1_{(0,1)}(t)$, then (1.2) gives rise to a mapping Φ_0, which was studied (in the case $d = 1$) in Barndorff-Nielsen et al. (2004). The domain of Φ_0 is the class of infinitely divisible laws, for which the Lévy measure has finite logarithmic moment, and the range of Φ_0 is the class of selfdecomposable laws (see Examples 7.7 below).
(c) Transformations in terms of random integrals. Under certain restrictions on γ, including the condition $\int_0^\infty (1 \vee x^2)\,\gamma(\mathrm{d}x) < \infty$, the mapping Υ^γ described above may be given a stochastic interpretation via random integrals: $\Upsilon^\gamma(\mu)$ may be realised as the distribution of the random integral

$$\int f_\gamma(t)\,\mathrm{d}Z_t,$$

for a suitable deterministic function f_γ (depending on γ), and where (Z_t) is a Lévy process with Z_1 having law μ. Mappings of this kind were introduced by Jurek (1990) under the name of λ-mixtures of dilations of measures on Banach spaces. The random integral point of view is not the focus of the present paper, but it will be discussed briefly at the end of the paper (Section 9), with reference in particular to extensive recent works of Sato (2006b), (2006a) and (2007).

The paper is organised as follows: Section 2 gives the definition of the Upsilon transformations of Lévy measures, discusses their regularising effect and provides some examples. In that section we also establish the commutativity of the Upsilon transformations and the relation of this to multiplicative convolution. Questions relating to the domains of the transformations are discussed in Section 3, partly based on an auxiliary function ψ, introduced in that section. Section 4 is concerned with composition and ranges of the transformations, and Section 5 considers their continuity properties. Injectivity is discussed in Section 6. The two penultimate sections discuss Upsilon transformations of $\mathit{ID}(\mathbb{R}^d)$. In Sections 7 we give their precise definition and establish their algebraic properties, and Section 8 is concerned with their continuity properties. The final Section 9 discusses how the Upsilon transformations, in somewhat less generality, are representable as random integrals with respect to Lévy processes.

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2. Definition, first properties and examples

2.1. Notation and definition.

By $\mathfrak{M}(\mathbb{R}^d)$ we denote the set of all (positive) Borel measures on $\mathbb{R}^d$, and by $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d)$ we denote the set of all Borel σ-finite measures ρ on $\mathbb{R}^d$ with $\rho(\{0\}) = 0$. Furthermore, $\mathfrak{M}_L(\mathbb{R}^d)$ stands for the subset of $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d)$ consisting of the Lévy measures, i.e.

$$\mathfrak{M}_L(\mathbb{R}^d) = \left\{\rho \in \mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d) \,\middle|\, \textstyle\int_{\mathbb{R}^d}(1 \wedge \|x\|^2)\,\rho(\mathrm{d}x) < \infty\right\},$$

with $\|\cdot\|$ the usual Euclidean norm on $\mathbb{R}^d$. The classes $\mathfrak{M}((0,\infty)^d)$, $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty)^d)$ and $\mathfrak{M}_L((0,\infty)^d)$ are defined analogously; and we use $\mathfrak{M}_L^+((0,\infty))$ to denote the class of Lévy measures for infinitely divisible distributions concentrated on $(0,\infty)$, i.e.

$$\mathfrak{M}_L^+((0,\infty)) = \left\{\rho \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty)) \,\middle|\, \textstyle\int_0^\infty(1 \wedge x)\,\rho(\mathrm{d}x) < \infty\right\}.$$

Elements of $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d)$ will be denoted by ρ, σ, or τ, and γ and η will denote members of $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Finally, we introduce the class $\mathfrak{M}_{02}(\mathbb{R}^d)$ of finite Borel measures on $\mathbb{R}^d$ with finite second moment:

$$\mathfrak{M}_{02}(\mathbb{R}^d) = \left\{\rho \in \mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d) \,\middle|\, \textstyle\int_{\mathbb{R}^d}(1 \vee \|x\|^2)\,\rho(\mathrm{d}x) < \infty\right\}.$$

Definition 2.1. For any $\gamma \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$, let $\Upsilon_\gamma : \mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d) \to \mathfrak{M}(\mathbb{R}^d)$ be the mapping determined by

$$[\Upsilon_\gamma(\rho)](B) = \int_0^\infty \rho(x^{-1}B)\,\gamma(\mathrm{d}x),$$

for all Borel sets B. We refer to Υ_γ as the Upsilon transformation with dilation measure γ.

We shall also use ρ_γ as a shorthand notation for $\Upsilon_\gamma(\rho)$, and if γ is absolutely continuous with a density g we occasionally write $\Upsilon_g(\rho)$ and ρ_g. Note that a measure γ from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ gives rise to an Upsilon transformation for each value of the dimension d. We shall sometimes use the notation $\Upsilon_\gamma^{(d)}$ for this mapping, when it is appropriate to emphasise d. In case ρ is a measure on $\mathbb{R}\setminus\{0\}$ then we shall write $\underleftarrow{\rho}$ for the transformation of ρ by the reciprocity mapping $x \mapsto x^{-1}$.

2.2. Commutativity and connection to multiplicative convolution.

The proofs of the following two propositions are straightforward, thus omitted. The latter result indicates that in wide generality Υ_γ has a regularising effect.

Proposition 2.2. *Let ρ and γ be measures in $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R})$ and $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$, respectively. Then for any Borel subset A of $\mathbb{R}\setminus\{0\}$,*

$$\rho_\gamma(A) = \int_0^\infty \rho(yA)\,\underleftarrow{\gamma}(\mathrm{d}y) = \int_{\mathbb{R}} \gamma(yA)\,\underleftarrow{\rho}(\mathrm{d}y)\,. \tag{2.1}$$

Proposition 2.3. *Suppose γ is a measure in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ which is absolutely continuous with respect to Lebesgue measure and let g denote the density of γ. Let further ρ be a measure in $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R})$. Then ρ_γ is absolutely continuous with respect to Lebesgue measure, and the density r_γ is given by*

$$r_\gamma(t) = \begin{cases} \int_0^\infty g\,(ty)\, y\, \underset{\leftarrow}{\rho}\,(\mathrm{d}y)\,, & \text{if } t > 0, \\ \int_{-\infty}^0 g\,(ty)\, |y|\, \underset{\leftarrow}{\rho}\,(\mathrm{d}y)\,, & \text{if } t < 0. \end{cases} \tag{2.2}$$

Examples 2.4. The following examples of Upsilon transformations with dilation density $g(x) = \frac{\mathrm{d}\gamma}{\mathrm{d}x}$ have previously been discussed in the literature (see the papers Barndorff-Nielsen and Thorbjørnsen, 2004, 2006, 2005; Barndorff-Nielsen et al., 2004; Barndorff-Nielsen and Pérez-Abreu, 2007). We return to these examples in the following sections.

(1) Setting

$$g(x) = \mathrm{e}^{-x}, \qquad (x \in (0,\infty)),$$

produces the Upsilon mapping Υ_0 which was introduced in Barndorff-Nielsen and Thorbjørnsen (2004) and studied further in Barndorff-Nielsen et al. (2004) and Barndorff-Nielsen and Thorbjørnsen (2006). Proposition 2.3 reveals that for any measure ρ in $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R})$, the density of $\Upsilon_0(\rho)$ is the Laplace transform of the measure $y\, \underset{\leftarrow}{\rho}\,(\mathrm{d}y)$.

(2) For α in $(0,1)$ we put

$$g\,(x) = \alpha^{-1} x^{-1-1/\alpha} \sigma_\alpha\big(x^{-1/\alpha}\big), \qquad (x \in (0,\infty)),$$

where σ_α is the density of the positive α-stable law having Laplace transform $\mathrm{e}^{-\theta^\alpha}$. We write Υ_α for the associated Upsilon transformation. In the limiting case $\alpha = 0$ we recover the mapping Υ_0 from (1) above, for $\alpha = 1$ the identity mapping, and the family $\{\Upsilon_\alpha \mid \alpha \in [0,1]\}$ interpolates smoothly between these two cases, see Barndorff-Nielsen and Thorbjørnsen (2006).

For any ρ in $\mathfrak{M}_{\sigma\mathrm{f}}((0,+\infty))$, it follows from Proposition 2.3 that $\Upsilon_\alpha(\rho)$ has Lebesgue-density

$$r_\alpha\,(t) = \alpha^{-1} t^{-1-1/\alpha} \int_0^\infty \xi^{-1/\alpha} \sigma_\alpha\big((t\xi)^{-1/\alpha}\big)\, \underset{\leftarrow}{\rho}\,(\mathrm{d}\xi)\,, \qquad (t > 0).$$

(3) For any λ in $(-2,\infty)$, let

$$g(x) = x^{\lambda-1}\mathrm{e}^{-x} \qquad (x \in (0,\infty)).$$

The corresponding Upsilon mappings Ξ_λ were introduced and studied in Sato (2005) and Barndorff-Nielsen and Pérez-Abreu (2007); see also Sato (2006b). For an extension to Upsilon mappings of Lévy measures on the cone of positive definite matrices, see Barndorff-Nielsen and Pérez-Abreu (2007).

(4) For $\lambda > -2$, consider the Lévy density given by

$$g(x) = x^{\lambda-1} 1_{(0,1)}(x), \qquad (x \in (0,\infty)).$$

We denote the corresponding Υ-mapping by Φ_λ. The mapping Φ_0 was introduced and studied in Barndorff-Nielsen et al. (2004). In this particular case, it follows from Proposition 2.3 and direct computation that for ρ in $\mathfrak{M}_{\sigma f}(\mathbb{R})$, $\Phi_0(\rho)$ has Lebesgue-density

$$r_0(t) = \begin{cases} t^{-1}\rho((t,\infty)), & \text{if } t > 0, \\ |t|^{-1}\rho((-\infty,t)), & \text{if } t < 0. \end{cases}$$

(5) For an arbitrary α in $(0,2)$, consider the Lévy density of the elemental tempered stable law, i.e.

$$g(x) = x^{-\alpha-1}\mathrm{e}^{-x}, \qquad (x \in (0,\infty)).$$

Such a Lévy measure is obtained as the image of the Lévy measure having density

$$r(\xi) = \frac{1}{\Gamma(\alpha)} 1_{(0,1)}(\xi)\xi^{-\alpha-1}(1-\xi)^{\alpha-1}$$

under the transformation Ξ_{-1}.

Given two σ-finite measures γ and η on the multiplicative group $(0,\infty)$ we consider their convolution $\gamma \circledast \eta$ given by

$$\gamma \circledast \eta(B) = \int_{(0,\infty)^2} 1_B(xy)\,\gamma(\,\mathrm{d}x)\eta(\,\mathrm{d}y), \tag{2.3}$$

for any Borel subset B of $(0,\infty)$. Clearly the operation $\circledast$ is commutative, i.e., $\gamma \circledast \eta = \eta \circledast \gamma$, and the multiplicative convolution $\circledast$ is converted into ordinary convolution by log transformation.

It is easy to verify that

$$\Upsilon_\gamma(\eta) = \gamma \circledast \eta = \eta \circledast \gamma = \Upsilon_\eta(\gamma). \tag{2.4}$$

Moreover, if $\gamma(\mathrm{d}t) = f_\gamma(t)\,\mathrm{d}t$, then $(\eta \circledast \gamma)(\mathrm{d}t) = f_{\eta\circledast\gamma}(t)\,\mathrm{d}t$, where

$$f_{\eta\circledast\gamma}(t) = \int_0^\infty f_\gamma(ts^{-1})s^{-1}\,\eta(\mathrm{d}s). \tag{2.5}$$

If in addition $\eta(\mathrm{d}t) = f_\eta(t)\,\mathrm{d}t$, then

$$f_{\eta\circledast\gamma}(t) = \int_0^\infty f_\gamma(ts^{-1})s^{-1}f_\eta(s)\,\mathrm{d}s. \tag{2.6}$$

Example 2.5. Notice that multiplicative convolution of σ-finite measures need not be σ-finite. Indeed, let $f_\gamma(t) = f_\eta(t) = t^{-1-\alpha}$, $\alpha \in \mathbb{R}$. Then

$$f_{\eta\circledast\gamma}(t) = \infty \quad \text{for every } t > 0.$$

Hence $\eta \circledast \gamma$ is infinite on every set of positive Lebesgue measure.

If η and γ are probability measures on $(0,\infty)$ and X and Y are independent random variables with distributions η and γ respectively, then $\eta \circledast \gamma$ is the distribution of the product XY. This provides a further link to infinite divisibility, which gives rise to a concept of “semigroups of Upsilon transformations”, see Barndorff-Nielsen and Maejima (2007).

3. Discussion of domains.

3.1. Lévy Domain: Definition, examples and first properties.

For any Upsilon mapping Υ_γ we define its *Lévy domain* by

$$\mathsf{dom}_L \Upsilon_\gamma = \big\{\rho \in \mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d) \bigm| \rho_\gamma \in \mathfrak{M}_L(\mathbb{R}^d)\big\},$$

where dom stands for domain. In other words, $\mathsf{dom}_L \Upsilon_\gamma$ is the set of measures from $\mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}^d)$ that are mapped to Lévy measures by Υ_γ. We also define

$$\mathsf{dom}_L^+ \Upsilon_\gamma = \big\{\rho \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty)^d) \bigm| \rho_\gamma \in \mathfrak{M}_L^+((0,\infty)^d)\big\},$$

so that $\mathsf{dom}_L^+ \Upsilon_\gamma$ is the pre-image for Υ_γ of the class of Lévy measures for subordinators.

Examples 3.1. We adopt the notation from Example 2.4, and assume for simplicity that $d = 1$.

(1) For the mapping Υ_0 we have $\mathsf{dom}_L \Upsilon_0 = \mathfrak{M}_L(\mathbb{R})$, as was shown in Barndorff-Nielsen and Thorbjørnsen (2004). This also follows immediately from Theorem 3.4 below.

(2) For the mappings Υ_α it was shown in Barndorff-Nielsen and Thorbjørnsen (2006) that $\mathsf{dom}_L \Upsilon_\alpha = \mathfrak{M}_L(\mathbb{R})$ for all α in $(0, 1)$. Again, this may be seen as an immediate consequence of Theorem 3.4.

(3) For the Ξ_λ-mappings it is easily established (see Barndorff-Nielsen et al., 2004) that

$$\mathsf{dom}_L \Xi_\lambda = \begin{cases} \mathfrak{M}_L(\mathbb{R}), & \text{if } \lambda > 0 \\ \mathfrak{M}_{\log}(\mathbb{R}), & \text{if } \lambda = 0 \\ \mathfrak{M}_\lambda(\mathbb{R}), & \text{if } \lambda \in (-2, 0) \end{cases}$$

where the classes $\mathfrak{M}_{\log}(\mathbb{R})$ and $\mathfrak{M}_\lambda(\mathbb{R})$, $\lambda \in (0, 1)$, are defined by:

$$\mathfrak{M}_{\log}(\mathbb{R}) = \big\{\rho \in \mathfrak{M}(\mathbb{R}) \bigm| \textstyle\int_1^\infty \log y \,\rho(\mathrm{d}y) < \infty\big\}$$

and

$$\mathfrak{M}_\lambda(\mathbb{R}) = \big\{\rho \in \mathfrak{M}(\mathbb{R}) \bigm| \textstyle\int_1^\infty y^{-\lambda} \,\rho(\mathrm{d}y) < \infty\big\},$$

respectively.

(4) For the Upsilon mappings Φ_λ, it is easy to check that for all λ in $(-2, \infty)$ we have

$$\mathsf{dom}_L \Phi_\lambda = \mathsf{dom}_L \Xi_\lambda,$$

with Ξ_λ as in (3).

Proposition 3.2. *For any nonzero measure γ in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$, we have*

$$\mathsf{dom}_L \Upsilon_\gamma^{(d)} \subseteq \mathfrak{M}_L\big(\mathbb{R}^d\big) \quad \textit{and} \quad \mathsf{dom}_L^+ \Upsilon_\gamma^{(d)} \subseteq \mathfrak{M}_L^+((0,\infty)^d). \tag{3.1}$$

Proof: Let $a > 0$ be such that $\gamma([a,\infty)) = b > 0$. Then for every $\rho \in \mathsf{dom}_L \Upsilon_\gamma$

$$\begin{aligned} \infty > \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\,\rho_\gamma(\mathrm{d}x) &= \int_{\mathbb{R}^d} \int_0^\infty (t^2\|x\|^2 \wedge 1)\,\gamma(\mathrm{d}t)\rho(\mathrm{d}x) \\ &\geq b \int_{\mathbb{R}^d} (a^2\|x\|^2 \wedge 1)\,\rho(\mathrm{d}x) \geq b(a^2 \wedge 1) \int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\,\rho(\mathrm{d}x) \end{aligned}$$

which shows that $\rho \in \mathfrak{M}_L(\mathbb{R}^d)$. The second inclusion follows similarly by replacing $\|x\|^2 \wedge 1$ by $\|x\| \wedge 1$ in the argument above.

Proposition 3.2 is valid even when $\mathbb{R}^d$ is replaced by a Banach space, see Proposition 2 in Jurek (1990). However, since Lévy measures on a general Banach space are not determined by an integrability condition, the above simple proof does not apply.

Remark 3.3. (a) Suppose γ and η are σ-finite measures on $(0,\infty)$ and consider their multiplicative convolution $\gamma \circledast \eta$ (cf. Subsection 2.2). Then from (2.4) we infer that

$$\gamma \circledast \eta \in \mathfrak{M}_L((0,\infty)) \iff \eta \in \mathsf{dom}_L \Upsilon_\gamma \iff \gamma \in \mathsf{dom}_L \Upsilon_\eta. \tag{3.2}$$

Assuming that $\gamma, \eta \neq 0$, Proposition 3.2 together with (3.2) then asserts that

$$\gamma \circledast \eta \in \mathfrak{M}_L((0,\infty)) \Longrightarrow \gamma, \eta \in \mathfrak{M}_L((0,\infty)). \tag{3.3}$$

(b) Let ρ and γ be measures in $\mathfrak{M}_L(\mathbb{R}^d)$ and $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$, respectively, and let $\|\rho\|$ denote the transformation of ρ under the mapping $x \mapsto \|x\|$. Using Tonelli's theorem we note then that

$$\begin{aligned}\int_{\mathbb{R}^d} (\|x\|^2 \wedge 1)\, \rho_\gamma(\mathrm{d}x) &= \int_{(0,\infty)} \Big(\int_{\mathbb{R}^d} (t^2 \|x\|^2 \wedge 1)\, \rho(\mathrm{d}x) \Big)\, \gamma(\mathrm{d}t) \\ &= \int_{(0,\infty)} \Big(\int_{(0,\infty)} (t^2 s^2 \wedge 1)\, \|\rho\|(\mathrm{d}s) \Big)\, \gamma(\mathrm{d}t) \\ &= \int_{\mathbb{R}^d} (t^2 \wedge 1)\, \gamma_{\|\rho\|}(\mathrm{d}t),\end{aligned} \tag{3.4}$$

so that

$$\rho \in \mathsf{dom}_L \Upsilon_\gamma \iff \gamma \in \mathsf{dom}_L \Upsilon_{\|\rho\|}. \tag{3.5}$$

Taking then Proposition 3.2 into account, it follows that

$$\forall \gamma \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))\colon \mathsf{dom}_L \Upsilon_\gamma \neq \{0\} \Longrightarrow \gamma \in \mathfrak{M}_L((0,\infty)), \tag{3.6}$$

which shows that Υ_γ is only interesting as a mapping on the class of Lévy measures if γ is itself a Lévy measure.

The following theorem has also been noted, independently, by K. Sato (cf. Sato, 2005). In the following section we obtain a proof of the theorem as a result of a comparison of domains for two Υ transformations.

Theorem 3.4. (i) *Let γ be a non-zero measure from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Then for any positive integer d we have*

$$\mathsf{dom}_L \Upsilon_\gamma^{(d)} = \mathfrak{M}_L\big(\mathbb{R}^d\big) \tag{3.7}$$

if and only if $\gamma \in \mathfrak{M}_{02}((0,\infty))$, i.e. if and only if

$$\gamma((0,\infty)) < \infty \quad \textit{and} \quad \int_0^\infty t^2\, \gamma(\mathrm{d}t) < \infty. \tag{3.8}$$

(ii) *Let γ be a non-zero measure from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Then for any positive integer d,*

$$\mathsf{dom}_L^+ \Upsilon_\gamma^{(d)} = \mathfrak{M}_L^+((0,\infty)^d) \tag{3.9}$$

if and only if

$$\int_{(0,\infty)} (1 \vee t)\, \gamma(\mathrm{d}t) < \infty. \tag{3.10}$$

Remark 3.5. Combining Theorem 3.4 with (3.5) it follows that

$$\forall \gamma \in \mathfrak{M}_L((0,\infty))\colon \mathsf{dom}_L \Upsilon_\gamma^{(d)} \supseteq \mathfrak{M}_{02}(\mathbb{R}^d) \tag{3.11}$$

and also that (cf. (3.6))

$$\forall \gamma \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))\colon \mathsf{dom}_L \Upsilon_\gamma^{(d)} \neq \{0\} \Longrightarrow \mathsf{dom}_L \Upsilon_\gamma^{(d)} \supseteq \mathfrak{M}_{02}(\mathbb{R}^d).$$

3.2. **An auxiliary function: Definition and applications.**

For a number of the calculations to follow, it is helpful to introduce an auxiliary function ψ_γ by

Definition 3.6. For a measure γ in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ we define the function $\psi_\gamma\colon [0,\infty) \to [0,\infty]$ by

$$\psi_\gamma(s) = \int_0^\infty (s^2t^2 \wedge 1)\,\gamma(\mathrm{d}t), \quad (s \in [0,\infty)). \tag{3.12}$$

It follows immediately from the calculation (3.4) that for $\gamma \neq 0$

$$\mathsf{dom}_L \Upsilon_\gamma = \big\{\rho \in \mathfrak{M}_L(\mathbb{R}^d) \;\big|\; \textstyle\int_{\mathbb{R}^d} \psi_\gamma(\|x\|)\,\rho(\mathrm{d}x) < \infty\big\}. \tag{3.13}$$

We mention in passing that for a non-zero Lévy measure γ on $(0,\infty)$, ψ_γ is a non-decreasing continuous function with $\psi_\gamma(0) = 0$ and $\psi_\gamma(s) > 0$, whenever $s > 0$. Moreover, $\lim_{s\to\infty} \psi_\gamma(s) = \gamma((0,\infty))$.

Remark 3.7. The characterisation (3.13) of $\mathsf{dom}_L \Upsilon_\gamma$ remains valid when $\mathbb{R}^d$ is replaced by a Hilbert space but is invalid for general Banach spaces. Jurek (1990) obtained some characterisations of $\mathsf{dom}_L \Upsilon_\gamma$ for Banach spaces in cases where either γ or ρ have restricted support.

Comparison of domains.

Theorem 3.8. *Let γ_1 and γ_2 be measures from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Then $\mathsf{dom}_L \Upsilon_{\gamma_2}^{(d)} \subseteq \mathsf{dom}_L \Upsilon_{\gamma_1}^{(d)}$ for all d, if and only if*

$$\exists C > 0\colon \psi_{\gamma_1}(s) \le C\psi_{\gamma_2}(s), \quad (s \in [0,\infty)). \tag{3.14}$$

Proof: We note first that we may assume that both γ_1 and γ_2 are Lévy measures. Indeed, if $\gamma \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$, then the inequalities

$$(1 \vee s^2)\int_{(0,\infty)} (t^2 \wedge 1)\,\gamma(\mathrm{d}t) \ge \psi_\gamma(s) \ge (1 \wedge s^2)\int_{(0,\infty)} (t^2 \wedge 1)\,\gamma(\mathrm{d}t)$$

verify the statement

$$\gamma \notin \mathfrak{M}_L((0,\infty)) \iff \psi_\gamma(s) = \infty, \quad \text{for all } s \text{ in } (0,\infty).$$

Moreover, for any γ in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ we have

$$\mathsf{dom}_L \Upsilon_\gamma = \{0\} \iff \gamma \notin \mathfrak{M}_L((0,\infty)),$$

where "$\Leftarrow$" follows from (3.6) and "$\Rightarrow$" follows from the fact that $\delta_1 \in \mathsf{dom}_L \Upsilon_\gamma$ for any Lévy measure γ. From these observations the proposition follows readily if one of the measures γ_1 or γ_2 is not a Lévy measure. In a similar manner we may assume that γ_1 and γ_2 are both non-zero. Indeed, for γ in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ we have

$$\psi_\gamma = 0 \iff \gamma = 0 \iff \mathsf{dom}_L \Upsilon_\gamma = \mathfrak{M}_{\sigma\mathrm{f}}(\mathbb{R}),$$

where, in the latter bi-implication, the implication "$\Leftarrow$" is a consequence of Proposition 3.2.

So assume in the following that γ_1, γ_2 are both non-zero Lévy measures on $(0,\infty)$. It follows immediately from (3.13) that condition (3.14) implies that $\mathsf{dom}_L \Upsilon_{\gamma_2} \subseteq \mathsf{dom}_L \Upsilon_{\gamma_1}$. Conversely, assume that (3.14) is *not* satisfied. We then construct, for each d in $\mathbb{N}$, a measure ρ in $\mathfrak{M}_L(\mathbb{R}^d)$ such that $\rho \in \mathsf{dom}_L \Upsilon_{\gamma_2} \setminus \mathsf{dom}_L \Upsilon_{\gamma_1}$. Indeed, since (3.14) is not satisfied we may, for each n in $\mathbb{N}$, choose a number s_n in $(0,\infty)$ such that

$$\psi_{\gamma_1}(s_n) > n\psi_{\gamma_2}(s_n).$$

Then choose a fixed unit vector u in $\mathbb{R}^d$ and define the measure ρ on $\mathbb{R}^d$ by

$$\rho = \sum_{n=1}^{\infty} \frac{1}{n\psi_{\gamma_1}(s_n)} \delta_{s_n u}.$$

Note then that

$$\int_{\mathbb{R}^d} \psi_{\gamma_2}(\|x\|)\, \rho(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{\psi_{\gamma_2}(s_n)}{n\psi_{\gamma_1}(s_n)} \le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

Thus, by (3.13), $\rho \in \mathsf{dom}_L \Upsilon_{\gamma_2}$, so in particular $\rho \in \mathfrak{M}_L(\mathbb{R}^d)$ according to Proposition 3.2. Note next that

$$\int_{\mathbb{R}^d} \psi_{\gamma_1}(\|x\|)\, \rho(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{\psi_{\gamma_1}(s_n)}{n\psi_{\gamma_1}(s_n)} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so that $\rho \notin \mathsf{dom}_L \Upsilon_{\gamma_1}$.

Based on Theorem 3.8 we present next the proclaimed proof of Theorem 3.4.

Proof of Theorem 3.4. (i) Suppose first that $\mathsf{dom}_L \Upsilon_{\gamma} = \mathfrak{M}_L(\mathbb{R}) = \mathsf{dom}_L \Upsilon_{\delta_1}$. Then it follows from Theorem 3.8 that

$$\int_0^{\infty} (1 \wedge s^2 t^2)\, \gamma(\mathrm{d}t) = \psi_{\gamma}(s) \le C\psi_{\delta_1}(s) = C(1 \wedge s^2), \qquad (s \in (0,\infty)) \tag{3.15}$$

for some positive constant C. For s in $(0,1)$, (3.15) says that

$$\int_0^{\infty} (s^{-2} \wedge t^2)\, \gamma(\mathrm{d}t) \le C,$$

and letting then $s \searrow 0$, we obtain by monotone convergence that $\int_0^{\infty} t^2\, \gamma(\mathrm{d}t) \le C$.

For s in $[1,\infty)$, (3.15) says that

$$\int_0^{\infty} (1 \wedge s^2 t^2)\, \gamma(\mathrm{d}t) \le C,$$

and letting $s \nearrow \infty$, we obtain by monotone convergence that

$$\gamma((0,\infty)) = \int_0^{\infty} 1\, \gamma(\mathrm{d}t) \le C.$$

Altogether $\gamma \in \mathfrak{M}_{02}((0,\infty))$. Conversely assume that $\gamma \in \mathfrak{M}_{02}((0,\infty))$. Then

$$\psi_{\gamma}(s) = \int_0^{\infty} (1 \wedge s^2 t^2)\, \gamma(\mathrm{d}t) \le (1 \wedge s^2) \int_0^{\infty} (1 \vee t^2)\, \gamma(\mathrm{d}t) = C\psi_{\delta_1}(s),$$

where $C = \int_0^{\infty} (1 \vee t^2)\, \gamma(\mathrm{d}t) < \infty$. Hence it follows from Theorem 3.8 that for any d in $\mathbb{N}$,

$$\mathfrak{M}_L(\mathbb{R}^d) \supseteq \mathsf{dom}_L(\Upsilon_{\gamma}^{(d)}) \supseteq \mathsf{dom}_L(\Upsilon_{\delta_1}^{(d)}) = \mathfrak{M}_L(\mathbb{R}^d),$$

as desired.

(ii) Let ρ be a measure in $\mathfrak{M}_L((0,\infty)^d)$ and let γ be a measure in $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Then we denote by $\|\rho\|^2$ and $\sqrt{\gamma}$ the transformations of ρ and γ by the mappings $x \mapsto \|x\|^2$ and $t \mapsto \sqrt{t}$, respectively. Note then that

$$\begin{aligned}\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \rho_{\sqrt{\gamma}}(\mathrm{d}x) &= \int_{(0,\infty)} \Big(\int_{\mathbb{R}^d} (1 \wedge t^2\|x\|^2)\, \rho(\mathrm{d}x) \Big) \sqrt{\gamma}(\mathrm{d}t) \\ &= \int_{(0,\infty)} \Big(\int_{(0,\infty)} (1 \wedge ts)\, \|\rho\|^2(\mathrm{d}s) \Big) \gamma(\mathrm{d}t) \\ &= \int_{(0,\infty)} (1 \wedge s)\, \|\rho\|^2_\gamma(\mathrm{d}s),\end{aligned}$$

which shows that

$$\rho \in \mathsf{dom}_L \Upsilon^{(d)}_{\sqrt{\gamma}} \iff \|\rho\|^2 \in \mathsf{dom}^+_L \Upsilon^{(1)}_\gamma. \tag{3.16}$$

In the case $\gamma = \delta_1$, note that $\mathsf{dom}_L \Upsilon^{(d)}_{\sqrt{\delta_1}} = \mathfrak{M}_L(\mathbb{R}^d)$ and that $\mathsf{dom}^+_L \Upsilon^{(1)}_{\delta_1} = \mathfrak{M}^+_L((0,\infty))$, and therefore (3.16) implies that

$$\{\|\rho\|^2 \mid \rho \in \mathfrak{M}_L(\mathbb{R}^d)\} = \mathfrak{M}^+_L((0,\infty)). \tag{3.17}$$

Indeed, the inclusion "$\subseteq$" follows immediately from (3.16). Conversely, let σ be a measure from $\mathfrak{M}^+_L((0,\infty))$, and let ρ be the transformation of σ under the mapping $t \mapsto \sqrt{t}u\colon (0,\infty) \to \mathbb{R}^d$ for some unit vector u in $\mathbb{R}^d$. Now, $\|\rho\|^2 = \sigma$ and, by (3.16) (with $\gamma = \delta_1$), $\rho \in \mathfrak{M}_L(\mathbb{R}^d)$.

Using then (3.16), (3.17), Proposition 3.2 and part (i) it follows that

$$\begin{aligned}\mathsf{dom}^+_L \Upsilon^{(1)}_\gamma = \mathfrak{M}^+_L((0,\infty)) &\iff \mathsf{dom}_L \Upsilon^{(d)}_{\sqrt{\gamma}} = \mathfrak{M}_L(\mathbb{R}^d) \\ &\iff \int_{(0,\infty)} (1 \vee t^2)\sqrt{\gamma}(\mathrm{d}t) < \infty \\ &\iff \int_{(0,\infty)} (1 \vee s)\, \gamma(\mathrm{d}s) < \infty,\end{aligned} \tag{3.18}$$

as desired. $\square$

How small can the domain get?

If $\rho \in \mathfrak{M}_L(\mathbb{R}^d) \setminus \mathfrak{M}_{02}(\mathbb{R}^d)$, or, equivalently, $\|\rho\| \in \mathfrak{M}_L((0,\infty)) \setminus \mathfrak{M}_{02}((0,\infty))$, then it follows from Theorem 3.4 that there is a measure η in $\mathfrak{M}_L((0,\infty))$ such that $\eta \notin \mathsf{dom}_L \Upsilon_{\|\rho\|}$. This implies by (3.5) that $\rho \notin \mathsf{dom}_L \Upsilon^{(d)}_\eta$, and hence, taking also (3.11) into account, we may conclude that

$$\bigcap_{\eta \in \mathfrak{M}_L((0,\infty))} \mathsf{dom}_L \Upsilon^{(d)}_\eta = \mathfrak{M}_{02}(\mathbb{R}^d).$$

One may then ask whether there is a single measure η from $\mathfrak{M}_L((0,\infty))$ such that $\mathsf{dom}_L \Upsilon^{(d)}_\eta = \mathfrak{M}_{02}(\mathbb{R}^d)$. This will be answered in the negative in Proposition 3.9 below.

Proposition 3.9. *For any Lévy measure γ on $(0,\infty)$ and for any positive integer d we have that*

$$\mathsf{dom}_L \Upsilon^{(d)}_\gamma \supsetneqq \mathfrak{M}_{02}(\mathbb{R}^d).$$

Proof: Clearly we may assume that $\gamma \neq 0$. Since ψ_γ is continuous and $\psi_\gamma(0) = 0$, we can choose a sequence (s_n) in $(0,1)$ such that

$$\forall n \in \mathbb{N}: \ \psi_\gamma(s_n) \leq \tfrac{1}{n}.$$

Consider the measure ρ on $\mathbb{R}^d$ given by

$$\rho = \sum_{n=1}^{\infty} \frac{1}{n} \delta_{s_n u},$$

where u is a fixed unit vector in $\mathbb{R}^d$. Now,

$$\int_{\mathbb{R}^d} (1 \vee \|x\|^2)\, \rho(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty,$$

so that ρ is not in $\mathfrak{M}_{02}(\mathbb{R}^d)$. At the same time

$$\int_{\mathbb{R}^d} \psi_\gamma(\|x\|)\, \rho(\mathrm{d}x) = \sum_{n=1}^{\infty} \frac{1}{n} \psi_\gamma(s_n) \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,$$

so that $\rho \in \mathsf{dom}_L \Upsilon_\gamma^{(d)}$ (in particular ρ must be a Lévy measure; cf. Proposition 3.2).

The case of regularly varying tails.

In order to characterise $\mathsf{dom}_L \Upsilon_\gamma$ we need to know the behaviour of $\psi_\gamma(s)$ (defined by formula (3.12)) at zero and infinity, cf. formula (3.13). This is possible when the tail of γ is regularly varying in the sense that we can specify the tail behaviours of ψ_γ in terms of the behaviour of the tail measure of γ at 0 and infinity.

Recall that a function $L\colon (0,\infty) \to [0,\infty)$ is slowly varying at infinity (resp. at 0) if

$$\frac{L(tx)}{L(t)} \longrightarrow 1, \quad \text{as } t \to \infty \text{ (resp. as } t \to 0\text{)},$$

for any positive number x. A function $U\colon (0,\infty) \to [0,\infty)$ is regularly varying with index α at infinity (resp. at 0), if it has the form

$$U(x) = x^\alpha L(x),$$

with L slowly varying at infinity (resp. at 0). Recall also that for γ in $\mathfrak{M}_L((0,\infty))$ we have

$$\psi_\gamma(s), \gamma([s^{-1},\infty)) \longrightarrow 0, \quad \text{as } s \to 0$$

and

$$\psi_\gamma(s), \gamma((s^{-1},\infty)) \longrightarrow \gamma((0,\infty)), \quad \text{as } s \to \infty.$$

Proposition 3.10. *Let γ be a non-zero Lévy measure on $(0,\infty)$, and suppose that the function $\gamma^+(t) = \gamma([t,\infty))$ is regularly varying with index $-\alpha$ at zero (infinity, resp.), where $\alpha < 2$. Then*

$$\frac{\psi_\gamma(s)}{\gamma([s^{-1},\infty))} \to \frac{2}{2-\alpha} \quad \textit{as } s \to \infty \ (0, \ \textit{resp.}). \tag{3.19}$$

Proof: We have

$$\begin{aligned}
\psi_\gamma(s) &= s^2 \int_{(0,s^{-1})} x^2\,\gamma(\mathrm{d}x) + \gamma([s^{-1},\infty)) \\
&= s^2 \int_{(0,s^{-1})} \int_0^x 2t\,\mathrm{d}t\,\gamma(\mathrm{d}x) + \gamma([s^{-1},\infty)) \\
&= s^2 \int_0^\infty 2t\gamma([t, s^{-1}\vee t))\,\mathrm{d}t + \gamma([s^{-1},\infty)) \\
&= s^2 \int_0^{s^{-1}} 2t\gamma([t,\infty))\,\mathrm{d}t. \qquad (3.20)
\end{aligned}$$

We first consider the case of $\gamma([t,\infty))$ regularly varying at zero. From (3.20) we get

$$\psi_\gamma(s) = 2s^2 \int_s^\infty x^{-3}\gamma([x^{-1},\infty))\,\mathrm{d}x.$$

By our assumption we can write $\gamma([x^{-1},\infty)) = x^\alpha \ell(x)$, where $\ell(x)$ is slowly varying at infinity. By Proposition 1.5.10 in Bingham et al. (1987) we have

$$\frac{\psi_\gamma(s)}{\gamma([s^{-1},\infty))} = \frac{2\int_s^\infty x^{\alpha-3}\ell(x)\,\mathrm{d}x}{s^{\alpha-2}\ell(s)} \to \frac{2}{2-\alpha} \qquad (3.21)$$

as $s\to\infty$.

Now we consider the case of $\gamma([t,\infty))$ regularly varying at infinity. We can write $\gamma([t,\infty)) = t^{-\alpha}\ell(t)$, where $\ell(t)$ is slowly varying at infinity. Using Proposition 1.5.8 Bingham et al. (1987) and (3.20) we get

$$\frac{\psi_\gamma(x^{-1})}{\gamma([x,\infty))} = \frac{2\int_0^x t^{1-\alpha}\ell(t)\,\mathrm{d}t}{x^{2-\alpha}\ell(x)} \to \frac{2}{2-\alpha}$$

as $x\to\infty$. This concludes the proof.

Remark 3.11. Suppose γ is a non-zero Lévy measure on $(0,\infty)$ and that the function $\gamma^+(t) = \gamma([t,\infty))$ is regularly varying at both 0 and infinity with indexes respectively $-\alpha$ and $-\beta$ from $(-2,0)$. Then it follows from Proposition 3.10 that there are positive constants c and C such that

$$c\gamma^+(s^{-1}) \le \psi_\gamma(s) \le C\gamma^+(s^{-1}), \qquad (s>0). \qquad (3.22)$$

Indeed, it is a consequence of Proposition 3.10 that there exist positive numbers ϵ, K, C' such that

$$\gamma^+(s^{-1}) \le C'\psi_\gamma(s), \quad \text{for all } s \text{ in } [\epsilon,K]^c.$$

Putting then $C'' = \gamma^+(K^{-1})/\psi_\gamma(\epsilon)$, we have for s in $[\epsilon,K]$ that

$$\gamma^+(s^{-1}) \le \gamma^+(K^{-1}) = C''\psi_\gamma(\epsilon) \le C''\psi_\gamma(s).$$

Thus, the constant $c = 1/(C'\vee C'')$ satisfies the first inequality in (3.22), and a similar argument produces a constant C satisfying the second inequality.

Corollary 3.12. *Suppose γ and η are non-zero measures from $\mathfrak{M}_L((0,\infty))$ such that the functions $\gamma^+(s) = \gamma([s,\infty))$ and $\eta^+(s) = \eta([s,\infty))$ are regularly varying at both 0 and infinity with indexes in $(-2,0)$. Then the following two assertions are equivalent:*

(i) $\mathsf{dom}_L\Upsilon_\gamma \subseteq \mathsf{dom}_L\Upsilon_\eta$.
(ii) $\exists C>0\ \forall s>0\colon \eta^+(s) \le C\gamma^+(s)$.

Proof: Suppose $\mathsf{dom}_L \Upsilon_\gamma \subseteq \mathsf{dom}_L \Upsilon_\eta$. Then by Theorem 3.4 there is a positive constant C' such that $\psi_\eta \le C'\psi_\gamma$ and combined with Remark 3.11 this provides a constant C such that $\eta^+ \le C\gamma^+$. The converse implication follows similarly.

4. Composition and ranges

For two measures γ and η from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ we may consider the composition $\Upsilon_\gamma^{(d)} \circ \Upsilon_\eta^{(d)}$ with Lévy domain defined naturally by

$$\mathsf{dom}_L\big(\Upsilon_\gamma^{(d)} \circ \Upsilon_\eta^{(d)}\big) = \big\{\rho \in \mathsf{dom}_L \Upsilon_\eta^{(d)} \;\big|\; \Upsilon_\eta^{(d)}(\rho) \in \mathsf{dom}_L \Upsilon_\gamma^{(d)}\big\}.$$

Proposition 4.1. *Let η and γ be non-zero measures from $\mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. Then for any d in $\mathbb{N}$,*

$$\mathsf{dom}_L\big(\Upsilon_\gamma^{(d)} \circ \Upsilon_\eta^{(d)}\big) = \mathsf{dom}_L \Upsilon_{\gamma\circledast\eta}^{(d)} = \mathsf{dom}_L\big(\Upsilon_\eta^{(d)} \circ \Upsilon_\gamma^{(d)}\big) \tag{4.1}$$

and

$$\Upsilon_\gamma^{(d)} \circ \Upsilon_\eta^{(d)} = \Upsilon_{\gamma\circledast\eta}^{(d)} = \Upsilon_\eta^{(d)} \circ \Upsilon_\gamma^{(d)}. \tag{4.2}$$

Proof: For a measure ρ from $\mathfrak{M}_L((0,\infty))$ we note first that by (3.2)

$$\begin{aligned}\rho \in \mathsf{dom}_L \Upsilon_{\gamma\circledast\eta}^{(d)} &\iff \|\rho\| \in \mathsf{dom}_L \Upsilon_{\gamma\circledast\eta}^{(1)} \iff (\gamma\circledast\eta)\circledast\|\rho\| \in \mathfrak{M}_L((0,\infty)) \\ &\iff \gamma\circledast(\eta\circledast\|\rho\|) \in \mathfrak{M}_L((0,\infty)) \iff \eta\circledast\|\rho\| \in \mathsf{dom}_L \Upsilon_\gamma^{(1)}.\end{aligned} \tag{4.3}$$

In particular, by virtue of (3.3),

$$\rho \in \mathsf{dom}_L \Upsilon_{\gamma\circledast\eta}^{(d)} \Longrightarrow \eta\circledast\|\rho\| \in \mathfrak{M}_L((0,\infty)) \iff \rho \in \mathsf{dom}_L \Upsilon_\eta^{(d)}. \tag{4.4}$$

Moreover, assuming that $\rho \in \mathsf{dom}_L \Upsilon_\eta^{(d)}$, note that

$$\begin{aligned}&\int_0^\infty \int_{\mathbb{R}^d} (1 \wedge (t^2\|x\|^2))\, \Upsilon_\eta^{(d)}(\rho)(\mathrm{d}x)\, \gamma(\mathrm{d}t) \\ &\qquad = \int_0^\infty \int_0^\infty \int_{\mathbb{R}^d} (1 \wedge (s^2t^2\|x\|^2))\, \rho(\mathrm{d}x)\, \eta(\mathrm{d}s)\, \gamma(\mathrm{d}t) \\ &\qquad = \int_0^\infty \int_0^\infty \int_0^\infty (1 \wedge (s^2t^2u^2))\, \|\rho\|(\mathrm{d}u)\, \eta(\mathrm{d}s)\, \gamma(\mathrm{d}t) \\ &\qquad = \int_0^\infty \int_0^\infty (1 \wedge (t^2u^2))\, (\|\rho\|\circledast\eta)(\mathrm{d}u)\, \gamma(\mathrm{d}t),\end{aligned}$$

which verifies that

$$\forall \rho \in \mathsf{dom}_L \Upsilon_\eta^{(d)} \colon \Upsilon_\eta^{(d)}(\rho) \in \mathsf{dom}_L \Upsilon_\gamma^{(d)} \iff \|\rho\|\circledast\eta \in \mathsf{dom}_L \Upsilon_\gamma^{(1)}. \tag{4.5}$$

Combining now (4.3), (4.4) and (4.5) establishes the first equality in (4.1), and the second one follows by symmetry.

Turning now to (4.2), assume that $\rho \in \mathsf{dom}_L \Upsilon_{\gamma\circledast\eta}^{(d)}$, and note then for any Borel subset B of $\mathbb{R}^d$ that

$$\begin{aligned}\Upsilon_{\eta\circledast\gamma}(\rho)(B) &= \int_0^\infty \rho(t^{-1}B)\, \eta\circledast\gamma(\mathrm{d}t) = \int_0^\infty \int_0^\infty \rho((st)^{-1}B)\, \eta(\mathrm{d}t)\, \gamma(\mathrm{d}s) \\ &= \int_0^\infty \rho_\eta(s^{-1}B)\, \gamma(\mathrm{d}s) = [\Upsilon_\gamma \circ \Upsilon_\eta(\rho)](B),\end{aligned}$$

as desired.

Example 4.2. Adopting the notation from Example 2.4, a direct calculation shows that

$$\Phi_0 \circ \Upsilon_0 = \Upsilon_0 \circ \Phi_0 = \Xi_0.$$

The first of these equalities was noted in Barndorff-Nielsen et al. (2004). It is a special case of formula (4.2).

For a measure γ in $\mathfrak{M}_{\sigma f}((0,\infty))$ we define the *Lévy range* $\mathsf{ran}_L \Upsilon_\gamma^{(d)}$ of $\Upsilon_\gamma^{(d)}$ by

$$\mathsf{ran}_L \Upsilon_\gamma^{(d)} = \bigl\{ \Upsilon_\gamma^{(d)}(\rho) \bigm| \rho \in \mathsf{dom}_L \Upsilon_\gamma^{(d)} \bigr\}.$$

Corollary 4.3. *Let γ_1 and γ_2 be non-zero measures from $\mathfrak{M}_L((0,\infty))$. Then the following assertions are equivalent:*

(i) $\mathsf{ran}_L \Upsilon_{\gamma_2}^{(d)} \subseteq \mathsf{ran}_L \Upsilon_{\gamma_1}^{(d)}$ *for all d in $\mathbb{N}$.*
(ii) $\mathsf{ran}_L \Upsilon_{\gamma_2}^{(1)} \subseteq \mathsf{ran}_L \Upsilon_{\gamma_1}^{(1)}$.
(iii) $\gamma_2 = \gamma_1 \circledast \gamma = \Upsilon_{\gamma_1}(\gamma)$ *for some measure γ from $\mathfrak{M}_L((0,\infty))$.*

Proof: Assume first that $\gamma_2 = \gamma_1 \circledast \gamma$ for some measure γ from $\mathfrak{M}_L((0,\infty))$. Then by Proposition 4.1 it follows that

$$\mathsf{ran}_L \Upsilon_{\gamma_2}^{(d)} = \mathsf{ran}_L \bigl(\Upsilon_{\gamma_1}^{(d)} \circ \Upsilon_{\gamma}^{(d)} \bigr) \subseteq \mathsf{ran}_L \Upsilon_{\gamma_1}^{(d)},$$

for all d in $\mathbb{N}$. Assume conversely that $\mathsf{ran}_L \Upsilon_{\gamma_2}^{(1)} \subseteq \mathsf{ran}_L \Upsilon_{\gamma_1}^{(1)}$. Since $\gamma_2 \in \mathfrak{M}_L((0,\infty))$, the Dirac measure $\delta_1 \in \mathsf{dom}_L \Upsilon_{\gamma_2}^{(1)}$, so that

$$\gamma_2 = \Upsilon_{\gamma_2}^{(1)}(\delta_1) = \Upsilon_{\gamma_1}^{(1)}(\rho)$$

for some measure ρ in $\mathsf{dom}_L \Upsilon_{\gamma_1}$. Since $\gamma_1 \neq 0$, $\rho \in \mathfrak{M}_L(\mathbb{R})$ according to Proposition 3.2. Moreover, since

$$0 = \gamma_2((-\infty,0)) = \int_0^\infty \rho((-\infty,0))\,\gamma_1(\mathrm{d}t) = \rho((-\infty,0)) \cdot \gamma_1((0,\infty)),$$

and since $\gamma_1 \neq 0$, it follows that $\rho((-\infty,0)) = 0$, so that actually $\rho \in \mathfrak{M}_L((0,\infty))$. Therefore

$$\gamma_2 = \Upsilon_{\gamma_1}(\rho) = \gamma_1 \circledast \rho,$$

as desired.

Remark 4.4.

(i) Suppose γ_1, γ_2 are non-zero measures from $\mathfrak{M}_L((0,\infty))$ and that $\mathsf{ran}_L \Upsilon_{\gamma_2} \subseteq \mathsf{ran}_L \Upsilon_{\gamma_1}$. Then Corollary 4.3 and Proposition 4.1 assert that $\Upsilon_{\gamma_2}^{(d)} = \Upsilon_{\gamma}^{(d)} \circ \Upsilon_{\gamma_1}^{(d)}$ for some measure γ from $\mathfrak{M}_L((0,\infty))$. By the definition of $\mathsf{dom}_L \bigl(\Upsilon_{\gamma}^{(d)} \circ \Upsilon_{\gamma_1}^{(d)}\bigr)$, this in particular implies that

$$\mathsf{dom}_L \Upsilon_{\gamma_2}^{(d)} \subseteq \mathsf{dom}_L \Upsilon_{\gamma_1}^{(d)},$$

for all d.

(ii) Let γ be a non-zero measure from $\mathfrak{M}_L((0,\infty))$. Then by Proposition 4.1 we have for any positive integer d

$$\begin{aligned}
\mathsf{dom}_L \Upsilon_\gamma^{(d)} \supseteq \mathsf{ran}_L \Upsilon_\gamma^{(d)} &\iff \forall \rho \in \mathsf{dom}_L \Upsilon_\gamma^{(d)} \colon \Upsilon_\gamma^{(d)}(\rho) \in \mathsf{dom}_L \Upsilon_\gamma^{(d)} \\
&\iff \mathsf{dom}_L \bigl(\Upsilon_\gamma^{(d)} \circ \Upsilon_\gamma^{(d)} \bigr) = \mathsf{dom}_L \Upsilon_\gamma^{(d)} \\
&\iff \mathsf{dom}_L \Upsilon_{\gamma \circledast \gamma} = \mathsf{dom}_L \Upsilon_\gamma.
\end{aligned}$$

In other words, the mapping Υ_γ may be iterated without precaution on all of its domain, if and only if $\mathsf{dom}_L \Upsilon_{\gamma \circledast \gamma} = \mathsf{dom}_L \Upsilon_\gamma$.

(iii) Let γ and η be non-zero measures from $\mathfrak{M}_L((0,\infty))$. Then using e.g. (2.3) it is straightforward to check that

$$\gamma \circledast \eta \in \mathfrak{M}_{02}((0,\infty)) \iff \gamma, \eta \in \mathfrak{M}_{02}((0,\infty)).$$

This may in fact also be extracted from Proposition 4.1, which, in the affirmative case, asserts that

$$\Upsilon_{\gamma \circledast \eta} = \Upsilon_\gamma \circ \Upsilon_\eta = \Upsilon_\eta \circ \Upsilon_\gamma,$$

on all of $\mathfrak{M}_L(\mathbb{R}^d)$.

5. Continuity Properties of Υ_γ

For measures $\rho, \rho_1, \rho_2, \rho_3, \ldots$ from $\mathfrak{M}_L(\mathbb{R}^d)$, we define *Lévy-weak convergence* of ρ_n to ρ, denoted $\rho_n \xrightarrow{\text{lw}} \rho$, as follows:

$$\rho_n \xrightarrow[n\to\infty]{\text{lw}} \rho \iff (1 \wedge \|x\|^2)\, \rho_n(\mathrm{d}x) \xrightarrow[n\to\infty]{\text{w}} (1 \wedge \|x\|^2)\, \rho(\mathrm{d}x).$$

The corresponding topology τ_L on $\mathfrak{M}_L(\mathbb{R}^d)$ is the weakest topology on $\mathfrak{M}_L(\mathbb{R}^d)$ making the mapping

$$\rho \mapsto (1 \wedge \|x\|^2)\, \rho(\mathrm{d}x) \colon \mathfrak{M}_L(\mathbb{R}^d) \to \mathfrak{M}_{\mathrm{f}}(\mathbb{R}^d)$$

continuous, when the class $\mathfrak{M}_{\mathrm{f}}(\mathbb{R}^d)$ of finite Borel measures on $\mathbb{R}^d$ is equipped with the topology for usual weak convergence. It is straightforward to check that $\mathfrak{M}_{02}(\mathbb{R}^d)$ is dense in $\mathfrak{M}_L(\mathbb{R}^d)$ with respect to τ_L, and hence Remark 3.5 asserts that Υ_γ is densely defined on $\mathfrak{M}_L(\mathbb{R}^d)$ for any Lévy measure γ on $(0,\infty)$. By Theorem 3.4, Υ_γ can be defined on all of $\mathfrak{M}_L(\mathbb{R}^d)$ if and only if $\gamma \in \mathfrak{M}_{02}((0,\infty))$.

Theorem 5.1. *Let γ be a Lévy measure on $(0,\infty)$ and let d be a positive integer. Then the following statements are equivalent:*

(i) $\gamma \in \mathfrak{M}_{02}((0,\infty))$.

(ii) $\Upsilon_\gamma^{(d)} \colon \mathsf{dom}_L \Upsilon_\gamma^{(d)} \to \mathfrak{M}_L(\mathbb{R}^d)$ *is continuous in the topology for Lévy weak convergence.*

(iii) $\Upsilon_\gamma^{(d)}$ *is continuous at* $0 \in \mathfrak{M}_L(\mathbb{R}^d)$ *in the topology for Lévy weak convergence.*

Proof: Assume first that γ belongs to $\mathfrak{M}_{02}((0,\infty))$, and let $\rho, \rho_1, \rho_2, \rho_3, \ldots$ be measures from $\mathfrak{M}_L(\mathbb{R}^d)$ such that $\rho_n \to \rho$ Lévy-weakly as $n \to \infty$. In order to show that $\Upsilon_\gamma(\rho_n) \to \Upsilon_\gamma(\rho)$ Lévy-weakly, we must establish that

$$\int_{\mathbb{R}^d} f(x)(1 \wedge \|x\|^2)\, \Upsilon_\gamma(\rho_n)(\mathrm{d}x) \xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} f(x)(1 \wedge \|x\|^2)\, \Upsilon_\gamma(\rho)(\mathrm{d}x), \tag{5.1}$$

for any continuous bounded function $f \colon \mathbb{R}^d \to \mathbb{R}$. Note here that

$$\int_{\mathbb{R}} f(x)(1 \wedge \|x\|^2)\, \Upsilon_\gamma(\rho_n)(\mathrm{d}x) = \int_0^\infty \int_{\mathbb{R}^d} f(sx)(1 \wedge s^2\|x\|^2)\, \rho_n(\mathrm{d}x)\, \gamma(\mathrm{d}s), \tag{5.2}$$

and that for a fixed s

$$\begin{aligned}\int_{\mathbb{R}^d} f(sx)(1\wedge s^2\|x\|^2)\,\rho_n(\mathrm{d}x) &= \int_{\mathbb{R}^d} f_s(x)(1\wedge \|x\|^2)\,\rho_n(\mathrm{d}x)\\ &\xrightarrow[n\to\infty]{} \int_{\mathbb{R}^d} f_s(x)(1\wedge \|x\|^2)\,\rho(\mathrm{d}x) \qquad (5.3)\\ &= \int_{\mathbb{R}^d} f(sx)(1\wedge s^2\|x\|^2)\,\rho(\mathrm{d}x),\end{aligned}$$

since the function

$$f_s(x) = \begin{cases} f(sx)\frac{1\wedge s^2\|x\|^2}{1\wedge\|x\|^2}, & \text{if } x\in\mathbb{R}^d\setminus\{0\},\\ s^2 f(0), & \text{if } x=0,\end{cases} \tag{5.4}$$

is continuous and bounded. Note also that for any n

$$\begin{aligned}\Big|\int_{\mathbb{R}^d} f(sx)(1\wedge s^2\|x\|^2)\,\rho_n(\mathrm{d}x)\Big| &\le \|f\|_u(1\vee s^2)\int_{\mathbb{R}^d}(1\wedge\|x\|^2)\,\rho_n(\mathrm{d}x),\\ &\le C\|f\|_u(1\vee s^2),\end{aligned} \tag{5.5}$$

where $\|f\|_u = \sup_{x\in\mathbb{R}^d}|f(x)| < \infty$ and $C = \sup_{n\in\mathbb{N}}\int_{\mathbb{R}^d}(1\wedge\|x\|^2)\,\rho_n(\mathrm{d}x) < \infty$. Since $\gamma\in\mathfrak{M}_{02}((0,\infty))$ we have $\int_0^\infty(1\vee s^2)\,\gamma(\mathrm{d}s)<\infty$, and hence by dominated convergence in combination with (5.2)-(5.5) we obtain (5.1).

It remains to show that continuity of $\Upsilon_\gamma^{(d)}$ at 0 implies that γ belongs to $\mathfrak{M}_{02}((0,\infty))$. Consider first the sequence (ρ_n) of measures from $\mathfrak{M}_L(\mathbb{R}^d)$ given by

$$\rho_n = \epsilon_n n^2\delta_{n^{-1}u}, \qquad (n\in\mathbb{N}),$$

where u is a fixed unit vector in $\mathbb{R}^d$ and (ϵ_n) is an arbitrary sequence of positive numbers such that $\epsilon_n\searrow 0$ as $n\to\infty$. Note then that

$$\int_{\mathbb{R}^d}(1\wedge\|x\|^2)\,\rho_n(\mathrm{d}x) = \epsilon_n n^2(1\wedge n^{-2}) = \epsilon_n,$$

so that $\rho_n \xrightarrow{\text{lw}} 0$ as $n\to\infty$. At the same time we have

$$\begin{aligned}\int_{\mathbb{R}^d}(1\wedge\|x\|^2)\,\Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x) &= \int_0^\infty \epsilon_n n^2(1\wedge t^2n^{-2})\,\gamma(\mathrm{d}t)\\ &= \epsilon_n\int_0^n t^2\,\gamma(\mathrm{d}t) + \epsilon_n n^2\gamma([n,\infty)).\end{aligned}$$

From the calculation above it follows that $\Upsilon_\gamma^{(d)}(\rho_n)\xrightarrow{\text{lw}} 0$ for all choices of (ϵ_n) as prescribed above if and only if $\int_0^\infty t^2\,\gamma(\mathrm{d}t)<\infty$, which is thus a necessary condition for continuity at 0 of $\Upsilon_\gamma^{(d)}$. Consider next the sequence (ρ_n) defined by

$$\rho_n = \epsilon_n\delta_{nu}, \qquad (n\in\mathbb{N}),$$

with u and (ϵ_n) as above. Then

$$\int_{\mathbb{R}^d}(1\wedge\|x\|^2)\,\rho_n(\mathrm{d}x) = \epsilon_n(1\wedge n^2) = \epsilon_n,$$

so that $\rho_n \xrightarrow{\text{lw}} 0$ as $n \to \infty$. Furthermore

$$\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x) = \int_0^\infty \epsilon_n (1 \wedge t^2 n^2)\, \gamma(\mathrm{d}t)$$
$$= \epsilon_n n^2 \int_0^{1/n} t^2\, \gamma(\mathrm{d}t) + \epsilon_n \gamma([1/n, \infty)),$$

and it follows that $\Upsilon_\gamma(\rho_n) \xrightarrow{\text{lw}} 0$ for all choices of (ϵ_n) if and only if $\gamma((0,\infty)) < \infty$. Thus, γ must also be finite in order for Υ_γ to be continuous at 0. This completes the proof.

Remark 5.2. When dealing with an upsilon transform $\Upsilon_\gamma^{(d)} \colon \mathsf{dom}_L \Upsilon_\gamma^{(d)} \to \mathfrak{M}_L(\mathbb{R}^d)$, it is natural to have in mind the setting of (unbounded) linear operators defined on subspaces of a Banach space. From this point of view, Theorem 5.1 corresponds to the fact that a linear, densely defined operator on a Banach space is bounded on its domain if and only if it has a bounded extension to the full Banach space. In addition, this condition is equivalent to continuity of the operator at 0 and also to continuity on all of the domain.

The next theorem is essential for studying the topological properties of $\mathsf{ran}_L \Upsilon_\gamma$ and of the inverse mapping of Υ_γ in case Υ_γ is one-to-one.

Theorem 5.3. *Let γ be a non-zero measure from $\mathfrak{M}_L((0,\infty))$, and let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence of measures from $\mathsf{dom}_L \Upsilon_\gamma^{(d)}$ such that $\Upsilon_\gamma^{(d)}(\rho_n) \xrightarrow{\text{lw}} \sigma$ for some measure σ from $\mathfrak{M}_L(\mathbb{R}^d)$. Then there is a subsequence $(\rho_{n_p})_{p\in\mathbb{N}}$ and a Lévy measure ρ in $\mathsf{dom}_L \Upsilon_\gamma^{(d)}$ such that $\rho_{n_p} \xrightarrow{\text{lw}} \rho$. Moreover, $\sigma \geq \Upsilon_\gamma^{(d)}(\rho)$ and these measures are equal when*

$$\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \sigma(\mathrm{d}x) = \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \Upsilon_\gamma^{(d)}(\rho)(\mathrm{d}x). \tag{5.6}$$

Before the proof, note that if $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of Lévy measures on $\mathbb{R}^d$, then it is certainly possible that $(1 \wedge \|x\|^2)\, \rho_n(\mathrm{d}x)$ converge weakly, as $n \to \infty$, to a finite measure ν on $\mathbb{R}^d$ with positive mass at 0. For instance, setting $\rho_n = n^2 \delta_{1/n}$, we have that $(1 \wedge x^2)\, \rho_n(\mathrm{d}x) \to \delta_0(\mathrm{d}x)$ weakly, as $n \to \infty$. According to the theorem above, the sequence $(\Upsilon_\gamma(\rho_n))_{n\in\mathbb{N}}$ does not have any cluster point with respect to the Lévy weak topology.

Proof of Theorem 5.3. We show first that the sequence

$$\nu_n(\mathrm{d}x) = (1 \wedge \|x\|^2)\, \rho_n(\mathrm{d}x), \qquad (n \in \mathbb{N}),$$

is precompact. By Ash and Doléans-Dade (2000, Theorem 7.8.7) it suffices to show that $(\nu_n)_{n\in\mathbb{N}}$ is tight and that $(\nu_n(\mathbb{R}^d))_{n\in\mathbb{N}}$ is bounded. Regarding the latter aspect, note that

$$\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x)$$
$$\geq \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\, \rho_n(\mathrm{d}x) \int_0^\infty (1 \wedge s^2)\, \gamma(\mathrm{d}t) = \nu_n(\mathbb{R}^d) \int_0^\infty (1 \wedge s^2)\, \gamma(\mathrm{d}t). \tag{5.7}$$

Since $(1\wedge\|x\|^2)\Upsilon_\gamma^{(d)}(\rho_n)\to(1\wedge\|x\|^2)\,\sigma$ weakly, the left hand side of (5.7) is bounded in n, and since $\gamma\neq 0$, (5.7) thus implies boundedness of $(\nu_n(\mathbb{R}^d))_{n\in\mathbb{N}}$. Regarding tightness of (ν_n), we find similarly for l in $(0,\infty)$ and ϵ in $(0,1)$ that

$$\begin{aligned}\int_{\{\|x\|>l\}}(1\wedge\|x\|^2)\,\Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x)&=\int_0^\infty\int_{\mathbb{R}^d}1_{(l/s,\infty)}(\|x\|)(1\wedge s^2\|x\|^2)\,\rho_n(\mathrm{d}x)\,\gamma(\mathrm{d}s)\\&\geq\int_\epsilon^\infty\int_{\mathbb{R}^d}1_{(l/\epsilon,\infty)}(\|x\|)\epsilon^2(1\wedge\|x\|^2)\,\rho_n(\mathrm{d}x)\,\gamma(\mathrm{d}s)\\&=\epsilon^2\gamma([\epsilon,\infty))\nu_n\big(\{\|x\|>l/\epsilon\}\big).\end{aligned}$$

Choosing then ϵ so small that $\gamma([\epsilon,\infty))>0$ and using the substitution $l=r\epsilon$, we find that

$$\nu_n\big(\{\|x\|>r\}\big)\leq\epsilon^{-2}\gamma([\epsilon,\infty))^{-1}\int_{\{\|x\|>r\epsilon\}}(1\wedge\|x\|^2)\,\Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x),$$

and since the sequence $(1\wedge\|x\|^2)\,\Upsilon_\gamma^{(d)}(\rho_n)(\mathrm{d}x)$ is tight by assumption, this implies tightness of (ν_n).

Having established precompactness of $(\nu_n)_{n\in\mathbb{N}}$, we may infer the existence of a subsequence $(\nu_{n_p})_{p\in\mathbb{N}}$ and a finite measure ν on $\mathbb{R}^d$ such that

$$(1\wedge\|x\|^2)\,\rho_{n_p}(\mathrm{d}x)=\nu_{n_p}(\mathrm{d}x)\xrightarrow{\mathrm{w}}\nu(\mathrm{d}x),\qquad\text{as } p\to\infty.$$

Let f be a function from $C_b(\mathbb{R}^d)$ and note that

$$\int_{\mathbb{R}^d}f(x)(1\wedge\|x\|^2)\,\Upsilon_\gamma^{(d)}(\rho_{n_p})(\mathrm{d}x)=\int_0^\infty\int_{\mathbb{R}^d}f(sx)(1\wedge s^2\|x\|^2)\,\rho_{n_p}(\mathrm{d}x)\,\gamma(\mathrm{d}s).$$

For fixed s in $(0,\infty)$ consider the continuous bounded function $f_s\colon\mathbb{R}^d\to\mathbb{R}$ introduced in (5.4). Then by assumption

$$\int_{\mathbb{R}^d}f(sx)(1\wedge s^2\|x\|^2)\,\rho_{n_p}(\mathrm{d}x)=\int_{\mathbb{R}^d}f_s(x)(1\wedge\|x\|^2)\,\rho_{n_p}(\mathrm{d}x)\xrightarrow[p\to\infty]{}\int_{\mathbb{R}^d}f_s(x)\,\nu(\mathrm{d}x).$$

Assuming now that $f\geq 0$, it results from Fatou's lemma that

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^d}f_s(x)\,\nu(\mathrm{d}x)\,\gamma(\mathrm{d}s)&\leq\liminf_{p\to\infty}\int_0^\infty\int_{\mathbb{R}^d}f(sx)(1\wedge s^2\|x\|^2)\,\rho_{n_p}(\mathrm{d}x)\,\gamma(\mathrm{d}s)\\&=\liminf_{p\to\infty}\int_{\mathbb{R}^d}f(x)(1\wedge\|x\|^2)\,\Upsilon_\gamma(\rho_{n_p})(\mathrm{d}x)\\&=\int_{\mathbb{R}^d}f(x)(1\wedge\|x\|^2)\,\sigma(\mathrm{d}x).\end{aligned}\tag{5.8}$$

Note next that ν may be decomposed as

$$\nu(\mathrm{d}x)=(1\wedge\|x\|^2)\,\rho(\mathrm{d}x)+\nu(\{0\})\delta_0(\mathrm{d}x),$$

with ρ a (uniquely determined) Lévy measure on $\mathbb{R}^d$. Hence

$$\begin{aligned}\int_0^\infty \int_{\mathbb{R}^d} & f_s(x)\,\nu(\mathrm{d}x)\,\gamma(\mathrm{d}s) \\ &= \int_0^\infty \int_{\mathbb{R}^d} f_s(x)(1\wedge\|x\|^2)\,\rho(\mathrm{d}x)\,\gamma(\mathrm{d}s) + \int_0^\infty \nu(\{0\}) f_s(0)\,\gamma(\mathrm{d}s) \\ &= \int_0^\infty \int_{\mathbb{R}^d} f(sx)(1\wedge s^2\|x\|^2)\,\rho(\mathrm{d}x)\,\gamma(\mathrm{d}s) + \nu(\{0\}) f(0) \int_0^\infty s^2\,\gamma(\mathrm{d}s).\end{aligned} \tag{5.9}$$

According to (5.8), the left hand side of (5.9) is finite, and hence, by considering the first term in the resulting expression of (5.9) in the case $f \equiv 1$, it follows that $\rho \in \mathsf{dom}_L \Upsilon_\gamma^{(d)}$. Combining this observation with (5.8) and (5.9) we obtain the estimate

$$\int_{\mathbb{R}^d} f(x)(1\wedge\|x\|^2)\,\sigma(\mathrm{d}x) \geq \int_{\mathbb{R}^d} f(x)(1\wedge\|x\|^2)\Upsilon_\gamma^{(d)}(\rho)(\mathrm{d}x) + \nu(\{0\}) f(0) \int_0^\infty s^2\,\gamma(\mathrm{d}s), \tag{5.10}$$

which holds for all non-negative f from $C_b(\mathbb{R}^d)$. Now choose a sequence $(g_i)_{i\in\mathbb{N}}$ from $C_b(\mathbb{R}^d)$ such that $0 \leq g_i \leq 1$ and $g_i(0) = 1$ for all i and such that $g_i \to 1_{\{0\}}$ point-wise as $i \to \infty$. Then by dominated convergence

$$\int_{\mathbb{R}^d} g_i(x)(1\wedge\|x\|^2)\,\sigma(\mathrm{d}x) \xrightarrow[i\to\infty]{} \int_{\mathbb{R}^d} 0\,\sigma(\mathrm{d}x) = 0,$$

and hence (5.10) implies that

$$0 \geq \nu(\{0\}) \int_0^\infty s^2 \gamma(\mathrm{d}s),$$

and since $\gamma \neq 0$, we must then have $\nu(\{0\}) = 0$. Consequently, $\nu(\mathrm{d}x) = (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x)$, which yields $\rho_{n_p} \xrightarrow{\mathrm{lw}} \rho$.

Now we will prove the last statement of the theorem. Formula (5.10) with $\nu(\{0\}) = 0$ gives

$$\int_{\mathbb{R}^d} f(x)(1\wedge\|x\|^2)\,\sigma(\mathrm{d}x) \geq \int_{\mathbb{R}^d} f(x)(1\wedge\|x\|^2)\Upsilon_\gamma^{(d)}(\rho)(\mathrm{d}x) \tag{5.11}$$

for any non-negative function f from $C_b(\mathbb{R}^d)$, which implies $\sigma \geq \Upsilon_\gamma^{(d)}(\rho)$. Let $M_f = \sup_x f(x)$, where f is as above. Using (5.11) for $M_f - f$ in place of f and (5.6) we get the reverse inequality in (5.11). Hence $\sigma = \Upsilon_\gamma^{(d)}(\rho)$ and the proof is completed. □

Corollary 5.4. *Let γ be a measure from $\mathfrak{M}_{02}((0,\infty))$, and let d be a positive integer.*

(i) *The mapping $\Upsilon_\gamma^{(d)}$ is closed in the following sense: For any subset F of $\mathfrak{M}_L(\mathbb{R}^d)$, which is closed in the topology for Lévy weak convergence, the same holds for the range $\Upsilon_\gamma^{(d)}(F) = \{\Upsilon_\gamma^{(d)}(\rho) \mid \rho \in F\}$. In particular the full range $\mathsf{ran}_L \Upsilon_\gamma^{(d)}$ is a closed subset of $\mathfrak{M}_L(\mathbb{R}^d)$.*

(ii) *If $\Upsilon_\gamma^{(d)}$ is injective, then it is automatically a homeomorphism with respect to Lévy weak convergence, i.e. the inverse mapping $(\Upsilon_\gamma^{(d)})^{-1}\colon \mathsf{ran}_L \Upsilon_\gamma \to \mathfrak{M}_L(\mathbb{R}^d)$ is continuous in the corresponding topology.*

Proof: (i) Let F be a subset of $\mathfrak{M}_L(\mathbb{R}^d)$, which is closed in the topology for Lévy weak convergence, and let σ be a measure from the closure of $\Upsilon_\gamma(F)$. Then we may choose a sequence (ρ_n) of measures from F, such that $\Upsilon_\gamma^{(d)}(\rho_n) \xrightarrow{\text{lw}} \sigma$ as $n \to \infty$. According to Theorem 5.3, there is a subsequence $(\rho_{n_p})_{p\in\mathbb{N}}$ converging Lévy weakly to a measure ρ necessarily in F. Since $\Upsilon_\gamma^{(d)}$ is continuous, and since the topology for Lévy weak convergence is Hausdorff, we may then conclude that

$$\sigma = \lim_{p\to\infty} \Upsilon_\gamma^{(d)}(\rho_{n_p}) = \Upsilon_\gamma^{(d)}(\rho) \in \Upsilon^{(d)}(F),$$

as desired.

(ii) Suppose that Υ_γ is injective. Then (i) informs us that the pre-image of any closed subset of $\mathfrak{M}_L(\mathbb{R}^d)$ by the inverse mapping $(\Upsilon_\gamma^{(d)})^{-1}$ is again a closed subset of $\mathfrak{M}_L(\mathbb{R}^d)$ and hence of $\mathsf{ran}_L \Upsilon_\gamma^{(d)}$. This means that $(\Upsilon_\gamma^{(d)})^{-1}$ is continuous on $\mathsf{ran}_L \Upsilon_\gamma^{(d)}$.

Pursuing further the analogy to operators on a Banach space mentioned in Remark 5.2, we introduce next the graph $\mathsf{graph}_L \Upsilon_\gamma^{(d)}$ of $\Upsilon_\gamma^{(d)}$ defined by

$$\mathsf{graph}_L \Upsilon_\gamma^{(d)} = \big\{(\rho, \Upsilon_\gamma(\rho)) \bigm| \rho \in \mathsf{dom}_L \Upsilon_\gamma^{(d)}\big\}.$$

We shall view $\mathsf{graph}_L \Upsilon_\gamma^{(d)}$ as a subset of $\mathfrak{M}_L(\mathbb{R}^d) \times \mathfrak{M}_L(\mathbb{R}^d)$ equipped with the product topology.

Proposition 5.5. *For any measure γ from $\mathfrak{M}_L((0,\infty))$, we have the implications:*

$$\Upsilon_\gamma^{(d)} \textit{ is continuous} \implies \mathsf{graph}_L \Upsilon_\gamma^{(d)} \textit{ is closed} \implies \mathsf{ran}_L \Upsilon_\gamma^{(d)} \textit{ is closed.}$$

Proof: Since $\mathsf{dom}_L \Upsilon_\gamma^{(d)} = \mathfrak{M}_L(\mathbb{R}^d)$, when $\Upsilon_\gamma^{(d)}$ is continuous, the first implication is straightforward. To prove the second one, assume (without loss of generality) that $\gamma \neq 0$ and that $\mathsf{graph}_L \Upsilon_\gamma^{(d)}$ is a closed subset of $\mathfrak{M}_L(\mathbb{R}^d) \times \mathfrak{M}_L(\mathbb{R}^d)$. Then let σ be an element of the closure of $\mathsf{ran}_L \Upsilon_\gamma^{(d)}$ in $\mathfrak{M}_L(\mathbb{R}^d)$, and choose a sequence $(\rho_n)_{n\in\mathbb{N}}$ from $\mathsf{dom}_L \Upsilon_\gamma^{(d)}$, such that $\Upsilon_\gamma^{(d)}(\rho_n) \to \sigma$ Lévy weakly as $n \to \infty$. According to Theorem 5.3, there is a subsequence $(\rho_{n_p})_{p\in\mathbb{N}}$ and a measure ρ from $\mathsf{dom}_L \Upsilon_\gamma^{(d)}$ such that $\rho_{n_p} \to \rho$ Lévy weakly as $p \to \infty$. Now $(\rho_{n_p}, \Upsilon_\gamma^{(d)}(\rho_{n_p})) \to (\rho, \sigma)$ in the product topology on $\mathfrak{M}_L(\mathbb{R}^d) \times \mathfrak{M}_L(\mathbb{R}^d)$, and hence $(\rho, \sigma) \in \mathsf{graph}_L \Upsilon_\gamma^{(d)}$, by our assumption. This means that $\sigma = \Upsilon_\gamma^{(d)}(\rho) \in \mathsf{ran}_L \Upsilon_\gamma^{(d)}$, as desired.

Example 5.6. In this example we exhibit a measure γ from $\mathfrak{M}_L((0,\infty))$ such that $\mathsf{ran}_L \Upsilon_\gamma$ is not closed. By Proposition 5.5 $\mathsf{graph}_L \Upsilon_\gamma^{(d)}$ can not be closed either. Specifically, let γ be the Lévy measure on $(0,\infty)$ given by

$$\gamma(\mathrm{d}t) = t^{-2} 1_{[1,\infty)}(t)\,\mathrm{d}t,$$

and consider the sequence $(\rho_n)_{n\in\mathbb{N}}$ from $\mathfrak{M}_{02}(\mathbb{R})$ given by

$$\rho_n = n\delta_{1/n}, \qquad (n \in \mathbb{N}).$$

Then it is straightforward to check that $\rho_n \to 0$ Lévy weakly as $n \to \infty$, and that

$$\Upsilon_\gamma(\rho_n)(\mathrm{d}t) = t^{-2} 1_{[1/n,\infty)}(t)\,\mathrm{d}t.$$

From the latter expression it is also straightforward to check that $\Upsilon_\gamma(\rho_n)(\mathrm{d}t) \to t^{-2}1_{(0,\infty)}(t)\,\mathrm{d}t$ Lévy weakly as $n \to \infty$. Since $\Upsilon_\gamma(0) = 0$, these observations show that $\mathsf{graph}_L \Upsilon_\gamma$ is not closed in $\mathfrak{M}_L(\mathbb{R}) \times \mathfrak{M}_L(\mathbb{R})$. To see that $\mathsf{ran}_L \Upsilon_\gamma$ is not closed (in $\mathfrak{M}_L(\mathbb{R})$) either, we show that

$$\sigma(\mathrm{d}t) := t^{-2} 1_{(0,\infty)}(t)\,\mathrm{d}t \notin \mathsf{ran}_L \Upsilon_\gamma.$$

We obtain this by proving that for any measure ρ from $\mathsf{dom}_L\Upsilon_\gamma$ supported on $(0,\infty)$, we have that

$$\alpha\Upsilon_\gamma(\rho)((\alpha,\infty)) \to 0, \qquad \text{as } \alpha \searrow 0. \tag{5.12}$$

Since $\Upsilon_\gamma(\rho)$ is supported on $(0,\infty)$ if and only if ρ is, and since $\sigma((\alpha,\infty)) = \alpha^{-1}$ for all α, the statement asserted above verifies that $\Upsilon_\gamma(\rho) \neq \sigma$ for all ρ in $\mathsf{dom}_L\Upsilon_\gamma$.

To establish (5.12), we note first that by direct calculation

$$\psi_\gamma(s) = \int_0^\infty (1 \wedge s^2t^2)\,\gamma(\mathrm{d}t) = (2s - s^2)1_{(0,1)}(s) + 1_{[1,\infty)}(s),$$

and hence (cf. (3.13))

$$\mathsf{dom}_L\Upsilon_\gamma = \big\{\rho \in \mathfrak{M}_L(\mathbb{R}) \bigm| \textstyle\int_{(-1,1)} |s|\,\rho(\mathrm{d}s) < \infty\big\}. \tag{5.13}$$

Now, let ρ be a measure from $\mathsf{dom}_L\Upsilon_\gamma$ which is supported on $(0,\infty)$. Then for any α in $(0,1)$,

$$\begin{aligned}
\alpha\Upsilon_\gamma(\rho)((\alpha,\infty)) &= \alpha\Upsilon_\rho(\gamma)((\alpha,\infty)) = \alpha\int_0^\infty \gamma((s^{-1}\alpha,\infty))\,\rho(\mathrm{d}s) \\
&= \alpha\int_0^\infty\int_{s^{-1}\alpha\vee 1}^\infty t^{-2}\,\mathrm{d}t\,\rho(\mathrm{d}s) = \alpha\int_0^\infty (s\alpha^{-1}\wedge 1)\,\rho(\mathrm{d}s) \\
&= \int_0^\alpha s\,\rho(\mathrm{d}s) + \alpha\rho([\alpha,\infty)) = \int_0^\alpha s\,\rho(\mathrm{d}s) + \alpha\rho([\alpha,1)) + \alpha\rho([1,\infty)).
\end{aligned}$$

Here, obviously $\alpha\rho([1,\infty)) \to 0$ as $\alpha \to 0$, and $\int_0^\alpha s\,\rho(\mathrm{d}s) \to 0$ as $\alpha \to 0$ by dominated convergence (cf. (5.13)). Finally

$$\alpha\rho([\alpha,1)) = \int_0^1 \alpha 1_{[\alpha,1)}(t)\,\rho(\mathrm{d}t) \longrightarrow 0, \qquad \text{as } \alpha \to 0,$$

again by dominated convergence, since $\alpha 1_{[\alpha,1)}(t) \le t$ for all t in $[0,1]$. This completes the proof of (5.12).

6. Injectivity

We now consider the question of when $\Upsilon_\gamma^{(d)}$ is injective for fixed $\gamma \in \mathfrak{M}_L((0,\infty))$ and $d \ge 1$. It is possible that the answer may depend on the domain on which $\Upsilon_\gamma^{(d)}$ is considered. We are naturally interested in the Lévy domain $\mathsf{dom}_L\Upsilon^{(d)}$, and henceforth the term injectivity refers to a property of $\Upsilon_\gamma^{(d)}$ on that domain. It was established in Barndorff-Nielsen and Thorbjørnsen (2004) and Barndorff-Nielsen and Thorbjørnsen (2006) that the injectivity is held by the Upsilon mappings introduced in Example 2.4(1) and (2). As the following example shows, Υ_γ cannot in general be expected to have this property.

Example 6.1. Consider the Lévy measure γ on $(0,\infty)$ given by

$$\gamma(\mathrm{d}t) = t^{-2}1_{(0,\infty)}(t)\,\mathrm{d}t,$$

and for any positive number c, consider the measure

$$\rho_c = c\delta_{1/c} \in \mathfrak{M}_{02}(\mathbb{R}).$$

For any Borel subset B of $\mathbb{R}$ note then that

$$\begin{aligned}\big[\Upsilon_\gamma(\rho_c)\big](B) &= c\int_0^\infty \delta_{1/c}(t^{-1}B)\,\gamma(\mathrm{d}t) = \int_0^\infty 1_B(t/c)\cdot\big(t/c\big)^{-2}\cdot c^{-1}\,\mathrm{d}t\\ &= \int_0^\infty 1_B(u)u^{-2}\,\mathrm{d}u = \gamma(B),\end{aligned}$$

so that $\Upsilon_\gamma(\rho_c) = \gamma$ for all c. In particular, Υ_γ is far from being injective.

The next proposition reduces the problem of uniqueness to $d = 1$ and measures on $(0,\infty)$.

Proposition 6.2. *Let $\gamma \in \mathfrak{M}_L((0,\infty))$. Then $\Upsilon_\gamma^{(d)}$ is one-to-one on $\mathsf{dom}_L\Upsilon_\gamma^{(d)}$ for all d in $\mathbb{N}$, if and only $\Upsilon_\gamma^{(1)}$ is one-to-one on $\mathsf{dom}_L\Upsilon_\gamma^{(1)} \cap \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$.*

Proof: Obviously, we only need to prove the proposition in one direction. Suppose that $\Upsilon_\gamma^{(1)}$ is one-to-one on $\mathsf{dom}_L\Upsilon_\gamma \cap \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$.

First we will show that $\Upsilon_\gamma^{(1)}$ is one-to-one on $\mathsf{dom}_L\Upsilon_\gamma^{(1)}$. Let $\rho_i \in \mathsf{dom}_L\Upsilon_\gamma^{(1)}$, $i = 1,2$. Define for a Borel subset A of $(0,\infty)$

$$\rho_i^+(A) = \rho_i(A\cap(0,\infty)), \quad \rho_i^-(A) = \rho_i(-A\cap(-\infty,0)).$$

Then $\rho_i^+, \rho_i^- \in \mathsf{dom}_L\Upsilon_\gamma^{(1)} \cap \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$. If $\Upsilon_\gamma^{(1)}(\rho_1) = \Upsilon_\gamma^{(1)}(\rho_2)$, then $\Upsilon_\gamma^{(1)}(\rho_1^+) = \Upsilon_\gamma^{(1)}(\rho_2^+)$ and $\Upsilon_\gamma^{(1)}(\rho_1^-) = \Upsilon_\gamma^{(1)}(\rho_2^-)$. By the assumption $\rho_1^+ = \rho_2^+$ and $\rho_1^- = \rho_2^-$. Thus $\rho_1 = \rho_2$.

Now let $d > 1$ and $\rho_i \in \mathsf{dom}_L\Upsilon_\gamma^{(d)}$, $i = 1,2$. For $y \in \mathbb{R}^d$ define $\rho_i^y \in \mathsf{dom}_L\Upsilon_\gamma^{(1)}$ by

$$\rho_i^y(A) = \rho_i(\{x \in \mathbb{R}^d \mid \langle y,x\rangle \in A\setminus\{0\}\}).$$

If $\Upsilon_\gamma^{(d)}(\rho_1) = \Upsilon_\gamma^{(d)}(\rho_2)$, then for every $u \in \mathbb{R}$

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{i}u\langle y,tx\rangle} - 1 - \frac{\mathrm{i}u\langle y,tx\rangle}{1+u^2\langle y,tx\rangle^2})\,\rho_1(\mathrm{d}x)\gamma(\mathrm{d}t)\\ &\qquad= \int_0^\infty\int_{\mathbb{R}^d}(\mathrm{e}^{\mathrm{i}u\langle y,tx\rangle} - 1 - \frac{\mathrm{i}u\langle y,tx\rangle}{1+u^2\langle y,tx\rangle^2})\,\rho_2(\mathrm{d}x)\gamma(\mathrm{d}t)\end{aligned}$$

or

$$\begin{aligned}&\int_0^\infty\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}ust} - 1 - \frac{\mathrm{i}ust}{1+u^2(st)^2})\,\rho_1^y(\mathrm{d}s)\gamma(\mathrm{d}t)\\ &\qquad= \int_0^\infty\int_{\mathbb{R}}(\mathrm{e}^{\mathrm{i}ust} - 1 - \frac{\mathrm{i}ust}{1+u^2(st)^2})\,\rho_2^y(\mathrm{d}s)\gamma(\mathrm{d}t)\end{aligned}$$

Hence $\Upsilon_\gamma^{(1)}(\rho_1^y) = \Upsilon_\gamma^{(1)}(\rho_2^y)$. From the already established case $d = 1$, we get $\rho_1^y = \rho_2^y$ for every $y \in \mathbb{R}^d$. We conclude that $\rho_1 = \rho_2$.

According to this proposition, the injectivity property of $\Upsilon_\gamma^{(d)}$ is shared by all dimensions d. If it holds, we will simply say that Υ_γ is injective. The injectivity of Υ_γ is equivalent to the *cancellation property* of the multiplicative convolution: for every $\rho_1,\rho_2 \in \mathsf{dom}_L\Upsilon_\gamma \cap \mathfrak{M}((0,\infty))$

$$\gamma \circledast \rho_1 = \gamma \circledast \rho_2 \implies \rho_1 = \rho_2. \tag{6.1}$$

If γ has density f_γ then the map

$$t \mapsto f_{\gamma\circledast\rho}(t) = \int_0^\infty f_\gamma(ts^{-1})\,s^{-1}\rho(\mathrm{d}s) \tag{6.2}$$

can be viewed as a transform of measures $\rho \in \mathsf{dom}_L \Upsilon_\gamma$. If this transform is one-to-one on $\mathsf{dom}_L \Upsilon_\gamma$, then (6.1) holds and Υ_γ is injective. We will give a couple of examples where this method works.

In case ρ is a measure on $\mathbb{R}\setminus\{0\}$ recall that we use the notation $\underleftarrow{\rho}$ for the transformation of ρ by the mapping $x \mapsto x^{-1}$.

Examples 6.3. (1) $\gamma(\mathrm{d}s) = s^{\lambda-1} 1_{[0,1]}(s)\,\mathrm{d}s$, $\lambda > -2$. Adapting the notation from Example 2.4(4), recall that

$$\mathsf{dom}_L \Phi_\lambda = \begin{cases} \mathfrak{M}_L(\mathbb{R}), & \text{if } \lambda > 0 \\ \mathfrak{M}_{\log}(\mathbb{R}), & \text{if } \lambda = 0 \\ \mathfrak{M}_\lambda(\mathbb{R}), & \text{if } \lambda \in (-2, 0) \end{cases}$$

(see Example 3.1). If $\rho \in \mathsf{dom}_L \Phi_\lambda$ then

$$f_{\gamma \circledast \rho}(t) = t^{\lambda-1} \int_{[t,\infty)} s^{-\lambda}\,\rho(\mathrm{d}s).$$

Obviously, this type of function determines ρ uniquely from $\mathsf{dom}_L \Phi_\lambda$, so that Φ_λ is injective. The cases of $\lambda = 0$ and $\lambda = 1$ are of special interest. Indeed, $\mathsf{ran}_L \Phi_\lambda^{(d)}$ equals the class of selfdecomposable Lévy measures when $\lambda = 0$ (see Barndorff-Nielsen et al., 2004), and the class of s-selfdecomposable Lévy measures when $\lambda = 1$ (see Jurek, 1985).

(2) $\gamma(\mathrm{d}s) = s^{\lambda-1} \mathrm{e}^{-s} 1_{(0,\infty)}(s)\,\mathrm{d}s$, $\lambda > -2$. Recall from Example 3.1 that $\mathsf{dom}_L \Xi_\lambda = \mathsf{dom}_L \Phi_\lambda$ for all λ. We get with $\alpha = -\lambda$

$$f_{\gamma \circledast \rho}(t) = t^{-1-\alpha} \int_0^\infty s^\alpha \mathrm{e}^{-t/s}\,\rho(\mathrm{d}s) = t^{-1-\alpha} \int_0^\infty \mathrm{e}^{-ts}\, s^{-\alpha}\,\underleftarrow{\rho}(\mathrm{d}s).$$

Again, $f_{\gamma \circledast \rho}$ determines ρ uniquely from $\mathsf{dom}_L \Xi_\lambda$, so that $\Xi_\lambda^{(d)}$ is injective. The cases of $\lambda = 1$, $\lambda = 0$ and $-2 < \lambda < 0$ are of special importance. When $\lambda = 1$, we get the mapping Υ_0 introduced in Example 2.4(1). In the cases $\lambda = 0$ and $-2 < \lambda < 0$, $\mathsf{ran}_L \Xi_\lambda^{(d)}$ equals the classes of Lévy measures corresponding to Thorin and to tempered α-stable distributions on $\mathbb{R}^d$, respectively (see Barndorff-Nielsen et al., 2004 and Rosiński, 2007).

Remark 6.4. For λ in $(-2,\infty)$ the mapping $\Upsilon_\gamma^{(d)}$ is not injective when γ is given by $\gamma(\mathrm{d}x) = x^{\lambda-1} 1_{(0,\infty)}(x)\,\mathrm{d}x$, which is the Lévy measure of a stable distribution. Indeed, since

$$f_{\gamma \circledast \rho}(t) = t^{\lambda-1} \int_0^\infty s^{-\lambda}\,\rho(\mathrm{d}s),$$

$\gamma \circledast \rho$ is the same measure for all ρ having equal $-\lambda$'th moment. It is also easy to see that $\Upsilon_\gamma^{(d)}$ is non-injective when γ is the Lévy measure of a semistable distribution.

Besides (6.2) we may use other integral transforms to identify Lévy measures. They are determined by a kernel $K : (0,\infty) \mapsto \mathbb{R}(\text{or } \mathbb{C})$ as follows. For a measure $\gamma \in \mathfrak{M}_{\sigma\mathrm{f}}((0,\infty))$ define

$$\mathbb{L}_\gamma(\theta) = \int_0^\infty K(\theta x)\,\gamma(\mathrm{d}x), \tag{6.3}$$

where $\theta \in \mathsf{dom}\mathbb{L}_\gamma := \{\theta \mid \int_0^\infty |K(\theta x)|\,\gamma(\mathrm{d}x) < \infty\}$. Then

$$\mathbb{L}_{\gamma \circledast \rho}(\theta) = \int_0^\infty \mathbb{L}_\gamma(\theta x)\,\rho(\mathrm{d}x) \qquad \theta \in \mathsf{dom}\mathbb{L}_{\gamma \circledast \rho}. \tag{6.4}$$

Below we give three examples of K and of the resulting integral transforms. These transforms each identify measures from $\mathfrak{M}_L((0,\infty))$, but the choice of which one to apply may depend on the type of measure γ (cf. Example 6.5 below).

(1) $K(x) = 1 - \cos x$, $x > 0$. Then $\mathbb{L}_\gamma(\theta) = \int_0^\infty (1 - \cos(\theta x))\,\gamma(\mathrm{d}x)$ is the Lévy exponent of an infinitely divisible distribution generated by symmetrisation of γ. We will call it the *Lévy transform* of γ.

(2) $K(x) = x^2 \exp(-x)$, $x > 0$. Then

$$\mathbb{L}_\gamma(\theta) = \theta^2 \int_0^\infty x^2 \mathrm{e}^{-\theta x}\,\gamma(\mathrm{d}x), \quad \theta > 0.$$

(3) $K(x) = \exp(-x^{-q})$, $x > 0$, $q > 0$ (fixed). Then

$$\mathbb{L}_\gamma(\theta) = \int_0^\infty \mathrm{e}^{-\theta^{-q}x^{-q}}\,\gamma(\mathrm{d}x) = \int_0^\infty \mathrm{e}^{-\theta^{-q}x}\,\underleftarrow{\gamma}(\mathrm{d}x^{1/q}), \quad \theta > 0.$$

In this case $\mathbb{L}_\gamma$ is expressible as the Laplace transform of another measure.

Example 6.5. Let γ be as in Example 2.4(2). That is,

$$f_\gamma(s) = \alpha^{-1} s^{-1-1/\alpha} \sigma_\alpha\big(s^{-1/\alpha}\big), \qquad s > 0,$$

where σ_α is the density of the positive α-stable law having Laplace transform $\mathrm{e}^{-\theta^\alpha}$, $0 < \alpha < 1$. Take transformation (3) with $q = 1/\alpha$. Using (2.5) and (6.4) we get

$$\begin{aligned}
\mathbb{L}_{\gamma\circledast\rho}(\theta) &= \int_0^\infty \mathrm{e}^{-\theta^{-1/\alpha}t^{-1/\alpha}} \int_0^\infty \alpha^{-1}(ts^{-1})^{-1-1/\alpha}\sigma_\alpha((ts^{-1})^{-1/\alpha})\, s^{-1}\rho(\mathrm{d}s)\,\mathrm{d}t \\
&= \int_0^\infty \int_0^\infty \mathrm{e}^{-\theta^{-1/\alpha}s^{-1/\alpha}x}\sigma_\alpha(x)\,\mathrm{d}x\,\rho(\mathrm{d}s) \\
&= \int_0^\infty \mathrm{e}^{-\theta^{-1}s^{-1}}\,\rho(\mathrm{d}s) = \int_0^\infty \mathrm{e}^{-\theta^{-1}s}\,\underleftarrow{\rho}(\mathrm{d}s).
\end{aligned}$$

Thus $\mathbb{L}_{\gamma\circledast\rho}$ identifies ρ. We conclude that Υ_γ is injective. This was established in Barndorff-Nielsen and Thorbjørnsen (2006) by a closely similar argument.

A more detailed and deeper study of the injectivity problem will appear in a separate paper.

7. Upsilon Mappings of $\mathcal{ID}(\mathbb{R}^d)$

The Upsilon transformations discussed in the foregoing give rise to regularising mappings from the class $\mathcal{ID}(\mathbb{R}^d)$ of infinitely divisible laws on $\mathbb{R}^d$ into itself. These mappings are one-to-one when the corresponding Upsilon transformation of Lévy measures are. The material discussed in this Section extends results obtained previously in the special case $d = 1$ and $\gamma(\mathrm{d}x) = \mathrm{e}^{-x}\,\mathrm{d}x$; cf. Barndorff-Nielsen and Thorbjørnsen (2004) and Barndorff-Nielsen and Thorbjørnsen (2006).

Before proceeding with the formal definition of the mentioned mappings of $\mathcal{ID}(\mathbb{R}^d)$, we recall for convenience the version of the Lévy-Khintchine representation for measures in $\mathcal{ID}(\mathbb{R}^d)$ that we shall make use of: A probability measure μ on $\mathbb{R}^d$ belongs to $\mathcal{ID}(\mathbb{R}^d)$ if and only if its characteristic function f_μ can be represented in the form $f_\mu(z) = \exp(C_\mu(z))$, where the cumulant C_μ of μ is given by

$$C_\mu(z) = \mathrm{i}\langle z, \eta\rangle - \tfrac{1}{2}\langle Az, z\rangle + \int_{\mathbb{R}^d} \big(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z, x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x), \quad (z \in \mathbb{R}^d), \tag{7.1}$$

where η is a vector in $\mathbb{R}^d$, A is a symmetric, non-negative definite $d \times d$ matrix (with real entries) and ρ is a Lévy measure on $\mathbb{R}^d$. The triplet (A, ρ, η) is uniquely determined by μ and is called the *characteristic triplet* for μ.

Definition 7.1. Let γ be a measure from $\mathfrak{M}_{02}((0,\infty))$, and consider the mapping $\Upsilon_\gamma \colon \mathfrak{M}_L(\mathbb{R}^d) \to \mathfrak{M}_L(\mathbb{R}^d)$. We then define the mapping $\Upsilon^\gamma \colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ in the following way:

If $\mu \in \mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (A, ρ, η), then $\Upsilon^\gamma(\mu)$ is the measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet $(M_2(\gamma)A, \Upsilon_\gamma(\rho), M_1(\gamma)\tilde{\eta})$, where $M_i(\gamma)$ denotes the i'th moment of γ $(i = 1, 2)$, and where

$$\tilde{\eta} = M_1(\gamma)\eta + \int_0^\infty \int_{\mathbb{R}^d} tx\Big(1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|)\Big)\,\rho(\mathrm{d}x)\,\gamma(\mathrm{d}t). \tag{7.2}$$

The well-definedness of the vector-valued double integral in (7.2) is ensured by part (i) of the following:

Lemma 7.2. *Let ρ be a Borel measure on $\mathbb{R}^d$.*

(i) *For any t in $(0,\infty)$ we have*

$$\int_{\mathbb{R}^d} t\|x\|\big|1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|)\big|\,\rho(\mathrm{d}x) \le (1 \vee t^2) \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x).$$

(ii) *For any vector z in $\mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} \big|\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_{[0,1]}(\|x\|)\big|\,\rho(\mathrm{d}x) \le \big(2 + \tfrac{1}{2}\|z\|^2\big) \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x).$$

Proof: In the case $d = 1$, (i) may be extracted easily from the proof of Lemma 3.13 Barndorff-Nielsen and Thorbjørnsen (2005). For $d \ge 2$ we note then for t in $(0,\infty)$ that

$$\int_{\mathbb{R}^d} t\|x\|\big|1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|)\big|\,\rho(\mathrm{d}x) = \int_0^\infty ts\big|1_{[0,1]}(ts) - 1_{[0,1]}(s)\big|\,\|\rho\|(\mathrm{d}s),$$

and hence (ii) follows by applying the case $d = 1$ to the measure $\|\rho\|$.

To prove (ii), we note first that for any x, z in $\mathbb{R}^d$ we have the well-known estimate

$$\big|\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle\big| \le \tfrac{1}{2}|\langle z,x\rangle|^2 \le \tfrac{1}{2}\|z\|^2\|x\|^2,$$

so that

$$\int_{\{\|x\|\le 1\}} \big|\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle\big|\,\rho(\mathrm{d}x) \le \frac{\|z\|^2}{2} \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x).$$

Moreover,

$$\int_{\{\|x\|>1\}} \big|\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1\big|\,\rho(\mathrm{d}x) \le 2\rho\big(\{\|x\| > 1\}\big) \le 2\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x).$$

Combining the two estimates above, (ii) follows readily.

The following proposition motivates the choice of the constant $\tilde{\eta}$ in the definition of Υ^γ.

Proposition 7.3. *Let γ be a measure from $\mathfrak{M}_{02}((0,\infty))$, and consider the mapping $\Upsilon^\gamma \colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ defined above. Then for any μ in $\mathcal{ID}(\mathbb{R}^d)$ we have the following relation between the cumulant transforms of μ and $\Upsilon^\gamma(\mu)$:*

$$C_{\Upsilon^\gamma(\mu)}(z) = \int_0^\infty C_\mu(tz)\,\gamma(\mathrm{d}t), \qquad (z \in \mathbb{R}^d). \tag{7.3}$$

Proof: Let μ be a measure in $\mathcal{ID}(\mathbb{R}^d)$ with characteristic triplet (A,ρ,η). For any vector y in $\mathbb{R}^d$ we get from (ii) in Lemma 7.2 that

$$|C_\mu(y)| \le |\langle y,\eta\rangle| + \tfrac{1}{2}\langle Ay,y\rangle + \int_{\mathbb{R}^d} \left|\mathrm{e}^{\mathrm{i}\langle y,x\rangle} - 1 - \mathrm{i}\langle y,x\rangle 1_{[0,1]}(\|x\|)\right| \rho(\mathrm{d}x)$$

$$\le \|\eta\|\|y\| + \tfrac{1}{2}\|A\|\|y\|^2 + \big(2+\tfrac{1}{2}\|y\|^2\big)\int_{\mathbb{R}} (1\wedge\|x\|^2)\,\rho(\mathrm{d}x).$$

Since $\int_0^\infty (1\vee t^2)\,\gamma(\mathrm{d}t) < \infty$, it thus follows for any vector z in $\mathbb{R}^d$ that

$$\int_0^\infty |C_\mu(tz)|\,\gamma(\mathrm{d}t) \le \|\eta\|\|z\|M_1(\gamma) + \tfrac{1}{2}\|A\|\|z\|^2 M_2(\gamma)$$
$$+ \int_{\mathbb{R}^d} (1\wedge\|x\|^2)\,\rho(\mathrm{d}x)\int_0^\infty \big(2+\tfrac{1}{2}\|z\|^2t^2\big)\,\gamma(\mathrm{d}t) < \infty,$$

which justifies the following calculations:

$$\begin{aligned}
&\int_0^\infty C_\mu(tz)\,\gamma(\mathrm{d}t)\\
&\quad= \mathrm{i}\langle z,\eta\rangle M_1(\gamma) - \tfrac{1}{2}\langle Az,z\rangle M_2(\gamma)\\
&\qquad+ \int_0^\infty \Big(\int_{\mathbb{R}^d} \big(\mathrm{e}^{\mathrm{i}t\langle z,x\rangle} - 1 - \mathrm{i}t\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x)\Big)\,\gamma(\mathrm{d}t)\\
&\quad= \mathrm{i}\langle z,\eta\rangle M_1(\gamma) - \tfrac{1}{2}\langle Az,z\rangle M_2(\gamma)\\
&\qquad+ \int_0^\infty \Big(\int_{\mathbb{R}^d} \big(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(t^{-1}\mathrm{d}x)\Big)\,\gamma(\mathrm{d}t)\\
&\qquad+ \int_0^\infty \Big(\int_{\mathbb{R}} \mathrm{i}t\langle z,x\rangle\big(1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x)\Big)\gamma(\mathrm{d}t)\\
&\quad= \mathrm{i}\langle z,\tilde{\eta}\rangle - \tfrac{1}{2}\langle Az,z\rangle M_2(\gamma) + \int_{\mathbb{R}} \big(\mathrm{e}^{\mathrm{i}\langle z,x\rangle} - 1 - \mathrm{i}\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\Upsilon_\gamma(\rho)(\mathrm{d}x)\\
&\quad= C_{\Upsilon^\gamma(\mu)}(z),
\end{aligned} \tag{7.4}$$

as desired.

Recall that for a $d\times d$ matrix B, we denote by $T_B\colon \mathbb{R}^d \to \mathbb{R}^d$ the corresponding linear transformation. For a Borel measure μ on $\mathbb{R}^d$, we let furthermore $T_B\mu$ denote the transformation of μ by the mapping T_B.

Corollary 7.4. *Let γ be a measure from $\mathfrak{M}_{02}((0,\infty))$, and consider the mapping $\Upsilon^\gamma\colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ defined above. Then Υ^γ has the following properties:*

(i) $\Upsilon^\gamma(\mu_1 * \mu_2) = \Upsilon^\gamma(\mu_1) * \Upsilon^\gamma(\mu_2), \qquad (\mu_1,\mu_2 \in \mathcal{ID}(\mathbb{R}^d))$.
(ii) $\Upsilon^\gamma(T_B\mu) = T_B\Upsilon^\gamma(\mu), \qquad (B \in M_d(\mathbb{R}),\ \mu \in \mathcal{ID}(\mathbb{R}^d))$.
(iii) $\Upsilon^\gamma(\delta_c) = \delta_{M_1(\gamma)c}, \qquad (c \in \mathbb{R}^d)$.

Proof: (i) Assume that $\mu_1,\mu_2 \in \mathcal{ID}(\mathbb{R}^d)$ and then note that

$$C_{\Upsilon^\gamma(\mu_1*\mu_2)}(z) = \int_0^\infty C_{\mu_1*\mu_2}(tz)\,\gamma(\mathrm{d}t) = \int_0^\infty C_{\mu_1}(tz)\,\gamma(\mathrm{d}t) + \int_0^\infty C_{\mu_2}(tz)\,\gamma(\mathrm{d}t)$$
$$= C_{\Upsilon^\gamma(\mu_1)}(z) + C_{\Upsilon^\gamma(\mu_2)}(z) = C_{\Upsilon^\gamma(\mu_1)*\Upsilon^\gamma(\mu_2)}(z),$$

for any vector z in $\mathbb{R}^d$. Clearly this implies (i).

(ii) Let B be a $d \times d$ matrix and let B^* denote the transposed of B. Then for any vector z in $\mathbb{R}^d$ we find that

$$\begin{aligned} C_{\Upsilon^\gamma(T_B\mu)}(z) &= \int_0^\infty C_{T_B\mu}(tz)\,\gamma(\mathrm{d}t) = \int_0^\infty C_\mu(tB^*z)\,\gamma(\mathrm{d}t) \\ &= C_{\Upsilon^\gamma(\mu)}(B^*z) = C_{T_B\Upsilon^\gamma(\mu)}(z), \end{aligned}$$

which implies (ii).

(iii) Let c be a fixed vector in $\mathbb{R}^d$. Then for any z in $\mathbb{R}^d$ we find that

$$C_{\Upsilon^\gamma(\delta_c)}(z) = \int_0^\infty \mathrm{i}\langle tz, c\rangle\,\gamma(\mathrm{d}t) = \mathrm{i}\langle z, c\rangle \int_0^\infty t\,\gamma(\mathrm{d}t) = \mathrm{i}\langle z, M_1(\gamma)c\rangle = C_{\delta_{M_1(\gamma)c}}(z),$$

which proves (iii).

Corollary 7.5. *Let γ be a measure from $\mathfrak{M}_{02}((0,\infty))$, and consider the mapping $\Upsilon^\gamma\colon \mathit{ID}(\mathbb{R}^d) \to \mathit{ID}(\mathbb{R}^d)$ defined above. We then have*

$$\Upsilon^\gamma(\mathcal{S}(\mathbb{R}^d)) \subseteq \mathcal{S}(\mathbb{R}^d) \quad \textit{and} \quad \Upsilon^\gamma(\mathcal{L}(\mathbb{R}^d)) \subseteq \mathcal{L}(\mathbb{R}^d),$$

where $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{L}(\mathbb{R}^d)$ denote, respectively, the class of d-dimensional stable and selfdecomposable laws.

Proof: Recall first that $\mathcal{S}(\mathbb{R}^d)$ is the class of probability measures μ on $\mathbb{R}^d$ satisfying that (cf. Samorodnitsky and Taqqu, 1994, Definition 2.1.1)

$$\forall \alpha, \alpha' > 0\ \exists \alpha'' > 0\ \exists \beta \in \mathbb{R}^d\colon D_\alpha\mu * D_{\alpha'}\mu = D_{\alpha''}\mu * \delta_\beta,$$

where $D_c\mu$ denotes the scaling of μ by the scalar c, i.e. $D_c\mu = T_{c\mathbf{1}_n}\mu$. Now, for any μ in $\mathcal{S}(\mathbb{R}^d)$ and α, α' from $(0,\infty)$ it follows by application of (i)-(iii) of Corollary 7.4 that

$$D_\alpha\Upsilon^\gamma(\mu) * D_{\alpha'}\Upsilon^\gamma(\mu) = \Upsilon^\gamma\big((D_\alpha\mu)*(D_{\alpha'}\mu)\big) = \Upsilon^\gamma\big(D_{\alpha''}\mu*\delta_\beta\big) = D_{\alpha''}\Upsilon^\gamma(\mu)*\delta_{M_1(\gamma)\beta},$$

for suitable α'' from $(0,\infty)$ and β from $\mathbb{R}^d$. This shows that $\Upsilon^\gamma(\mu) \in \mathcal{S}(\mathbb{R}^d)$ too. The inclusion $\Upsilon^\gamma(\mathcal{L}(\mathbb{R}^d)) \subseteq \mathcal{L}(\mathbb{R}^d)$ follows similarly from (i) and (ii) of Corollary 7.4 by recalling that $\mathcal{L}(\mathbb{R}^d)$ may be characterised as the class of probability measures in $\mathit{ID}(\mathbb{R}^d)$ satisfying that

$$\forall c \in (0,1)\ \exists \mu_c \in \mathit{ID}(\mathbb{R}^d)\colon \mu = D_c\mu * \mu_c,$$

(cf. Sato, 1999, Proposition 15.5).

Remark 7.6. If $\gamma \in \mathfrak{M}_L((0,\infty)) \setminus \mathfrak{M}_{02}((0,\infty))$, then Definition 7.1 does not make sense, even if we restrict attention to the class $\mathit{ID}_0(\mathbb{R})$ of infinitely divisible laws μ with no drift and no Gaussian part, i.e. laws with characteristic triplets of the form $(0,\rho,0)$. For one thing we need to require that ρ is in the Lévy domain $\mathsf{dom}_L\Upsilon_\gamma$, but this generally does not ensure that the integral in (7.2) is well-defined. To remedy this situation we introduce the subclass (cf. Proposition 7.3)

$$\mathsf{dom}_{\mathit{ID}}\Upsilon^\gamma = \big\{\mu \in \mathit{ID}_0(\mathbb{R}) \ \big|\ \forall z \in \mathbb{R}\colon \textstyle\int_0^\infty |C_\mu(zt)|\,\gamma(\mathrm{d}t) < \infty\big\}.$$

For a given γ, Definition 7.1 then makes sense for all μ in $\mathsf{dom}_{\mathit{ID}}\Upsilon^\gamma$ and gives rise to a mapping $\Upsilon^\gamma\colon \mathsf{dom}_{\mathit{ID}}(\Upsilon^\gamma) \to \mathit{ID}(\mathbb{R}^d)$ with (algebraic) properties similar to those derived below in the case $\gamma \in \mathfrak{M}_{02}((0,\infty))$. In the present paper we restrict attention to the mappings Υ^γ, where γ is assumed in $\mathfrak{M}_{02}((0,\infty))$, and we merely indicate by an example (cf. Example 7.7(4) below) that the more general setting outlined above gives rise to important and interesting mappings as well.

Examples 7.7. We adopt the notation from Examples 2.4.

(1) Consider the mapping Υ_0 introduced in Example 2.4(1). The corresponding mapping $\Upsilon^0 \colon I\!D(\mathbb{R}) \to I\!D(\mathbb{R})$ is one-to-one and is related to free probability via the formula

$$C_{\Upsilon^0(\mu)}(z) = \mathcal{C}_{\Lambda(\mu)}(\mathrm{i}z), \qquad (z \in \mathbb{R}),$$

where Λ is the so-called Bercovici-Pata bijection from $I\!D(\mathbb{R})$ onto the class of infinitely divisible probability measures with respect to (additive) convolution in free probability theory. In addition, $\mathcal{C}$ is the analog of the cumulant transform in free probability (see Barndorff-Nielsen and Thorbjørnsen, 2004 for details). The range of Υ^0 was identified as the so-called Goldie-Steutel-Bondesson class in Barndorff-Nielsen et al. (2004). Furthermore, Υ^0 maps the class of stable laws onto itself and the class of selfdecomposable laws onto the so-called Thorin class (see Barndorff-Nielsen and Thorbjørnsen, 2006).

(2) For α in $[0,1]$ consider the mapping Υ_α introduced in Example 2.4(2). The associated mapping $\Upsilon^\alpha \colon I\!D(\mathbb{R}) \to I\!D(\mathbb{R})$ was introduced and studied in Barndorff-Nielsen and Thorbjørnsen (2006). For all α, Υ^α is one-to-one. For $\alpha = 0$, Υ^α agrees with the mapping Υ^0 described in (1) and Υ^1 is the identity mapping on $I\!D(\mathbb{R})$. The family $(\Upsilon_\alpha)_{\alpha\in[0,1]}$ thus, in a certain sense, interpolates smoothly between infinite divisibility in classical and free probability (see Barndorff-Nielsen and Thorbjørnsen, 2004).

(3) Consider for λ in $(-2,\infty)$ the mapping Ξ_λ introduced in Example 2.4(3), i.e. the Upsilon transformations corresponding to the measures $\gamma_\lambda(\mathrm{d}t) = t^{\lambda-1}\mathrm{e}^{-t}\,\mathrm{d}t$. When $\lambda > 0$, $\gamma_\lambda \in \mathfrak{M}_{02}((0,\infty))$ and we obtain a mapping $\Xi^\lambda \colon I\!D(\mathbb{R}) \to I\!D(\mathbb{R})$ via Definition 7.1. When $\lambda \in (-2,0]$, $\gamma_\lambda \notin \mathfrak{M}_{02}((0,\infty))$ and Definition 7.1 does not apply. However, the construction outlined in Remark 7.6 gives rise to mappings $\Xi^\lambda \colon \mathsf{dom}_{I\!D}\Xi^\lambda \to I\!D(\mathbb{R})$, where

$$\mathsf{dom}_{I\!D}\Xi^\lambda = \big\{\mu \in I\!D_0(\mathbb{R}) \,\big|\, \forall z \in \mathbb{R} \colon \textstyle\int_0^\infty |C_\mu(zt)| t^{\lambda-1}\mathrm{e}^{-t}\,\mathrm{d}t < \infty\big\}.$$

Questions related to the random integral representations of these mappings Ξ^λ have been studied by Sato in Sato (2006b).

(4) Consider for λ in $(-2,\infty)$ the mapping Φ_λ introduced in Example 2.4(4), i.e. the Upsilon transformations corresponding to the measures $\gamma_\lambda(\mathrm{d}t) = t^{\lambda-1}1_{(0,1)}(t)\,\mathrm{d}t$. As in (3) we obtain mappings $\Phi^\lambda \colon I\!D(\mathbb{R}) \to I\!D(\mathbb{R})$ via Definition 7.1 when $\lambda > 0$, and for λ in $(-2,0]$ the construction outlined in Remark 7.6 gives rise to mappings $\Phi^\lambda \colon \mathsf{dom}_{I\!D}\Phi^\lambda \to I\!D(\mathbb{R})$. The particular case $\lambda = 0$ was studied in Barndorff-Nielsen et al. (2004), where it was established that

$$\mathsf{dom}_{I\!D}\Phi^0 = \{\mu \in I\!D_0(\mathbb{R}) \mid \rho(\mu) \in \mathfrak{M}_{\log}(\mathbb{R})\},$$

where $\rho(\mu)$ denotes the Lévy measure of μ. Thus, the condition that the integral $\int_0^\infty |C_\mu(zt)|\,\gamma(\mathrm{d}t)$ is finite for all z, is, in this case, equivalent to the requirement that $\rho(\mu) \in \mathfrak{M}_{\log}(\mathbb{R}) = \mathsf{dom}_L\Phi_0$, but, as indicated in Remark 7.6, this is not a general feature. The range of Φ^0 is the class of all selfdecomposable laws; cf. for instance Barndorff-Nielsen et al. (2004).

We close this section by giving a Lévy-Khintchine type representation of $\Upsilon^\gamma(\mu)$.

Proposition 7.8. *Let γ be a measure in $\mathfrak{M}_{02}((0,\infty))$, and consider the mapping $\Upsilon^\gamma\colon \mathit{ID}(\mathbb{R}^d) \to \mathit{ID}(\mathbb{R}^d)$ defined above. Let further μ be a measure in $\mathit{ID}(\mathbb{R}^d)$ with characteristic triplet (A,ρ,η). Then for any z in $\mathbb{R}^d$*

$$\begin{aligned} C_{\Upsilon^\gamma(\mu)}(z) &= \mathrm{i}M_1(\gamma)\langle z,\eta\rangle - \tfrac{1}{2}M_2(\gamma)\langle Az,z\rangle \\ &\quad + \int_{\mathbb{R}^d} \big(\phi_\gamma(\langle z,x\rangle) - M_0(\gamma) - \mathrm{i}M_1(\gamma)\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x), \end{aligned}$$

where $\phi_\gamma(u) = \int_0^\infty \mathrm{e}^{\mathrm{i}ut}\,\gamma(\mathrm{d}t)$ for u in $\mathbb{R}$, and $M_j(\gamma) = \int_0^\infty t^j\,\gamma(\mathrm{d}t)$ $(j=0,1,2)$.

Proof: Using the calculation (7.4) from the proof of Proposition 7.3 we find that

$$\begin{aligned} C_{\Upsilon^\gamma(\mu)}(z) &= \mathrm{i}\langle z,\eta\rangle M_1(\gamma) - \tfrac{1}{2}\langle Az,z\rangle M_2(\gamma) \\ &\quad + \int_0^\infty \Big(\int_{\mathbb{R}^d} \big(\mathrm{e}^{\mathrm{i}t\langle z,x\rangle} - 1 - \mathrm{i}t\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x)\Big)\,\gamma(\mathrm{d}t). \end{aligned} \tag{7.5}$$

By Lemma 7.2(ii) we may change the order of integration in the double integral, so that

$$\begin{aligned} &\int_0^\infty \Big(\int_{\mathbb{R}^d} \big(\mathrm{e}^{\mathrm{i}t\langle z,x\rangle} - 1 - \mathrm{i}t\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x)\Big)\,\gamma(\mathrm{d}t) \\ &\qquad = \int_{\mathbb{R}^d} \big(\phi_\gamma(\langle z,x\rangle) - M_0(\gamma) - \mathrm{i}M_1(\gamma)\langle z,x\rangle 1_{[0,1]}(\|x\|)\big)\,\rho(\mathrm{d}x), \end{aligned}$$

which inserted in (7.5) yields the desired formula.

8. Continuity properties of Υ^γ

In this section we establish continuity results for upsilon transforms Υ^γ under the assumption that $\gamma \in \mathfrak{M}_{02}((0,\infty))$. The derived results may be seen as counterparts to the results accomplished in Section 5.1 for Υ_γ, also in the $\mathfrak{M}_{02}((0,\infty))$-case. We shall need the following well-known lemma (see e.g. the proof of Barndorff-Nielsen et al., 2004, Proposition 2.4(v)).

Lemma 8.1. *Let (μ_n) be a sequence of measures from $\mathit{ID}(\mathbb{R}^d)$, and for each n let (A_n,ρ_n,η_n) be the characteristic triplet for μ_n. Then (μ_n) is precompact if and only if the following four conditions are satisfied:*

(a) $\sup_{n\in\mathbb{N}} \|A_n\| < \infty$.
(b) $\sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} (1\wedge\|x\|^2)\rho_n(\mathrm{d}x) < \infty$.
(c) $\forall\epsilon>0\ \exists K>0\colon \sup_{n\in\mathbb{N}} \rho_n\big(\{\|x\|>K\}\big) < \epsilon$.
(d) $\sup_{n\in\mathbb{N}} \|\eta_n\| < \infty$.

Proposition 8.2. *For any γ in $\mathfrak{M}_{02}((0,\infty))$ the mapping $\Upsilon^\gamma\colon \mathit{ID}(\mathbb{R}^d)\to\mathit{ID}(\mathbb{R}^d)$ is continuous with respect to weak convergence, i.e. for any sequence (μ_n) of measures in $\mathit{ID}(\mathbb{R}^d)$ and any measure μ in $\mathit{ID}(\mathbb{R}^d)$ we have*

$$\mu_n \xrightarrow[n\to\infty]{\mathrm{w}} \mu \implies \Upsilon^\gamma(\mu_n) \xrightarrow[n\to\infty]{\mathrm{w}} \Upsilon^\gamma(\mu).$$

Proof: Let (μ_n) be a sequence of measures from $\mathit{ID}(\mathbb{R}^d)$ such that $\mu_n \xrightarrow{\mathrm{w}} \mu$ as $n\to\infty$ for some measure μ (necessarily) in $\mathit{ID}(\mathbb{R}^d)$. Then by Sato (1999, Lemma 7.7)

$$C_{\mu_n}(y) \xrightarrow[n\to\infty]{} C_\mu(y), \qquad \text{for all } y \text{ in } \mathbb{R}^d,$$

and it suffices to establish that

$$C_{\Upsilon^\gamma(\mu_n)}(y) \xrightarrow[n\to\infty]{} C_{\Upsilon^\gamma(\mu)}(y), \qquad \text{for all } y \text{ in } \mathbb{R}^d.$$

By Proposition 7.3 and Lebesgue's theorem on dominated convergence it suffices to verify, for each fixed z in $\mathbb{R}$, the existence of a Borel function $g_z\colon (0,\infty) \to [0,\infty)$ such that

$$\forall n \in \mathbb{N}\colon \left|C_{\mu_n}(zt)\right| \le g_z(t), \qquad (t \in (0,\infty)), \tag{8.1}$$

and

$$\int_0^\infty g_z(t)\,\gamma(\mathrm{d}t) < \infty. \tag{8.2}$$

For each positive integer n, let (A_n, ρ_n, η_n) be the generating triplet for μ_n. Combining then (7.1) with (ii) in Lemma 7.2, we find that

$$|C_{\mu_n}(y)| \le \|\eta_n\|\|y\| + \tfrac{1}{2}\|A_n\|\|y\|^2 + \big(2 + \tfrac{1}{2}\|y\|^2\big)\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho_n(\mathrm{d}x),$$

for any vector y in $\mathbb{R}^d$. Since $\mu_n \xrightarrow{\mathrm{w}} \mu$ as $n \to \infty$, it follows that (cf. Lemma 8.1)

$$H := \sup_{n\in\mathbb{N}} \|\eta_n\| < \infty,\ \ A := \sup_{n\in\mathbb{N}} \|A_n\| < \infty \text{ and } R := \sup_{n\in\mathbb{N}} \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho(\mathrm{d}x) < \infty.$$

Thus, if we put

$$g_z(t) = H\|z\|t + \tfrac{1}{2}A\|z\|^2t^2 + R\big(2 + \tfrac{1}{2}\|z\|^2t^2\big),$$

it follows that g_z satisfies both (8.1) and (8.2), since $\gamma \in \mathfrak{M}_{02}((0,\infty))$.

Lemma 8.3. *Let γ be a non-zero measure from $\mathfrak{M}_{02}((0,\infty))$, and let (μ_n) be a sequence of measures from $\mathcal{ID}(\mathbb{R}^d)$ such that $\Upsilon^\gamma(\mu_n) \xrightarrow{\mathrm{w}} \nu$ as $n \to \infty$ for some measure ν in $\mathcal{ID}(\mathbb{R}^d)$. Then the sequence $(\mu_n)_{n\in\mathbb{N}}$ is precompact.*

Proof: We proceed as in the proof of Barndorff-Nielsen et al. (2004, Proposition 2.4(v)):

Denoting by $(\tilde{A}_n, \tilde{\rho}_n, \tilde{\eta}_n)$ the characteristic triplet for $\Upsilon^\gamma(\mu_n)$, let (ã)–($\tilde{\text{d}}$) be the conditions obtained by replacing (A_n, ρ_n, η_n) by $(\tilde{A}_n, \tilde{\rho}_n, \tilde{\eta}_n)$ in (a)–(d) of Lemma 8.1. Then our assumption implies that (ã)–($\tilde{\text{d}}$) are satisfied. By definition of Υ^γ we have that $\tilde{A}_n = M_2(\gamma)A_n$, $\tilde{\rho}_n = \Upsilon_\gamma(\rho_n)$ and

$$\tilde{\eta} = M_1(\gamma)\eta + \int_0^\infty \int_{\mathbb{R}^d} tx\Big(1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|)\Big)\,\rho(\mathrm{d}x)\,\gamma(\mathrm{d}t).$$

Hence, since $\gamma \neq 0$, (a) is an immediate consequence of (ã), and (b) follows from ($\tilde{\text{b}}$) and the estimate

$$\begin{aligned}
\int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\tilde{\rho}_n(\mathrm{d}x) &= \int_0^\infty \int_{\mathbb{R}^d} (1 \wedge t^2\|x\|^2)\,\rho_n(\mathrm{d}x)\,\gamma(\mathrm{d}t) \\
&\ge \int_0^\infty \int_{\mathbb{R}^d} (1 \wedge t^2)(1 \wedge \|x\|^2)\,\rho_n(\mathrm{d}x)\,\gamma(\mathrm{d}t) \\
&= \int_0^\infty (1 \wedge t^2)\,\gamma(\mathrm{d}t) \int_{\mathbb{R}^d} (1 \wedge \|x\|^2)\,\rho_n(\mathrm{d}x),
\end{aligned}$$

recalling again that $\gamma \neq 0$. To verify (c), note that for any positive numbers L and δ we have

$$\tilde{\rho}_n\big(\{\|x\| > L\}\big) = \int_0^\infty \rho_n\big(\{\|x\| > L/t\}\big)\,\gamma(\mathrm{d}t) \geq \gamma([\delta,\infty))\rho_n\big(\{\|x\| > L/\delta\}\big).$$

Choosing now δ such that $\gamma([\delta,\infty)) > 0$ and using the substitution $L = K\delta$, we find that

$$\rho_n\big(\{\|x\| > K\}\big) \leq \gamma([\delta,\infty))^{-1}\tilde{\rho}_n\big(\{\|x\| > K\delta\}\big),$$

for any positive number K and any n in $\mathbb{N}$. Therefore (c) is a consequence of ($\tilde{\text{c}}$). Finally, to establish (d), note that

$$\sup_{n\in\mathbb{N}} \int_0^\infty \int_{\mathbb{R}^d} t\|x\| \Big| 1_{[0,1]}(t\|x\|) - 1_{[0,1]}(\|x\|) \Big| \,\rho(\mathrm{d}x)\,\gamma(\mathrm{d}t) < \infty,$$

as a result of Lemma 7.2 in conjunction with (b). Therefore (d) follows from ($\tilde{\text{d}}$).

Proposition 8.4. *Let γ be a measure in $\mathfrak{M}_{02}((0,\infty))$.*

(i) *The mapping $\Upsilon^\gamma\colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ is closed in the following sense: For any subset F of $\mathcal{ID}(\mathbb{R}^d)$, which is closed in the topology for weak convergence, the same holds for $\Upsilon^\gamma(F) = \{\Upsilon^\gamma(\mu) \mid \mu \in F\}$.*

(ii) *Assume that the mapping $\Upsilon^\gamma\colon \mathcal{ID}(\mathbb{R}^d) \to \mathcal{ID}(\mathbb{R}^d)$ is injective. Then it is automatically a homeomorphism onto its range $\mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$.*

Proof: (i) We may clearly assume that $\gamma \neq 0$. Let F be a closed subset of $\mathcal{ID}(\mathbb{R}^d)$ and let ν be a measure from the closure of $\Upsilon^\gamma(F)$. Then we may choose a sequence (μ_n) from F such that $\Upsilon^\gamma(\mu_n) \to \nu$ as $n \to \infty$, and by Lemma 8.3 (μ_n) is necessarily precompact. In particular there exists a subsequence $(\mu_{n_p})_{p\in\mathbb{N}}$ converging weakly to some μ, which must belong to F, since F is closed. Now by Proposition 8.2, $\Upsilon^\gamma(\mu_{n_p}) \xrightarrow{\mathrm{w}} \Upsilon^\gamma(\mu)$ as $n \to \infty$, and since also $\Upsilon^\gamma(\mu_{n_p}) \xrightarrow{\mathrm{w}} \nu$ as $n \to \infty$, we conclude that $\nu = \Upsilon^\gamma(\mu) \in \Upsilon^\gamma(F)$, as desired.

(ii) This follows from (i) as in the proof of (ii) in Corollary 5.4.

Corollary 8.5. *Let γ be a non-zero measure from $\mathfrak{M}_{02}((0,\infty))$ and consider the full range*

$$\mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma := \big\{\Upsilon^\gamma(\mu) \;\big|\; \mu \in \mathcal{ID}(\mathbb{R}^d)\big\}.$$

This subclass of $\mathcal{ID}(\mathbb{R}^d)$ has the following properties:

(i) *$\nu_1 * \nu_2 \in \mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$, whenever $\nu_1, \nu_2 \in \mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$.*
(ii) *$T_B\nu \in \mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$, whenever $\nu \in \mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$ and $B \in M_d(\mathbb{R})$.*
(iii) *$\delta_c \in \mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$ for all c in $\mathbb{R}^d$.*
(iv) *$\mathsf{ran}_{\mathcal{ID}}\Upsilon^\gamma$ is a closed subset of $\mathcal{ID}(\mathbb{R}^d)$ in the topology for weak convergence.*

Proof: These properties follow readily from Corollary 7.4 and Proposition 8.4.

Examples 8.6. (1) The Upsilon mapping $\Upsilon^0\colon \mathcal{ID}(\mathbb{R}) \to \mathcal{ID}(\mathbb{R})$ considered in Example 7.7(1) is one-to-one and corresponds to the measure $\gamma(\mathrm{d}t) = \mathrm{e}^{-t}\,\mathrm{d}t$ from $\mathfrak{M}_{02}((0,\infty))$. Thus, by Corollary 8.4(ii), it is a homeomorphism onto its range (which is the Goldie-Steutel-Bondesson class). This was established directly in Barndorff-Nielsen et al. (2004).

(2) The Upsilon mappings $\Upsilon^\alpha\colon \mathcal{ID}(\mathbb{R}) \to \mathcal{ID}(\mathbb{R})$ considered in Example 7.7(2) are also injective and correspond to measures γ_α from $\mathfrak{M}_{02}((0,\infty))$. Hence these mappings are also homeomorphisms onto their ranges.

(3) For $\lambda > 0$ the mappings Ξ^{λ} considered in Example 7.7(3) correspond to measures γ_λ from $\mathfrak{M}_{02}((0,\infty))$, and they are injective according to Example 6.3(ii). Thus, these mappings are homeomorphisms as well.

(4) By virtue of Example 6.3(i), it follows as in (3) that for positive λ the mappings $\Phi^{\lambda}\colon \mathcal{ID}(\mathbb{R}) \to \mathcal{ID}(\mathbb{R})$ introduced in Example 7.7(4) are homeomorphisms onto their ranges.

9. Random Integral Representation

In many cases the Upsilon transformations introduced in Section 7 can be represented as random integrals, in the following sense. (Here we consider only one-dimensional integrators; for some results on the multivariate case cf. Barndorff-Nielsen et al., 2004.)

Suppose that γ has finite upper tail measure and let

$$\varepsilon_\gamma(\xi) = \gamma([\xi,\infty)).$$

Then $\gamma(\mathrm{d}\xi) = -\mathrm{d}\varepsilon_\gamma(\xi)$. The inverse function of ε_γ, denoted ε_γ^*, is defined by

$$\varepsilon_\gamma^*(t) = \inf\left\{\xi > 0 \,\middle|\, \varepsilon_\gamma(\xi) \le t\right\}. \tag{9.1}$$

Both functions $\xi \to \varepsilon_\gamma(\xi)$ and $t \to \varepsilon_\gamma^*(t)$ are decreasing and càglàd.

Now, given a Lévy measure ρ and an $\eta \in \mathbb{R}$, let $Z = \{Z_t\}$ be the Lévy process for which the cumulant function of Z_1 is given by

$$C_\rho(z) = \mathrm{i}\eta z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt1_{[-1,1]}(t)\right)\rho(\mathrm{d}t), \tag{9.2}$$

and consider the random integral

$$Y = \int_0^{\varepsilon_\gamma(0)} \varepsilon_\gamma^*(s)\,\mathrm{d}Z_s. \tag{9.3}$$

Definition 9.1. We say that (9.3) is a *random integral representation* (RIR) of Υ_γ at $\rho \in \mathsf{dom}_L\Upsilon_\gamma$ provided the integral (9.3) exists as the limit in probability of the Riemann sums and the random variable Y (which is then necessarily infinitely divisible) has Lévy measure $\rho_\gamma = \Upsilon_\gamma(\rho)$ and cumulant function

$$C_{\rho_\gamma}(z) = \mathrm{i}\tilde{\eta}z + \int_{\mathbb{R}} \left(e^{\mathrm{i}zt} - 1 - \mathrm{i}zt1_{[-1,1]}(t)\right)\rho_\gamma(\mathrm{d}t) \tag{9.4}$$

where

$$\tilde{\eta} = \int_0^\infty x\Big(\eta + \int_{\mathbb{R}} y\left(1_{[-1,1]}(xy) - 1_{[-1,1]}(y)\right)\rho(\mathrm{d}y)\Big)\gamma(\mathrm{d}x).$$

For Υ_γ to have RIR at $\rho(\in \mathsf{dom}_L\Upsilon_\gamma)$ it suffices that $\gamma \in \mathfrak{M}_{02}((0,\infty))$ and ε_γ^* is continuous. In that case it moreover holds that

$$\int_0^\infty |C_\rho(tz)|\,\gamma(\mathrm{d}t) < \infty \tag{9.5}$$

and that we have the important relation

$$C_{\rho_\gamma}(z) = \int_0^\infty C_\rho(tz)\,\gamma(\mathrm{d}t), \tag{9.6}$$

which in fact is the same as (7.3).

This result was established for the case $\gamma(\mathrm{d}x) = \mathrm{e}^{-x}\,\mathrm{d}x$ in Barndorff-Nielsen and Thorbjørnsen (2004), and for the measures introduced in Example 2.4(2) in

Barndorff-Nielsen and Thorbjørnsen (2006). The proofs given in those cases extend directly to the present setting.

Remark 9.2. The measures γ in Example 3.1(1)–(3) all have second moment and continuous ε_γ^*, and thus the RIR.

Remark 9.3. If we take Z to be the Lévy process with characteristic triplet (a, ρ, η) then, again provided that $\gamma \in \mathfrak{M}_{02}\left((0, \infty)\right)$ and ε_γ^* is continuous, we have that (9.5) and (9.6) hold (cf. Proposition 7.3) and, furthermore, that Y has triplet $(\tilde{a}, \rho_\gamma, \tilde{\eta})$ with $\tilde{a} = aM_2(\gamma)$ (where $M_2(\gamma)$ denotes the second moment of γ). Otherwise put, (9.3) is then a random integral representation of the transformation Υ^γ discussed in Sections 7 and 8.

Extensions and ramifications of the original results (in Barndorff-Nielsen and Thorbjørnsen, 2004 and Barndorff-Nielsen and Thorbjørnsen, 2006) are also discussed in Barndorff-Nielsen et al. (2004), Sato (2006b), Sato (2006a) and Sato (2007). The latter three papers develop the theory of integration of deterministic functions with respect to Lévy processes and related RIR results in great generality and detail.

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