Alea 5, 3–20 (2009)

Functional moderate deviations for triangular arrays and applications

Florence Merlevède and Magda Peligrad

Université Paris 6, LPMA and C.N.R.S UMR 7599, 175 rue du Chevaleret, 75013 Paris, FRANCE

Department of Mathematical Sciences, University of Cincinnati, PO Box 210025, Cincinnati, Oh 45221-0025

Abstract. Motivated by the study of dependent random variables by coupling with independent blocks of variables, we obtain first sufficient conditions for the moderate deviation principle in its functional form for triangular arrays of independent random variables. Under some regularity assumptions our conditions are also necessary in the stationary case. The results are then applied to derive moderate deviation principles for linear processes, kernel estimators of a density and some classes of dependent random variables.

1. Introduction

In recent years substantial progress was achieved in obtaining necessary and sufficient condition for the moderate deviations behavior of sums of independent identically distributed random variables. Papers by Ledoux (1992) and Arcones (2003a,b,c), among others, are steps in this direction. These works show that the moderate deviation principle can be applied for i.i.d. sequences, even when the moment generating function is not defined in a neighborhood of zero. Due to its invariant nature, a natural question is to treat triangular arrays of random variables. Some sufficient conditions for bounded triangular arrays are contained in Lemma 2.3 in Arcones (2003c) and also in the results by Puhalskii (1994) about triangular arrays of martingale differences. Djellout (2002) studied this problem for not necessarily stationary martingale differences sequences.

In this paper we derive sufficient conditions for the moderate deviation principle in its functional form for triangular arrays of independent random variables. In the stationary case and under some regularity conditions, the condition is necessary

Received by the editors December 12 2007, accepted April 15 2008.

2000 *Mathematics Subject Classification.* 60F10, 60G50.

Key words and phrases. triangular arrays, independent random variables, strong mixing, moderate deviations, invariance principle.

Supported in part by a Charles Phelps Taft Memorial Fund grant and NSA grant, H98230-07-1-0016.

as well. These results open the way to address the moderate deviation principle for classes of dependent random variables that were not studied so far, by dividing the variables in blocks that are further approximated by a triangular array of independent random variables. As a matter of fact this was the initial motivation of our study. The results are used to treat general linear processes, Kernel estimators of a density, and some dependent structures including classes of strong mixing sequences.

The moderate deviation principle is an intermediate estimation between central limit theorem and large deviation. We shall assume for the moment that we have a triangular array of independent, centered and square integrable random variables $(X_{n1}, X_{n2}, ..., X_{nk_n})$, where k_n is a sequence of integers. Denote by

$$S_{n,0} = 0,\ S_{nl} = \sum_{j=1}^{l} X_{nj},\ S_n = \sum_{j=1}^{k_n} X_{nj},\ \sigma_{nj}^2 = \mathrm{Var}(X_{nj}),\ s_n^2 = \sum_{j=1}^{k_n} \sigma_{nj}^2 \text{ and } s_{ni}^2 = \sum_{j=1}^{i} \sigma_{nj}^2 .$$

In the rest of the paper MDP stays for Moderate Deviation Principle.

Definition 1.1. *We say that the MDP holds for $s_n^{-1} S_n$ with the speed $a_n \to 0$ and rate function $I(t)$ if for each A Borelian,*

$$\begin{aligned} -\inf_{t \in A^o} I(t) &\leq \liminf_n a_n \log \mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n} S_n \in A\Big) \\ &\leq \limsup_n a_n \log \mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n} S_n \in A\Big) \leq -\inf_{t \in \bar{A}} I(t)\,. \end{aligned} \tag{1.1}$$

We are also interested to give a more general result concerning the Donsker process associated to the partial sums.

Definition 1.2. *Let $\{W_n, n > 0\}$ be the family of random variables on $D[0,1]$ defined as follows:*

$W_n(t) = S_{n,i-1}/s_n$ for $t \in [s_{n,i-1}^2/s_n^2, s_{ni}^2/s_n^2)$, where $1 \leq i \leq k_n$ and $W_n(1) = S_n/s_n$.

We say that the family of random variables $\{W_n, n > 0\}$ satisfies the functional Moderate Deviation Principle (MDP) in $D[0,1]$ endowed with uniform topology, with speed $a_n \to 0$ and good rate function $I(.)$, if the level sets $\{x, I(x) \leq \alpha\}$ are compact for all $\alpha < \infty$, and for all Borel sets $\Gamma \in \mathcal{B}$

$$\begin{aligned} -\inf_{t \in \Gamma^0} I(t) &\leq \liminf_n a_n \log \mathbf{P}(\sqrt{a_n} W_n \in \Gamma) \\ &\leq \limsup_n a_n \log \mathbf{P}(\sqrt{a_n} W_n \in \Gamma) \leq -\inf_{t \in \bar{\Gamma}} I(t)\,. \end{aligned} \tag{1.2}$$

Our first result is:

Theorem 1.3. *Assume that $(X_{n1}, X_{n2}, ..., X_{nk_n})$ is a triangular array of independent centered and square integrable random variables. Assume $a_n \to 0$ and that for any $\beta > 0$*

$$\limsup_{n \to \infty} a_n \sum_{j=1}^{k_n} \mathbf{E}([\exp \beta \frac{|X_{nj}|}{\sqrt{a_n} s_n}] I(\sqrt{a_n} s_n < |X_{nj}| < s_n/\sqrt{a_n}) = 0\,, \tag{1.3}$$

$$\limsup_n a_n \log \mathbf{P}(\max_{1 \leq j \leq k_n} |X_{nj}| \geq s_n/\sqrt{a_n}) = -\infty \tag{1.4}$$

and for any $\epsilon > 0$

$$\frac{1}{s_n^2}\sum_{j=1}^{k_n}\mathbf{E}[X_{nj}^2 I(|X_{nj}| \geq \epsilon s_n\sqrt{a_n})] \to 0\ . \tag{1.5}$$

Then $\{W_n, n > 0\}$ *satisfies MDP in* $D[0,1]$ *with speed* a_n *and rate function* $I(.)$ *defined by*

$$I(z) = \frac{1}{2}\int_0^1 (z'(u))^2 du \text{ if } z(0) = 0 \text{ and } z \text{ is absolutely continuous} \tag{1.6}$$

and ∞ *otherwise.*

Comment 1.4. *Under the assumptions of the theorem, we have in particular that* $\{s_n^{-1}\sum_{j=1}^{k_n} X_{nj}\}$ *satisfies the MDP with speed* a_n *and rate* $I(t) = t^2/2$.

Standard computations show that all the conditions of Theorem 1.3 are satisfied if we impose the unique condition (1.7) below, that can be viewed as a generalized Lindeberg's condition. So we can state:

Corollary 1.5. *Assume* $(X_{n1}, X_{n2}, ..., X_{nk_n})$ *is a triangular array of independent centered and square integrable random variables. Assume* $a_n \to 0$*, and for any* $\epsilon > 0$ *and any* $\beta > 0$*,*

$$\limsup_{n\to\infty} a_n \sum_{j=1}^{k_n}\mathbf{E}([\exp\beta\frac{|X_{nj}|}{\sqrt{a_n}s_n}]I(|X_{nj}| > \epsilon\sqrt{a_n}s_n) = 0\,. \tag{1.7}$$

Then the conclusion of Theorem 1.3 is satisfied.

Simple computations involving Chebyshev's inequality and integration by parts (see Appendix) lead to conditions imposed to the tails distributions of the random variables involved.

Comment 1.6. *Condition (1.3) is equivalent to : There is a constant* C_1 *with the following property: for any* $\beta > 0$ *there is* $N(\beta)$ *such that for* $n > N(\beta)$

$$a_n\sum_{j=1}^{k_n}\mathbf{P}(|X_{nj}| > u\sqrt{a_n}s_n) \leq C_1\exp(-\beta u) \text{ for all } 1 \leq u \leq 1/a_n\,. \tag{1.8}$$

Condition (1.7) is equivalent to : There is a constant C_1 *with the property that for any* $\epsilon > 0$ *and any* $\beta > 0$*, there is* $N(\epsilon,\beta)$ *such that for* $n > N(\epsilon,\beta)$*, the inequality in relation (1.8) is satisfied for all* $u \geq \epsilon$*.*

If we impose some regularity assumptions the conditions simplify.

RC The functions $f(n) = s_n^2 a_n$ and $g(n) = s_n^2/a_n$ are strictly increasing to infinite, and the function $l(n) = s_n^2/k_n$ is nondecreasing.

Assuming **RC**, we construct the strictly increasing continuous function $f(x)$ that is formed by the line segments from $(n, f(n))$ to $(n+1, f(n+1))$. Similarly we define $g(x)$ and denote by $c(x) = f^{-1}(g(x))$.

Corollary 1.7. *Assume* $(X_{n1}, X_{n2}, ..., X_{nk_n})$ *is a triangular array of independent, centered and square integrable random variables. Assume* $a_n \to 0$*, the regularity conditions* ***RC*** *hold and*

$$a_n\log(\sup_{n\leq m\leq c(n+1)}\sup_{1\leq i\leq k_m} k_n\mathbf{P}(|X_{mi}| > s_n/\sqrt{a_n})) \to -\infty \text{ as } n\to\infty\,. \tag{1.9}$$

Assume in addition that (1.5) is satisfied. Then the conclusion of Theorem 1.3 holds.

In the sequel we shall denote by $[x]$ the integer part of x.

Remark 1.8. *In the case where $(X_n)_{n\geq 0}$ is a sequence of i.i.d. r.v's with mean zero and finite second moment the conditions of corollary 1.7 simplify. If $a_n \searrow 0$, $na_n \nearrow \infty$ and*

$$a_n \log n \mathbf{P}\Big(|X_0| > \frac{\sigma\sqrt{n}}{\sqrt{a_n}}\Big) \to -\infty \tag{1.10}$$

then, the conclusion of Theorem 3 holds with $W_n(t) = n^{-1/2}\sum_{j=1}^{[nt]} X_j$. Moreover, condition (1.10) is necessary for the moderate deviation principle in this case. This result for i.i.d. is contained in Arcones (Theorem 2.4, 2003-a).

For the sake of applications we give a sufficient condition in terms of the moments of $X_{n,i}$.

Proposition 1.9. *Assume $(X_{n1},\ X_{n2}, ..., X_{nk_n})$ is a triangular array of independent centered and square integrable random variables. Assume that there exists n_0 such that for each $n \geq n_0$ and $1 \leq k \leq k_n$ there are nonnegative numbers A_{nk} and B_n such that for each $m \geq 3$*

$$\mathbf{E}|X_{n,k}|^m \leq m! A_{nk}^m B_n \,. \tag{1.11}$$

Assume in addition that $a_n \to 0$,

$$A_{n,k} = o(\sqrt{a_n} s_n) \text{ as } n \to \infty \text{ uniformly in } k \tag{1.12}$$

and there is a positive constant C such that

$$\frac{B_n}{s_n^2}\sum_{j=1}^{k_n} |A_{nj}|^2 \leq C \text{ for all } n \geq n_0 \,. \tag{1.13}$$

Then the conclusion of Theorem 1.3 holds.

2. Applications

2.1. *A class of Linear processes.* In this section, we consider a sequence $\{\xi_k\}_{k\in\mathbf{Z}}$ of i.i.d. and centered random variables such that $\mathbf{E}(\xi_0)^2 = \sigma^2 > 0$ and let $\{c_{ni}, 1 \leq i \leq k_n\}$ be a triangular array of numbers. Many statistical procedures produce estimators of the type

$$S_n = \sum_{i=1}^{k_n} c_{ni}\xi_i \,. \tag{2.1}$$

For instance, consider the fixed design regression problem $Z_k = \theta q_k + \xi_k$, where the fixed design points are of the form $q_k = 1/g(k/n)$ where $g(.)$ is a function. To analyze the error of the estimator $\hat{\theta} = n^{-1}\sum_{k=1}^n Z_k g(k/n)$, we are led to study the behavior of processes of the form (2.1).
Setting

$$s_n^2 = \sigma^2 \sum_{i=1}^{k_n} c_{ni}^2 \,, \tag{2.2}$$

we are interested to give sufficient conditions for the moderate deviation principle for $\dfrac{S_n}{s_n}$ and also for the stochastic process $W_n(\cdot)$ defined in Definition 1.2 with

$X_{n,i} = c_{ni}\xi_i$, $s_{ni}^2 = \sigma^2 \sum_{j=1}^{i} c_{nj}^2$. In order for the Lindeberg's condition (1.5) to be satisfied we shall impose the following condition

$$\frac{1}{\sqrt{a_n} s_n} \max_{1 \le j \le k_n} |c_{nj}| \to 0 \quad \text{as } n \to \infty . \tag{2.3}$$

By applying Corollary 1.7 we easily obtain the following result

Proposition 2.1. *Let S_n and s_n^2 be defined by (2.1) and (2.2). Assume that $a_n \to 0$, condition (2.3) holds and the regularity conditions $\boldsymbol{RC}$. Denote by $C_n = \sup_{n \le m \le c(n+1)} \sup_{1 \le i \le k_m} |c_{m,i}|$ and assume that the following condition holds*

$$a_n \log(k_n \mathbf{P}(|\xi_0| > s_n / C_n \sqrt{a_n})) \to -\infty \text{ as } n \to \infty . \tag{2.4}$$

Then $\{W_n(\cdot)\}$ satisfies the MDP in $D[0,1]$ with speed a_n and rate $I(\cdot)$ defined in Theorem 1.3.

Notice that the variable ξ_0 is not required to have moment generating functions. As a matter of fact, by using Proposition 1.9 we can easily derive.

Proposition 2.2. *Let S_n and s_n^2 be defined by (2.1) and (2.2). Assume that $a_n \to 0$ and condition (2.3) holds. Assume that for some positive constant K,*

$$\mathbf{E}(|\xi_0|^m) \le m! K^m \text{ for all } m \in \mathbf{N} . \tag{2.5}$$

Then $\{W_n(\cdot)\}$ satisfies the MDP in $D[0,1]$ with speed a_n and rate $I(\cdot)$ defined in Theorem 1.3.

To give a few examples, notice that if the double sequence $|c_{m,i}|_{m,i}$ is uniformly bounded by a constant, condition (2.3) is verified provided $\lim_{n\to\infty} a_n s_n^2 = \infty$. Moreover, if for each n fixed, the sequence $\{|c_{nj}|\}_{j \ge 1}$ is increasing and satisfies the regularity assumption $\sum_{i=1}^{n} c_{ni}^2 \sim n c_{n,n}^2$, then condition (2.3) is satisfied if $n a_n \to \infty$. This is the case for instance when $c_{ni}^2 = c_i^2 = h(i)$ with $h(x)$ a slowly varying increasing function.

Of course, if smaller classes of random variables $(\xi_k)_{k \in \mathbf{Z}}$ are considered, such as bounded or sub-gaussian variables, a requirement weaker then (2.3) may guaranty MDP. We give here an example showing that condition (2.3) of Proposition 2.2 is necessary when the random variables $(\xi_k)_{k \in \mathbf{Z}}$ satisfy only a condition of type (2.5).

Assume $(X_i, i \in \mathbf{Z})$ is a sequence of independent identically distributed random variables with exponential law with mean 1, ($\mathbf{P}(X_0 > x) = e^{-x}$), denote $\xi_n = X_n - 1$ and assume the sequence of constants has the property $\max_{1 \le j \le n} |c_{nj}| = 1$. Notice first that, for any $t > 0$, we have that

$$\mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n} |\xi_0| \ge t\Big) \le \mathbf{P}\Big(\max_{1 \le i \le n} \frac{\sqrt{a_n}}{s_n} |c_{ni}\xi_i| \ge t\Big) .$$

Also, by standard symmetrization arguments and Levy's inequality (see for instance Proposition 2.3 in Ledoux and Talagrand (1991)), we get that for any $t > 0$ and n large enough (such that $a_n \le t^2/8$),

$$\mathbf{P}\Big(\max_{1 \le i \le n} \frac{\sqrt{a_n}}{s_n} |c_{ni}\xi_i| \ge t\Big) \le 2\Big(1 - \frac{4a_n}{t^2}\Big)^{-1} \mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n} |S_n| \ge t/2\Big) .$$

Now if $\{s_n^{-1} S_n\}$ satisfies the MDP, then the previous inequalities entail that necessarily

$$\limsup_{n\to\infty} a_n \log \mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n}(X_0 - 1) \ge t\Big) \le -\frac{t^2}{8} .$$

On an other hand for any $t>0$

$$a_n \log \mathbf{P}\Big(\frac{\sqrt{a_n}}{s_n}(X_0-1)\geq t\Big) = -a_n - t\sqrt{a_n}s_n\,.$$

In order for $\limsup_{n\to\infty}(-a_n - t\sqrt{a_n}s_n) \leq -\frac{t^2}{8}$ for all $t>0$ we see that necessarily $a_n s_n^2 \to \infty$, which implies that condition (2.3) is satisfied since $\max_{1\leq j\leq n}|c_{nj}|=1$.

Proof of Proposition 2.2. We shall apply Proposition 1.9 with $X_{n,k}=c_{nk}\xi_k$. Notice that for all positive integers m,

$$\mathbf{E}\big(|X_{nk}|^m\big) \leq |c_{nk}|^m \mathbf{E}\big(|\xi_0|^m\big)\,.$$

Whence, by using (2.5), we get for all positive integers m,

$$\mathbf{E}\big(|X_{nk}|^m\big) \leq m!|c_{nk}|^m K^m\,.$$

Then, the conditions of Proposition 1.9 are satisfied and the result follows.

2.2. *Kernel Estimators of the density.* In this section we apply our results to obtain a simple MDP in its functional form for the Kernel estimator at a fix point. Different and further pointing problems related to the moderate deviation principle for kernel estimators of the density or of the regression function were addressed in several papers. For instance, for kernel density estimator, Louani (1998) addresses the problem of large deviations, Gao (2003) studies the MDP uniformly in x, while Mokkadem, Pelletier and Worms (2005) give the large and moderate deviation principles for partial derivatives of a multivariate density. Concerning the kernel estimators of the multivariate regression, Mokkadem, Pelletier and Thiam (2007) study their large and moderate deviation principles.

Let $X=(X_k, k\in\mathbf{Z})$ be a sequence of i.i.d. random variables. We now impose the following conditions:

(A.1) The density function of X_0 is bounded and continuous at a fixed point x.

(A.2) The kernel K is a function such that $\int_{\mathbf{R}} K(x)dx=1$ and there exists a positive constant C such that for all positive integers m

$$\int_{\mathbf{R}} |K(u)|^m du \leq m!C^m\,.$$

This requirement on the kernel is weaker than the exponential moment condition imposed by Gao (2003, Relation (1.6)).

For each real number x, each positive integer n and each $t\in[0,1]$, let us define

$$f_{[nt]}(x) = \frac{1}{nh_n}\sum_{k=1}^{[nt]} K\Big(\frac{x-X_k}{h_n}\Big)\,,$$

where for all $n\geq 1$, h_n is a strictly positive real number. Obviously when $t=1$, this is the usual kernel-type estimator of f. In this section we are interested in the moderate deviation principle for the following processes considered as elements of $D([0,1])$. For fixed real number x, each positive integer n and each $t\in[0,1]$, let us define

$$U_n(t) := \sqrt{nh_n}(f_{[nt]}(x) - \mathbf{E}(f_{[nt]}(x)))\,.$$

Proposition 2.3. *Suppose (A.1) and (A.2) hold. Then, assuming that $a_n\to 0$ and $a_n nh_n\to\infty$, the processes $U_n(.)$ satisfy (1.2) with the good rate function $I_f(\cdot)=(f(x)\int K^2(u)du)^{-1}I(\cdot)$ where $I(\cdot)$ is defined by (1.6).*

Proof of Proposition 2.3 Let us define

$$Y_{n,k}(x)=\frac{1}{\sqrt{h_n}}\Big(K\Big(\frac{x-X_k}{h_n}\Big)-\mathbf{E}K\Big(\frac{x-X_k}{h_n}\Big)\Big)\,.$$

Then

$$\frac{\sum_{k=1}^{[nt]}Y_{k,n}(x)}{\sqrt{n}}=\sqrt{nh_n}(f_{[nt]}(x)-\mathbf{E}(f_{[nt]}(x)))\,.$$

By stationarity, for any $1\le j\le n$, $\sum_{k=1}^{j}\mathrm{Var}(Y_{k,n}(x))=j\mathrm{Var}(Y_{1,n}(x))$. Hence the conclusion follows provided the triangular array of independent centered random variables $\{Y_{n,k}(x)\}$ satisfies the conditions of Proposition 1.9. Notice that for each $m\ge 1$, our conditions imply

$$\begin{aligned}\mathbf{E}|Y_{n,k}(x)|^m &\le 2^m\Big(\frac{1}{h_n}\Big)^{m/2}\mathbf{E}\Big|K\Big(\frac{x-X_k}{h_n}\Big)\Big|^m\\ &\le m!2^m h_n\|f\|_\infty\Big(\frac{1}{h_n}\Big)^{m/2}C^m\,.\end{aligned}$$

Setting $A_{n,k}=2Ch_n^{-1/2}$ and $B_n=h_n\|f\|_\infty$, the assumptions of Proposition 1.9 hold since $na_nh_n\to\infty$. It follows that

$$\Big\{\frac{\sum_{k=1}^{[nt]}Y_{k,n}(x)}{\sqrt{n}\sqrt{\mathrm{Var}(Y_{n,1})}},t\in[0,1]\Big\}\text{ satisfies the MDP.}$$

The proof ends by noticing that the dominated convergence theorem ensures that

$$\mathrm{Var}(Y_{n,1})\to f(x)\int_{\mathbf{R}}K^2(u)du\,.$$

2.3. *Application to a class of dependent variables.* In the recent years MDP was obtained for classes of dependent random variables by using various martingale approximation techniques. For example, papers by Gao (1996), Djellout (2002), Dedecker, Merlevède, Peligrad and Utev (2007), used this approach to obtain MDP for classes of ϕ-mixing sequences with polynomial rates. In this section we treat other classes of mixing sequences by another method: approximating the sums of variables in blocks with triangular array of independent random variables and applying then our Theorem 1.3 to prove the MDP. The measure of dependence, called τ, that we shall use in this section has been introduced by Dedecker and Prieur (2004) and it can easily be computed in many situations such as causal Bernoulli shifts, functions of strong mixing sequences, iterated random functions and so on. We refer to papers by Dedecker and Prieur (2004) or Dedecker and Merlevède (2006) for precise estimation of τ for these examples. Since the rate of convergence in the next corollary is geometric, we would like also to mention that the result in this section can be also applied to ARCH models whose coefficients a_j are zero for large enough $j\ge J$, since these models are geometrically τ-dependent (see Proposition 5.1 in Comte, Dedecker and Taupin (2007)).

Let us now introduce the dependence coefficients used in what follows.

For any real random variable X in $\mathbf{L}^1$ and any σ-algebra $\mathcal{M}$ of $\mathcal{A}$, let $\mathbf{P}_{X|\mathcal{M}}$ be a conditional distribution of X given $\mathcal{M}$ and let $\mathbf{P}_X$ be the distribution of X. We consider the coefficient $\tau(\mathcal{M}, X)$ of weak dependence (Dedecker and Prieur, 2004) which is defined by

$$\tau(\mathcal{M}, X) = \Big\| \sup_{f \in \Lambda_1(\mathbf{R})} \Big| \int f(x)\mathbf{P}_{X|\mathcal{M}}(dx) - \int f(x)\mathbf{P}_X(dx) \Big| \Big\|_1 , \tag{2.6}$$

where $\Lambda_1(\mathbf{R})$ is the set of 1-Lipschitz functions from $\mathbf{R}$ to $\mathbf{R}$.

The τ-coefficient has the following coupling property: If Ω is rich enough then the coefficient $\tau(\mathcal{M}, X)$ is the infimum of $\|X - Y\|_1$ where Y is independent of $\mathcal{M}$ and distributed as X (see Lemma 5 in Dedecker and Prieur (2004)). This coupling property allows to relate the τ-coefficient with the strong mixing coefficient Rosenblatt (1956) defined by

$$\alpha(\mathcal{M}, \sigma(X)) = \sup_{A \in \mathcal{M}, B \in \sigma(X)} |\mathbf{P}(A \cap B) - \mathbf{P}(A)P(B)| \ .$$

as shown in Rio (2000), page 161 (see Peligrad, 2002 for the unbounded case). In case when X is bounded, we have

$$\tau(\mathcal{M}, X) \le 4\|X\|_\infty \alpha(\mathcal{M}, \sigma(X)) \, .$$

For equivalent definitions of the strong mixing coefficient we refer for instance to Bradley (2007, Lemma 4.3 and Theorem 4.4).

If Y is a random variable with values in $\mathbf{R}^k$, the coupling coefficient τ is defined as follows: If $Y \in \mathbf{L}^1(\mathbf{R}^k)$,

$$\tau(\mathcal{M}, Y) = \sup\{\tau(\mathcal{M}, f(Y)), f \in \Lambda_1(\mathbf{R}^k)\} \, , \tag{2.7}$$

where $\Lambda_1(\mathbf{R}^k)$ is the set of 1-Lipschitz functions from $\mathbf{R}^k$ to $\mathbf{R}$.

We can now define the coefficient τ for a sequence $(X_i)_{i \in \mathbf{Z}}$ of real valued random variables.

For a strictly sequence $(X_i)_{i \in \mathbf{Z}}$ of real-valued random variables and for any positive integer i, define

$$\tau(i) = \sup_{k \ge 0} \max_{1 \le \ell \le k} \frac{1}{\ell} \sup \Big\{ \tau(\mathcal{M}_0, (X_{j_1}, \cdots, X_{j_\ell})), \, i \le j_1 < \cdots < j_\ell \Big\} , \tag{2.8}$$

where $\mathcal{M}_0 = \sigma(X_j, j \le 0)$ and supremum also extends for all $i \le j_1 < \cdots < j_\ell$.

On an other hand, the sequence of strong mixing coefficients $(\alpha(i))_{i>0}$ is defined by:

$$\alpha(i) = \alpha(\mathcal{M}_0, \sigma(X_j, j \ge i)) \, .$$

In the case where the variables are bounded the following bound is valid

$$\tau(i) \le 4\|X_0\|_\infty \alpha(i) \tag{2.9}$$

(see Dedecker and Prieur, 2004, Lemma 7).

In the next proposition, we consider a strictly stationary sequence whose τ-dependence coefficients are geometrically decreasing.

Proposition 2.4. *Let $(X_i)_{i \in \mathbf{Z}}$ be a strictly stationary sequence of centered random variables such that $\|X_0\|_\infty < \infty$. Let $S_n = \sum_{i=1}^n X_i$ and $\sigma_n^2 = \mathrm{Var}(S_n)$. Let $(\tau(n))_{n \ge 1}$ be the sequence of dependence coefficients of $(X_i)_{i \in \mathbf{Z}}$ defined by (2.8). Assume that $\sigma_n^2 \to \infty$ and that there exists $\rho \in]0, 1[$ such that $\tau(n) \le \rho^n$. Then, for all positive sequences a_n with $a_n \to 0$ and $n a_n^2 \to \infty$, the normalized partial sums processes $\{\sigma_n^{-1} \sum_{i=1}^{[nt]} X_i, t \in [0, 1]\}$ satisfy (1.2) with the good rate function given in Theorem 1.3.*

Remark 2.5. *Taking into account the bound (2.9), Proposition 2.4 directly applies to strongly mixing sequences of bounded random variables with geometric mixing rate* ($\alpha(n) \leq \rho^n$).

Remark 2.6. *Notice that, since the variables are centered and bounded, we get that* $|\mathrm{Cov}(X_0, X_k)| \leq \|X_0\|_\infty \|\mathbf{E}(X_k|\mathcal{M}_0)\|_1$. *Now from the definition of the* τ*-dependence coefficient we clearly have that* $\|\mathbf{E}(X_k|\mathcal{M}_0)\|_1 \leq \tau(k)$. *It follows that the condition on the sequence of coefficients* $\tau(n)$ *implies* $\sum_k k|\mathrm{Cov}(X_0, X_k)| < \infty$. *This condition together with the fact that* $\sigma_n^2 \to \infty$ *entail that* $n^{-1}\mathrm{Var}(S_n)$ *converges to a finite number* $\sigma^2 > 0$ *(see Bradley, 2007, Lemma 1).*

Proof of Proposition 2.4. Let $\varepsilon_n^2 \to 0$ in such a way that $\varepsilon_n^2 n a_n^2 \to \infty$ (this is possible because $na_n^2 \to \infty$) and $\varepsilon_n^2 n a_n / \log(a_n n) \to \infty$. Take $p_n = \varepsilon_n n a_n$ and $q_n := \varepsilon_n^2 n a_n$.

We now divide the variables $\{X_i\}$ in big blocks of size p_n and small blocks of size q_n in the following way : Let us set $k_n = [n(p_n + q_n)^{-1}]$. For a given positive integer n, the set $1, 2, \cdots, n$ is being partitioned into blocks of consecutive integers, the blocks being $I_1, J_1, ..., I_{k_n}, J_{k_n}$, such that for each $1 \leq j \leq k_n$, I_j contains p_n integers and J_j contains q_n integers.

Denote by $Y_{j,n} := \sum_{i \in I_j} X_i$ and $Z_{j,n} := \sum_{i \in J_j} X_i$ for $1 \leq j \leq k_n$. Now we consider the following decomposition: for any $t \in [0, 1]$,

$$\sum_{i=1}^{[nt]} X_i = \sum_{j=1}^{[k_n t]} Y_{j,n} + \sum_{j=1}^{[k_n t]} Z_{j,n} + R_{n,t} \,, \tag{2.10}$$

where

$$R_{n,t} := \sum_{i=1}^{[nt]} X_i - \Big(\sum_{j=1}^{[k_n t]} Y_{j,n} + \sum_{j=1}^{[k_n t]} Z_{j,n} \Big) .$$

The idea of the proof is the following: Using Lemma 5 in Dedecker and Prieur (2004), we get the existence of independent random variables $(Y_{i,n}^*)_{1 \leq i \leq k_n}$ with the same distribution as the random variables $Y_{i,n}$ such that

$$\mathbf{E}|Y_{i,n} - Y_{i,n}^*| \leq p_n \tau(q_n) \,. \tag{2.11}$$

Then we show that the partial sums processes $\{\sigma_n^{-1} \sum_{j=1}^{[k_n t]} Y_{j,n}^*, t \in [0, 1]\}$ satisfy (1.2) with the good rate function given in Theorem 1.3, while the remainder is negligible in the sense of moderate deviations, i.e for all $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \Big(\mathbf{P} \big(\sup_{0 \leq t \leq 1} \frac{\sqrt{a_n}}{\sigma_n} \big| \sum_{i=1}^{[nt]} X_i - \sum_{j=1}^{[k_n t]} Y_{j,n}^* \big| \geq \eta \big) \Big) = -\infty \,. \tag{2.12}$$

By stationarity, $\mathrm{Var}(Y_{j,n}^*) = \sigma_{p_n}^2$ for any $1 \leq j \leq k_n$ and that, by Remark 2.6, $k_n \sigma_{p_n}^2 / \sigma_n^2 \to 1$. Also for any $k \in [1, k_n]$, $\sum_{j=1}^{k} \mathrm{Var}(Y_{j,n}^*)/(k_n \sigma_{p_n}^2) = k/k_n$. Hence, by taking into account these considerations, we shall verify the conditions of Theorem 1.3 for the variables $\{Y_{j,n}^*\}_{1 \leq j \leq k_n}$. According to Comment 1.6 and using stationarity, it suffices to verify that there is a constant C_1 with the property that any $\epsilon > 0$ and any $\beta > 0$ there is $N(\epsilon, \beta)$ such that for $n > N(\epsilon, \beta)$

$$a_n k_n \mathbf{P}(|S_{p_n}| > u\sqrt{a_n}\sigma_n) \leq C_1 \exp(-\beta u) \text{ for any } u \geq \epsilon \,. \tag{2.13}$$

Applying Lemma 4.2 in the Appendix, we derive that there exist two positive constants C_1 and C_2 depending only on $\|X_0\|_\infty$ and ρ such that

$$a_n k_n \mathbf{P}(|S_{p_n}| > u\sqrt{a_n}\sigma_n) \le C_1 \frac{a_n n}{p_n} \exp(-C_2 \frac{u\sqrt{a_n}\sigma_n}{\sqrt{p_n}}) .$$

Since $\sigma_n^2/n \to \sigma^2 > 0$ and by the selection of p_n we have that $p_n = o(na_n)$ which proves (2.13) and we conclude that the process $\{\sigma_n^{-1} \sum_{j=1}^{[k_n t]} Y_{j,n}^*, t \in [0,1]\}$ satisfies the conclusion of Theorem 1.3.

Hence it remains to show (2.12). We shall decompose the proof of this negligibility in several steps.

Using again Lemma 5 in Dedecker and Prieur (2004), there are independent random variables $(Z_{i,n}^*)_{1\le i\le k_n}$ with the same distribution as the random variables $Z_{i,n}$ such that $\mathbf{E}|Z_{i,n} - Z_{i,n}^*| \le q_n \tau(p_n)$. By the same arguments as for the sequence $\{Y_{j,n}^*\}_{1\le j\le k_n}$, we get that the Donsker process $\{(k_n \sigma_{q_n}^2)^{-1/2} \sum_{j=1}^{[k_n t]} Z_{j,n}^*, t \in [0,1]\}$ satisfies (1.2) with the good rate function given in Theorem 1.3. Now since $k_n \sigma_{q_n}^2/\sigma_n^2 \sim q_n/p_n$ converges to zero as $n \to \infty$ we easily deduce that for all $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \Big(\mathbf{P}\big(\sup_{0\le t\le 1} \frac{\sqrt{a_n}}{\sigma_n} \big|\sum_{j=1}^{[k_n t]} Z_{j,n}^*\big| \ge \eta\big)\Big) = -\infty .$$

Consequently, to prove (2.12), it remains to prove that for all $\eta > 0$,

$$\limsup_{n\to\infty} a_n \log \Big(\mathbf{P}\big(\sup_{0\le t\le 1} \frac{\sqrt{a_n}}{\sigma_n} \big|\sum_{j=1}^{[k_n t]} (Y_{j,n} - Y_{j,n}^* + Z_{j,n} - Z_{j,n}^*)\big| \ge \eta\big)\Big) = -\infty , \quad (2.14)$$

and

$$\limsup_{n\to\infty} a_n \log \Big(\mathbf{P}\big(\sup_{0\le t\le 1} \frac{\sqrt{a_n}|R_{n,t}|}{\sigma_n} \ge \eta\big)\Big) = -\infty . \quad (2.15)$$

By using Markov inequality, we clearly have that

$$\mathbf{P}\left(\sup_{0\le t\le 1} \frac{\sqrt{a_n}}{\sigma_n} \big|\sum_{j=1}^{[k_n t]} (Y_{j,n} - Y_{j,n}^* + Z_{j,n} - Z_{j,n}^*)\big| \ge \eta\right) \le \frac{\sqrt{a_n}}{\eta\sigma_n} k_n \left(\mathbf{E}|Y_{1,n} - Y_{1,n}^*| + \mathbf{E}|Z_{1,n} - Z_{1,n}^*|\right)$$
$$\le \frac{2}{\eta} \frac{n\sqrt{a_n}}{\sigma_n} \tau(q_n) \le \frac{2}{\eta} \frac{n\sqrt{a_n}}{\sigma_n} e^{-q_n \log(1/\rho)} ,$$

which proves (2.14) by using the selection of ε_n and q_n and the fact that $\sigma_n^2/n \to \sigma^2 > 0$.

Since for any $t \in [0,1]$, $R_{n,t}$ contains at most $2(p_n + q_n)$ terms, by stationarity we have

$$\mathbf{P}\big(\sup_{0\le t\le 1} \frac{\sqrt{a_n}|R_{n,t}|}{\sigma_n} \ge \eta\big) \le (k_n + 1)\mathbf{P}\big(\max_{1\le j\le 2(p_n+q_n)} \frac{\sqrt{a_n}|\sum_{i=1}^j X_i|}{\sigma_n} \ge \eta\big) .$$

Applying Lemma 4.2 in the Appendix, we derive that there exist positive constants C_1 and C_2 depending only on $\|X_0\|_\infty$ and ρ such that

$$\mathbf{P}\big(\max_{1\le j\le 2(p_n+q_n)} \frac{\sqrt{a_n}|\sum_{i=1}^j X_i|}{\sigma_n} \ge \eta\big) \le C_1 \exp\Big(- C_2 \frac{\eta\sigma_n}{\sqrt{a_n}\sqrt{2(p_n + q_n)}}\Big) .$$

It follows that

$$a_n \log \Big(\mathbf{P}\big(\sup_{0\le t\le 1} \frac{\sqrt{a_n}|R_{n,t}|}{\sigma_n} \ge \eta\big)\Big) \le a_n \log(k_n+1) + a_n \log(C_1) - C_2 \frac{\eta\sqrt{a_n}\sigma_n}{\sqrt{2(p_n+q_n)}} .$$

Since $\sigma_n^2/n \to \sigma^2 > 0$ and $p_n = o(na_n)$, we get that $\sqrt{a_n}\sigma_n/\sqrt{p_n+q_n} \to \infty$. In addition $k_n \sim a_n^{-1}$ implying that $a_n \log(k_n+1) \to 0$. Hence (2.15) is proved which completes the proof of (2.12) and then of the proposition.

3. Proofs

3.1. ***Proof of Theorem 1.3***. To prove this theorem we shall use a truncation argument. Without restricting the generality we shall assume in this proof $s_n^2 = \sum_{j=1}^{k_n} \mathbf{E}(X_{nj})^2 = 1$. This is possible by dividing all variables by s_n^2 and redenoting them also by X_{nj}. We truncate the variables in the following way: For all $1 \le j \le k_n$, let

$$X_{nj}^{'} := X_{nj} I(|X_{nj}| \le \sqrt{a_n}) - \mathbf{E}(X_{nj} I(|X_{nj}| \le \sqrt{a_n}) ,$$

$$\begin{aligned} X_{nj}^{"} &:= X_{nj} I(\sqrt{a_n} < |X_{nj}| \le 1/\sqrt{a_n}) - \mathbf{E}(X_{nj} I(\sqrt{a_n} < |X_{nj}| \le 1/\sqrt{a_n})) \\ &:= \bar{X}_{nj} I(|\bar{X}_{nj}| > \sqrt{a_n}) - \mathbf{E}(\bar{X}_{nj} I(|\bar{X}_{nj}| > \sqrt{a_n}))) , \end{aligned}$$

and

$$X_{nj}^{'''} := X_{nj} I(|X_{nj}| > 1/\sqrt{a_n}) - \mathbf{E}(X_{nj} I(|X_{nj}| > 1/\sqrt{a_n})) .$$

Above we used also the notation: $\bar{X}_{nj} = X_{nj} I(|X_{nj}| \le 1/\sqrt{a_n})$. Notice first that, since $s_n^2 = 1$

$$\sqrt{a_n} \sum_{j=1}^{k_n} \mathbf{E}(|X_{nj}| I(|X_{nj}| > 1/\sqrt{a_n})) \le a_n \to 0 ,$$

and that for any $\delta > 0$,

$$a_n \log \mathbf{P}\big(\sum_{j=1}^{k_n} |X_{nj}| I(|X_{nj}| > 1/\sqrt{a_n}) \ge \delta\big) \le a_n \log \mathbf{P}\big(\max_{1\le j\le k_n} |X_{nj}| > 1/\sqrt{a_n}\big) .$$

Hence, by taking into account condition (1.4), the variables $X_{nj}^{'''}$ have a negligible contribution to the MDP (see Dembo and Zeitouni, 1998, Theorem 4.2.13). Consequently, without restricting the generality we have just to consider the sums $S_{nl}^{'} = \sum_{j=1}^{l} X_{nj}^{'}$ and $S_{nl}^{"} = \sum_{j=1}^{l} X_{nj}^{"}$. We denote by $W_n^{'}(t)$ (respectively by $W_n^{"}(t)$) the random function on $[0,1]$ that is linear on each interval $[s_{n,i-1}^2,\ s_{ni}^2]$ and has the values $W_n^{'}(s_{ni}^2) = S_{ni}^{'}$ (respectively $W_n^{"}(s_{ni}^2) = S_{ni}^{"}$) at the points of division. Then

$$W_n(t) \approx W_n^{'}(t) + W_n^{"}(t) .$$

We first show that the sequence $W_n^{"}(t)$ is also negligible, that is for any $\delta > 0$,

$$\lim_{n\to\infty} a_n \log \Big(\mathbf{P}\big(\sup_{0\le t\le 1} \sqrt{a_n}|W_n^{"}(t)| \ge \delta\big)\Big) = -\infty . \tag{3.1}$$

Notice that, $\sqrt{a_n} \sum_{k=1}^{k_n} |\mathbf{E}(\bar{X}_{nk} I(|\bar{X}_{nk}| > \sqrt{a_n})|$ converges to zero as a consequence of condition (1.5). Hence we have to establish for any $\delta > 0$

$$\lim_{n\to\infty} a_n \log \Big(\mathbf{P}\Big(\sqrt{a_n} \sum_{k=1}^{k_n} \big|\bar{X}_{nk} I(|\bar{X}_{nk}| > \sqrt{a_n})\big| \ge \delta\Big)\Big) = -\infty .$$

Clearly, for any $\lambda > 0$

$$a_n \log\Big(\mathbf{P}\Big(\sqrt{a_n}\sum_{k=1}^{k_n}\big|\bar{X}_{nk} I(|\bar{X}_{nk}| > \sqrt{a_n})\big| \geq \delta\Big)\Big) \leq -\lambda\delta +$$

$$a_n \sum_{k=1}^{k_n} \log \mathbf{E}\big(\exp(\frac{\lambda|\bar{X}_{nk}|}{\sqrt{a_n}} I(|\bar{X}_{nk}| > \sqrt{a_n})\big)\,,$$

which shows that it is enough to prove that there is a positive constant C such that for each $\lambda > 0$

$$\limsup_{n\to\infty} a_n \sum_{k=1}^{\ell_n(j)} \log \mathbf{E}\big(\exp(\frac{\lambda|\bar{X}_{nk}|}{\sqrt{a_n}} I(|\bar{X}_{nk}| > \sqrt{a_n})\big) \leq C\,.$$

Since $e^{xI(A)} - 1 = (e^x - 1)I(A)$ and also $\log(1 + x) \leq x$ the above inequality is implied by

$$\limsup_{n\to\infty} a_n \sum_{k=1}^{k_n} \mathbf{E}\big([\exp(\frac{\lambda|\bar{X}_{nk}|}{\sqrt{a_n}}) - 1] I(|\bar{X}_{nk}| > \sqrt{a_n})\big) \leq C\,,$$

which is a consequence of condition (1.3).

In order to prove that the sequence $W_n^{'}(t)$ satisfies the moderate deviation principle, according to Theorem 3.2. in Arcones (2003a), it is enough to show that, for a fixed integer m, and each $0 = t_0 \leq t_1 \leq \cdots \leq t_m \leq 1$,

$$\begin{aligned}&\big(W_n'(t_1), \cdots, W_n'(t_m)\big) \text{ satisfies the MDP in } \mathbf{R}^m \text{ with speed } a_n \text{ and the}\\ &\text{good rate function } I_m(u_1, \cdots, u_m) = \sum_{\ell=1}^{m} \frac{1}{2}\frac{(u_\ell - u_{\ell-1})^2}{(t_\ell - t_{\ell-1})} \text{ with } u_0 = 0,\end{aligned} \tag{3.2}$$

and for each $\delta > 0$

$$\lim_{\eta\to 0}\limsup_{n\to\infty} a_n \log\Big(\mathbf{P}\Big\{\sup_{|s-t|\leq\eta, 0\leq s,t\leq 1} \sqrt{a_n}|W_n'(t) - W_n'(s)| \geq \delta\Big\}\Big) = -\infty\,. \tag{3.3}$$

By the contraction principle (see Dembo and Zeitouni, 1998, Theorem 4.2.1) to prove the convergence of the finite dimensional distributions we have to show that $Y_n^{'} := \big(W_n^{'}(t_1), W_n^{'}(t_2) - W_n^{'}(t_1), \cdots, W_n^{'}(t_m) - W_n^{'}(t_{m-1})\big)$ satisfies the MDP in $\mathbf{R}^m$ with speed a_n and the good rate function given by

$$I_m'(u_1, \cdots, u_m) = \sum_{\ell=1}^{m} \frac{1}{2}\frac{u_\ell^2}{(t_\ell - t_{\ell-1})}\,. \tag{3.4}$$

According to Theorem II.2 in Ellis (1984) and independence we have to verify for each $j, 1 \leq j \leq m$,

$$\lim_{n\to\infty} a_n \log\Big(\mathbf{E}\Big\{\exp\big(\frac{1}{\sqrt{a_n}}\lambda_j(W_n^{'}(t_j) - W_n^{'}(t_{j-1}))\Big\}\Big) = \frac{1}{2}\lambda_j^2(t_j - t_{j-1})\,. \tag{3.5}$$

Notice that

$$W_n^{'}(t_j) - W_n^{'}(t_{j-1}) = \sum_{k=\ell_n(t_{j-1})+1}^{\ell_n(t_j)} X_{nk}^{'}\,, \tag{3.6}$$

with $\ell_n(t_j)$ the maximum k for which $s_{nk}^2 \leq t_j$ (this difference is understood to be 0 if $\ell_n(t_{j-1}) = \ell_n(t_j)$)). We shall verify the conditions of Lemma 2.3 in Arcones

(2003c), given for convenience in Appendix, to the real valued random sum of independent random variables: $Y_{n1} = X'_{n,\ell_n(t_{j-1})+1}, \ldots, Y_{nk_n} = X'_{n,\ell_n(t_j)}$. Since the random variables X'_{nk} are uniformly bounded by $\sqrt{a_n}$, condition (4.1) holds. Now the Lindeberg's condition (1.5) clearly implies (4.2). Hence it remains to verify that for any $1 \leq j \leq m$,

$$\lim_{n\to\infty} \sum_{k=\ell_n(t_{j-1})+1}^{\ell_n(t_j)} \mathbf{E}(X'_{nk})^2 = (t_j - t_{j-1})\,. \tag{3.7}$$

By condition (1.5), (3.7) holds provided that

$$\lim_{n\to\infty} \sum_{k=\ell_n(t_{j-1})+1}^{\ell_n(t_j)} \mathbf{E}(X_{nk})^2 = (t_j - t_{j-1})\,. \tag{3.8}$$

For n sufficiently large $\ell_n(t_{j-1}) \neq \ell_n(t_j)$ and $\sum_{k=\ell_n(t_{j-1})+1}^{\ell_n(t_j)} \mathbf{E}(X_{nk})^2 = s^2_{n,\ell_n(j)} - s^2_{n,\ell_n(j-1)}$. Also, condition (1.5) implies that for all k, $s^2_{nk} \to 0$, therefore

$$\lim_{n\to\infty} s^2_{n,\ell_n(t_j)} = t_j\,. \tag{3.9}$$

Hence (3.8) holds, and so does (3.7). This ends the proof of (3.5) and of (3.2).

To prove (3.3), we notice that by Theorem 7.4 in Billingsley (1999), for each $\delta > 0$,

$$\mathbf{P}\left(\sup_{d(s,t)\leq\eta} \sqrt{a_n}\left|W'_n(t) - W'_n(s)\right| \geq 3\delta\right) \leq \sum_{i=1}^{m} \mathbf{P}\left(\sup_{\frac{i-1}{m}\leq s<\frac{i}{m}} \sqrt{a_n}\left|W'_n(s) - W'_n(\frac{i-1}{m})\right| \geq \delta\right),$$

where $m = [\eta^{-1}]$. In terms of partial sums and above notation, we get

$$\mathbf{P}\left(\sup_{\frac{i-1}{m}\leq s<\frac{i}{m}} \sqrt{a_n}\left|W'_n(s) - W'_n\left(\frac{i-1}{m}\right)\right| \geq \delta\right) \leq \mathbf{P}\left(\max_{\ell(\frac{i-1}{m})\leq k\leq \ell(\frac{i}{m})+1} \sqrt{a_n}\left|\sum_{j=\ell(\frac{i-1}{m})}^{k} X'_{nj}\right| \geq \delta\right).$$

By Lindeberg's condition (1.5) and (3.9) and with the notation

$$B^2_{i,m} = \sum_{j=\ell(\frac{i-1}{m})+1}^{\ell(\frac{i}{m})} \mathbf{E}(X'_{nj})^2$$

we have

$$\lim_{n\to\infty} B^2_{i,m} = \lim_{n\to\infty} \sum_{j=\ell(\frac{i-1}{m})}^{\ell(\frac{i}{m})} B^2_{i,m} = \frac{1}{m}. \tag{3.10}$$

This limit along with Kolmogorov maximal inequality (2.13) in Petrov (1995) and the fact that $a_n \to 0$ as $n \to \infty$ gives

$$\mathbf{P}\big(\max_{\ell(\frac{i-1}{m})\leq k\leq \ell(\frac{i}{m})+1} \sqrt{a_n}|\sum_{j=\ell(\frac{i-1}{m})}^{k} X'_{nj}| \geq \delta\big) \leq 2\mathbf{P}(\sqrt{a_n}|\sum_{j=\ell(\frac{i-1}{m})}^{\ell(\frac{i}{m})+1} X'_{nj}| \geq 2^{-1}\delta)$$

for all n sufficiently large. Now we apply Prokhorov's inequality (see Lemma 4.3 in the Appendix) with $B = 2\sqrt{a_n}$ and $t = \frac{1}{2}\delta a_n^{-1/2}$ to obtain

$$\mathbf{P}(\sqrt{a_n}|\sum_{j=\ell(\frac{i-1}{m})}^{\ell(\frac{i}{m})+1} X'_{nj}| \geq 2^{-1}\delta) \leq \exp(-\frac{\delta}{4a_n}\ \operatorname{arcsinh}\frac{\delta}{2B^2_{i,m}})\,.$$

Therefore by (3.10),

$$\limsup_{n\to\infty} a_n \log \mathbf{P}(\sqrt{a_n}|\sum_{j=\ell(\frac{i-1}{m})}^{\ell(\frac{i}{m})+1} X'_{nj}| \geq 2^{-1}\delta) \leq -\frac{\delta}{4}\ \operatorname{arcsinh}\frac{m\delta}{2}\,,$$

which converges to $-\infty$ when $m \to \infty$. This convergence implies (3.3).

3.2. *Proof of Corollary 1.7.* We just have to prove that Condition (1.9) implies Condition (1.3) (condition (1.4) being obviously satisfied). The proof is straightforward but delicate and it is inspired by the type of arguments developed by Arcones (2003c, Theorem 2.4) and Djellout (2002). So, according to Comment 1.6, we shall verify that condition (1.8) holds for all $1 \leq u \leq 1/a_n$.

Recall that $f(x)$ and $g(x)$ are strictly increasing continuous functions and for any positive integer m, $f(m) = a_m s_m^2$ and $g(m) = s_m^2/a_m$. Fix an integer n_0 and for any $n \geq n_0$ and $1 \leq u \leq 1/a_n$ define $N = N(u,n)$ as $N = g^{-1}(u^2 f(n))$. So $n = f^{-1}(u^{-2}g(N))$. Notice that N might not be an integer. Obviously, by monotonicity

$$g^{-1}(\ f(n)) \leq N \leq g^{-1}(\ f(n)/a_n^2) = n$$

and

$$N \leq n \leq f^{-1}(g(N))\,.$$

Notice that by the above relations and the assumption $a_n s_n^2 \to \infty$, $N(n) \to \infty$ as $n \to \infty$ uniformly in $u \geq 1$.

Now, by the definitions of $f(x)$, $g(x)$, and N, for any $1 \leq i \leq k_n$ we have

$$P(|X_{n,i}| \geq u\sqrt{a_n}s_n) \leq \sup_{N\leq n\leq f^{-1}(g(N))}\ \sup_{1\leq i\leq k_n} P(|X_{n,i}| \geq s_{[N]}/\sqrt{a_{[N]}})\,.$$

Whence, by condition (1.9) for any $\lambda > 0$, and $n \geq n_0(\lambda)$

$$k_{[N]}P(|X_{n,i}| \geq u\sqrt{a_n}s_n) \leq \exp(-\lambda/a_{[N]})\,.$$

So, for $u \geq 1$ and $n \geq n_0(\lambda)$ and taking into account that both $s_n^2 a_n$ and s_n^2/k_n are nondecreasing along with condition (1.5), we easily derive the sequence of inequalities

$$\begin{aligned}
&a_n \sum_{i=1}^{k_n} P(|X_{n,i}| \geq u\sqrt{a_n}s_n) \leq a_n k_n \frac{1}{k_{[N]}} \exp(-\lambda/a_{[N]})\\
&= \frac{f(n)}{s_n^2}\frac{k_n}{k_{[N]}} \exp(-\lambda/a_{[N]}) \leq \frac{g(N)}{s_n^2}\frac{k_n}{k_{[N]}} \exp(-\lambda/a_{[N]})\\
&\leq \frac{s^2_{[N]+1}}{s_n^2 a_{[N]+1}}\frac{k_n}{k_{[N]}} \exp(-\lambda/a_{[N]}) \leq 2\exp((-\lambda+1)/a_{[N]})\,.
\end{aligned}$$

Hence, it remains to compare u with $1/a_{[N]}$. Recall that $u^2 = g(N)/f(n)$ and by monotonicity of $g(x)$ and $f(n)$ and condition (1.5), we have

$$u \leq \frac{1}{s_n\sqrt{a_n}} \frac{s_{[N]+1}}{\sqrt{a_{[N]+1}}} \leq \frac{1}{s_n\sqrt{a_n}} \frac{s^2_{[N]+1}}{\sqrt{s^2_{[N]}a_{[N]}}} \leq 2\frac{1}{a_{[N]}} \frac{s_{[N]}\sqrt{a_{[N]}}}{s_n\sqrt{a_n}} .$$

Since $[N] \leq n$, we get $u \leq 2/a_{[N]}$ and therefore condition (1.8) holds for all $1 \leq u \leq 1/a_n$.

3.3. *Proof of Proposition 1.9.* We just have to verify that condition (1.7) holds. By Maclaurin expansion and condition (1.11) we get that for $n \geq n_0$

$$a_n \sum_{j=1}^{k_n} \mathbf{E}([\exp \frac{\beta|X_{nj}|}{\sqrt{a_n}s_n}] I(\frac{|X_{nj}|}{\sqrt{a_n}s_n} > \epsilon) \leq \frac{1}{\epsilon^3\sqrt{a_n}s_n^3} \sum_{j=1}^{k_n}\sum_{p=0}^{\infty} \frac{1}{p!} E\Big(\frac{\beta^p|X_{nj}|^{3+p}}{(\sqrt{a_n}s_n)^p}\Big)$$

$$\leq \frac{24B_n}{\epsilon^3\sqrt{a_n}s_n^3} \sum_{j=1}^{k_n}\sum_{p=0}^{\infty} p^3|A_{nj}|^3\Big(\beta\frac{|A_{nj}|}{\sqrt{a_n}s_n}\Big)^p \leq \frac{24}{\beta\epsilon^3}\frac{B_n}{s_n^2}\sum_{j=1}^{k_n}|A_{nj}|^2\sum_{p=0}^{\infty}\Big(\frac{8\beta|A_{nj}|}{\sqrt{a_n}s_n}\Big)^{p+1}$$

which converges to 0 by (1.12) together with (1.13).

4. Appendix

We first state lemma 2.3 in Arcones (2003c).

Lemma 4.1. *Assume that* $(Y_{n1}, Y_{n2}, \ldots, Y_{nk_n})$ *is a triangular array of independent random variables, with mean zero and such that*

$$\lim_{n\to\infty} \mathbf{E}(\sum_{j=1}^{k_n} Y_{nj}^2) = \sigma^2.$$

Let $(a_n)_{n\geq 1}$ *be a sequence of real numbers converging to zero. Assume there is a constant* τ *such that*

$$\sup_{1\leq i\leq k_n} |Y_{nj}| \leq \tau\sqrt{a_n} \ a.s. \tag{4.1}$$

and for each $\delta > 0$

$$a_n \sum_{j=1}^{k_n} \mathbf{P}(|Y_{nj}| \geq \delta\sqrt{a_n}) \to 0 \ \text{as } n \to \infty\,. \tag{4.2}$$

Then, for any $t \in \mathbf{R}$,

$$a_n \log \mathbf{E} \exp\Big(t\frac{\sum_{j=1}^{k_n} Y_{nj}}{\sqrt{a_n}}\Big) \to \frac{t^2\sigma^2}{2} \ \text{as } n \to \infty$$

and therefore $\{\sum_{j=1}^{k_n} Y_{nj}\}$ *satisfies the MDP in with speed* a_n *and rate function* $I(t) = \frac{t^2}{2\sigma^2}$.

Now we give the following consequence of Corollary 3 in Dedecker and Doukhan (2003).

Lemma 4.2. *Let $(X_i)_{i\in\mathbf{Z}}$ be a strictly stationary sequence of centered real random variables such that $\|X_0\|_\infty < \infty$. Let $(\tau(n))_{n\geq 1}$ be the sequence of dependence coefficients of $(X_i)_{i\in\mathbf{Z}}$ defined by (2.8). Assume that there exist $\rho \in]0,1[$ such that $\tau(n) \leq \rho^n$. Let $S_k = \sum_{i=1}^k X_i$. Then there exist constants C_1 and C_2 depending only on ρ and $\|X_0\|_\infty$ such that the following inequality holds for any integer $m \geq 1$:*

$$\mathbf{P}(\max_{1\leq j\leq m} |S_j| > x) \leq C_1 \exp(-C_2 x/\sqrt{m}) \,.$$

Proof of Lemma 4.2 First we notice that by the definition of the $\tau-$dependence coefficient

$$\gamma(n) = \|\mathbf{E}(X_n|\mathcal{M}_0)\|_1 \leq \tau(n) \ \leq \rho^n.$$

By stationarity and applying Corollary 3 in Dedecker and Doukhan (2003), we get that for any $1 \leq i \leq j \leq m$, there exists a constant K depending only on ρ and $\|X_0\|_\infty$ such that

$$\mathbf{P}\big(|\sum_{\ell=i}^{j} X_\ell| > x\big) \leq K \exp\Big(\frac{-x\sqrt{\log(1/\rho)}}{e\|X_0\|_\infty\sqrt{j-i+1}}\Big)\,.$$

Hence the lemma follows by taking into account Theorem 2.2 in Móricz, Serfling and Stout (1982) together with the remark (ii) stated page 1033 in their paper.

Now we recall the Prokhorov's inequality (1959) that we used in the paper.

Lemma 4.3. *Assume that we have an independent random vector (not necessarily Stationary) $(X_1,\ X_2, ...X_m)$, centered such that*

$$\max_{1\leq i\leq m} |X_i| \leq B \ \ a.s.$$

Denote by $s_n^2 = \sum_{j=1}^n E(X_j^2)$. Then for all $t > 0$, the following inequality holds

$$\mathbf{P}(|\sum_{j=1}^{n} X_j| \geq t) \leq \exp\big(-\frac{t}{2B}\ \ arcsinh\frac{Bt}{2s_n^2}\big)\,.$$

We turn now to the proof of the Comment 1.6.

Proof of Comment 1.6.

Denote $\bar{X}_{nj} = X_{nj} I[|X_{nj}| < s_n/\sqrt{a_n}]$. We show that (1.3) is equivalent to the following condition: There is a constant C_1 with the property: for any $\beta > 0$ there is $N(\beta)$ such that for $n > N(\beta)$

$$a_n \sum_{j=1}^{k_n} \mathbf{P}(|\bar{X}_{nj}| > u\sqrt{a_n}s_n) \leq C_1 \exp(-\beta u) \text{ for all } u \geq 1\,, \tag{4.3}$$

which is equivalent to (1.8).

For any $u \geq 1$

$$a_n \sum_{j=1}^{k_n} \mathbf{P}(|\bar{X}_{nj}| > u\sqrt{a_n}s_n) \leq a_n \sum_{j=1}^{k_n} \exp(-\beta u)\mathbf{E}(\exp(\beta\frac{|\bar{X}_{nj}|}{\sqrt{a_n}s_n})I(|\bar{X}_{nj}| > u\sqrt{a_n}s_n)\,.$$

Hence (1.3) implies (4.3). On the other hand,

$$\mathbf{E}\left(\left[\exp\frac{\beta|\bar{X}_{nj}|}{2\sqrt{a_n}s_n}\right]I(|\bar{X}_{nj}|>\sqrt{a_n}s_n)\right)$$
$$=e^{\beta/2}\mathbf{P}(|\bar{X}_{nj}|>\sqrt{a_n}s_n)+\frac{\beta}{2}\int_1^\infty e^{\beta u/2}\mathbf{P}(|\bar{X}_{nj}|>u\sqrt{a_n}s_n)du$$

Now if (4.3) holds then for n sufficiently large,

$$a_n\sum_{j=1}^{k_n}\mathbf{E}\left(\left[\exp\beta\frac{|\bar{X}_{nj}|}{2\sqrt{a_n}s_n}\right]I(|\bar{X}_{nj}|>\sqrt{a_n}s_n)\right)$$
$$\leq C_1\left(e^{-\beta/2}+\frac{\beta}{2}\int_1^\infty e^{-\beta u/2}du\right)\leq 2C_1e^{-\beta/2}\,,$$

proving that (1.3) is satisfied.

References

M.A. Arcones. The large deviation principle for stochastic processes I. *Theory of Probability and its Applications* **47**, 567–583 (2003a).

M.A. Arcones. The large deviation principle for stochastic processes II. *Theory of Probability and its Applications* **48**, 19–44 (2003b).

M.A. Arcones. Moderate deviations of empirical processes. In *Stochastic inequalities and applications*, volume 56 of *Progr. Probab.*, pages 189–212. Birkhäuser, Basel (2003c).

P. Billingsley. *Convergence of Probability Measures.* Wiley, New York (1999).

R.C. Bradley. *Introduction to strong mixing conditions*, volume 1,2,3. Kendrick Press (2007).

F. Comte, J. Dedecker and M.L. Taupin. Adaptive density estimation for general ARCH models (2007).
http://www.math-info.univ-paris5.fr/∼comte/publi.html.

J. Dedecker and P. Doukhan. A new covariance inequality and applications. *Stoch. Processes Appl.* **106**, 63–80 (2003).

J. Dedecker and F. Merlevède. Inequalities for partial sums of Hilbert-valued dependent sequences and applications. *Math. Methods Statist.* **15**, 176–206 (2006).

J. Dedecker, F. Merlevède, M. Peligrad and S. Utev. *Moderate deviations for stationary sequences of bounded random variables.* Prépublication 1183. LPMA. Université Paris 6 (2007).

J. Dedecker and C. Prieur. Coupling for τ-dependent sequences and applications. *J. Theoret. Probab.* **17**, 861–885 (2004).

A. Dembo and O. Zeitouni. *Large Deviations Techniques and Applications.* Springer, New York, 2nd edition (1998).

H. Djellout. Moderate deviations for Martingale Differences and applications to $\phi-$mixing sequences. *Stoch. Stoch. Rep.* **73** (1-2), 37–63 (2002).

R. S. Ellis. Large deviations for a general class of random vectors. *Ann. Probab.* **12**, 1–12 (1984).

F-Q. Gao. Moderate deviations for martingales and mixing random processes. *Stochastic Process. Appl.* **61**, 263–275 (1996).

F-Q. Gao. Moderate deviations and large deviations for kernel density estimators. *J. Theoret. Probab.* **16**, 401–418 (2003).

M. Ledoux. Sur les déviations modérées de sommes de variables aléatoires vectorielles indépendantes de même loi. *Ann. Inst. Henri Poincaré* **28**, 267–280 (1992).

M. Ledoux and M. Talagrand. *Probability in Banach spaces.* Springer-Verlag, Berlin (1991).

D. Louani. Large deviations limit theorems for kernel density estimator. *Scand.J. Statist.* **25**, 243–253 (1998).

A. Mokkadem, M. Pelletier and B. Thiam. Large and moderate deviations principles for kernel estimators of a multivariate regression (2007). *hal-00136115. http://arxiv.org/abs/math/0703341.*

A. Mokkadem, M. Pelletier and J. Worms. Large and moderate deviations principles for kernel estimators of a multivariate density and its partial derivatives. *Aust.N.Z.J.Stat.* **47**, 489–502 (2005).

F.A. Móricz, R.J. Serfling and W.F. Stout. Moment and probability bounds with quasi-superadditive structure for the maximum partial sum. *Ann. Probab.* **10**, 1032–1040 (1982).

M. Peligrad. Some remarks on coupling of dependent random variables. *Stat. and Prob. Lett.* **60**, 201–209 (2002).

V. Petrov. *Limit theorems in probability theory.* Oxford Studies in Probability Series. Clarendon press, Oxford (1995).

Yu. V. Prokhorov. An extremal problem in probability theory. *Theory of Probability and its Applications* **4**, 201–203 (1959).

A. Puhalskii. Large deviations of semimartingales via convergence of the predictable characteristics. *Stoch. Stoch. Rep.* **49**, 27–85 (1994).

E. Rio. Théorie asymptotique des processus aléatoires faiblement dépendants. In *Mathématiques & Application*, volume 31. Springer-Verlag, Berlin Heidelberg (2000).

M. Rosenblatt. A central limit theorem and a strong mixing condition. *Proc. Nat. Acad. Sci. U. S. A.* **42**, 43–47 (1956).