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Fluctuation bounds for the asymmetric simple exclusion process

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Abstract. We give a partly new proof of the fluctuation bounds for the second class particle and current in the stationary asymmetric simple exclusion process. One novelty is a coupling that preserves the ordering of second class particles in two systems that are themselves ordered coordinatewise.

1. Introduction

The asymmetric simple exclusion process (ASEP) is a Markov process that describes the motion of particles on the one-dimensional integer lattice $\mathbb{Z}$. Each particle executes a continuous-time nearest-neighbor random walk on $\mathbb{Z}$ with jump rate p to the right and q to the left. Particles interact through the exclusion rule which means that at most one particle is allowed at each site. Any attempt to jump onto an already occupied site is ignored. The asymmetric case is $p \neq q$. We assume $0 \leq q < p \leq 1$ and also $p + q = 1$.

In this paper we give a partially new proof of the fluctuation bounds for ASEP first proved by Balázs and Seppäläinen (2007, 2008). For a more thorough explanation of the context and related work we refer the reader to these earlier papers. The present paper has two novelties.

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(i) Both the earlier and the present approach are based on identities that link together the current, space-time covariances, and the second class particle. Balázs and Seppäläinen (2007) proved these identities with martingale techniques and generator computations. Section 2 of the present paper shows that these identities are consequences of not much more than simple counting of particles. These arguments are elementary and should work very generally. Without much effort we extend the identities to more general exclusion dynamics that allow bounded jumps and rates that depend on the local configuration. However, product invariant distributions remains a key assumption.

(ii) Sophisticated couplings introduced by Ferrari et al. (1991) and some complicated estimation were used in Balázs and Seppäläinen (2008) to bound the positions of certain second class particles. In Section 3 of the present paper we introduce a coupling for ASEP that keeps the second class particle of a denser system behind the second class particle of a comparison system. Since the macroscopic speed of a second class particle is $H'(\rho)$, the slope of the flux at the given density ρ, concavity of the flux suggests that second class particles travel slower in a denser system. Since this is what the coupling achieves, we think of this coupling as a form of *microscopic concavity.*

Fix two parameters $0 \le q < p \le 1$ such that $p + q = 1$. We run quickly through the fundamentals of (p, q)-ASEP. We refer the reader to standard references (Liggett, 1985, 1999) for further details.

Definition and graphical construction. ASEP represents the motion of particles on the integer lattice $\mathbb{Z}$. The particles are subject to an exclusion rule which means that each site of $\mathbb{Z}$ can contain at most one particle. The state of the system at time t is a configuration $\eta(t) = (\eta_i(t))_{i\in\mathbb{Z}} \in \{0, 1\}^{\mathbb{Z}}$ of zeroes and ones. The value $\eta_i(t) = 1$ means that site i is occupied by a particle at time t, while the value $\eta_i(t) = 0$ means that site i is vacant at time t.

The motion of the particles is controlled by independent Poisson processes (*Poisson clocks*) $\{N^{i\to i+1}, N^{i\to i-1} : i \in \mathbb{Z}\}$. Each Poisson clock $N^{i\to i+1}$ has rate p and each $N^{i\to i-1}$ has rate q. If t is a jump time for $N^{i\to i+1}$ and if $(\eta_i(t-), \eta_{i+1}(t-)) = (1, 0)$ then at time t the particle from site i moves to site $i+1$ and the new values are $(\eta_i(t), \eta_{i+1}(t)) = (0, 1)$. Similarly if t is a jump time for $N^{i\to i-1}$ a particle is moved from i to $i-1$ at time t, provided the configuration at time $t-$ permits this move. If the jump prompted by a Poisson clock is not permitted by the state of the system, this jump attempt is simply ignored and the particles resume waiting for the next prompt coming from the Poisson clocks.

This construction of the process is known as the *graphical construction* or the *Harris construction*. When the initial state is a fixed configuration η, P^η denotes the distribution of the process.

We write η, ω, etc for elements of the state space $\{0, 1\}^{\mathbb{Z}}$, but also for the entire process so that η-process stands for $\{\eta_i(t) : i \in \mathbb{Z}, 0 \le t < \infty\}$. The configuration δ_i is the state that has a single particle at position i but otherwise the lattice is vacant.

Invariant distributions. A basic fact is that i.i.d. Bernoulli distributions $\{\nu^\rho\}_{\rho\in[0,1]}$ are extremal invariant distributions for ASEP. For each density value $\rho \in [0, 1]$, ν^ρ is the probability measure on $\{0, 1\}^{\mathbb{Z}}$ under which the occupation variables $\{\eta_i\}$ are i.i.d. with common mean $\int \eta_i \, d\nu^\rho = \rho$. When the process η is stationary with time-marginal ν^ρ, we write P^ρ for the probability distribution of the entire process. The *stationary density-ρ process* means the ASEP η that is stationary in time and has marginal distribution $\eta(t) \sim \nu^\rho$.

Basic coupling and second class particles. The *basic coupling* of two exclusion processes η and ω means that they obey a common set of Poisson clocks $\{N^{i\to i+1}, N^{i\to i-1}\}$. Suppose the two processes η and η^+ satisfy $\eta^+(0) = \eta(0) + \delta_{Q(0)}$ at time zero, for some position $Q(0) \in \mathbb{Z}$. This means that $\eta_i^+(0) = \eta_i(0)$ for all $i \neq Q(0)$, $\eta^+_{Q(0)}(0) = 1$ and $\eta_{Q(0)}(0) = 0$. Then throughout the evolution in the basic coupling there is a single discrepancy between $\eta(t)$ and $\eta^+(t)$ at some position $Q(t)$: $\eta^+(t) = \eta(t) + \delta_{Q(t)}$. From the perspective of $\eta(t)$, $Q(t)$ is called a second class particle. By the same token, from the perspective of $\eta^+(t)$ $Q(t)$ is a second class *anti*particle. In particular, we shall call the pair (η, Q) a (p,q)-*ASEP with a second class particle*.

We write a boldface $\mathbf{P}$ for the probability measure when more than one process are coupled together. In particular, $\mathbf{P}^\rho$ represents the situation where the initial occupation variables $\eta_i(0) = \eta_i^+(0)$ are i.i.d. mean-ρ Bernoulli for $i \neq 0$, and the second class particle Q starts at $Q(0) = 0$.

More generally, if two processes η and ω are in basic coupling and $\omega(0) \geq \eta(0)$ (by which we mean coordinatewise ordering $\omega_i(0) \geq \eta_i(0)$ for all i) then the ordering $\omega(t) \geq \eta(t)$ holds for all $0 \leq t < \infty$. The effect of the basic coupling is to give priority to the η particles over the $\omega - \eta$ particles. Consequently we can think of the ω-process as consisting of first class particles (the η particles) and second class particles (the $\omega - \eta$ particles).

Current. For $x \in \mathbb{Z}$ and $t > 0$, $J_x(t)$ stands for the net left-to-right particle current across the straight-line space-time path from $(1/2, 0)$ to $(x + 1/2, t)$. More precisely, $J_x(t) = J_x(t)^+ - J_x(t)^-$ where $J_x(t)^+$ is the number of particles that lie in $(-\infty, 0]$ at time 0 but lie in $[x+1, \infty)$ at time t, while $J_x(t)^-$ is the number of particles that lie in $[1, \infty)$ at time 0 and in $(-\infty, x]$ at time t. When more than one process (ω, η, etc) is considered in a coupling, the currents of the processes are denoted by $J_x^\omega(t)$, $J_x^\eta(t)$, etc.

Flux and characteristic speed. The average net rate at which particles in the stationary (p,q)-ASEP at density ρ move across a fixed edge $(i, i+1)$ is the *flux*

$$H(\rho) = (p-q)\rho(1-\rho). \tag{1.1}$$

The characteristic speed at density ρ is

$$V^\rho = H'(\rho) = (p-q)(1-2\rho). \tag{1.2}$$

In the stationary process the expected current is

$$E^\rho[J_x(t)] = tH(\rho) - x\rho \tag{1.3}$$

as can be seen by noting that particles that crossed the edge $(0,1)$ either also crossed $(x, x+1)$ and contributed to $J_x(t)$ or did not. Another important and less obvious expectation is

$$\mathbf{E}^\rho[Q(t)] = tH'(\rho) \tag{1.4}$$

for the second class particle that starts at the origin. We derive (1.4) in Section 2.

We can now state the main result, the moment bounds on the second class particle.

Theorem 1.1. *Consider (p,q)-ASEP with rates such that $0 \le q = 1-p < p \le 1$.*

(Upper bound) *For densities $\rho \in (0,1)$ there exists a constant $0 < C_1(\rho) < \infty$ such that for $1 \le m < 3$ and $t \ge 1$*

$$\mathbf{E}^\rho\big\{\,|Q(t) - V^\rho t|^m\,\big\} \;\le\; \frac{C_1(\rho)}{3-m} t^{2m/3}. \tag{1.5}$$

$C_1(\rho)$ is a continuous function of ρ.

(Lower bound) *For densities $\rho \in (0,1)$ there exist constants $0 < t_0(\rho), C_2(\rho) < \infty$ such that for $t \ge t_0(\rho)$ and all $m \ge 1$*

$$\mathbf{E}^\rho\big\{\,|Q(t) - V^\rho t|^m\,\big\} \;\ge\; C_2(\rho) t^{2m/3}. \tag{1.6}$$

$C_2(\rho)$ and $t_0(\rho)$ are continuous functions of $\rho \in (0,1)$.

The constants C_1, t_0, C_2 depend also on p but since p is regarded as fixed we omit this dependence from the notation. The boundary dependence on ρ is as follows: as $\rho \to \{0,1\}$, $C_1(\rho) \to \infty$, $C_2(\rho) \to 0$ and $t_0(\rho) \to \infty$. The convergence $C_2(\rho) \to 0$ as $\rho \to \{0,1\}$ is natural because for $\rho \in \{0,1\}$ $Q(t)$ is a random walk. The divergences of $C_1(\rho)$ and $t_0(\rho)$ appear to be artifacts of our proof.

For the third moment and beyond our upper bound argument gives these bounds: for $0 < \rho < 1$

$$\mathbf{E}^\rho\big\{\,|Q(t) - V^\rho t|^m\,\big\} \le \begin{cases} B(\rho,3)t^2\log t & \text{for } m = 3 \text{ and } t \ge e, \\ B(\rho,m)t^{m-1} & \text{for } 3 < m < \infty \text{ and } t \ge 1. \end{cases} \tag{1.7}$$

The constant $B(\rho, m)$ is finite and continuous in ρ for each fixed $3 \le m < \infty$.

A key identity proved in the next section states that

$$\mathrm{Var}^\rho[J_{\lfloor V^\rho t\rfloor}(t)] = \rho(1-\rho)\mathbf{E}^\rho|Q(t) - \lfloor V^\rho t\rfloor|.$$

From this and Theorem 1.1 we get the order of the variance of the current as seen by an observer traveling at the characteristic speed V^ρ: for large enough t,

$$C_1(\rho)t^{2/3} \le \mathrm{Var}^\rho[J_{\lfloor V^\rho t\rfloor}(t)] \le C_2(\rho)t^{2/3}.$$

In the case of TASEP (*totally* asymmetric simple exclusion process with $p = 1 = 1-q$) Ferrari and Spohn (2006) proved a distributional limit for the current $J_{\lfloor V^\rho t\rfloor}(t)$. At the time of this writing this precise distributional limit has not yet been proved for ASEP. If the observer chooses a speed $V \ne V^\rho$ then $\mathrm{Var}^\rho[J_{\lfloor Vt\rfloor}(t)]$ is of order t and in fact $J_{\lfloor Vt\rfloor}(t)$ satisfies a central limit theorem that records Gaussian fluctuations of the initial particle configuration (Ferrari and Fontes (1994)).

Organization of the paper. Section 2 proves the identities that connect the current and the second class particle. In Section 3 we develop the coupling that keeps two second class particles in different densities ordered. Section 4 contains a tail bound on a biased random walk in an inhomogeneous environment that is needed for the last section. Section 5 proves Theorem 1.1.

Further notation. $\mathbb{Z}_+ = \{0,1,2,\dots\}$ and $\mathbb{N} = \{1,2,3,\dots\}$. Constants denoted by C or C_i $(i = 1,2,3,\dots)$ can change from line to line. Centering of a random variable is denoted by $\widetilde{X} = X - EX$. Shift on $\{0,1\}^{\mathbb{Z}}$ is denoted by $(\theta_i\omega)_j = \omega_{i+j}$.

2. Covariance identities

In this section we can consider more general exclusion processes with bounded jumps and rates that depend on the configuration around the jump location.

Let R be a constant that bounds the range of admissible jumps. For $1 \le k \le R$ let p_k and q_k be functions of particle configurations that depend only on coordinates $(\omega_i : -R \le i \le R)$ and satisfy $0 \le p_k(\omega), q_k(\omega) \le 1$. The rule for the evolution is that for all $i \in \mathbb{Z}$ and all $k \in \{1, \dots, R\}$, independently of everything else, the exchange $\omega \mapsto \omega^{i,i+k}$ happens at rate

$$p_k(\theta_i\omega)\omega_i(1-\omega_{i+k}) + q_k(\theta_{i+k}\omega)\omega_{i+k}(1-\omega_i).$$

The configuration $\omega^{i,i+k}$ is the result of exchanging the contents of sites i and $i+k$:

$$\omega_j^{i,i+k} = \begin{cases} \omega_{i+k}, & \text{if } j = i \\ \omega_i, & \text{if } j = i+k \\ \omega_j, & \text{if } j \notin \{i, i+k\}. \end{cases}$$

The rule of evolution can be restated as follows: whenever possible a particle jumps from i to $i+k$ at rate $p_k(\theta_i\omega)$ and from i to $i-k$ at rate $q_k(\theta_i\omega)$.

Key assumption. Bernoulli distributions $\{\nu^\rho\}_{\rho\in[0,1]}$ are invariant for the process. As throughout the paper, P^ρ, E^ρ, Var^ρ and Cov^ρ refer to the density-ρ invariant process.

We state formulas that tie together the current, space-time covariance (also called "two-point function"), and the second class particle. We begin with the well-known formula that connects the two-point function with the second class particle.

Let $\mathbf{P}^\rho$ be the probability distribution of two coupled processes $\omega \le \omega^+$ that start with identical Bernoulli-ρ occupation variables $\omega_i(0) = \omega_i^+(0)$ at $i \ne 0$ and a single discrepancy Q that starts at the origin. In other words $Q(0) = 0$ and $\omega^+(t) = \omega(t) + \delta_{Q(t)}$. Then for $0 < \rho < 1$

$$\mathrm{Cov}^\rho[\omega_j(t),\, \omega_0(0)] = \rho(1-\rho)\mathbf{P}^\rho\{Q(t) = j\}. \tag{2.1}$$

Note that the two sides of the identity come from different processes: the left-hand side is a covariance in a stationary process, while the right-hand side is in terms of processes perturbed at the origin at time 0. For the sake of completeness we give below a proof of (2.1).

Theorem 2.1. *For any density $0 \le \rho \le 1$, $z \in \mathbb{Z}$ and $t > 0$ we have these formulas:*

$$\mathrm{Var}^\rho[J_z(t)] = \sum_{j\in\mathbb{Z}} |j-z| \,\mathrm{Cov}^\rho[\omega_j(t), \omega_0(0)] = \rho(1-\rho)\mathbf{E}^\rho|Q(t) - z| \tag{2.2}$$

and

$$\frac{d}{d\rho} E^\rho[J_z(t)] = \mathbf{E}^\rho[Q(t)] - z. \tag{2.3}$$

The terms in the series in (2.2) decay exponentially in $|j|$, uniformly over ρ. All members of these identities are continuous functions of $\rho \in [0,1]$. At $\rho = 0$ and $\rho = 1$ the left-hand side of (2.3) is a one-sided derivative.

Some comments and consequences follow.

At $\rho = 0$ the second class particle $Q(t)$ is a random walk that takes jumps of size k at rate $p_k(\delta_0)$ and jumps of size $-k$ at rate $q_k(\delta_0)$. At the other extreme, $\rho = 1$, $Q(t)$ is a random walk that takes jumps of size k at rate $q_k(\mathbf{1} - \delta_{-k})$ and jumps of size $-k$ at rate $p_k(\mathbf{1}-\delta_k)$. Here $\mathbf{1}$ denotes the configuration $\omega \equiv 1$. In particular, in (p, q)-ASEP, as ρ goes from 0 to 1, Q interpolates between nearest-neighbor random walks with rates (p, q) and (q, p).

The equilibrium current past the origin satisfies

$$E^\rho[J_0(t)] = tH(\rho) \tag{2.4}$$

where the flux $H(\rho)$ is the expected rate of particle motion across any fixed edge in the stationary density-ρ process:

$$H(\rho) = \sum_{k=1}^{R} kE^\rho(p_k - q_k).$$

Combining (2.3) for $z = 0$ and (2.4) gives the useful identity

$$\mathbf{E}^\rho[Q(t)] = tH'(\rho). \tag{2.5}$$

We turn to the proofs, beginning with (2.1).

Proof of equation (2.1)*:* This is a straight-forward calculation.

$$\begin{aligned}
&\mathrm{Cov}^\rho[\omega_j(t), \omega_0(0)] = E^\rho[\omega_j(t)\omega_0(0)] - \rho^2 = \rho E^\rho[\omega_j(t)\,|\,\omega_0(0) = 1] - \rho^2 \\
&= \rho\Big(E^\rho[\omega_j(t)\,|\,\omega_0(0) = 1] - \rho E^\rho[\omega_j(t)\,|\,\omega_0(0) = 1] - (1-\rho)E^\rho[\omega_j(t)\,|\,\omega_0(0) = 0]\Big) \\
&= \rho(1-\rho)\Big(E^\rho[\omega_j(t)\,|\,\omega_0(0) = 1] - E^\rho[\omega_j(t)\,|\,\omega_0(0) = 0]\Big) \\
&= \rho(1-\rho)\Big(\mathbf{P}^\rho[\omega_j^+(t) = 1] - \mathbf{P}^\rho[\omega_j(t) = 1]\Big) = \rho(1-\rho)\mathbf{P}^\rho[\omega_j^+(t) = 1\,,\, \omega_j(t) = 0] \\
&= \rho(1-\rho)\mathbf{P}^\rho[Q(t) = j].
\end{aligned}$$

□

The remainder of this section proves Theorem 2.1. The second equality in (2.2) comes from (2.1).

Let ω be a stationary exclusion process satisfying the assumptions made in this section, with i.i.d. Bernoulli(ρ) distributed occupations $\{\omega_i(t)\}$ at any fixed time t. To approximate the infinite system with finite systems, for each $N \in \mathbb{N}$ let process ω^N have initial configuration

$$\omega_i^N(0) = \omega_i(0)\mathbf{1}_{\{-N \le i \le N\}}. \tag{2.6}$$

We assume that all these processes are coupled through a Harris-type construction, with jump attempts prompted by Poisson clocks, with appropriate rates, attached to directed edges (i, j) for $|j - i| \le R$. Let $J_z^N(t)$ denote the current in process ω^N.

Let $z(0) = 0$, $z(t) = z$, and introduce the counting variables

$$I_+^N(t) = \sum_{n > z(t)} \omega_n^N(t)\,, \quad I_-^N(t) = \sum_{n \le z(t)} \omega_n^N(t). \tag{2.7}$$

Then the current can be expressed as

$$J_z^N(t) = I_+^N(t) - I_+^N(0) = I_-^N(0) - I_-^N(t),$$

and its variance as

$$\begin{aligned}
\operatorname{Var} J_z^N(t) &= \operatorname{Cov}\bigl(I_+^N(t) - I_+^N(0),\, I_-^N(0) - I_-^N(t)\bigr) \\
&= \operatorname{Cov}\bigl(I_+^N(t), I_-^N(0)\bigr) + \operatorname{Cov}(I_+^N(0), I_-^N(t)) \\
&\qquad - \operatorname{Cov}(I_+^N(0), I_-^N(0)) - \operatorname{Cov}(I_+^N(t), I_-^N(t)) \\
&= \sum_{k\le 0,\, m>z} \operatorname{Cov}[\omega_m^N(t), \omega_k^N(0)] + \sum_{k\le z,\, m>0} \operatorname{Cov}[\omega_k^N(t), \omega_m^N(0)] \\
&\qquad\qquad - \operatorname{Cov}(I_+^N(0), I_-^N(0)) - \operatorname{Cov}(I_+^N(t), I_-^N(t)).
\end{aligned}$$

Independence gives

$$\operatorname{Cov}(I_+^N(0), I_-^N(0)) = 0$$

and the identity above simplifies to

$$\begin{aligned}
\operatorname{Var} J_z^N(t) = &\sum_{k\le 0,\, m>z} \operatorname{Cov}[\omega_m^N(t), \omega_k^N(0)] \\
&+ \sum_{k\le z,\, m>0} \operatorname{Cov}[\omega_k^N(t), \omega_m^N(0)] - \operatorname{Cov}(I_+^N(t), I_-^N(t)).
\end{aligned} \tag{2.8}$$

We show that identity (2.8) converges to identity (2.2).

To take advantage of the decaying correlations that result from the bounded jump range, we introduce another family of auxiliary processes. Let $\eta^{a,b}$ and $\eta^{N,a,b}$ be exclusion processes defined from initial conditions

$$\eta_i^{a,b}(0) = \omega_i(0)\mathbf{1}_{\{a<i<b\}} \quad \text{and} \quad \eta_i^{N,a,b}(0) = \omega_i^N(0)\mathbf{1}_{\{a<i<b\}} \tag{2.9}$$

and with a "reduced" Poisson construction: all Poisson jumps that involve any site outside (a,b) are deleted. The point of this definition is that for disjoint intervals (a,b) and (u,v), processes $\eta^{a,b}$ and $\eta^{u,v}$ are independent because they do not share initial occupation variables or Poisson clocks.

We write $\mathbf{P}$ for the probability measure under which all these coupled processes $\{\omega, \omega^N,\ \eta^{a.b}, \eta^{N,a,b}\}$ live. The next lemma is valid for completely general initial occupations $\{\omega_i(0)\}$.

Lemma 2.2. *Let $\{\omega_i(0)\}$ be an arbitrary deterministic or random initial configuration, and define initial configurations $\omega^N(0)$, $\eta^{a,b}(0)$ and $\eta^{N,a,b}(0)$ by (2.6) and (2.9). For a fixed $0<t<\infty$ there is a constant $C(t)<\infty$ such that, for all indices N, m, k, and all times $s\in[0,t]$*

$$\mathbf{E}|\omega_k^N(s) - \eta_k^{N,k-m,k+m}(s)| \le e^{-C(t)m} \tag{2.10}$$

$$\textit{and} \quad \mathbf{E}|\omega_k(s) - \eta_k^{k-m,k+m}(s)| \le e^{-C(t)m}. \tag{2.11}$$

Proof: We give the argument for a particular N and k. But the argument and the resulting bound have no dependence on N, k or the initial configurations. The only issue is the disconnectedness of the graph created by the Poisson clocks in the Harris construction during time interval $[0,t]$.

For any given integer $i>0$, there is a fixed positive probability that no site in the intervals $[k-i-R+1, k-i]$ and $[k+i, k+i+R-1]$ is involved in any Poisson jump during time interval $[0,t]$. Hence the probability that this event fails at each integer $i=3R\ell$ for $1\le\ell\le n$ is exponentially small in n. (For distinct ℓ the clocks involved are independent.)

If this event succeeds for $i = 3R\ell$, then $\omega_k^N(s) = \eta_k^{N,k-m,k+m}(s)$ for all $m \geq 3R\ell + R$ and $s \in [0,t]$ by the following reasoning. This m is large enough to ensure that the initial configurations $\omega^N(0)$ and $\eta^{N,k-m,k+m}(0)$ agree on $(k-i-R, k+i+R)$. The absence of Poisson jumps has frozen the configurations in $(k-i-R, k-i] \cup [k+i, k+i+R)$ up to time t. Also, up to time t no Poisson jump connects the process inside the interval $(k-i, k+i)$ to the outside. The rates of the jumps inside the interval $(k-i, k+i)$ are determined by the configuration in the larger interval $(k-i-R, k+i+R)$. Consequently inside $(k-i, k+i)$, up to time t, the processes ω^N and $\eta^{N,k-m,k+m}$ execute the same moves. □

Now combine the previous lemma with independent initial occupations.

Corollary 2.3. *Let $\{\omega_i(0)\}$ be i.i.d. Bernoulli(ρ) and $\omega^N(0)$ defined by (2.6). Fix $0 < t < \infty$. There exists a constant $C(t)$ such that for all indices N, i, k, all densities $0 \leq \rho \leq 1$ and times $s \in [0,t]$,*

$$\left| \mathbf{E}\widetilde{\omega}_i^N(s)\widetilde{\omega}_k^N(t) \right| \leq 4e^{-C(t)|k-i|} \tag{2.12}$$

and

$$\left| \operatorname{Cov}^\rho[\omega_i(s), \omega_k(t)] \right| \;=\; \left| \mathbf{E}\widetilde{\omega}_i(s)\widetilde{\omega}_k(t) \right| \leq 4e^{-C(t)|k-i|}. \tag{2.13}$$

For any function g on $\mathbb{Z}$ that grows at most polynomially,

$$[0,1] \ni \rho \mapsto \sum_k g(k) \operatorname{Cov}^\rho[\omega_k(t), \omega_0(0)] \tag{2.14}$$

is a continuous function.

Proof: Let $m = |i-k|$. Variables $\eta_i^{N,i-m/2,i+m/2}(s)$ and $\eta_{i+m}^{N,k-m/2,k+m/2}(t)$ are independent and so their covariance is zero. (2.12) follows from (2.10). Similarly for (2.13).

For fixed k, m the function

$$\rho \mapsto \operatorname{Cov}^\rho[\eta_k^{k-m,k+m}(t), \omega_0(0)] \tag{2.15}$$

is continuous because the expectation depends on ρ through the finitely many variables $\{\eta_i^{k-m,k+m}(0) : k-m < i < k+m, \omega_0(0)\}$. These can be coupled simultaneously for all values of ρ through i.i.d. uniform $U_i \sim \mathrm{Unif}(0,1)$ by writing

$$\eta_i^{k-m,k+m}(0) = \mathbf{1}\{0 < U_i < \rho\}. \tag{2.16}$$

Continuity in (2.15) follows by dominated convergence. Estimate (2.11) then shows the continuity of the individual terms in the series in (2.14). The continuity of the whole series follows from the ρ-uniformity of the tail bounds in (2.13). □

Once $N > |a| \vee |b|$, $\eta^{N,a,b}(t) = \eta^{a,b}(t)$. From this and Lemma 2.2 one concludes that

$$\omega_i^N(t) \to \omega_i(t) \quad \text{as } N \to \infty \text{, a.s. and in } L^2 \text{, for any } i. \tag{2.17}$$

From these estimations we derive this lemma:

Lemma 2.4. *The right-hand side of (2.8) converges as $N \to \infty$ to the middle member of (2.2).*

Proof: We begin by showing the convergence of the last term of (2.8):

$$\mathbf{Cov}(I_+^N(t), I_-^N(t)) \to 0. \tag{2.18}$$

This follows from writing, for a fixed $K > 0$,

$$\begin{aligned}
\mathbf{Cov}(I_+^N(t), I_-^N(t)) &= \sum_{k>z}\sum_{j\le z} \mathbf{Cov}\big(\omega_k^N(t),\, \omega_j^N(t)\big) \\
&= \sum_{0<k-z\le K}\ \sum_{-K\le j-z\le 0} \mathbf{Cov}\big(\omega_k^N(t),\, \omega_j^N(t)\big) \\
&\quad + \sum_{k>z}\sum_{j<z-K} \mathbf{E}\widetilde{\omega}_k^N(t)\widetilde{\omega}_j^N(t) + \sum_{k>z+K}\ \sum_{-K\le j-z\le 0} \mathbf{E}\widetilde{\omega}_k^N(t)\widetilde{\omega}_j^N(t) \\
&= \sum_{0<k-z\le K}\ \sum_{-K\le j-z\le 0} \mathbf{Cov}\big(\omega_k^N(t),\, \omega_j^N(t)\big) \\
&\quad + O\Big(\sum_{k>z, j<z-K} e^{-C(k-j)} + \sum_{k>z+K, -K\le j-z\le 0} e^{-C(k-j)} \Big) \\
&= \sum_{0<k-z\le K}\ \sum_{-K\le j-z\le 0} \mathbf{Cov}\big(\omega_k^N(t),\, \omega_j^N(t)\big) + O(e^{-CK}).
\end{aligned}$$

For a fixed K as $N \to \infty$

$$\begin{aligned}
&\sum_{0<k-z\le K}\ \sum_{-K\le j-z\le 0} \mathbf{Cov}\big(\omega_k^N(t),\, \omega_j^N(t)\big) \\
&\qquad \longrightarrow \sum_{0<k-z\le K}\ \sum_{-K\le j-z\le 0} \mathbf{Cov}\big(\omega_k(t),\, \omega_j(t)\big) \ = \ 0
\end{aligned}$$

where the vanishing is due to the assumption of i.i.d. variables at time t. Letting $K \nearrow \infty$ finishes the proof of (2.18).

A similar argument, approximating with the variables $\eta_k^{N,\, k-|k|/2\,,\, k+|k|/2}(t)$ in the tails of the series and taking the $N \to \infty$ limit in a finite sum, proves

$$\begin{aligned}
&\sum_{k\le 0,\, m>z} \mathbf{Cov}[\omega_m^N(t), \omega_k^N(0)] + \sum_{k\le z,\, m>0} \mathbf{Cov}[\omega_k^N(t), \omega_m^N(0)] \\
&\longrightarrow \sum_{k\le 0,\, m>z} \mathbf{Cov}[\omega_m(t), \omega_k(0)] + \sum_{k\le z,\, m>0} \mathbf{Cov}[\omega_k(t), \omega_m(0)] \\
&= \sum_{n\in\mathbb{Z}} |n-z| \mathbf{Cov}[\omega_n(t), \omega_0(0)].
\end{aligned} \tag{2.19}$$

The last equality used shift-invariance. This proves Lemma (2.4). □

To complete the proof of (2.2) it only remains to show that the left-hand side of (2.8) converges as $N \to \infty$ to the first member of (2.2). The same line of reasoning works again because up to an exponentially small probability in m, the current is not altered by removal of all jumps that involve sites outside $(-m, m)$.

Identity (2.2) is now proved. The statement about the uniform convergence of the series and the continuity in ρ come from Corollary 2.3.

We turn to the proof of (2.3).

Lemma 2.5. *For densities $0<\lambda<\rho<1$, currents in stationary processes satisfy*

$$E^{\rho}[J_z(t)]-E^{\lambda}[J_z(t)]=\int_{\lambda}^{\rho}\frac{1}{\theta(1-\theta)}\sum_{j\in\mathbb{Z}}(j-z)\operatorname{Cov}^{\theta}[\omega_j(t),\,\omega_0(0)]\,d\theta. \tag{2.20}$$

Proof: Let again process with superscript N be the one whose initial configuration is (2.6) with i.i.d. occupations on $[-N,N]$ and vacant sites elsewhere, and $\mathbf{P}$ the measure for all the coupled processes. Let $I^N=\sum_i\omega_i^N(t)$ be the number of particles in the process ω^N. I^N is a Binomial$(2N+1,\rho)$ random variable. For $0<\rho<1$

$$\begin{aligned}\frac{d}{d\rho}\mathbf{E}[J_z^N(t)]&=\frac{d}{d\rho}\sum_{m=0}^{2N+1}\binom{2N+1}{m}\rho^m(1-\rho)^{2N+1-m}E[J_z^N(t)|I^N=m]\\&=\sum_{m=0}^{2N+1}P(I^N=m)\Big(\frac{m}{\rho}-\frac{2N+1-m}{1-\rho}\Big)E[J_z^N(t)|I^N=m]\\&=\frac{1}{\rho(1-\rho)}\mathbf{E}\Big[J_z^N(t)\cdot\sum_i(\omega_i^N(0)-\rho)\Big]\end{aligned}$$

[recall definitions (2.7)]

$$\begin{aligned}&=\frac{1}{\rho(1-\rho)}\mathbf{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_-^N(0)+I_+^N(0)\big]\\&=\frac{1}{\rho(1-\rho)}\Big(\mathbf{Cov}\big[I_+^N(t)\,,\,I_-^N(0)\big]\\&\qquad\qquad+\mathbf{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_+^N(0)\big]\Big).\end{aligned} \tag{2.21}$$

The last equality used $\mathbf{Cov}[I_+^N(0),I_-^N(0)]=0$ that comes from the i.i.d. distribution of initial occupations. The first covariance on line (2.21) write directly as

$$\mathbf{Cov}\big[I_+^N(t)\,,\,I_-^N(0)\big]=\sum_{k\le0,\,m>z}\mathbf{Cov}[\omega_m^N(t),\omega_k^N(0)].$$

The second covariance on line (2.21) write as

$$\begin{aligned}\mathbf{Cov}\big[I_+^N(t)-I_+^N(0)\,,\,I_+^N(0)\big]&=\mathbf{Cov}\big[I_-^N(0)-I_-^N(t)\,,\,I_+^N(0)\big]\\&=\;-\mathbf{Cov}\big[I_-^N(t)\,,\,I_+^N(0)\big]\;=\;-\sum_{k\le z,\,m>0}\mathbf{Cov}[\omega_k^N(t),\omega_m^N(0)].\end{aligned}$$

Inserting these back on line (2.21) gives

$$\begin{aligned}\frac{d}{d\rho}\mathbf{E}[J_z^N(t)]=\frac{1}{\rho(1-\rho)}\Big(&\sum_{k\le0,\,m>z}\mathbf{Cov}[\omega_m^N(t),\omega_k^N(0)]\\&-\sum_{k\le z,\,m>0}\mathbf{Cov}[\omega_k^N(t),\omega_m^N(0)]\Big).\end{aligned}$$

Thus compared to line (2.19) we have the difference instead of the sum. Integrate over the density ρ and take $N\to\infty$ as was taken on the line following (2.19). This proves the proposition. $\square$

This lemma together with (2.1) gives

$$E^\rho[J_z(t)] - E^\lambda[J_z(t)] = \int_\lambda^\rho (\mathbf{E}^\theta[Q(t)] - z)\,d\theta \tag{2.22}$$

for $0 < \lambda < \rho < 1$. Couplings show the continuity of these expectations:

$$E^\lambda[J_z(t)] \to E^\rho[J_z(t)] \quad \text{and} \quad \mathbf{E}^\lambda[Q(t)] \to \mathbf{E}^\rho[Q(t)] \quad \text{as } \lambda \to \rho \text{ in } [0,1].$$

Precisely speaking, if the initial configurations are coupled as indicated in (2.16), the integrands converge a.s. to the corresponding integrands. Then bounds in terms of Poisson processes give uniform integrability that makes the expectations converge.

Thus the right-hand side of (2.22) can be differentiated in ρ and identity (2.3) for $0 < \rho < 1$ follows.

For the one-sided derivatives, consider the case $\rho = 0$. If we take $\lambda \searrow 0$ in (2.22) then

$$E^\rho[J_z(t)] = \int_0^\rho (\mathbf{E}^\theta[Q(t)] - z)\,d\theta.$$

Continuity of the integrand now allows us to differentiate from the right at $\rho = 0$. Similar argument for the left derivative at $\rho = 1$ completes the proof of identity (2.3). The continuity of the right-hand side of (2.3) was also argued along the way.

3. A coupling for microscopic concavity

As observed in (2.5) the speed of the second class particle in a density-ρ ASEP is $H'(\rho)$. Thus by the concavity of the flux H a defect travels slower in a denser system (recall that we assume $p > q$ throughout). However, in ASEP the basic coupling does not respect this, except in the totally asymmetric ($p = 1, q = 0$) case. To see this, consider two pairs of processes (ω^+, ω) and (η^+, η) such that both pairs have one discrepancy: $\omega^+(t) = \omega(t) + \delta_{Q^\omega(t)}$ and $\eta^+(t) = \eta(t) + \delta_{Q^\eta(t)}$. Assume that $\omega(t) \ge \eta(t)$. In basic coupling the jump from state

$$\begin{bmatrix} \omega_i^+ & \omega_{i+1}^+ \\ \omega_i & \omega_{i+1} \\ \eta_i^+ & \eta_{i+1}^+ \\ \eta_i & \eta_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to state} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

happens at rate q and results in $Q^\omega = i + 1 > i = Q^\eta$.

In this section we construct a different coupling that combines the basic coupling with auxiliary clocks for second class particles. The idea is to think of a single "special" second class particle as performing a random walk on the process of $\omega - \eta$ second class particles. This coupling preserves the expected ordering of the special second class particles, hence it can be regarded as a form of *microscopic concavity*.

This theorem summarizes the outcome.

Theorem 3.1. *Assume given two initial configurations $\{\zeta_i(0)\}$ and $\{\xi_i(0)\}$ and two not necessarily distinct positions $Q^\zeta(0)$ and $Q^\xi(0)$ on $\mathbb{Z}$. Suppose the coordinatewise ordering $\zeta(0) \ge \xi(0)$ holds, $Q^\zeta(0) \le Q^\xi(0)$, and $\zeta_i(0) = \xi_i(0) + 1$ for $i \in \{Q^\zeta(0), Q^\xi(0)\}$. Define the configuration $\zeta^-(0) = \zeta(0) - \delta_{Q^\zeta(0)}$.*

Then there exists a coupling of processes $(\zeta^-(t), Q^\zeta(t), \xi(t), Q^\xi(t))_{t\ge 0}$ with initial state $(\zeta^-(0), Q^\zeta(0), \xi(0), Q^\xi(0))$ given in the previous paragraph, such that both pairs (ζ^-, Q^ζ) and (ξ, Q^ξ) are (p,q)-ASEP's with a second class particle, and $Q^\zeta(t) \le Q^\xi(t)$ for all $t \ge 0$.

To begin the construction, put two exclusion processes ζ and ξ in basic coupling, obeying Poisson clocks $\{N^{i\to i\pm 1}\}$. They are ordered so that $\zeta \geq \xi$. The $\zeta - \xi$ second class particles are labeled in increasing order $\cdots < X_{m-1}(t) < X_m(t) < X_{m+1}(t) < \cdots$. We assume there is at least one such second class particle, but beyond that we make no assumption about their number. Thus there is some finite or infinite subinterval $I \subseteq \mathbb{Z}$ of indices such that the positions of the $\zeta - \xi$ second class particles are given by $\{X_m(t) : m \in I\}$.

We introduce two dynamically evolving labels $a(t)$, $b(t) \in I$ in such a manner that $X_{a(t)}(t)$ is the position of a second class antiparticle in the ζ-process, $X_{b(t)}(t)$ is the position of a second class particle in the ξ-process, and the ordering

$$X_{a(t)}(t) \leq X_{b(t)}(t) \tag{3.1}$$

is preserved by the dynamics.

The labels $a(t)$, $b(t)$ are allowed to jump from m to $m\pm 1$ only when particle $X_{m\pm 1}$ is adjacent to X_m. The labels do not take jump commands from the Poisson clocks $\{N^{i\to i\pm 1}\}$ that govern (ξ, ζ). Instead, the directed edges $(i, i+1)$ and $(i, i-1)$ are given another collection of independent Poisson clocks so that the following jump rates are realized.

(i) If $a = b$ and $X_{a+1} = X_a + 1$ then

$$(a, b) \text{ jumps to } \begin{cases} (a, b+1) & \text{with rate } p - q \\ (a+1, b+1) & \text{with rate } q. \end{cases}$$

(ii) If $a = b$ and $X_{a-1} = X_a - 1$ then

$$(a, b) \text{ jumps to } \begin{cases} (a-1, b) & \text{with rate } p - q \\ (a-1, b-1) & \text{with rate } q. \end{cases}$$

(iii) If $a \neq b$ then a and b jump independently with these rates:

$$a \text{ jumps to } \begin{cases} a+1 & \text{with rate } q \text{ if } X_{a+1} = X_a + 1 \\ a-1 & \text{with rate } p \text{ if } X_{a-1} = X_a - 1; \end{cases}$$

$$b \text{ jumps to } \begin{cases} b+1 & \text{with rate } p \text{ if } X_{b+1} = X_b + 1 \\ b-1 & \text{with rate } q \text{ if } X_{b-1} = X_b - 1. \end{cases}$$

Let us emphasize that the pair process (ξ, ζ) is still governed by the old clocks $\{N^{i\to i\pm 1}\}$ in the basic coupling. The new clocks that realize rules (i)–(iii) are not observed except when sites $\{i, i+1\}$ are both occupied by X-particles and at least one of X_a or X_b lies in $\{i, i+1\}$.

First note that if initially $a(0) \leq b(0)$ then jumps (i)–(iii) preserve the inequality $a(t) \leq b(t)$ which gives (3.1). (Since the jumps in point (iii) happen independently, there cannot be two simultaneous jumps. So it is not possible for a and b to cross each other with a $(a, b) \to (a+1, b-1)$ move.)

Define processes $\zeta^-(t) = \zeta(t) - \delta_{X_{a(t)}(t)}$ and $\xi^+(t) = \xi(t) + \delta_{X_{b(t)}(t)}$. In other words, to produce ζ^- remove particle X_a from ζ, and to produce ξ^+ add particle X_b to ξ. The second key point is that, even though these new processes are no longer defined by the standard graphical construction, distributionwise they are still ASEP's with second class particles. We argue this point for (ζ^-, X_a) and leave the argument for (ξ, X_b) to the reader.

Lemma 3.2. *The pair (ζ^-, X_a) is a (p, q)-ASEP with a second class particle.*

Proof: We check that the jump rates for the process (ζ^-, X_a), produced by the combined effect of the basic coupling with clocks $\{N^{i\to i\pm1}\}$ and the new clocks, are the same jump rates that result from defining an (ASEP, second class particle) pair in terms of the graphical construction as explained in the Introduction.

To have notation for the possible jumps, let 0 denote an empty site, 1 a ζ^--particle, and 2 particle X_a. Consider a fixed pair $(i, i+1)$ of sites and write xy with $x, y \in \{0, 1, 2, \}$ for the contents of sites $(i, i+1)$ before and after the jump. Then here are the possible moves across the edge $\{i, i+1\}$, and the rates that these moves would have in the basic coupling.

$$\begin{array}{lll} \text{Type 1} & 10 \longrightarrow 01 & \text{with rate } p \\ & 01 \longrightarrow 10 & \text{with rate } q \\ \text{Type 2} & 20 \longrightarrow 02 & \text{with rate } p \\ & 02 \longrightarrow 20 & \text{with rate } q \\ \text{Type 3} & 12 \longrightarrow 21 & \text{with rate } p \\ & 21 \longrightarrow 12 & \text{with rate } q \end{array}$$

Our task is to check that the construction of (ζ^-, X_a) actually realizes these rates.

Jumps of types 1 and 2 are prompted by the clocks $\{N^{i\to i\pm1}\}$ of the graphical construction of (ξ, ζ), and hence have the correct rates listed above.

Jumps of type 3 occur in two distinct ways.

(Type 3.1) First there can be a ξ-particle next to X_a, and then the rates shown above are again realized by the clocks $\{N^{i\to i\pm1}\}$ because in the basic coupling the ξ-particles have priority over the X-particles.

(Type 3.2) The other alternative is that both sites $\{i, i+1\}$ are occupied by X-particles and one of them is X_a. The clocks $\{N^{i\to i\pm1}\}$ cannot interchange the X-particles across the edge $\{i, i+1\}$ because in the (ξ, ζ)-graphical construction these are lower priority ζ-particles that do not jump on top of each other. The otherwise missing jumps are now supplied by the "new" clocks that govern the jumps described in rules (i)–(iii).

Combining (i)–(iii) we can read that if $X_a = i+1$ and $X_{a-1} = i$, then a jumps to $a-1$ with rate p. This is the first case of type 3 jumps above, corresponding to a ζ^--particle moving from i to $i+1$ with rate p, and the second class particle X_a yielding. On the other hand, if $X_a = i$ and $X_{a+1} = i+1$ then a jumps to $a+1$ with rate q. This is the second case in type 3, corresponding to a ζ^--particle moving from $i+1$ to i with rate q and exchanging places with the second class particle X_a.

We have verified that the process (ζ^-, X_a) operates with the correct rates.

To argue from the rates to the correct distribution of the process, we can make use of the process (ζ^-, ζ). The processes (ζ^-, X_a) and (ζ^-, ζ) determine each other uniquely. The virtue of (ζ^-, ζ) is that it has a compact state space and only nearest-neighbor jumps with bounded rates. Hence by the basic theory of semigroups and generators of particle systems as developed in Liggett (1985), given the initial configuration, the distribution of the process is uniquely determined by the action of the generator on local functions. Thus it suffices to check that individual jumps have the correct rates across each edge $\{i, i+1\}$. This is exactly what we did above in the language of (ζ^-, X_a). $\square$

Similar argument shows that (ξ, X_b) is a (p,q)-ASEP with a second class particle. To prove Theorem 3.1 take $Q^\zeta = X_a$ and $Q^\xi = X_b$. This gives the coupling whose existence is claimed in the theorem.

To conclude, let us observe that the four processes $(\xi, \xi^+, \zeta^-, \zeta)$ are not in basic coupling. For example, the jump from state

$$\begin{bmatrix} \zeta_i & \zeta_{i+1} \\ \zeta_i^- & \zeta_{i+1}^- \\ \xi_i^+ & \xi_{i+1}^+ \\ \xi_i & \xi_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{to state} \quad \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

happens at rate q (second case of rule (i)), while in basic coupling this move is impossible.

4. Biased random walk in an inhomogeneous environment

This section proves an estimate that is needed for the proof of Theorem 1.1. Let $Z(t)$ be a continuous-time nearest-neighbor random walk on state-space $S \subseteq \mathbb{Z}$ that contains $\mathbb{Z}_- = \{\dots, -2, -1, 0\}$. Initially $Z(0) = 0$. Z attempts to jump from x to $x+1$ with rate p for $x \le -1$, and from x to $x-1$ with rate q for $x \le 0$. Assume $p > q = 1-p$ and let $\theta = p - q$. The rates on $S \setminus \mathbb{Z}_-$ need not be specified.

Whether jumps are permitted or not is determined by a fixed environment expressed in terms of $\{0,1\}$-valued functions $\{u(x,t) : x \in S,\ 0 \le t < \infty\}$. A jump across edge $\{x-1, x\}$ in either direction is permitted at time t if $u(x,t) = 1$, otherwise not. In other words, $u(x,t)$ is the indicator of the event that edge $\{x-1,x\}$ is open at time t.

Assumption. Assume that for all $x \in S$ and $T < \infty$, $u(x,t)$ flips between 0 and 1 only finitely many times during $0 \le t \le T$. Assume for convenience right-continuity: $u(x, t+) = u(x,t)$.

Lemma 4.1. *For all $t \ge 0$ and $k \ge 0$,*

$$P\{Z(t) \le -k\} \le e^{-2\theta k}.$$

This bound holds for any fixed environment $\{u(x,t)\}$ subject to the assumption above.

Proof: Let $Y(t)$ be a walk that operates exactly as $Z(t)$ on $\mathbb{Z}_-$ but is rectricted to remain in $\mathbb{Z}_-$ by setting the rate of jumping from 0 to 1 to zero. Give $Y(t)$ geometric initial distribution

$$P\{Y(0) = -j\} = \pi(j) \equiv \Bigl(1 - \frac{q}{p}\Bigr)\Bigl(\frac{q}{p}\Bigr)^j \quad \text{for } j \ge 0.$$

The initial points satisfy $Y(0) \le Z(0)$ a.s. Couple the walks through Poisson clocks so that the inequality $Y(t) \le Z(t)$ is preserved for all time $0 \le t < \infty$.

Without the inhomogeneous environment $Y(t)$ would be a stationary, reversible birth and death process. We argue that even with the environment the time marginals $Y(t)$ still have distribution π. This suffices for the conclusion, for then

$$P\{Z(t) \le -k\} \le P\{Y(t) \le -k\} = (q/p)^k = \exp\bigl(k \log \tfrac{1-\theta}{1+\theta}\bigr) \le e^{-2\theta k}.$$

To justify the claim about $Y(t)$, consider approximating processes $Y^{(m)}(t)$, $m \in \mathbb{N}$, with the same initial value $Y^{(m)}(0) = Y(0)$. $Y^{(m)}(t)$ evolves so that the environments $\{u(x,t)\}$ restrict its motion only on edges $\{x-1, x\}$ for $-m+1 \le x \le 0$.

In other words, for walk $Y^{(m)}(t)$ we set $u(x,t) \equiv 1$ for $x \le -m$ and $0 \le t < \infty$. We couple the walks together so that $Y(t) = Y^{(m)}(t)$ until the first time one of the walks exits the interval $\{-m+1, \dots, 0\}$.

Fixing m for a moment, let $0 = s_0 < s_1 < s_2 < s_3 < \dots$ be a partition of the time axis so that $s_j \nearrow \infty$ and the environments $\{u(x,t) : -m < x \le 0\}$ are constant on each interval $t \in [s_i, s_{i+1})$. Then on each time interval $[s_i, s_{i+1})$ $Y^{(m)}(t)$ is a continuous time Markov chain with time-homogeneous jump rates

$$c(x, x+1) = \begin{cases} pu(x+1, s_i), & -m \le x \le 0 \\ p, & x \le -m-1 \end{cases}$$

and

$$c(x, x-1) = \begin{cases} qu(x, s_i), & -m+1 \le x \le 0 \\ q, & x \le -m. \end{cases}$$

One can check that detailed balance $\pi(x)c(x, x+1) = \pi(x+1)c(x+1, x)$ holds for all $x \le -1$. Thus π is a reversible measure for walk $Y^{(m)}(t)$ on each time interval $[s_i, s_{i+1})$, and we conclude that $Y^{(m)}(t)$ has distribution π for all $0 \le t < \infty$.

The coupling ensures that $Y^{(m)}(t) \to Y(t)$ almost surely as $m \to \infty$, and consequently also $Y(t)$ has distribution π for all $0 \le t < \infty$. $\square$

5. Moment bounds for the second class particle

In this section we prove Theorem 1.1.

5.1. *Upper bound.* We prove the upper bound of Theorem 1.1 by proving bounds for tail probabilities. We do this first for the right tail of $Q(t)$. Throughout we assume fixed rates $p > q = 1-p$ and abbreviate

$$\theta = p - q. \tag{5.1}$$

Introduce also the notation

$$\Psi(t) = \mathbf{E}^\rho |Q(t) - V^\rho t|. \tag{5.2}$$

Lemma 5.1. *Let $r \ge 1 \vee 8\sqrt{\theta}$. Then for each density $0 < \rho < 1$ there exists a constant $C(\rho) \in (0, \infty)$ such that, for $t \ge 1$ and $u \ge rt^{2/3}$,*

$$\mathbf{P}^\rho\{Q(t) \ge V^\rho t + u\} \le C(\rho)\Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big). \tag{5.3}$$

The constant $C(\rho)$ is continuous in $\rho \in (0,1)$, and $\lim_{\rho \searrow 0} C(\rho) = \infty$.

Proof: Assume for convenience that u is a positive integer. Since $\lfloor u \rfloor \ge u/2$ for $u \ge 1$, (5.3) extends from integers u to real u by an adjustment of the constant $C(\rho)$.

Fix a density $0 < \rho < 1$ and an auxiliary density $0 < \lambda < \rho$ that will vary in the argument. Start with the basic coupling of three exclusion processes $\omega \ge \omega^- \ge \eta$ with this initial set-up:

(a) Initially $\{\omega_i(0) : i \ne 0\}$ are i.i.d. Bernoulli(ρ) distributed and $\omega_0(0) = 1$.

(b) Initially $\omega^-(0) = \omega(0) - \delta_0$.

(c) Initially variables $\{\eta_i(0) : i \ne 0\}$ are i.i.d. Bernoulli(λ) and $\eta_0(0) = 0$. The coupling of the initial occupations is such that $\omega_i(0) \ge \eta_i(0)$ for all $i \ne 0$.

Recall that basic coupling meant that these processes obey common Poisson clocks.

Let $Q(t)$ be the position of the single second class particle between $\omega(t)$ and $\omega^-(t)$, initially at the origin. Let $\{X_i(t) : i \in \mathbb{Z}\}$ be the positions of the $\omega - \eta$ second class particles, initially labeled so that

$$\cdots < X_{-2}(0) < X_{-1}(0) < X_0(0) = 0 < X_1(0) < X_2(0) < \cdots$$

These second class particles preserve their labels in the dynamics and stay ordered. Thus the $\omega(t)$ configuration consists of first class particles (the $\eta(t)$ process) and second class particles (the $X_j(t)$'s). $\mathbf{P}$ denotes the probability measure under which all these coupled processes live. Note that the marginal distribution of (ω, ω^-, Q) under $\mathbf{P}$ is exactly as it would be under $\mathbf{P}^\rho$.

For $x \in \mathbb{Z}$, $J^\omega_x(t)$ is the net current in the ω-process between space-time positions $(1/2, 0)$ and $(x + 1/2, t)$. Similarly $J^\eta_x(t)$ in the η-process, and $J^{\omega-\eta}_x(t)$ is the net current of second class particles. Current in the ω-process is a sum of the first class particle current and the second class particle current:

$$J^\omega_x(t) = J^\eta_x(t) + J^{\omega-\eta}_x(t). \tag{5.4}$$

$Q(t)$ is included among the $\{X_j(t)\}$ for all time because the basic coupling preserves the coordinatewise ordering $\omega^-(t) \geq \eta(t)$. Define the label $m_Q(t)$ by $Q(t) = X_{m_Q(t)}(t)$ with initial value $m_Q(0) = 0$. We insert a bound on the label.

Lemma 5.2. *For all $t \geq 0$ and $k \geq 0$,*

$$\mathbf{P}\{m_Q(t) \geq k\} \leq e^{-2\theta k}.$$

Proof of Lemma 5.2: In the basic coupling the label $m_Q(t)$ evolves as follows. When X_{m_Q-1} is adjacent to X_{m_Q}, m_Q jumps down by one at rate p. And when X_{m_Q+1} is adjacent to X_{m_Q}, m_Q jumps up by one at rate q. When X_{m_Q} has no X-particle in either neighboring site, the label m_Q cannot jump. Thus the situation is like that in Lemma 4.1 (with a reversal of lattice directions) with environment given by the adjacency of X-particles: $u(m, t) = \mathbf{1}\{X_m(t) = X_{m-1}(t) + 1\}$. However, the basic coupling mixes together the evolution of the environment and the walk m_Q, so the environment is not specified in advance as required by Lemma 4.1.

We can get around this difficulty by imagining an alternative but distributionally equivalent construction for the joint process (η, ω^-, ω). Let (η, ω) obey basic coupling with the given Poisson clocks $\{N^{x\to x\pm1}\}$ attached to directed edges $(x, x\pm1)$. Divide the $\omega - \eta$ particles further into class II consisting of the particles $\omega^- - \eta$ and class III that consists only of the single particle $\omega - \omega^- = \delta_Q$. Let class II have priority over class III. Introduce another independent set of Poisson clocks $\{\widetilde{N}^{x\to x\pm1}\}$, also attached to directed edges $(x, x \pm 1)$ of the space $\mathbb{Z}$ where particles move. Let clocks $\{\widetilde{N}^{x\to x\pm1}\}$ govern the exchanges between classes II and III. In other words, for each edge $\{x, x + 1\}$ clocks $\widetilde{N}^{x\to x+1}$ and $\widetilde{N}^{x+1\to x}$ are observed if sites $\{x, x+1\}$ are both occupied by $\omega - \eta$ particles. All other jumps are prompted by the original clocks.

The rates for individual jumps are the same in this alternative construction as in the earlier one where all processes were together in basic coupling. Thus the same distribution for the process (η, ω^-, ω) is created.

To apply Lemma 4.1 perform the construction in two steps. First construct the process (η, ω) for all time. This determines the environment $u(m,t) = \mathbf{1}\{X_m(t) = X_{m-1}(t)+1\}$. Then run the dynamics of classes II and III in this environment. Now Lemma 4.1 gives the bound for m_Q. $\square$

The preliminaries are ready and we begin to develop a series of inequalities. Let u and k be positive integers.

$$\begin{aligned} &\mathbf{P}\{Q(t) \ge V^\rho t + u\} \\ &\qquad \le \mathbf{P}\{m_Q(t) \ge k\} + \mathbf{P}\{J^\omega_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^\eta_{\lfloor V^\rho t\rfloor+u}(t) > -k\}. \end{aligned} \tag{5.5}$$

To explain the inequality above, if $Q(t) \ge V^\rho t + u$ and $m_Q(t) < k$ then $X_k(t) > \lfloor V^\rho t\rfloor + u$. This puts the bound

$$J^{\omega-\eta}_{\lfloor V^\rho t\rfloor+u}(t) > -k$$

on the second class particle current, because at most particles $X_1, \dots, X_{k-1}$ could have made a negative contribution to this current.

Lemma 5.2 takes care of the first probability on line (5.5). We work on the second probability on line (5.5).

Here is a simple observation that will be used repeatedly. Process ω can be coupled with a stationary density-ρ process $\omega^{(\rho)}$ so that the coupled pair $(\omega, \omega^{(\rho)})$ has at most 1 discrepancy. In this coupling

$$|J^\omega_x(t) - J^{\omega^{(\rho)}}_x(t)| \le 1. \tag{5.6}$$

This way we can use computations for stationary processes at the expense of small errors.

Recall that $V^\rho = H'(\rho)$. Let c_1 below be a constant that absorbs the errors from using means of stationary processes and from ignoring integer parts. It satisfies $|c_1| \le 3$.

$$\begin{aligned} &\mathbf{E}J^\omega_{\lfloor V^\rho t\rfloor+u}(t) - \mathbf{E}J^\eta_{\lfloor V^\rho t\rfloor+u}(t) && (5.7)\\ &\qquad = tH(\rho) - (H'(\rho)t+u)\rho - tH(\lambda) + (H'(\rho)t+u)\lambda + c_1 \\ &\qquad = -\tfrac12 tH''(\rho)(\rho-\lambda)^2 - u(\rho-\lambda) + c_1 \\ &\qquad = t\theta(\rho-\lambda)^2 - u(\rho-\lambda) + c_1. && (5.8) \end{aligned}$$

For more general fluxes with nonvanishing $H''(\rho)$ the Taylor expansion would produce more terms above.

The discussion splits into three cases according to the range of u-values. Only the first case requires substantial work.

Case 1. $rt^{2/3} \le u \le \rho\theta t$.

Choose

$$\lambda = \rho - \frac{u}{2\theta t} \quad \text{and} \quad k = \left\lfloor \frac{u^2}{8\theta t} \right\rfloor - 3. \tag{5.9}$$

The assumptions $r \ge 1 \vee 8\sqrt{\theta}$, $t \ge 1$ and $u \ge rt^{2/3}$ ensure that

$$k \ge \frac{u^2}{16\theta t} \ge 1. \tag{5.10}$$

In the next line below the -3 in k absorbs c_1 from line (5.8). Recall that $\widetilde{X} = X - EX$ stands for a centered random variable. We continue bounding the second probability from line (5.5).

$$\begin{aligned}
&\mathbf{P}\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) > -k\} \\
&\le \mathbf{P}\Big\{\widetilde{J}^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; \widetilde{J}^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) \ge \frac{u^2}{8t\theta}\Big\} \\
&\le \frac{64\theta^2t^2}{u^4}\mathbf{Var}\Big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\Big\} \\
&\le \frac{128\theta^2t^2}{u^4}\Big(\mathbf{Var}\big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t)\big\} \;+\; \mathbf{Var}\big\{J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\big\}\Big). \qquad (5.11)
\end{aligned}$$

We develop bounds on the variances above, first for J^ω. Pass to the stationary density-ρ process via (5.6) and apply (2.2) of Theorem 2.1:

$$\begin{aligned}
\mathbf{Var}\big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t)\big\} &\le 2\,\mathrm{Var}^\rho\big\{J_{\lfloor V^\rho t\rfloor+u}(t)\big\} + 2 \\
&= 2\rho(1-\rho)\mathbf{E}\big|Q(t) - \lfloor V^\rho t\rfloor - u\big| + 2 \\
&\le \mathbf{E}|Q(t) - V^\rho t\,| + u + 3.
\end{aligned}$$

As was already pointed out, as far as $Q(t)$ goes, the $\mathbf{E}^\rho$ expectation in the right member of (2.2) is the same as $\mathbf{E}$ in the coupling of this section. Recall the notation $\Psi(t)$ from (5.2), bound 3 by $3u$ (recall that $u \ge 1$), and write the above bound in the form

$$\mathbf{Var}\big\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t)\big\} \le \Psi(t) + 4u. \qquad (5.12)$$

In order to get the same bound for $\mathbf{Var}\big\{J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\big\}$ we utilize the coupling developed in Section 3. Let Var^λ denote variance in the stationary density-λ process and let $Q^\eta(t)$ denote the position of a second class particle added to a process η defined as at the beginning of Section 5.1.

Reasoning as was done for (5.12): switch to a stationary density-λ process and apply (2.2):

$$\begin{aligned}
\mathbf{Var}\big\{J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t)\big\} &\le 2\,\mathrm{Var}^\lambda\big\{J_{\lfloor V^\rho t\rfloor+u}(t)\big\} + 2 \\
&\le \mathbf{E}^\lambda\big|Q^\eta(t) - \lfloor V^\rho t\rfloor - u\big| + 2 \\
&\le \mathbf{E}^\lambda|Q^\eta(t) - V^\rho t\,| + 4u
\end{aligned}$$

Introduce process $(\zeta^-(t), Q^\zeta(t), \eta(t), Q^\eta(t))_{t\ge 0}$ coupled as in Theorem 3.1, where ζ starts with Bernoulli(ρ) occupations away from the origin and initially $Q^\zeta(0) = Q^\eta(0) = 0$. Below apply the triangle inequality and use inequality $Q^\zeta(t) \le Q^\eta(t)$ from Theorem 3.1. Thus continuing from above:

$$\begin{aligned}
&= \mathbf{E}|Q^\eta(t) - Q^\zeta(t) + Q^\zeta(t) - V^\rho t\,| + 4u \\
&\le \mathbf{E}\big\{Q^\eta(t) - Q^\zeta(t)\big\} + \mathbf{E}|Q^\zeta(t) - V^\rho t\,| + 4u \\
&= V^\lambda t - V^\rho t + \Psi(t) + 4u \\
&= 2\theta t(\rho - \lambda) + \Psi(t) + 4u \\
&= \Psi(t) + 5u. \qquad (5.13)
\end{aligned}$$

Marginally the process (ζ, Q^ζ) is the same as the process (ω, Q) in the coupling of this section, hence the appearance of $\Psi(t)$ above. Then we used (2.5) for the expectations of the second class particles and the choice (5.9) of λ.

Insert bounds (5.12) and (5.13) into (5.11) to get

$$\mathbf{P}\{J^{\omega}_{\lfloor V^\rho t\rfloor+u}(t) \;-\; J^{\eta}_{\lfloor V^\rho t\rfloor+u}(t) > -k\} \le C\theta^2\Big(\frac{t^2}{u^4}\Psi(t)+\frac{t^2}{u^3}\Big). \tag{5.14}$$

where $C = 1152$. By (5.10) and Lemma 5.2

$$\mathbf{P}\{m_Q(t) \ge k\} \le e^{-u^2/8t}$$

which we bound by (recalling $u \ge 1$)

$$e^{-u^2/8t} \le C\Big(\frac{t}{u^2}\Big)^2 \le C\frac{t^2}{u^3}.$$

Insert this and (5.14) into line (5.5) to get the upper tail bound

$$\mathbf{P}\{Q(t) \ge V^\rho t + u\} \le C(\theta)\Big(\frac{t^2}{u^4}\Psi(t)+\frac{t^2}{u^3}\Big) \tag{5.15}$$

which is exactly the goal (5.3) for **Case 1**.

Case 2. $\rho\theta t \le u \le 3t$.

This comes from the previous case. With $u \le 3t$, apply (5.15) to $v = \frac{1}{3}\rho\theta u \le \rho\theta t$ to get the bound

$$\begin{aligned}\mathbf{P}\{Q(t) \ge V^\rho t + u\} &\le \mathbf{P}\{Q(t) \ge V^\rho t + v\}\\ &\le C(\theta)\Big(\frac{t^2}{v^4}\Psi(t)+\frac{t^2}{v^3}\Big) \le \frac{C(\theta)}{\rho^4}\Big(\frac{t^2}{u^4}\Psi(t)+\frac{t^2}{u^3}\Big).\end{aligned} \tag{5.16}$$

Case 3. $u \ge 3t$.

This comes from a large deviation bound. $Q(t)$ is stochastically dominated by a rate 1 Poisson process Z_t and $|V^\rho| \le 1$. A straightforward exponential Chebyshev argument for the Poisson distribution gives the bound

$$\mathbf{P}^\rho\{Q(t) \ge V^\rho t + u\} \le \mathbf{P}\{Z_t \ge 2u/3\} \le e^{-Bu} \tag{5.17}$$

for $t \ge 1$ and $u \ge 3t$ with a constant B independent of all the parameters. The rightmost member of (5.17) is dominated by Ct^2/u^3 with constant C a fixed number. So it can be covered by the rightmost member of (5.16) by increasing the constant $C(\theta)$ because $\rho^{-4} \ge 1$. Renaming the constant in (5.16) gives the conclusion of Lemma 5.1. □

We extend the bound to both tails.

Lemma 5.3. *Let $r \ge 1 \vee 8\sqrt{\theta}$. Then for each density $0 < \rho < 1$ there exists a constant $C_1(\rho) \in (0,\infty)$ such that, for $t \ge 1$ and $u \ge rt^{2/3}$,*

$$\mathbf{P}^\rho\{\,|Q(t) - V^\rho t| \ge u\} \le C_1(\rho)\Big(\frac{t^2}{u^4}\Psi(t)+\frac{t^2}{u^3}\Big). \tag{5.18}$$

The constant $C_1(\rho)$ is continuous in $\rho \in (0,1)$, and $\lim_{\rho\to\{0,1\}} C_1(\rho) = \infty$.

Proof: Consider coupled processes (ω, ω^+) that start with i.i.d. density $\rho \in (0,1)$ occupations away from the origin and initially $\omega^+(0) = \omega(0) + \delta_0$. So there is one second class particle Q initially at the origin. Define new processes

$$\widehat{\omega}_i(t) = 1 - \omega_i(t) \quad \text{and} \quad \widehat{\omega}_i^-(t) = 1 - \omega_i^+(t).$$

Process $\widehat{\omega}$ records the dynamics of holes in the ω-process, and similarly for $\widehat{\omega}^-$. The discrepancy between $(\widehat{\omega}^-, \widehat{\omega})$ is exactly the same as the discrepancy between (ω, ω^+). That is, $\widehat{Q}(t) = Q(t)$. Processes $(\widehat{\omega}^-, \widehat{\omega})$ are instances of $(\widehat{p}, \widehat{q})$-ASEP with density $\widehat{\rho} = 1 - \rho$ and rates $\widehat{p} = q$ and $\widehat{q} = p$.

To recover the original rates (p, q) we reflect the lattice across the origin. Define

$$\omega_i^R(t) = \widehat{\omega}_{-i}(t) \quad \text{and} \quad \omega_i^{R-}(t) = \widehat{\omega}_{-i}^-(t).$$

Process (ω^{R-}, ω^R) is an instance of (p, q)-ASEP at density $1-\rho$ with a discrepancy, so the previous bound (5.16) applies. The discrepancy is now at $Q^R(t) = -\widehat{Q}(t) = -Q(t)$. The characteristic speed is $V^{1-\rho} = -V^\rho$. Hence also

$$\Psi(t) = \mathbf{E}^\rho|Q(t) - V^\rho t| = \mathbf{E}^\rho|Q^R(t) - V^{1-\rho} t|.$$

By an application of inequality (5.16) to the process $(\omega^{R-}, \omega^R, Q^R)$,

$$\begin{aligned}\mathbf{P}^\rho\{Q(t) \le V^\rho t - u\} &= \mathbf{P}^\rho\{Q^R(t) \ge V^{1-\rho} t + u\} \\ &\le \frac{C(\theta)}{(1-\rho)^4}\Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big).\end{aligned} \tag{5.19}$$

Combining (5.16) and (5.19) we have the conclusion (5.18) with constant

$$C_1(\rho) = \frac{C(\theta)}{\rho^4 \wedge (1-\rho)^4} \tag{5.20}$$

where $C(\theta)$ is a constant that depends on θ. □

Let us also record the two-sided large deviation bound, one side of which was argued above in (5.17).

Lemma 5.4. *There exists a constant B such that*

$$\mathbf{P}^\rho\{\,|Q(t) - V^\rho t| \ge u\} \le e^{-Bu} \tag{5.21}$$

for $t \ge 1$, $u \ge 3t$ and $0 < \rho < 1$.

Inequalities (5.18) and (5.21) give the upper bound on the moments of the second class particle in (1.5) via a two-step integration argument. We keep track of the precise constants for a while for use in the lower bound proof to come. First by (5.18)

$$\begin{aligned}\Psi(t) &= \int_0^\infty \mathbf{P}^\rho\{\,|Q(t) - V^\rho t| \ge u\}\,du \\ &\le rt^{2/3} + C_1(\rho)\int_{rt^{2/3}}^\infty \Big(\frac{t^2}{u^4}\Psi(t) + \frac{t^2}{u^3}\Big)\,du \\ &\le \frac{C_1(\rho)}{3r^3}\Psi(t) + \Big(r + \frac{C_1(\rho)}{2r^2}\Big)t^{2/3}.\end{aligned}$$

Fixing $r = \max\{1,\, 8\sqrt{\theta},\, C_1(\rho)^{1/3}\}$ shows that $\Psi(t) \le C_2(\rho)t^{2/3}$ for a new constant $C_2(\rho)$ with the same properties as $C_1(\rho)$.

Put this back into the estimate (5.18) to get

$$\mathbf{P}^{\rho}\{\,|Q(t)-V^{\rho}t|\geq u\}\leq C_3(\rho)\Big(\frac{t^{8/3}}{u^4}+\frac{t^2}{u^3}\Big)\leq C_3(\rho)\frac{t^2}{u^3} \tag{5.22}$$

with new constant $C_3(\rho)$. The second inequality used $u\geq rt^{2/3}$. Now take $m>1$ and this time use both (5.18) and (5.21):

$$\begin{aligned}\mathbf{E}^{\rho}|Q(t)-V^{\rho}t|^m &= m\int_0^{\infty}\mathbf{P}^{\rho}\{\,|Q(t)-V^{\rho}t|\geq u\}u^{m-1}\,du\\ &\leq r^m t^{2m/3}\;+\;C_3(\rho)\int_{rt^{2/3}}^{3t} t^2u^{m-4}\,du\;+\;m\int_{3t}^{\infty}e^{-Bu}u^{m-1}\,du.\end{aligned} \tag{5.23}$$

Performing the integrations gives these bounds:

$$\mathbf{E}^{\rho}|Q(t)-V^{\rho}t|^m\leq\begin{cases}\dfrac{C_4(\rho)}{3-m}t^{2m/3} & \text{for } 1<m<3 \text{ and } t\geq 1,\\ C_5(\rho)t^2\log t & \text{for } m=3 \text{ and } t\geq e,\\ C_6(\rho,m)t^{m-1} & \text{for } 3<m<\infty \text{ and } t\geq 1.\end{cases}$$

One can see that the constants C_i are continuous in ρ. C_6 diverges to ∞ as $m\searrow 3$ or $m\nearrow\infty$.

After renaming constants, this gives the upper bounds in Theorem 1.1 and (1.7). We record one case here for use in the lower bound proof.

Proposition 5.5. *There exists a continuous function $0<C_{UB}(\rho)<\infty$ of density $0<\rho<1$ such that for $t\geq 1$,*

$$\mathbf{E}^{\rho}|Q(t)-V^{\rho}t|\leq C_{UB}(\rho)t^{2/3}.$$

5.2. *Lower bound.* Fix a density $0<\rho<1$. For the lower bound we prove that $t^{-2/3}\Psi(t)$ has a positive lower bound for large enough t where $\Psi(t)$ was defined in (5.2). The lower bound (1.6) follows then for all $m\geq 1$ from Jensen's inequality.

Let a_1, a_2 be finite positive constants and set

$$b=a_2^2/(32\theta). \tag{5.24}$$

In the argument a_1 and a_2 will be chosen sufficiently large relative to p and relative to the constants $C_{UB}(\cdot)$ in Proposition 5.5. Define an auxiliary density $\lambda=\rho-bt^{-1/3}$. Fix $t_0\geq 1$ and large enough so that $\lambda\in(\rho/2,\rho)$ for $t\geq t_0$. Restrict all subsequent calculations to $t\geq t_0$. Define positive integers

$$u=\lfloor a_1t^{2/3}\rfloor\quad\text{and}\quad n=\lfloor V^{\lambda}t\rfloor-\lfloor V^{\rho}t\rfloor+u. \tag{5.25}$$

Start with a basic coupling of three processes $\eta\leq\eta^+\leq\zeta$:

(a) Initially η has i.i.d. Bernoulli(λ) occupations $\{\eta_i(0):i\neq -n\}$ and $\eta_{-n}(0)=0$.

(b) Initially $\eta^+(0)=\eta(0)+\delta_{-n}$. $Q^{(-n)}(t)$ is the location of the unique discrepancy between $\eta(t)$ and $\eta^+(t)$.

(c) Initially ζ has independent occupation variables, coupled with $\eta(0)$ as follows:
(c.1) $\zeta_i(0)=\eta_i(0)$ for $-n<i\leq 0$.
(c.2) $\zeta_{-n}(0)=1$.
(c.3) For $i<-n$ and $i>0$ variables $\zeta_i(0)$ are i.i.d. Bernoulli(ρ) and $\zeta_i(0)\geq\eta_i(0)$.

Thus the initial density of ζ is piecewise constant: on the segment $\{-n+1,\dots,0\}$ $\zeta(0)$ is i.i.d. with density λ, at site $-n$ $\zeta(0)$ has density 1, and elsewhere on $\mathbb{Z}$ $\zeta(0)$ is i.i.d. with density ρ.

Label the $\zeta-\eta$ second class particles as $\{Y_m(t): m\in\mathbb{Z}\}$ so that initially

$$\cdots < Y_{-1}(0) < Y_0(0) = -n = Q^{(-n)}(0) < 0 < Y_1(0) < Y_2(0) < \cdots$$

Let again $m_Q(t)$ be the label such that $Q^{(-n)}(t) = Y_{m_Q(t)}(t)$. Initially $m_Q(0)=0$. The inclusion $Q^{(-n)}(t)\in\{Y_m(t)\}$ persists for all time because the basic coupling preserves the ordering $\zeta(t)\ge\eta^+(t)$. Through the basic coupling m_Q jumps to the left with rate q and to the right with rate p, but only when there is a Y-particle adjacent to Y_{m_Q}. As in the proof of Lemma 5.2 we can apply Lemma 4.1 to prove this statement:

Lemma 5.6. *For all $t\ge 0$ and $k\ge 0$,*

$$\mathbf{P}\{m_Q(t)\le -k\}\le e^{-2\theta k}.$$

Fix $a_1>0$ large enough so that $u=\lfloor a_1 t^{2/3}\rfloor$ satisfies, for all the auxiliary densities λ,

$$\mathbf{P}\{Q^{(-n)}(t)\ge\lfloor V^\rho t\rfloor\} = \mathbf{P}\{Q^{(-n)}(t)\ge -n+\lfloor V^\lambda t\rfloor + u\}\le\tfrac12. \tag{5.26}$$

This is possible because the constant $C_{UB}(\lambda)$ in the upper bound in Proposition 5.5 is continuous in the density and we restricted to $t\ge t_0$ to ensure that $\lambda\in(\rho/2,\rho)$.

If $Q^{(-n)}(t)\le\lfloor V^\rho t\rfloor$ and $m_Q(t)>-k$ then $Y_{-k}(t)\le\lfloor V^\rho t\rfloor$. As was argued for (5.5) in Section 5.1 this implies a bound on the second class particle current:

$$J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t) = J^{\zeta-\eta}_{\lfloor V^\rho t\rfloor}(t)\le k.$$

So from (5.26)

$$\begin{aligned}\tfrac12 &\le \mathbf{P}\{Q^{(-n)}(t)\le\lfloor V^\rho t\rfloor\}\\ &\le \mathbf{P}\{m_Q(t)\le -k\} + \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t)\le k\}.\end{aligned} \tag{5.27}$$

Take $a_2>0$ large enough so that

$$a_2\ge 8+8\sqrt{C_{UB}(\alpha)}\quad\text{for }\alpha\in(\rho/2,\rho). \tag{5.28}$$

Increase t_0 further so that for $t\ge t_0$

$$\mathbf{P}\{m_Q(t)\le -a_2t^{1/3}+3\}\le\tfrac14 \tag{5.29}$$

(utilizing Lemma 5.6). Combine displays (5.27) and (5.29) with $k=\lfloor a_2t^{1/3}\rfloor-2$ to get the next inequality. Then split the probability.

$$\begin{aligned}\tfrac14 &\le \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) - J^\eta_{\lfloor V^\rho t\rfloor}(t)\le a_2t^{1/3}-2\}\\ &\le \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t)\le 2a_2t^{1/3}+t\theta(2\rho\lambda-\lambda^2)\}\\ &\qquad + \mathbf{P}\{J^\eta_{\lfloor V^\rho t\rfloor}(t)\ge a_2t^{1/3}+t\theta(2\rho\lambda-\lambda^2)+2\}.\end{aligned} \tag{5.30}$$

We treat line (5.30). Recall that P^λ denotes probabilities of a stationary density-λ process. As described above (5.6) we can imagine a basic coupling in which the

η-process differs from a stationary density-λ process by at most one discrepancy. Compute the mean current in the stationary process:

$$\begin{aligned} E^\lambda\{J_{\lfloor V^\rho t\rfloor}(t)\} &= tH(\lambda) - \lambda\lfloor V^\rho t\rfloor \\ &\le tH(\lambda) - \lambda V^\rho t + 1 = t\theta(2\rho\lambda - \lambda^2) + 1. \end{aligned}$$

From these comes the bound

$$\begin{aligned} \text{line (5.30)} &\le P^\lambda\{J_{\lfloor V^\rho t\rfloor}(t) \ge a_2 t^{1/3} + t\theta(2\rho\lambda - \lambda^2) + 1\} \\ &\le P^\lambda\{\widetilde{J}_{\lfloor V^\rho t\rfloor}(t) \ge a_2 t^{1/3}\} \le a_2^{-2} t^{-2/3} \operatorname{Var}^\lambda\{J_{\lfloor V^\rho t\rfloor}(t)\} \\ &\le \frac{\mathbf{E}^\lambda|Q(t) - \lfloor V^\rho t\rfloor|}{a_2^2 t^{2/3}} \le \frac{\mathbf{E}^\lambda|Q(t) - V^\lambda t|}{a_2^2 t^{2/3}} + \frac{2\theta b t^{2/3} + 1}{a_2^2 t^{2/3}} \\ &\le C_{UB}(\lambda) a_2^{-2} + \tfrac{1}{16} + \tfrac{1}{64} \le \tfrac{1}{8}. \end{aligned} \tag{5.31}$$

After Chebyshev above we used (2.2) and introduced a second class particle $Q(t)$ in a density-λ system under the measure $\mathbf{P}^\lambda$. $\lfloor V^\rho t\rfloor$ was replaced with $V^\lambda t$ at the cost of an error 1 for dropping integer parts, and $V^\rho - V^\lambda = 2\theta b t^{-1/3}$. Last we applied the upper bound from Proposition 5.5 and properties (5.24), (5.28) and $t \ge t_0 \ge 1$.

Put this last bound back into line (5.30). This leaves

$$\tfrac{1}{8} \le \mathbf{P}\{J^\zeta_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3} + t\theta(2\rho\lambda - \lambda^2)\} \tag{5.32}$$

To treat this probability we take the estimation back to a stationary density-ρ process by inserting the Radon-Nikodym factor. Let γ denote the distribution of the initial $\zeta(0)$ configuration described by (a)–(c) in the beginning of this section. As before ν^ρ is the density-ρ i.i.d. Bernoulli measure. Their Radon-Nikodym derivative is

$$f(\omega) = \frac{d\gamma}{d\nu^\rho}(\omega) = \frac{1}{\rho}\mathbf{1}\{\omega_{-n} = 1\} \cdot \prod_{i=-n+1}^{0} \Big(\frac{\lambda}{\rho}\mathbf{1}\{\omega_i = 1\} + \frac{1-\lambda}{1-\rho}\mathbf{1}\{\omega_i = 0\}\Big).$$

Recalling that $n = O(t^{2/3})$ and $\rho - \lambda = O(t^{-1/3})$,

$$E^\rho(f^2) = \frac{1}{\rho}\Big(1 + \frac{(\rho-\lambda)^2}{\rho(1-\rho)}\Big)^n \le \rho^{-1} e^{n(\rho-\lambda)^2/\rho(1-\rho)} \le c_1(\rho)^2, \tag{5.33}$$

where $c_1(\rho)$ is independent of t, continuous in ρ but diverges to ∞ as $\rho \to \{0,1\}$.

Let A denote the exclusion process event

$$A = \{J_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3} + t\theta(2\rho\lambda - \lambda^2)\}.$$

Then from (5.32)

$$\begin{aligned} \tfrac{1}{8} \le \mathbf{P}\{\zeta \in A\} &= \int P^\omega(A)\,\gamma(d\omega) = \int P^\omega(A) f(\omega)\,\nu^\rho(d\omega) \\ &\le \big(P^\rho(A)\big)^{1/2}\big(E^\rho(f^2)\big)^{1/2} \le c_1(\rho)\big(P^\rho(A)\big)^{1/2}. \end{aligned} \tag{5.34}$$

In the final calculation we bound $P^\rho(A)$ by Chebyshev to return to the current variance. Note that

$$E^\rho\big\{J_{\lfloor V^\rho t\rfloor}(t)\big\} = tH(\rho) - \rho\lfloor V^\rho t\rfloor = t\theta\rho^2 + \rho V^\rho t - \rho\lfloor V^\rho t\rfloor \ge t\theta\rho^2.$$

Also, if we choose a_2 large enough, namely $a_2 > (2048\theta)^{1/3}$, we can ensure that

$$c_2 \equiv b^2\theta - 2a_2 > 0. \tag{5.35}$$

From line (5.34) we have, utilizing $\lambda = \rho - bt^{-1/3}$ and (5.35),

$$\begin{aligned}(8c_1(\rho))^{-2} &\le P^\rho(A) = P^\rho\{J_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3} + t\theta(2\rho\lambda - \lambda^2)\}\\ &\le P^\rho\{\widetilde{J}_{\lfloor V^\rho t\rfloor}(t) \le 2a_2 t^{1/3} - t\theta(\rho-\lambda)^2\}\\ &\le P^\rho\{\widetilde{J}_{\lfloor V^\rho t\rfloor}(t) \le -(b^2\theta - 2a_2)t^{1/3}\}\\ &\le c_2^{-2} t^{-2/3}\,\mathrm{Var}^\rho\big\{J_{\lfloor V^\rho t\rfloor}(t)\big\}\\ &\le c_2^{-2} t^{-2/3}\Psi(t)\end{aligned}$$

by (2.2) again and the abbreviation (5.2). We have assumed $t \ge t_0$. The first and last lines of the calculation above show that $t^{-2/3}\Psi(t)$ has a positive lower bound for $t \ge t_0$. This lower bound depends on ρ through $c_1(\rho)$ in (5.33), and vanishes as $\rho \to \{0, 1\}$.

The lower bound of Theorem 1.1 is proved.

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