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Erratum to: "The Stochastic Heat Equation with Fractional-Colored Noise: Existence of the Solution"

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Abstract. We give a correction and an extension of Theorem 3.13 of Balan and Tudor (2008).

1. Correction of Theorem 3.13 of Balan and Tudor (2008)

Let $\dot{W}$ be a Gaussian noise, which is fractional in time (with Hurst index $H > 1/2$), and colored in space (with spatial covariance kernel f). Theorem 3.13 of Balan and Tudor (2008) gives the necessary and sufficient condition for the existence of the solution of the stochastic heat equation:

$$u_t = \frac{1}{2}\Delta u + \dot{W}, \quad (t,x) \in (0,T) \times \mathbb{R}^d, \tag{1.1}$$

with $u(0,\cdot) = 0$. This condition is equivalent to saying that $\|g_{tx}\|_{\mathcal{HP}} < \infty$, where $g_{tx}(s,y) = [2\pi(t-s)]^{-d/2}\exp\{-|x-y|^2/[2(t-s)]\} := p_{t-s}(x-y)$ and

$$\|\varphi\|^2_{\mathcal{HP}} := \alpha_H \int_0^T \int_0^T \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(u,x)\varphi(v,y)|u-v|^{2H-2} f(x-y) dy\, dx\, dv\, du.$$

The condition is incorrectly stated in the case of the Bessel kernel, the heat kernel, and the Poisson kernel. Here is the corrected result.

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Theorem 1.1. *(i) If f is the Riesz kernel of order α, or the Bessel kernel of order α, then $\|g_{tx}\|_{\mathcal{HP}} < \infty$ if and only if $H > (d-\alpha)/4$.*

(ii) If f is the heat kernel of order α, or the Poisson kernel of order α, then $\|g_{tx}\|_{\mathcal{HP}} < \infty$ for any $H > 1/2$ and $d \geq 1$.

Proof: Note that $\|g_{tx}\|^2_{\mathcal{HP}} = \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} I(r,s) drds$ where

$$I(r,s) := \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x_1 - y_1) p_{t-s}(x - x_1) p_{t-r}(x - y_1) dx_1 dy_1.$$

(i) In the case of the Riesz kernel, the result has been correctly proved in Balan and Tudor (2008). Suppose that f is the Bessel kernel of order α. Then

$$I(r,s) = \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} (w + 2t - r - s)^{-d/2} dw,$$

and

$$\begin{aligned}
&\|g_{tx}\|^2_{\mathcal{HP}} \\
= \; & \alpha_H \gamma'_\alpha \int_0^t \int_0^t |r-s|^{2H-2} \int_0^\infty w^{\alpha/2-1} e^{-w} (w+r+s)^{-d/2} dwdrds \\
\geq \; & \alpha_H \gamma'_\alpha 2^{-d/2} \int_0^t \int_0^t |r-s|^{2H-2} (r+s)^{-d/2} \left(\int_0^{r+s} w^{\alpha/2-1} e^{-w} dw \right) drds \\
= \; & \alpha_H \gamma'_\alpha 2^{-d/2+1} \int_0^t \int_0^r (r-s)^{2H-2} (r+s)^{-d/2} \left(\int_0^{r+s} w^{\alpha/2-1} e^{-w} dw \right) dsdr \\
= \; & \alpha_H \gamma'_\alpha 2^{-d/2+1} \int_0^t r^{2H-1-d/2} \int_0^1 (1-x)^{2H-2} (1+x)^{-d/2} \\
& \left(\int_0^{r(1+x)} w^{\alpha/2-1} e^{-w} dw \right) dxdr \\
\geq \; & 2\alpha_H \gamma'_\alpha \int_0^t r^{2H-1-d/2} \int_0^1 (1-x)^{2H-2} \left(\int_0^r w^{\alpha/2-1} e^{-w} dw \right) dxdr \\
= \; & 2H \gamma'_\alpha \int_0^t r^{2H-1-d/2} \gamma(\alpha/2, r) dr,
\end{aligned}$$

where

$$\gamma(a,x) = \int_0^x w^{a-1} e^{-w} dw, \quad a > 0, x > 0$$

is the *imcomplete Gamma function*. It is known that: (see Abramowitz and Stegun, 1964, Section 6.5, pages 260-263)

$$\lim_{x \to 0} \frac{\gamma(a,x)}{x^a} = 1. \tag{1.2}$$

From (1.2) it follows that the function $\gamma(\alpha/2, r)$ behaves as $r^{\alpha/2}$, for r close to zero. The fact that the integral $\int_0^t r^{2H-1-d/2} \gamma(\alpha/2, r) dr$ is finite forces the condition $d < 4H + \alpha$.

Suppose now that $H > (d-\alpha)/4$. We prove that $\|g_{tx}\|_{\mathcal{HP}} < \infty$. We have:

$$\begin{aligned}
&\|g_{tx}\|^2_{\mathcal{HP}} \\
= \;& 2\alpha_H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \int_0^t \int_s^t (r-s)^{2H-2}(w+r+s)^{-d/2} drdsdw \\
= \;& 2\alpha_H \gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \int_0^{2t} \int_0^t u^{2H-2}(w+v)^{-d/2} 1_{\{u\leq v\}} 1_{\{u \leq 2t-v\}} dudvdw \\
= \;& 2H\gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} \\
& \left[\int_0^t v^{2H-1}(w+v)^{-d/2} dv + \int_0^t v^{2H-1}(w+2t-v)^{-d/2} dv\right] dw \\
:= \;& 2H\gamma'_\alpha \int_0^\infty w^{\alpha/2-1} e^{-w} [I_1(w) + I_2(w)] dw,
\end{aligned}$$

and

$$\begin{aligned}
I_1(w) &\leq \int_0^t (w+v)^{2H-1-d/2} dv = \int_w^{w+t} v^{2H-1-d/2} dv \\
I_2(w) &\leq \int_0^t (w+2t-v)^{2H-1-d/2} dv = \int_{w+t}^{w+2t} v^{2H-1-d/2} dv.
\end{aligned}$$

If $H > d/4$, then $I_1(w) + I_2(w) \leq [(w+t)^{2H-d/2} + (w+2t)^{2H-d/2}]/(2H-d/2)$ and

$$\|g_{tx}\|^2_{\mathcal{HP}} \leq \frac{2H\gamma'_\alpha}{2H-d/2} \int_0^\infty w^{\alpha/2-1} e^{-w} [(w+t)^{2H-d/2} + (w+2t)^{2H-d/2}] dw < \infty.$$

If $(d-\alpha)/4 < H < d/4$, then $I_1(w) + I_2(w) \leq [w^{2H-d/2} + (w+t)^{2H-d/2}]/(d/2 - 2H)$ and

$$\|g_{tx}\|^2_{\mathcal{HP}} \leq \frac{2H\gamma'_\alpha}{d/2-2H} \int_0^\infty w^{\alpha/2-1} e^{-w} [w^{2H-d/2} + (w+t)^{2H-d/2}] dw < \infty.$$

If $H = d/4$, then $I_1(w) + I_2(w) \leq \ln(w+2t) - \ln w$ and

$$\|g_{tx}\|^2_{\mathcal{HP}} \leq C \int_0^\infty w^{\alpha/2-1} e^{-w} [\ln(w+2t) - \ln w] dw < \infty$$

(ii) If f is the heat kernel of order α, then $I(r,s) = (2\pi)^{-d/2}(\alpha + 2t - r - s)^{-d/2}$ and

$$\begin{aligned}
\|g_{tx}\|^2_{\mathcal{HP}} &= (2\pi)^{-d/2} \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} (\alpha + r + s)^{-d/2} drds \\
&\leq (2\pi\alpha)^{-d/2} \alpha_H \int_0^t \int_0^t |r-s|^{2H-2} drds = (2\pi\alpha)^{-d/2} t^{2H} < \infty.
\end{aligned}$$

If f is the Poisson kernel of order α, then:

$$I(r,s) = C_d \int_0^\infty w^{d/2-1} e^{-w/2} [2(2t-r-s) + \alpha^2]^{-(d+1)/2} dw$$

and

$$\begin{aligned}\|g_{tx}\|^2_{\mathcal{HP}} &= C_d\alpha_H \int_0^\infty w^{d/2-1}e^{-w/2} \\ &\int_0^t\int_0^t |r-s|^{2H-2}[2(r+s)+\alpha^2]^{-(d+1)/2}drdsdw \\ &\leq C_d\alpha_H\alpha^{-(d+1)}\int_0^\infty w^{d/2-1}e^{-w/2}\int_0^t\int_0^t |r-s|^{2H-2}drdsdw < \infty.\end{aligned}$$

□

2. An extension

The following result gives a sufficient condition for the existence of the solution of equation (1.1), in the case of an arbitrary covariance kernel f. It remains an open problem to see if this condition is necessary as well.

Theorem 2.1. *Let $H > 1/2$ and f be the Fourier transform of a tempered measure μ on $\mathbb{R}^d$, i.e. $f(x) = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\mu(d\xi)$ for any $x \in \mathbb{R}^d$. If*

$$\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{2H}\mu(d\xi) < \infty, \tag{2.1}$$

then $\|g_{tx}\|_{\mathcal{HP}} < \infty$ for any $(t,x) \in [0,T]\times\mathbb{R}^d$.

Proof: By Lemma 4.1 of Balan and Tudor (2009),

$$I(r,s) = (2\pi)^{-d}\int_{\mathbb{R}^d}\exp\left\{-\frac{1}{2}(2t-r-s)|\xi|^2]\right\}\mu(d\xi).$$

Using the fact that for any $\varphi \in L^{1/H}([0,t])$,

$$\alpha_H\int_0^t\int_0^t \varphi(r)\varphi(s)|r-s|^{2H-2}drds \leq b_H^2\left(\int_0^t|\varphi(s)|^{1/H}\right)^{2H},$$

where $b_H > 0$ is a constant depending on H, we obtain:

$$\begin{aligned}\|g_{tx}\|^2_{\mathcal{HP}} &= (2\pi)^{-d}\alpha_H\int_{\mathbb{R}^d}\left(\int_0^t\int_0^t|r-s|^{2H-2}e^{-(r+s)|\xi|^2/2}drds\right)\mu(d\xi) \\ &\leq (2\pi)^{-d}b_H^2\int_{\mathbb{R}^d}\left(\int_0^t e^{-s|\xi|^2/(2H)}ds\right)^{2H}\mu(d\xi) \\ &= (2\pi)^{-d}b_H^2\int_{\mathbb{R}^d}\left(\frac{1-e^{-t|\xi|^2/(2H)}}{|\xi|^2/(2H)}\right)^{2H}\mu(d\xi) \\ &\leq c_{d,H}\max\{t^{2H},(2H)^{2H}\}\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{2H}\mu(d\xi),\end{aligned}$$

where $c_{d,H}$ is a constant depending on d and H. For the last inequality above, we used the fact that for $\alpha > 0$,

$$I := \int_{\mathbb{R}^d}\left(\frac{1-e^{-\alpha|\xi|^2}}{\alpha|\xi|^2}\right)^{2H}\mu(d\xi) \leq 2^{2H}\max\{1,\alpha^{-2H}\}\int_{\mathbb{R}^d}\left(\frac{1}{1+|\xi|^2}\right)^{2H}\mu(d\xi),$$

which is proved by noting that $I = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \int_{\{|\xi|\leq 1\}} \left(\frac{1-e^{-\alpha|\xi|^2}}{\alpha|\xi|^2}\right)^{2H} \mu(d\xi) \leq \int_{\{|\xi|\leq 1\}} \mu(d\xi) \\ &\leq \int_{\{|\xi|\leq 1\}} \left(\frac{2}{1+|\xi|^2}\right)^{2H} \mu(d\xi), \\ I_2 &= \int_{\{|\xi|>1\}} \left(\frac{1-e^{-\alpha|\xi|^2}}{\alpha|\xi|^2}\right)^{2H} \mu(d\xi) \leq \int_{\{|\xi|>1\}} \left(\frac{1}{\alpha|\xi|^2}\right)^{2H} \mu(d\xi) \\ &\leq \int_{\{|\xi|>1\}} \left(\frac{2}{\alpha+\alpha|\xi|^2}\right)^{2H} \mu(d\xi). \end{aligned}$$

□

Remark 2.2. In the case of the Riesz kernel of order α or the Bessel kernel of order α, condition (2.1) becomes $H > (d-\alpha)/4$. In the case of the heat kernel or the Poisson kernel, condition (2.1) is satisfied for any $H > 1/2$ and $d \geq 1$.

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