

Limsup behaviors of multi-dimensional selfsimilar processes with independent increments

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Abstract. Laws of the iterated logarithm of “limsup” type are studied for multi-dimensional selfsimilar processes $\{X(t)\}$ with independent increments having exponent H . It is proved that, for any positive increasing function $g(t)$ with $\lim_{t \rightarrow \infty} g(t) = \infty$, there is $C \in [0, \infty]$ such that $\limsup |X(t)|/(t^H g(|\log t|)) = C$ a.s. as $t \rightarrow \infty$, in addition, as $t \rightarrow 0$. A necessary and sufficient condition for the existence of $g(t)$ with $C = 1$ is obtained. In the case where $g(t)$ with $C = 1$ does not exist, a criterion to classify functions $g(t)$ according to $C = 0$ or $C = \infty$ is given. Moreover, various “limsup” type laws with identification of the positive constants C are explicitly presented in several propositions and examples. The problems that exchange the roles of $\{X(t)\}$ and $g(t)$ are also discussed.

1. Introduction

Since Lamperti (1962), the study of selfsimilar processes has been made by many scholars. In a celebrated paper Sato (1991), K. Sato introduced and characterized selfsimilar processes with independent increments without assuming the stationarity of increments. As far as the authors know, M. Yor and D. Madan, in the second Lévy conference at Aarhus in 2002, called those processes *Sato processes*. This naming is used in applications to mathematical finance, for example, in Carr et al. (2005, 2007). K. Sato called those processes *selfsimilar additive processes* in his book Sato (1999) afterwards. In this paper, we discuss “limsup” type limit theorems for multi-dimensional selfsimilar additive processes as time goes to infinity, in addition, to zero. The four problems are proposed later on. They are motivated and started from a pioneer paper Pruitt (1990) for increasing random

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walks. Our aim is to solve them. It is known by [Sato \(1991\)](#) that the marginal distribution at any time of an H -selfsimilar additive process is selfdecomposable and conversely, for any selfdecomposable distribution μ and $H > 0$, there is a unique in law H -selfsimilar additive process such that the marginal distribution at time 1 is the same as the distribution μ . Thus the investigation on the tail behaviors of selfdecomposable distributions is crucial for the resolution of the problems. In particular, the dominated variation and the evolution of generalized moments of an infinitely divisible distribution play key roles. The greatest difficulty lies in the fact that an infinitely divisible distribution and its Lévy measure do not always have the same tail behaviors.

Historically, [Sato \(1991\)](#) started to study the rate of growth of selfsimilar additive processes in some increasing case. Following this, [Watanabe \(1996\)](#) investigated “limsup” and “liminf” type limit theorems for the general increasing case and [Yamamuro \(2003\)](#) treated the same problem in a certain class of two-sided processes. As closely related results, the sample path behaviors of selfsimilar Markov processes are investigated in [Chaumont and Pardo \(2006\)](#); [Pardo \(2006\)](#); [Rivero \(2003\)](#). A selfsimilar Markov process was introduced and characterized by [Lamperti \(1972\)](#). [Sato and Yamamuro \(1998, 2000\)](#) and [Yamamuro \(2000a,b\)](#) studied recurrence-transience for selfsimilar additive processes in detail. As a remarkable fact, [Yamamuro \(2000b\)](#) proved that all selfsimilar additive processes in dimensions greater than or equal to three are transient. In dimensions 1 and 2, the attempt to find a criterion of their recurrence-transience has not been successful. In recent years, problems analogous to the present paper were discussed in a series of works [Watanabe \(2002a,b, 2004, 2007\)](#) for shift selfsimilar additive random sequences. They were applied to solve some classical problems of random fractals.

In what follows, let \mathbb{R}^d be the d -dimensional Euclidean space and S^{d-1} be the $(d-1)$ -dimensional unit sphere with the understanding that $S^0 := \{-1, 1\}$. The symbol $|x|$ stands for the Euclidean norm of x in \mathbb{R}^d and $\langle z, x \rangle$ does for the Euclidean inner product of z and x in \mathbb{R}^d . Let $\mathbb{R}_+ = [0, \infty)$. We use the words “increase” and “decrease” in the wide sense allowing flatness. Thus the word “monotone” means either “increase” or “decrease”. A precise definition of a selfsimilar additive process is given as below.

Definition 1.1. An \mathbb{R}^d -valued stochastic process $\{X(t) : t \geq 0\}$ on a probability space (Ω, \mathcal{F}, P) is called a *selfsimilar additive process* (or Sato process) with exponent $H > 0$ if the following conditions are satisfied:

(d.1) $\{X(t)\}$ is selfsimilar with exponent H , that is,

$$\{X(ct)\} \stackrel{d}{=} \{c^H X(t)\} \text{ for every } c > 0,$$

where the symbol $\stackrel{d}{=}$ stands for the equality of finite dimensional distributions.

(d.2) $\{X(t)\}$ has independent increments, that is,

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \dots, X(t_n) - X(t_{n-1})$$

are independent for any n and any choice of $0 \leq t_0 < t_1 < \dots < t_n$.

(d.3) $\{X(t)\}$ has càdlàg path, that is, almost surely $X(t)$ is right-continuous in $t \geq 0$ and has left limits in $t > 0$.

Instead of a selfsimilar additive process with exponent H , we sometimes say an *H -selfsimilar additive process*.

Note that the stochastic continuity of $\{X(t)\}$ follows from (d.1) and (d.2) above and that we do not assume the stationarity of increments. If the process $\{X(t)\}$ has stationary increments, that is, if the process is a Lévy process, then it is a strictly α -stable Lévy process with $\alpha = 1/H$. If there is $c \in \mathbb{R}^d$ such that $X(t) = t^H c$, then $\{X(t)\}$ is called deterministic. Otherwise it is called non-deterministic. The process $\{X(t)\}$ is called symmetric if $\{X(t)\} \stackrel{d}{=} \{-X(t)\}$. The process $\{X(t)\}$ is called Gaussian if the distribution of $X(t)$ is Gaussian at any time $t > 0$, otherwise it is called non-Gaussian.

Before [Sato \(1991\)](#), [Gettoor \(1979\)](#) gave the following two examples of selfsimilar additive processes in relation to the d -dimensional Brownian motion. Let $\{B(t), t \geq 0\}$ be a Brownian motion on \mathbb{R}^d , starting at the origin. Define the hitting time process $\{T_r, r \geq 0\}$ and, for $d \geq 3$, the last exit time process $\{L_r, r \geq 0\}$ as follows: For $r \geq 0$,

$$T_r := \inf\{t \geq 0 : |B(t)| = r\} \text{ and } L_r := \sup\{t \geq 0 : |B(t)| = r\}.$$

Then it is proved by [Gettoor \(1979\)](#) that $\{T_r, r \geq 0\}$ and $\{L_r, r \geq 0\}$ are \mathbb{R}_+ -valued increasing 2-selfsimilar additive processes. Note that in the case $d = 3$, $\{L_r, r \geq 0\}$ is a one-sided 1/2-stable Lévy process and, except this case, $\{T_r, r \geq 0\}$ and $\{L_r, r \geq 0\}$ are not Lévy processes. The explicit representations of the distributions of T_1 and L_1 are already known by [Ciesielski and Taylor \(1962\)](#) and [Gettoor \(1979\)](#), respectively. Here we define a class \mathcal{G}_1 of functions g (the so-called g -functions) on \mathbb{R}_+ as

$$\mathcal{G}_1 := \{g(x) : g(x) \text{ is positive and increasing on } [0, \infty), \\ \lim_{x \rightarrow \infty} g(x) = \infty\}. \quad (1.1)$$

Directly applying the key lemmas (Lemmas 3.1, 3.2, 4.3, and 4.5) of [Watanabe \(1996\)](#), we obtain the following results both as $r \rightarrow 0$ and as $r \rightarrow \infty$. Assertion (i) below is also found in [Gruet and Shi \(1995\)](#).

Theorem A. (i) *We have*

$$\liminf \frac{\log |\log r|}{r^2} T_r = \frac{1}{2} \quad a.s.$$

(ii) *Let $k := d/2 - 1$. We have*

$$\limsup \frac{T_r}{r^2 \log |\log r|} = \frac{2}{j_k^2} \quad a.s.$$

where j_k is the first positive zero of the Bessel function $J_k(x)$ of the first kind.

(iii) *Let $d \geq 3$. We have*

$$\liminf \frac{\log |\log r|}{r^2} L_r = \frac{1}{2} \quad a.s.$$

(iv) *Let $d \geq 3$ and $g \in \mathcal{G}_1$. When we consider the case $r \rightarrow 0$, we make an additional assumption that $x^2 g(|\log x|)$ is increasing in x with $0 < x < 1$. If $\int_0^\infty (g(x))^{-(d/2-1)} dx < \infty$ (resp. $= \infty$), then*

$$\limsup \frac{L_r}{r^2 g(|\log r|)} = 0 \text{ (resp. } = \infty) \text{ a.s.}$$

It is obvious that the laws in Theorem A are equivalent to the well known laws of the iterated logarithm for the Brownian motion $\{B(t)\}$. Thus we can give another nice proof for the following classical laws both as $t \rightarrow 0$ and as $t \rightarrow \infty$.

Corollary A. (i) *We have*

$$\limsup \frac{1}{\sqrt{t \log |\log t|}} \sup_{0 \leq s \leq t} |B(s)| = \sqrt{2} \quad a.s.$$

(ii) *Let $k := d/2 - 1$. We have*

$$\liminf \sqrt{\frac{\log |\log t|}{t}} \sup_{0 \leq s \leq t} |B(s)| = \frac{j_k}{\sqrt{2}} \quad a.s.$$

(iii) *Let $d \geq 3$. We have*

$$\limsup \frac{1}{\sqrt{t \log |\log t|}} \inf_{t \leq s} |B(s)| = \sqrt{2} \quad a.s.$$

(iv) *Let $d \geq 3$ and $g \in \mathcal{G}_1$. When we consider the case $t \rightarrow 0$, we make an additional assumption that $x^2 g(|\log x|)$ is increasing in x with $0 < x < 1$. If $\int_0^\infty (g(x))^{-(d/2-1)} dx < \infty$ (resp. $= \infty$), then*

$$\liminf \sqrt{\frac{g(|\log t|)}{t}} \inf_{t \leq s} |B(s)| = \infty \text{ (resp. } = 0) \text{ a.s.}$$

In Corollary A, “ $\sup_{0 \leq s \leq t} |B(s)|$ ” in (i) and “ $\inf_{t \leq s} |B(s)|$ ” in (iv) can be replaced by $|B(t)|$. Assertion (i), (ii), (iii) and (iv) above are due to [Khinchine \(1933\)](#), [Taylor \(1967\)](#), [Khoshnevisan et al. \(1994\)](#) and [Dvoretzky and Erdős \(1951\)](#), respectively. In particular, (ii) is called the Chung type law of the iterated logarithm. The order of the assertions in Theorem A corresponds to that in Corollary A. Throughout this paper, let $\{X(t)\}$ be an H -selfsimilar additive process on \mathbb{R}^d . We investigate limsup behaviors of $\{X(t)\}$.

By virtue of our key theorem (Theorem 2.11 in Sect.2), we obtain that for any $g \in \mathcal{G}_1$, there is $C \in [0, \infty]$ such that

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H g(\log t)} = C \quad a.s. \quad (1.2)$$

When $0 < C < \infty$, we take $Cg(x)$ in place of $g(x)$. Hence the cases to consider are three cases where $C = 0$, $C = 1$ and $C = \infty$. The law (1.2) is called to be normal if $C = 1$. Now we propose four problems. As is seen in (i) of Corollary A, the Gaussian case is already known. In the present paper, we focus on the non-Gaussian case.

If we suppose that $\limsup_{x \rightarrow \infty} g(x+1)/g(x) < e^H$ for $g \in \mathcal{G}_1$, our key theorem (Theorem 2.11) remains true with (1.2) replaced by

$$\limsup_{t \rightarrow 0} \frac{|X_t|}{t^H g(|\log t|)} = C \quad a.s. \quad (1.3)$$

Hence our theorems and propositions hold with (1.2) replaced by (1.3).

Problem 1. (Normalizability)

Suppose that $\{X(t)\}$ is given. What is a necessary and sufficient condition for the existence of $g \in \mathcal{G}_1$ satisfying (1.2) with $C = 1$? Further, in what way can we give this $g \in \mathcal{G}_1$?

Problem 2. (Integral test for g)

Suppose that $\{X(t)\}$ is given and that there is no $g \in \mathcal{G}_1$ satisfying (1.2) with $C = 1$. What criterion classifies functions $g \in \mathcal{G}_1$ into those satisfying (1.2) with $C = 0$ and those satisfying (1.2) with $C = \infty$?

Problem 3. (Possible types of g -functions in the normal laws)

Suppose that $g \in \mathcal{G}_1$ is given. What is a necessary and sufficient condition in terms of $g \in \mathcal{G}_1$ for the existence of $\{X(t)\}$ satisfying (1.2) with $C = 1$? Moreover, in what way can we give this $\{X(t)\}$?

Problem 4. (Integral test for $\{X(t)\}$)

Suppose that $g \in \mathcal{G}_1$ is given and that there is no $\{X(t)\}$ satisfying (1.2) with $C = 1$. What criterion classifies processes $\{X(t)\}$ into those satisfying (1.2) with $C = 0$ and those satisfying (1.2) with $C = \infty$?

In Sect.2, we answer Problems 1 and 2 in Theorems 2.13 and 2.14, and partially but substantially answer Problems 3 and 4 in Theorems 2.18, 2.19 and 2.28. In addition, we present several interesting and explicit examples with identification of the constant C in (1.2). In Sect.3, we show some preliminary results on the tail behaviors of multivariate infinitely divisible distributions. In Sect.4, we prove the main results mentioned in Sect.2. In Sect.5, we prove Theorem 3.9 in Sect.3.

2. Answers to four problems

In this section, we answer the problems mentioned in Sect.1. We give the answers only as $t \rightarrow \infty$. Those to the case $t \rightarrow 0$ are similar and omitted.

Let $\mathbb{N} = \{1, 2, \dots\}$, $\mathbb{Z} = \{0, \pm 1, \pm 2, \dots\}$, and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. For positive measurable functions $f(x)$ and $g(x)$ on \mathbb{R}^1 , we define the relation $f(x) \sim g(x)$ by $\lim_{x \rightarrow \infty} f(x)/g(x) = 1$ and the relation $f(x) \asymp g(x)$ by $0 < \liminf_{x \rightarrow \infty} f(x)/g(x) \leq \limsup_{x \rightarrow \infty} f(x)/g(x) < \infty$. We denote the tail of a measure η on \mathbb{R}^d by $\eta(|x| > r)$, that is, $\eta(|x| > r) := \eta(\{x : |x| > r\})$ and the right tail of a measure ζ on \mathbb{R}_+ by $\zeta(x > r)$, namely, $\zeta(x > r) := \zeta(\{x : x > r\})$. For a measure η on \mathbb{R}^d , we define the probability measure $\tilde{\eta}$ by

$$\tilde{\eta}(dx) := (\eta(|x| > 1))^{-1} 1_{\{|x| > 1\}} \eta(dx) \quad (2.1)$$

only when $0 < \eta(|x| > 1) < \infty$. Denote by $\delta_a(dx)$ the Dirac mass at $a \in \mathbb{R}^d$, that is, the probability measure concentrated at $a \in \mathbb{R}^d$. Let μ and ρ be distributions on \mathbb{R}^d . Denote by $\mu * \rho$ the convolution of μ and ρ and by μ^{n*} the n th convolution power of μ . Furthermore, we denote by $\hat{\mu}(z)$ the characteristic function of μ . A distribution μ on \mathbb{R}^d is called non-degenerate if its support is not included in any $(d-1)$ -dimensional hyperplane in \mathbb{R}^d .

In what follows, we use the terminology in Sato (1999). Let μ be an infinitely divisible distribution on \mathbb{R}^d , $d \geq 1$, with generating triplet (A, ν, γ) . Here A is the Gaussian covariance matrix, ν is the Lévy measure, and γ is the location parameter. That is,

$$\hat{\mu}(z) := \int_{\mathbb{R}^d} \exp(i\langle z, x \rangle) \mu(dx) = \exp(\psi(z)), \quad z \in \mathbb{R}^d \quad (2.2)$$

with

$$\psi(z) = \int_{\mathbb{R}^d} (e^{i\langle z, x \rangle} - 1 - 1_{\{|x| \leq 1\}}(x) i\langle z, x \rangle) \nu(dx) + i\langle \gamma, z \rangle - \frac{1}{2} \langle Az, z \rangle, \quad (2.3)$$

where ν is a measure on \mathbb{R}^d satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}^d} (1 \wedge |x|^2) \nu(dx) < \infty$, $\gamma \in \mathbb{R}^d$, and A is a nonnegative-definite matrix. If $\nu \neq 0$, then μ is said to be non-Gaussian. Further, if $A = 0$, then μ is said to be purely non-Gaussian.

Definition 2.1. A distribution μ on \mathbb{R}^d is said to be *selfdecomposable* (or of class L) if, for each $b \in (0, 1)$, there is a distribution λ_b on \mathbb{R}^d such that

$$\widehat{\mu}(z) = \widehat{\mu}(bz) \widehat{\lambda}_b(z). \quad (2.4)$$

Note that λ_b in (2.4) is infinitely divisible. Selfdecomposable distribution were introduced by Lévy (1954). They are infinitely divisible and their convolutions are selfdecomposable again. Stable including Gaussian, Pareto, log-normal, logistic, gamma, F, t, hyperbolic, half-Cauchy distributions are known to be selfdecomposable. In addition, so is a Weibull distribution with parameter $0 < \alpha \leq 1$ (see Remark 8.12 of Sato, 1999, as to parameter α). The proofs of the selfdecomposability of those distributions are not trivial, because the Lévy measures are not always explicitly known. See Example 15.13 and Exercise 34.14 of Sato (1999) and Examples 2.4 and 9.16 in Chapter V and 12.8 in Chapter VI of Steutel and van Harn (2004). In the following lemma, assertions (i) and (ii) is due to Wolfe (1980) for $d = 1$ and Sato (1980) for $d \geq 2$, and assertion (iii) is due to Sato (1982).

Lemma 2.2. *Let μ be a selfdecomposable distribution on \mathbb{R}^d .*

(i) *Let μ be non-Gaussian but not purely non-Gaussian. Then there are Gaussian distribution μ_1 and purely non-Gaussian selfdecomposable distribution μ_2 such that $\mu = \mu_1 * \mu_2$.*

(ii) *Let μ be purely non-Gaussian. Then the Lévy measure ν is expressed as*

$$\nu(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) k_\xi(r) r^{-1} dr \quad (2.5)$$

for a Borel set B in \mathbb{R}^d . Here σ is a finite non-zero measure on S^{d-1} and $k_\xi(r)$ is a nonnegative function which is measurable in $\xi \in S^{d-1}$ and decreasing in $r > 0$. In the α -stable case with $0 < \alpha < 2$, $k_\xi(r) = r^{-\alpha}$.

(iii) *If μ is non-degenerate, then it is absolutely continuous with respect to the Lebesgue measure.*

The selfsimilar additive processes $\{X(t)\}$ were characterized by Sato (1991) as follows.

Lemma 2.3. (i) *Let $\{X(t)\}$ be an H -selfsimilar additive process on \mathbb{R}^d . For any $t > 0$, the distribution of $X(t)$ is selfdecomposable.*

(ii) *Fix $H > 0$. For any selfdecomposable distribution μ , there is a unique in law H -selfsimilar additive process $\{X(t)\}$ with distribution μ at time 1.*

In the rest of this paper, we assume that $\{X(t)\}$ is non-deterministic. Denote by μ the distribution of $\{X(1)\}$. In the non-Gaussian case, the Lévy measure ν of μ is expressed as (2.5). The process $\{X(t)\}$ is called non-degenerate if μ is non-degenerate. For $l \in \mathbb{N}$, we define

$$\rho_l(dx) := P(X(1) - X(e^{-l}) \in dx). \quad (2.6)$$

Then ρ_l is infinitely divisible. The equation (2.4) holds with $b = e^{-lH}$ and $\lambda_b = \rho_l$. Let η_l be the Lévy measure of ρ_l . Note that $\int_{|x|>1} \log |x| \rho_l(dx) < \infty$, that is, ρ_l has finite log-moment by Theorem 25.3 of Sato (1999). If μ is non-degenerate, then it

is absolutely continuous by Lemma 2.2 (iii). Then we denote by $p(x)$ the density of μ . Let $m(d\xi)$ be the uniform probability measure on S^{d-1} . Let $l \in \mathbb{N}$. Define four functions $G_l(r)$, $K(r)$, $L(r)$, and $M(r)$ on $(0, \infty)$ as follows:

$$G_l(r) := P(|X(1) - X(e^{-l})| > r) = \rho_l(|x| > r), \quad (2.7)$$

$$K(r) := \int_{S^{d-1}} k_\xi(r) \sigma(d\xi), \quad (2.8)$$

$$L(r) := P(r < |X(1)| \leq e^{3H}r) = \mu(r < |x| \leq e^{3H}r), \quad (2.9)$$

and in the case where the measure μ is non-degenerate, we define

$$M(r) := \int_{S^{d-1}} p(r\xi) m(d\xi). \quad (2.10)$$

Then the tail of the Lévy measure η_l of ρ_l is expressed as

$$\eta_l(|x| > r) = \int_r^{e^{1H}r} \frac{K(s)}{s} ds. \quad (2.11)$$

As $\{X(t)\}$ is non-deterministic, $G_l(r)$ and $L(r)$ are positive on $(0, \infty)$ but $K(r)$ can be 0 for some $r > 0$. In particular, it means the Gaussian case that $K(r) = 0$ for all $r > 0$.

Definition 2.4. Let $\mathcal{K} = \mathbb{R}^d$ or let $\mathcal{K} = \mathbb{R}_+$. A positive measurable function $h(x)$ on \mathcal{K} is called *submultiplicative* on \mathcal{K} if there is $c > 0$ such that $h(x+y) \leq ch(x)h(y)$ for every $x, y \in \mathcal{K}$.

Remark 2.5. (i) If $h(x)$ is submultiplicative on \mathbb{R}_+ , then so is $h(|x|)$ on \mathbb{R}^d .

(ii) If $h(x)$ is submultiplicative on \mathbb{R}^d , then there are $c_1, c_2 > 0$ such that $h(x) \leq c_1 \exp(c_2|x|)$ for every $x \in \mathbb{R}^d$. See Lemma 25.5 of Sato (1999). For example, we consider a function $h(x) = \exp[|c|x|^\alpha (\log(|x| + 1))^\beta]$ on \mathbb{R}^d , where $c > 0$, $\alpha > 0$ and $-\infty < \beta < \infty$. Then $h(x)$ is submultiplicative only in the case where $0 < \alpha < 1$ or the case where $\alpha = 1$ and $-\infty < \beta \leq 0$.

Definition 2.6. A positive measurable function $h(x)$ on \mathbb{R}_+ is said to belong to the class *OR* if $h(cx) \asymp h(x)$ for any $c > 0$. In particular, $h(x)$ is said of *dominated variation* if it is monotone and $h(2x) \asymp h(x)$. Then we write $h \in D$.

Here we remark that $h(x) \in D$ if and only if $h(x)$ is monotone and $h(x) \in OR$.

Definition 2.7. The functions

$$h(x) = \int_0^x q(t) dt, \quad f(x) = \int_0^x q^{-1}(t) dt,$$

on \mathbb{R}_+ are called *Young conjugate* functions if $q(t)$ is positive on $(0, \infty)$, right-continuous and increasing with $q(0) = 0$ and $\lim_{t \rightarrow \infty} q(t) = \infty$.

For example, refer to pages 54 and 65 in Bingham et al. (1987) and to Kruglov and Antonov (1983) as to Definitions 2.6 and 2.7, respectively. Lastly, we introduce the important classes of distributions.

Definition 2.8. Let ζ be a distribution on \mathbb{R}_+ .

- (a) The distribution ζ is called to belong to the class \mathcal{S} if $\zeta * \zeta(x > r) \sim 2\zeta(x > r)$ as $r \rightarrow \infty$.
 (b) The distribution ζ is called to belong to the class \mathcal{OS} if $\zeta * \zeta(x > r) \asymp \zeta(x > r)$ as $r \rightarrow \infty$.

Remark 2.9. A distribution ρ on \mathbb{R}_+ in the class \mathcal{S} is called subexponential. Distributions in the class \mathcal{S} are contained in the class \mathcal{OS} . See also [Embrechts et al. \(1997\)](#); [Shimura and Watanabe \(2005\)](#).

Definition 2.10. Let ρ be a distribution on \mathbb{R}^d .

- (a) Set $\zeta(x > r) := \rho(|x| > r)$ for any $r \geq 0$. The distribution ρ is called to belong to the class \mathcal{OS} if $\zeta \in \mathcal{OS}$ as a distribution on \mathbb{R}_+ .
 (b) The distribution ρ is called to belong to the class \mathcal{D} if $\rho(|x| > r) \in \mathcal{D}$.

Recall the definition (1.1) of the class \mathcal{G}_1 . For $g \in \mathcal{G}_1$, we define

$$g^{-1}(x) := \sup\{y : g(y) < x\} \text{ for } x \geq 0 \quad (2.12)$$

with the understanding that $\sup \emptyset = 0$. Now we present the key theorem.

Theorem 2.11. Let $g \in \mathcal{G}_1$.

- (i) There is $C \in [0, \infty]$ such that (1.2) holds.
 (ii) The equality (1.2) holds for some $C \in [0, \infty]$ if and only if

$$\int_0^\infty G_l(\delta g(x)) dx = \int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) \begin{cases} < \infty & \text{for } \delta > C \text{ and all } l \in \mathbb{N}, \\ = \infty & \text{for } 0 < \delta < C \text{ and some } l = l(\delta) \in \mathbb{N}. \end{cases} \quad (2.13)$$

(iii) Suppose that $g \in \mathcal{G}_1$ and $g^{-1}(|x|)$ is submultiplicative on \mathbb{R}^d . Then we have (1.2) if and only if

$$\int_1^\infty \frac{K(r) - K(e^H r)}{r} g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (2.14)$$

In the case where $\limsup_{r \rightarrow \infty} K(e^H r)/K(r) < 1$, (2.14) is also equivalent to the following :

$$\int_1^\infty K(r) \frac{g^{-1}(\delta^{-1}r)}{r} dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (2.15)$$

Remark 2.12. (i) Suppose further that $\limsup_{x \rightarrow \infty} g(x+1)/g(x) < e^H$ is satisfied for $g \in \mathcal{G}_1$. By virtue of Proposition 4.2 in Sect.4, the theorem holds with (1.2) replaced by (1.3).

(ii) We can choose $g \in \mathcal{G}_1$ such that $\int_{\mathbb{R}^d} g^{-1}(|x|) \rho_1(dx) = \infty$, because the support of ρ_1 is unbounded. It follows from Theorem 2.11 (ii) that

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H} = \infty \quad \text{a.s.} \quad (2.16)$$

If $\lim_{x \rightarrow \infty} g(x) < \infty$, then (1.2) always holds with $C = \infty$. Hence any problem is not left by assuming that $\lim_{x \rightarrow \infty} g(x) = \infty$ in the definition of \mathcal{G}_1 .

(iii) We see from (2.13) that we can replace $g(\log t)$ in (1.2) by $g(a \log t)$ for any $a > 0$ without changing the value of the constant C .

We answer the first part of Problem 1. In the non-Gaussian case, the tail behavior of $G_l(r)$ is determined by that of ν . Thus we see from Lemmas 2.2 and Theorem 2.11 that the answer should be given in terms of the pair (σ, k_ξ) or of the function $K(r)$. Moreover, by virtue of Lemma 2.3, the answer could be given by means of μ .

Let $F(r)$ be arbitrarily chosen out of $G_1(r)$, $K(r)$, or $L(r)$. Note that the condition $K(r) \notin OR$ includes the case where $K(r)$ vanishes on (c, ∞) with some $c \geq 0$. In particular, the case $c = 0$ is the Gaussian case.

Theorem 2.13. *There exists $g \in \mathcal{G}_1$ such that (1.2) holds with $C = 1$ if and only if $F(r) \notin OR$.*

We answer Problem 2.

Theorem 2.14. *Let $g \in \mathcal{G}_1$. Suppose that $F(r) \in OR$. Then (1.2) holds with $C = 0$ or $C = \infty$ according as*

$$\int_0^\infty F(g(x))dx < \infty \quad \text{or} \quad = \infty. \quad (2.17)$$

As a corollary, we give the strictly stable Lévy case which has been already shown by Khintchine (1938) for $d = 1$ and by Yamamuro (2005) for $d \geq 2$. See also Pruitt and Taylor (1983) for the non-strict case with $d = 1$. Denote by $\log_{(n)} r$ n -fold iteration of the logarithmic function. Here we define a function $L_{(n,\alpha,\varepsilon)}(r)$ for $n \in \mathbb{N}$, $\alpha > 0$, and $\varepsilon \in \mathbb{R}^1$ by

$$L_{(n,\alpha,\varepsilon)}(r) := (\log_{(n+1)} r)^{\varepsilon+1/\alpha} \prod_{k=1}^n (\log_{(k)} r)^{1/\alpha}. \quad (2.18)$$

Corollary 2.15. *Let $g \in \mathcal{G}_1$ and $\{X(t)\}$ be a strictly α -stable Lévy process with $0 < \alpha < 2$. Then (1.2) holds with $C = 0$ or $C = \infty$ according as*

$$\int_0^\infty (g(x))^{-\alpha} dx < \infty \quad \text{or} \quad = \infty. \quad (2.19)$$

Here $K(r) = cr^{-\alpha}$ with some $c > 0$ and $H = 1/\alpha$.

In particular, we have for $n \in \mathbb{N}$

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^{1/\alpha} L_{(n,\alpha,\varepsilon)}(t)} \begin{cases} = 0 \text{ a.s.} & \text{for } \varepsilon > 0, \\ = \infty \text{ a.s.} & \text{for } \varepsilon \leq 0. \end{cases} \quad (2.20)$$

By virtue of Theorems 2.13 and 2.14 and Proposition 3.14 in Sect.3, we obtain the following corollary. It is useful in the case where the density of μ is explicitly known with $d = 1$.

Corollary 2.16. *Suppose either that $d = 1$ or that $\{X(t)\}$ is symmetric and non degenerated with $d \geq 2$.*

(i) *There is $g \in \mathcal{G}_1$ such that (1.2) holds with $C = 1$ if and only if $M(r) \notin OR$.*

(ii) *Let $g \in \mathcal{G}_1$ and $M(r) \in OR$. Then (1.2) holds with $C = 0$ or $C = \infty$ according as*

$$\int_0^\infty (g(x))^d M(g(x))dx < \infty \quad \text{or} \quad = \infty. \quad (2.21)$$

We give an example of Corollary 2.16. In the example, the density of Lévy measure is not simple. It was given by Halgreen (1979).

Example 2.17. If μ is a t -distribution on \mathbb{R}^1 with parameter $m > 0$, namely,

$$\mu(dx) = \Gamma\left(\frac{m+1}{2}\right) \left(\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)\right)^{-1} \frac{1}{(1+x^2)^{(m+1)/2}} dx.$$

When $m = 1$, μ is the Cauchy distribution. Let $g \in \mathcal{G}_1$. Then (1.2) holds with $C = 0$ or $C = \infty$ according as $\int_0^\infty (g(x))^{-m} dx < \infty$ or $= \infty$. In particular, we have, for $n \in \mathbb{N}$,

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H L_{(n,m,\varepsilon)}(t)} = \begin{cases} 0 & \text{a.s. for } \varepsilon > 0, \\ \infty & \text{a.s. for } \varepsilon \leq 0. \end{cases}$$

We answer Problem 3, where the answer is partial but substantial. If $g^{-1}(|x|)$ is not submultiplicative on \mathbb{R}^d , the problem is not easy and not yet completely solved. See Theorem 2.28 below.

Theorem 2.18. *Let $g \in \mathcal{G}_1$. If there exists $\{X(t)\}$ such that (1.2) holds with $C = 1$, then $g^{-1}(x) + \log x \notin OR$. The converse is also true provided that $g^{-1}(|x|)$ is submultiplicative on \mathbb{R}^d .*

We answer Problem 4 substantially. The answer is similar to Corollary 5.2 of Watanabe (2002a).

Theorem 2.19. *Let $g \in \mathcal{G}_1$ such that $g^{-1}(x) + \log x \in OR$. Then (1.2) holds with $C = 0$ or $C = \infty$ according as*

$$\int_{\mathbb{R}^d} g^{-1}(|x|) \rho_1(dx) < \infty \quad \text{or} \quad = \infty. \quad (2.22)$$

Now we show two propositions useful for answering to the second parts of Problems 1 and 3 besides Theorem 2.11.

Proposition 2.20. *Let $C \in [0, \infty)$ and $g \in \mathcal{G}_1$.*

(i) (1.2) holds if

$$\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \mu(dx) \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (2.23)$$

The converse is also true provided that $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \mu(dx) < \infty$ for some $\delta \in (0, \infty)$.

(ii) Suppose that $g^{-1}(|x|)$ is submultiplicative on \mathbb{R}^d . Then (1.2) holds if (2.15) holds. The converse is also true provided that

$$\int_1^\infty K(r) g^{-1}(\delta^{-1}r) r^{-1} dr < \infty \quad \text{for some } \delta \in (0, \infty). \quad (2.24)$$

Remark 2.21. By virtue of Proposition 2.20, if (2.13) holds with $C = \infty$, then (2.23) holds with $C = \infty$. However, even if (2.13) holds for some C with $0 \leq C < \infty$, its C does not always satisfy (2.23). For example, let $g^{-1}(x) = \log(x+1)$ for $x \geq 1$. Then we can choose μ such that (2.13) holds with $C = 0$ but (2.23) does with $C = \infty$. Moreover, there is $K(r)$ such that $K(r) \notin OR$ but $\tilde{\nu} \in \mathcal{D}$, that is, $\mu \in \mathcal{D}$ by Corollary 3.11 in Sect.3. See Remark 4.1 of Watanabe and Yamamuro (2010). Thus we see from Theorem 2.13 that there is $g \in \mathcal{G}_1$ such that (2.13) holds with $0 < C < \infty$, but (2.23) holds with $C = \infty$.

Proposition 2.22. Let $C \in [0, \infty]$ and $g \in \mathcal{G}_1$.

(i) Suppose that $\rho_1 \in \mathcal{OS}$. Then (1.2) holds if and only if

$$\int_0^\infty G_1(\delta g(x)) dx \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (2.25)$$

(ii) Suppose that $\tilde{\eta}_1 \in \mathcal{OS}$. Then (1.2) holds if and only if (2.14) holds.

(iii) Suppose that $\tilde{\nu} \in \mathcal{OS}$. Then (1.2) holds if (2.15) holds. The converse is also true provided that (2.24) is satisfied.

Here we give an example of Propositions 2.20 and 2.22.

Example 2.23. Let $C \in [0, \infty]$ and $g \in \mathcal{G}_1$. Suppose that μ is the standard lognormal distribution, namely,

$$\mu(dx) = \frac{1}{\sqrt{2\pi x}} \exp(-(\log x)^2/2) dx \text{ on } \mathbb{R}_+.$$

Then (1.2) holds if and only if (2.23) holds. In particular, we have

$$\limsup_{t \rightarrow \infty} \frac{X(t)}{t^H \exp\left(\sqrt{2 \log_{(2)} t}\right)} = 1 \text{ a.s.}$$

We investigate the law (1.2) in detail in the case where the g -function of the law is expressed as $g(t) = (\log t)^{1/\alpha} / \varphi(\log_{(2)} t)$ with some $0 < \alpha \leq 1$ and some function φ . Put $a \vee b := \max\{a, b\}$.

Proposition 2.24. Let $C \in [0, \infty]$. Suppose that $f(x)$ is regularly varying as $x \rightarrow \infty$ and $\exp(r^\alpha f(\log(r \vee 1)))$ with $0 < \alpha \leq 1$ is increasing and submultiplicative on \mathbb{R}_+ . Then

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t)^{1/\alpha} / f(\alpha^{-1} \log_{(3)} t)^{1/\alpha}} = C \text{ a.s.} \quad (2.26)$$

if and only if

$$\liminf_{r \rightarrow \infty} \frac{-\log K(r)}{r^\alpha f(\log r)} = C^{-\alpha}, \quad (2.27)$$

equivalently

$$\liminf_{r \rightarrow \infty} \frac{-\log \mu(|x| > r)}{r^\alpha f(\log r)} = C^{-\alpha}. \quad (2.28)$$

Recall the definition of Young conjugate: The functions $h(x)$ and $f(x)$ are Young conjugate if $h(x) = \int_0^x q(t) dt$ and $f(x) = \int_0^x q^{-1}(t) dt$, where $q(t)$ is positive on $(0, \infty)$, right-continuous and increasing with $q(0) = 0$ and $\lim_{t \rightarrow \infty} q(t) = \infty$.

Proposition 2.25. Let $C \in [0, \infty)$. Suppose that h and f are Young conjugate functions. Then we have

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t) / f^{-1}(\log_{(3)} t)} = C \text{ a.s.} \quad (2.29)$$

if and only if

$$\liminf_{r \rightarrow \infty} \frac{h^{-1}(-\log K(r))}{r} = C^{-1}, \quad (2.30)$$

equivalently

$$\liminf_{r \rightarrow \infty} \frac{-\log \rho_1(|x| > r)}{r f^{-1}(\log r)} = C^{-1}. \quad (2.31)$$

In particular, if $h(r)$ is regularly varying with positive index, then (2.30) is equivalent to

$$\liminf_{r \rightarrow \infty} \frac{-\log \mu(|x| > r)}{r f^{-1}(\log r)} = C^{-1}. \quad (2.32)$$

Proposition 2.26. *Let $C \in [0, \infty]$. Then we have*

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t) / \log_{(3)} t} = C \quad \text{a.s.} \quad (2.33)$$

if and only if $C = \inf\{r > 0 : K(r) = 0\}$ with the understanding that $\inf \emptyset = \infty$, equivalently

$$\lim_{r \rightarrow \infty} \frac{-\log \mu(|x| > r)}{r \log r} = C^{-1}. \quad (2.34)$$

Remark 2.27. If $\nu \neq 0$, then from Proposition 2.26 it follows that

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t) / \log_{(3)} t} = C_1 \quad \text{a.s.}$$

with $C_1 \in (0, \infty]$. If $\nu = 0$, then we have by the Gaussian type law of the iterated logarithm

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H \sqrt{\log_{(2)} t}} = C_2 \quad \text{a.s.}$$

with $C_2 \in (0, \infty)$. Thus there is a big difference in the “limsup” behaviors of $\{X(t)\}$ between the Gaussian and the non-Gaussian case.

We supplement Theorem 2.18 with the following theorem. Pay attention to Remark 2.5 (ii). In the theorem, $g^{-1}(|x|)$ is not submultiplicative on \mathbb{R}^d for $g \in \mathcal{G}_1$. It shows that Problem 3 is more difficult than the analogous problem in Watanabe (2002a). See Theorem 5.2 of Watanabe (2002a).

Theorem 2.28. *Let $g \in \mathcal{G}_1$.*

(i) *There is $\{X(t)\}$ such that (1.2) holds with $C = 1$ in the following cases :*

(1) *For some $c \in (0, \infty)$,*

$$g^{-1}(x) \asymp \exp(cx^2). \quad (2.35)$$

(2) *For some $c \in (0, \infty)$,*

$$g^{-1}(x) \asymp \exp(cx \log x). \quad (2.36)$$

(3) *Suppose that $f(x)$ is expressed as $f(x) = \int_0^x q(t) dt$ on $[0, \infty)$, where $q(t)$ is positive on $(0, \infty)$, right-continuous and increasing with $q(0) = 0$ and $\lim_{t \rightarrow \infty} q(t) = \infty$.*

Further, $g(x)$ satisfies that

$$g^{-1}(x) \asymp \exp(x f^{-1}(\log x)). \quad (2.37)$$

(ii) *There is no $\{X(t)\}$ such that (1.2) holds with $C = 1$ in the following cases:*

(1) The function g satisfies that

$$\lim_{x \rightarrow \infty} \frac{g^{-1}(x)}{\exp(cx^2)} = \infty \text{ for all } c \in (0, \infty). \quad (2.38)$$

(2) The function g satisfies that

$$\lim_{x \rightarrow \infty} \frac{g^{-1}(x)}{\exp(cx \log x)} = \infty \text{ for all } c \in (0, \infty), \quad (2.39)$$

and

$$\lim_{x \rightarrow \infty} \frac{g^{-1}(x)}{\exp(cx^2)} = 0 \text{ for all } c \in (0, \infty). \quad (2.40)$$

Remark 2.29. There is not always $\{X(t)\}$ such that (1.2) holds with $C = 1$ even provided that $g^{-1}(x) + \log x \notin OR$. For example, consider a function $g \in \mathcal{G}_1$ such that $g^{-1}(x) \asymp \exp(c_0 x (\log x)^{\alpha_0})$ for some $\alpha_0 > 1$ and $c_0 \in (0, \infty)$, and apply Theorem 2.28 (ii).

Finally, we give an example of Propositions 2.24 and 2.25. It is also related with Proposition 2.26 and Theorem 2.28. The example shows that even if $K(r)$ has the same form of functions with parameters α and β , there is a delicate difference in the law (1.2) according as the parameters change their values. The difference essentially comes from whether $g^{-1}(x)$ is submultiplicative on \mathbb{R}_+ for the g -function of the law or not.

Example 2.30. Let $d \geq 1$, $c, c' \in [0, \infty]$, $0 < \alpha, \alpha' < \infty$, and $-\infty < \beta, \beta', \gamma' < \infty$. We consider the following two kinds of assumptions :

$$\liminf_{r \rightarrow \infty} \frac{-\log K(r)}{r^\alpha (\log r)^\beta} = c, \quad (2.41)$$

and

$$\liminf_{r \rightarrow \infty} \frac{-\log \mu(|x| > r)}{r^{\alpha'} (\log r)^{\beta'} (\log_{(2)} r)^{\gamma'}} = c'. \quad (2.42)$$

(i) Suppose either that $0 < \alpha < 1$ or that $\alpha = 1$ and $\beta \leq 0$. Only in this case, $g^{-1}(x)$ is submultiplicative for the g -function of the law (2.43) below. We have

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t)^{1/\alpha} / (\log_{(3)} t)^{\beta/\alpha}} = \alpha^{\beta/\alpha} c^{-1/\alpha} \text{ a.s.} \quad (2.43)$$

if and only if (2.41) holds, equivalently (2.42) holds with $c' = c$, $\alpha' = \alpha$, $\beta' = \beta$, and $\gamma' = 0$. In particular, if μ is a Weibull distribution with parameter $0 < \alpha \leq 1$, namely, $\mu(dx) = \alpha x^{\alpha-1} \exp(-x^\alpha) dx$ on \mathbb{R}_+ , then (2.43) holds with $c = 1$ and $\beta = 0$.

(ii) Suppose that $c \in (0, \infty]$ and either that $\alpha = 1$ and $\beta > 0$ or that $\alpha > 1$ and $-\infty < \beta < \infty$. Let $C_1 := c^{-1}$ for $\alpha = 1$ and $C_1 := \alpha^{(\beta-\alpha)/\alpha} (\alpha-1)^{(\alpha-1)/\alpha} c^{-1/\alpha}$ for $\alpha > 1$. Then we have

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H (\log_{(2)} t) / ((\log_{(3)} t)^{(\alpha-1)} (\log_{(4)} t)^\beta)^{1/\alpha}} = C_1 \text{ a.s.}$$

if and only if (2.41) holds, equivalently (2.42) holds with $c' = C_1^{-1}$, $\alpha' = 1$, $\beta' = (\alpha-1)/\alpha$, and $\gamma' = \beta/\alpha$.

3. Results on tail behaviors

In this section, we give several preliminary results on the tail behaviors of infinitely divisible distributions on \mathbb{R}^d . The results below except for the lemmas are new and of interest in themselves. Theorem 3.9 below is proved only in Sect.5. As in Sect.2, we denote by μ the distribution of $X(1)$. Then μ is a selfdecomposable distribution on \mathbb{R}^d except for the delta measures. We continue to use the notation of Sect.2. The following definition is due to [Dharmadhikari and Joag-Dev \(1988\)](#).

Definition 3.1. Let ρ be a distribution on \mathbb{R}^d .

(a) Let $d = 1$. The distribution ρ is called *unimodal* with mode a if

$$\rho(dx) = p(x)dx + c\delta_a(dx) \quad (3.1)$$

where $c \geq 0$, $p(x)$ is increasing on $(-\infty, a)$ and decreasing on (a, ∞) .

(b) Let $d \geq 1$. Suppose that ρ is absolutely continuous on \mathbb{R}^d . Then ρ is called *star-unimodal* about 0 if

$$\rho(dx) = p(x)dx \quad (3.2)$$

where $p(r\xi)$ is decreasing in r on $(0, \infty)$ for every $\xi \in S^{d-1}$.

Definition 3.2. Let $h(x)$ be a positive measurable function on \mathbb{R}^d . The function $h(x)$ is called *quasi-submultiplicative* on \mathbb{R}^d if the following conditions are satisfied:

- (d.1) For any $R > 0$, $\sup_{|x| \leq R} h(x) < \infty$
- (d.2) For each $\varepsilon > 0$, there are $c_1, c_2 > 0$ such that

$$h(x + y) \leq c_1 h((1 + \varepsilon)x) h(c_2 y) \text{ for every } x, y \in \mathbb{R}^d. \quad (3.3)$$

Remark 3.3. (i) Let $h(x)$ be quasi-submultiplicative on \mathbb{R}^d . Note that letting $y = 0$ in (3.3), we have the following :

(d.3) For any $\delta > 1$, there is $c_3 > 0$ such that $h(x) \leq c_3 h(\delta x)$ for every $x \in \mathbb{R}^d$.

(ii) Dividing into the two cases $|y| \leq \varepsilon|x|$ and $|y| > \varepsilon|x|$ for $\varepsilon > 0$, we see that if the function $h(x)$ is positive and increasing on \mathbb{R}_+ , then $h(|x|)$ is quasi-submultiplicative on \mathbb{R}^d .

The following celebrated results (i) and (ii) are due to [Yamazato \(1978\)](#) and [Wolfe \(1978\)](#), respectively.

Lemma 3.4. *Suppose that μ is non-degenerate.*

- (i) *If $d = 1$, then μ is unimodal.*
- (ii) *If $\{X(t)\}$ is symmetric, then μ is star-unimodal about 0.*

The following lemma is from Theorem 25.3 of [Sato \(1999\)](#). Refer also to [Kruglov \(1970, 1972\)](#) and [Sato \(1973\)](#).

Lemma 3.5. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Furthermore, let $h(x)$ be submultiplicative on \mathbb{R}^d . Then $\int_{\mathbb{R}^d} h(x)\rho(dx) < \infty$ if and only if $\int_{|x|>1} h(x)\eta(dx) < \infty$.*

Proposition 3.6. *Let $h(x)$ be quasi-submultiplicative on \mathbb{R}^d .*

(i) *If*

$$\int_{\mathbb{R}^d} h(x)\mu(dx) < \infty, \quad (3.4)$$

then for any $\varepsilon \in (0, 1)$ and any $l \in \mathbb{N}$,

$$\int_{\mathbb{R}^d} h((1 + \varepsilon)^{-1}x) \rho_l(dx) < \infty. \quad (3.5)$$

(ii) If

$$\int_{\mathbb{R}^d} h(cx) \mu(dx) < \infty \text{ and } \int_{\mathbb{R}^d} h(x) \mu(dx) = \infty \quad (3.6)$$

for some $c \in (0, \infty)$, then for any $\varepsilon \in (0, 1)$, there is $l(\varepsilon) \in \mathbb{N}$ such that for any integer $l \geq l(\varepsilon)$,

$$\int_{\mathbb{R}^d} h((1 + \varepsilon)x) \rho_l(dx) = \infty. \quad (3.7)$$

Proof: First we prove (i). Suppose that (3.4) holds. Define a subset Λ of $(\mathbb{R}^d)^\mathbb{N}$ by

$$\Lambda := \{(x_j)_{j=1}^\infty : |x_j| \leq Ne^{jHl/2} \text{ for } j \geq 1\}.$$

Here we choose $N > 0$ such that $\rho_l(\{x : |x| \leq N\}) > 0$. Since the log-moment of ρ_l is finite, we have

$$c_0 := \int_{\Lambda} \prod_{n=1}^\infty \rho_l(dx_n) > 0. \quad (3.8)$$

Moreover, since $h(x)$ is quasi-submultiplicative, we see from condition (d.1) of Definition 3.2 that

$$c_4 := \sup_{(x_n) \in \Lambda} h(-(1 + \varepsilon)^{-1}c_2 \sum_{n=1}^\infty e^{-nHl} x_n) < \infty.$$

Hence we have

$$\begin{aligned} & c_0 \int_{\mathbb{R}^d} h((1 + \varepsilon)^{-1}x_0) \rho_l(dx_0) \\ & \leq c_1 \int_{\mathbb{R}^d \times \Lambda} h\left(\sum_{n=0}^\infty e^{-nHl} x_n\right) h(-(1 + \varepsilon)^{-1}c_2 \sum_{n=1}^\infty e^{-nHl} x_n) \prod_{n=0}^\infty \rho_l(dx_n) \\ & \leq c_1 c_4 \int_{\mathbb{R}^d \times \Lambda} h\left(\sum_{n=0}^\infty e^{-nHl} x_n\right) \prod_{n=0}^\infty \rho_l(dx_n) \\ & \leq c_1 c_4 \int_{(\mathbb{R}^d)^{\mathbb{Z}_+}} h\left(\sum_{n=0}^\infty e^{-nHl} x_n\right) \prod_{n=0}^\infty \rho_l(dx_n) \\ & = c_1 c_4 \int_{\mathbb{R}^d} h(x) \mu(dx) < \infty. \end{aligned}$$

Thus we have (3.5).

Next we prove (ii). Let $\varepsilon > 0$. Suppose that (3.6) holds for some $c \in (0, \infty)$. We see that

$$\begin{aligned} E(h(X(1))) & \leq c_1 E(h((1 + \varepsilon)(X(1) - X(e^{-l})))h(c_2 X(e^{-l}))) \\ & = c_1 E(h((1 + \varepsilon)(X(1) - X(e^{-l}))) E(h(c_2 X(e^{-l}))) \\ & = c_1 E(h((1 + \varepsilon)(X(1) - X(e^{-l}))) E(h(c_2 e^{-Hl} X(1))). \end{aligned}$$

For all sufficiently large l , we have $E(h(c_2 e^{-Hl} X(1))) < \infty$ by (d.3) of Remark 3.3, so we obtain

$$E(h((1 + \varepsilon)(X(1) - X(e^{-l})))) = \infty.$$

Hence (3.7) holds. \square

Lemmas 3.7 and 3.8 below show that it is possible that an infinitely divisible distribution and its Lévy measure do not have the same tail behaviors. The following is due to Sato (1973). Let B_r be the closed ball with center 0 and radius r . Denote by $S(\eta)$ the support of a measure η on \mathbb{R}^d .

Lemma 3.7. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Let $C := \inf\{r \geq 0 : S(\eta) \subset B_r\}$. Then we have the following :*

- (i) *If $0 < a < 1/C$, then $\rho(|x| > r) = o(r^{-ar})$.*
- (ii) *If $a > 1/C$, then $r^{-ar} = o(\rho(|x| > r))$.*
- (iii) *We have*

$$\lim_{r \rightarrow \infty} \frac{-\log \rho(|x| > r)}{r \log r} = C^{-1}. \quad (3.9)$$

Let $q(t)$ be a positive and right-continuous function on $(0, \infty)$. Define the function $\varphi(r)$ by $\log r = \int_0^{\varphi(r)} q(t) dt$. If $q(t) = 1$ on \mathbb{R}_+ , then $\varphi(r) = \log r$. The limit of $(-\log \rho(|x| > r))/(r\varphi(r))$ is discussed by Kruglov and Antonov (1983) in the case where $q(t)$ is increasing:

Lemma 3.8. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Suppose that h and f are Young conjugate functions. Let $0 < \gamma \leq \infty$. Then we have*

$$\liminf_{r \rightarrow \infty} \frac{-\log \rho(|x| > r)}{r f^{-1}(\log r)} = \gamma \quad (3.10)$$

if and only if

$$\liminf_{r \rightarrow \infty} \frac{h^{-1}(-\log \eta(|x| > r))}{r} = \gamma. \quad (3.11)$$

We give a sufficient condition for which an infinitely divisible distribution and its Lévy measure have the same tail behaviors in the relation “ \asymp ”. The result is found also in Shimura and Watanabe (2005) in the case of an infinitely divisible distribution on \mathbb{R}_+ . It is proved in Sect.5. Recall the definition (2.1) of $\tilde{\eta}$ for a measure η .

Theorem 3.9. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Define a distribution ζ on \mathbb{R}_+ by $\zeta(x > r) := \tilde{\eta}(|x| > r)$ for $r \geq 0$.*

(i) *$\rho \in \mathcal{OS}$ on \mathbb{R}^d if and only if there is a positive integer n such that $\tilde{\eta}^{n*} \in \mathcal{OS}$ on \mathbb{R}^d . Moreover, if $\rho \in \mathcal{OS}$, then, for the above n ,*

$$\rho(|x| > r) \asymp \tilde{\eta}^{n*}(|x| > r). \quad (3.12)$$

(ii) *If $\tilde{\eta} \in \mathcal{OS}$, then we have*

$$\rho(|x| > r) \asymp \eta(|x| > r). \quad (3.13)$$

(iii) *$\rho \in \mathcal{OS}$ if and only if there is a positive integer m such that $\zeta^{m*} \in \mathcal{OS}$ on \mathbb{R}_+ and*

$$\rho(|x| > r) \asymp \zeta^{m*}(x > r). \quad (3.14)$$

(iv) *Let $h(x)$ be a nonnegative increasing function on \mathbb{R}_+ . Suppose that $\tilde{\eta} \in \mathcal{OS}$ on \mathbb{R}^d . Then $\int_{\mathbb{R}^d} h(|x|)\rho(dx) < \infty$ if and only if $\int_{|x|>1} h(|x|)\eta(dx) < \infty$.*

Remark 3.10. In (i) and (iii) of Theorem 3.9, n and m are not always 1. See Theorem 1.1 of Shimura and Watanabe (2005). In (iii), we do not know whether $\rho \in \mathcal{OS}$ provided that there is a positive integer $m \geq 2$ such that $\zeta^{m*} \in \mathcal{OS}$ on \mathbb{R}_+ .

We give a result as to the class \mathcal{D} . The class \mathcal{OS} includes the class \mathcal{D} . The fact is found also in Watanabe (1996) in the case of infinitely divisible distributions on \mathbb{R}_+ .

Corollary 3.11. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Then the following holds:*

- (i) $\rho \in \mathcal{D}$ if and only if $\tilde{\eta} \in \mathcal{D}$.
- (ii) If $\tilde{\eta} \in \mathcal{D}$, then $\rho(|x| > r) \asymp \eta(|x| > r)$.

Proof: Let ζ be the distribution on \mathbb{R}_+ defined in Theorem 3.9. Note that $\mathcal{D} \subset \mathcal{OS}$ on \mathbb{R}_+ and on \mathbb{R}^d . Thus assertion (ii) is obvious from Theorem 3.9 (ii). Next we prove assertion (i). See Lemma 5.3 (ii) in Sect.5. We find from Theorem 3.9 (i) that if $\tilde{\eta} \in \mathcal{D}$, then $\rho \in \mathcal{D}$. We obtain from Theorem 3.9 (iii) that if $\rho \in \mathcal{D}$, then $\zeta^{m*} \in \mathcal{D}$ on \mathbb{R}_+ for some $m \in \mathbb{N}$. Moreover we see from Proposition 1.1 (iii) and 2.5 (iii) of Shimura and Watanabe (2005) (Lemma 5.4 in Sect.5) that $\zeta^{m*} \in \mathcal{D}$ on \mathbb{R}_+ for some $m \in \mathbb{N}$ if and only if $\zeta \in \mathcal{D}$ on \mathbb{R}_+ . Thus, if $\rho \in \mathcal{D}$, then $\zeta \in \mathcal{D}$ and thereby assertion (i) is true. \square

Proposition 3.12. (i) $G_1(r) \in OR$ if and only if $L(r) \in OR$.
(ii) If $L(r) \in OR$, then $G_1(r) \asymp L(r)$.

Proof: Let $b := e^{-H}$ and $\delta := e^H - 1$. On the one hand, we see

$$\begin{aligned} \mu(|x| > r) &= P(|X(e^{-1}) + X(1) - X(e^{-1})| > r) \\ &\leq P(e^{-H}|X(1)| > \sqrt{b}r) + P(|X(1) - X(e^{-1})| > (1 - \sqrt{b})r) \\ &= \mu(|x| > r/\sqrt{b}) + \rho_1(|x| > (1 - \sqrt{b})r). \end{aligned}$$

Hence we have

$$\mu(r < |x| \leq r/\sqrt{b}) \leq \rho_1(|x| > (1 - \sqrt{b})r).$$

We obtain that

$$\begin{aligned} \mu(r < |x| \leq e^{3H}r) &\leq \sum_{j=0}^5 \mu(b^{-j/2}r < |x| \leq b^{-(j+1)/2}r) \\ &\leq \sum_{j=0}^5 \rho_1(|x| > (1 - \sqrt{b})b^{-j/2}r) \\ &\leq 6\rho_1(|x| > (1 - \sqrt{b})r). \end{aligned} \tag{3.15}$$

On the other hand, we see

$$\begin{aligned} \mu(|x| > r) &= P(|X(e^{-1}) + X(1) - X(e^{-1})| > r) \\ &\geq P(|X(1) - X(e^{-1})| > e^H r, |X(e^{-1})| \leq \delta r) \\ &\quad + P(|X(e^{-1}) + X(1) - X(e^{-1})| > r, |X(1) - X(e^{-1})| \leq e^H r) \\ &\geq P(|X(1) - X(e^{-1})| > e^H r)P(|X(e^{-1})| \leq \delta r) \\ &\quad + P(|X(e^{-1})| > e^{2H} r)P(|X(1) - X(e^{-1})| \leq e^H r) \\ &= \rho_1(|x| > e^H r)\mu(|x| \leq \delta r/b) + \mu(|x| > e^{3H} r)\rho_1(|x| \leq e^H r). \end{aligned}$$

Hence we have

$$\mu(r < |x| \leq e^{3H}r) \geq \rho_1(|x| > e^Hr) \{ \mu(|x| \leq \delta r/b) - \mu(|x| > e^{3H}r) \}. \quad (3.16)$$

By (3.15) and (3.16), $L(r) := \mu(r < |x| \leq e^{3H}r) \in OR$ if and only if $G_1(r) := \rho_1(|x| > r) \in OR$. Furthermore, if $L(r) \in OR$, then $L(r) \asymp G_1(r)$. \square

Theorem 3.13. (i) *The following are equivalent:*

- (1) $G_1(r) \in OR$, (2) $K(r) \in OR$, (3) $L(r) \in OR$.
- (ii) *If $K(r) \in OR$, then $G_1(r) \asymp K(r) \asymp L(r)$.*

Proof: Now (2.11) implies that $\eta_1(|x| > r) \in OR$ is equivalent to $K(r) \in OR$. Use Corollary 3.11 with $\rho = \rho_1$ and $\eta = \eta_1$. Hence $\eta_1(|x| > r) \in OR$, namely, $\tilde{\eta}_1 \in \mathcal{D}$ is equivalent to $\rho_1 \in \mathcal{D}$, namely, $G_1(r) := \rho_1(|x| > r) \in OR$. Furthermore, if $K(r) \in OR$, namely, $\tilde{\eta}_1 \in \mathcal{D}$, then $G_1(r) = \rho_1(|x| > r) \asymp \eta_1(|x| > r) \asymp K(r)$. The remaining proof is proved by Proposition 3.12. \square

Proposition 3.14. *Suppose either that $d = 1$ or that $\{X(t)\}$ is symmetric and non degenerated with $d \geq 2$.*

- (i) *$M(r) \in OR$ if and only if $L(r) \in OR$.*
- (ii) *If $M(r) \in OR$, then $r^d M(r) \asymp L(r)$.*

Proof: We have

$$L(r) := \mu(r < |x| \leq e^{3H}r) = \int_r^{e^{3H}r} M(u)u^{d-1}du,$$

By Lemma 3.4, the proposition is obvious. \square

4. Proof of the results

In this section, we prove the results mentioned in Sect.2. First of all, we present two important propositions which lead to the key theorem, that is, Theorem 2.11.

Proposition 4.1. *Let $g \in \mathcal{G}_1$ and $l \in \mathbb{N}$.*

- (i) *If*

$$\int_0^\infty P(|X(1) - X(e^{-l})| > g(x))dx = \infty, \quad (4.1)$$

then

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H g(\log t)} \geq 1 \quad a.s. \quad (4.2)$$

- (ii) *If*

$$\int_0^\infty P(|X(1) - X(e^{-l})| > g(x))dx < \infty, \quad (4.3)$$

then

$$\limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H g(\log t)} \leq (1 - e^{-lH})^{-1} \quad a.s. \quad (4.4)$$

Proposition 4.2. Let $g \in \mathcal{G}_1$ and $l \in \mathbb{N}$. Suppose that $\limsup_{x \rightarrow \infty} g(x+1)/g(x) < e^H$.

(i) If

$$\int_0^\infty P(|X(1) - X(e^{-l})| > g(x)) dx = \infty, \quad (4.5)$$

then

$$\limsup_{t \rightarrow 0} \frac{|X(t)|}{t^H g(|\log t|)} \geq 1 \quad a.s. \quad (4.6)$$

(ii) If

$$\int_0^\infty P(|X(1) - X(e^{-l})| > g(x)) dx < \infty, \quad (4.7)$$

then

$$\limsup_{t \rightarrow 0} \frac{|X(t)|}{t^H g(|\log t|)} \leq (1 - e^{-lH_1})^{-1} \quad a.s. \quad (4.8)$$

for any $H_1 \in (0, H)$.

In order to prove Propositions 4.1 and 4.2, we use some results of [Watanabe \(2002a\)](#) for shift selfsimilar additive random sequences defined below.

Definition 4.3. Let $c > 1$. An \mathbb{R}^d -valued random sequence $\{Y(n), n \in \mathbb{Z}\}$ on a probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})$ is called a *shift c -self-similar additive random sequence* if the following two conditions are satisfied:

(1) The sequence $\{Y(n), n \in \mathbb{Z}\}$ has shift c -self-similarity, that is,

$$\{Y(n+1), n \in \mathbb{Z}\} \stackrel{d}{=} \{cY(n), n \in \mathbb{Z}\},$$

where the symbol $\stackrel{d}{=}$ stands for equality in the finite-dimensional distributions.

(2) The sequence $\{Y(n), n \in \mathbb{Z}\}$ has independent increments (or additivity), that is, for every $n \in \mathbb{Z}$, $\{Y(k), k \leq n\}$ and $Y(n+1) - Y(n)$ are independent.

In the rest of this section, we define the random sequence $\{Y(n), n \in \mathbb{Z}\}$ by $Y(n) := X(e^n)$ for $n \in \mathbb{Z}$. Under the assumption that $\{X(t)\}$ is not purely Gaussian, we decompose $\{X(t)\}$ in law as the sum of two independent selfsimilar additive processes $\{X_j(t)\}$ for $j = 1, 2$ as follows: Denote by μ_j the distribution of $X_j(1)$ for $j = 1, 2$. Define $k_{(\xi, N)}(r) := k_\xi(r \vee N^{-1})$ for $N > 0$ and denote by $K_1(r)$ the function $K(r)$ in (2.8) replacing $k_\xi(r)$ by $k_{(\xi, N)}(r)$. Here we note that $K(r) - K_1(r) = 0$ for $r \geq N^{-1}$. By Lemma 2.3, we can define $\{X_j(t)\}$ by determining μ_j for $j = 1, 2$. Now μ satisfies (2.2), (2.3) and (2.5). Hence we take μ_1 and μ_2 such that $\mu = \mu_1 * \mu_2$ and for $j = 1, 2$,

$$\hat{\mu}_j(z) := \int_{\mathbb{R}^d} \exp(i\langle z, x \rangle) \mu_j(dx) = \exp(\psi_j(z)), \quad z \in \mathbb{R}^d \quad (4.9)$$

with

$$\psi_1(z) := \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty (e^{i\langle z, r\xi \rangle} - 1) k_{(\xi, N)}(r) r^{-1} dr, \quad (4.10)$$

and

$$\psi_2(z) := \psi(z) - \psi_1(z). \quad (4.11)$$

The following is obvious from the definition.

Lemma 4.4. *The sequence $\{Y(n), n \in \mathbb{Z}\}$ is a shift e^H -self-similar additive random sequence.*

The following lemma is from Lemma 5.2 in [Watanabe \(2002a\)](#). Remark 4.5 is needed in the proof of Lemma 4.6 (i) below.

Remark 4.5. Let $g \in \mathcal{G}_1$ and let $l \in \mathbb{N}$. If

$$\int_0^\infty P(|Y(0) - Y(-l)| > g(x))dx = \infty, \quad (4.12)$$

then, for all $k \in \mathbb{N}$ and all $\varepsilon \in (0, 1)$,

$$\int_0^\infty P(|Y(0) - Y(-kl)| > (1 - \varepsilon)g(x))dx = \infty. \quad (4.13)$$

Lemma 4.6. *Let $g \in \mathcal{G}_1$ and let $l \in \mathbb{N}$.*

(i) *If*

$$\int_0^\infty P(|Y(0) - Y(-l)| > g(x))dx = \infty, \quad (4.14)$$

then

$$\limsup_{n \rightarrow \infty} \frac{|Y(n)|}{e^{nH}g(n)} \geq 1 \quad \text{a.s.} \quad (4.15)$$

(ii) *Suppose that $\limsup_{x \rightarrow \infty} g(x+1)/g(x) < e^H$. If (4.14) is satisfied, then*

$$\limsup_{n \rightarrow \infty} \frac{|Y(-n)|}{e^{-nH}g(n)} \geq 1 \quad \text{a.s.} \quad (4.16)$$

In Lemmas 4.7 and 4.8 below, we assume that $\{X(t)\}$ is non-Gaussian. Further, we take sufficiently large N and let $\{X_1(t)\}$ be non-Gaussian.

Lemma 4.7. *Let $l \in \mathbb{N}$. There is $c_1 > 0$ such that, for $r \geq 0$,*

$$\sup_{t \in [e^{-l}, 1]} P(|X_1(t) - X_1(e^{-l})| > r) \leq c_1 P(|X_1(1) - X_1(e^{-l})| > r). \quad (4.17)$$

Proof: Note that the distribution of $X_1(t) - X_1(e^{-l})$ for $t \in (e^{-l}, 1]$ is compound Poisson. Denote by ν_t its Lévy measure. Then ν_t is represented as

$$\nu_t(B) = \int_{S^{d-1}} \sigma(d\xi) \int_0^\infty 1_B(r\xi) \frac{k_{(\xi, N)}(r/t^H) - k_{(\xi, N)}(e^{lH}r)}{r} dr$$

for any Borel set B in \mathbb{R}^d . We have, for $r \geq 0$,

$$P(|X_1(t) - X_1(e^{-l})| > r) = e^{-c(t)} \sum_{n=1}^\infty \frac{1}{n!} \nu_t^{n*}(|x| > r),$$

where $c(t) := \nu_t(\mathbb{R}^d) < \infty$. Then $\nu_t(B)$ is increasing in t , so is $\sum_{n=1}^\infty \nu_t^{n*}(B)/n!$. Thus we obtain that

$$\sup_{t \in [e^{-l}, 1]} P(|X_1(t) - X_1(e^{-l})| > r) \leq e^{c(1)} P(|X_1(1) - X_1(e^{-l})| > r).$$

Thus we have (4.17). □

Lemma 4.8. *Let $\delta > 0$ and $l \in \mathbb{N}$. If N of $k_{(\xi, N)}(r)$ is sufficiently large, we can take sufficiently large $\delta_0 > \delta$ such that*

$$\sup_{t \in [e^{-l}, 1]} P(|X_2(t) - X_2(e^{-l})| > r) = o(\exp(-\delta_0 r \log r)) \quad \text{as } r \rightarrow \infty. \quad (4.18)$$

Proof: We see that

$$\begin{aligned} & \sup_{t \in [e^{-l}, 1]} P(|X_2(t) - X_2(e^{-l})| > r) \\ & \leq \sup_{t \in [e^{-l}, 1]} (P(|X_2(t)| > r/2) + P(|X_2(e^{-l})| > r/2)) \\ & \leq 2P(|X_2(1)| > r/2). \end{aligned}$$

In the last inequality, we used the selfsimilarity. Now the support of the Lévy measure of $\{X_2(t)\}$ is the empty set or is contained in a small ball with center 0. Hence we obtain (4.18) from Lemma 3.7. \square

Lemma 4.9. *Let $l \in \mathbb{N}$. For any $\varepsilon \in (0, 1)$, there is $c_2 > 0$ such that, for $r \geq 0$,*

$$\sup_{t \in [e^{-l}, 1]} P(|X(t) - X(e^{-l})| > r) \leq c_2 P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r). \quad (4.19)$$

Proof: In the case where $\{X(t)\}$ is Gaussian, the lemma can be proved by straightforward calculations. Thus we only prove the case where $\{X(t)\}$ is non-Gaussian. Choose $\delta \in (0, 1)$ such that $1 - \varepsilon = (1 - \delta)^2$. By Lemmas 4.7 and 4.8, there is $c > 0$ such that for sufficiently large N and δ_0 ,

$$\begin{aligned} & \sup_{t \in [e^{-l}, 1]} P(|X(t) - X(e^{-l})| > r) \\ & \leq \sup_{t \in [e^{-l}, 1]} P(|X_1(t) - X_1(e^{-l})| + |X_2(t) - X_2(e^{-l})| > r) \\ & \leq \sup_{t \in [e^{-l}, 1]} P(|X_1(t) - X_1(e^{-l})| > (1 - \delta)r) + \sup_{t \in [e^{-l}, 1]} P(|X_2(t) - X_2(e^{-l})| > \delta r) \\ & \leq cP(|X_1(1) - X_1(e^{-l})| > (1 - \delta)r) + o(\exp(-\delta_0 \delta r \log(\delta r))). \end{aligned} \quad (4.20)$$

Further we have

$$\begin{aligned} & P(|X_1(1) - X_1(e^{-l})| > (1 - \delta)r) \\ & \leq P(|X(1) - X(e^{-l})| + |X_2(1) - X_2(e^{-l})| > (1 - \delta)r) \\ & \leq P(|X(1) - X(e^{-l})| > (1 - \delta)^2 r) + P(|X_2(1) - X_2(e^{-l})| > \delta(1 - \delta)r) \\ & = P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r) + o(\exp(-\delta_0 \delta(1 - \delta)r \log(\delta(1 - \delta)r))). \end{aligned} \quad (4.21)$$

Taking sufficiently large δ_0 , we find from Lemma 3.7 that

$$\exp(-\delta_0 \delta r \log(\delta r)) = o(P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r))$$

and

$$\exp(-\delta_0 \delta(1 - \delta)r \log(\delta(1 - \delta)r)) = o(P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r)).$$

Thus (4.20) and (4.21) yields that (4.19) holds for some $c_2 > 0$. \square

Lemma 4.10. *Let $0 \leq s < t$, $a, b > 0$, and $\varepsilon > 0$. Then*

$$P\left(\sup_{u \in [s, t]} |X(u) - X(s)| > 3\varepsilon\right) \leq 3 \sup_{u \in [s, t]} P(|X(u) - X(s)| > \varepsilon), \quad (4.22)$$

and

$$P\left(\sup_{u \in [s, t]} |X(u) - X(s)| > a + b\right) \leq \frac{P(|X(t) - X(s)| > a)}{P(\sup_{u \in [s, t]} |X(u) - X(s)| \leq b/2)}. \quad (4.23)$$

Proof: The inequality (4.22) is due to Lemma 2.1 of Yamamuro (2003). The proof of (4.23) follows along the lines of the proof of (4.22) from Remark 20.3 in Sato (1999). \square

Lemma 4.11. *For any $\varepsilon \in (0, 1)$, there is $R > 0$ such that, for $r > R$,*

$$P\left(\sup_{t \in [e^{-l}, 1]} |X(t) - X(e^{-l})| > r\right) \leq 2P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r). \quad (4.24)$$

Proof: We see from (4.23) of Lemma 4.10 that

$$P\left(\sup_{t \in [e^{-l}, 1]} |X(t) - X(e^{-l})| > r\right) \leq \frac{P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r)}{P(\sup_{t \in [e^{-l}, 1]} |X(t) - X(e^{-l})| \leq \varepsilon r/2)}.$$

We obtain from Lemma 4.9 and (4.22) of Lemma 4.10 that

$$\begin{aligned} P\left(\sup_{t \in [e^{-l}, 1]} |X(t) - X(e^{-l})| \leq \varepsilon r/2\right) &= 1 - P\left(\sup_{t \in [e^{-l}, 1]} |X(t) - X(e^{-l})| > \varepsilon r/2\right) \\ &\geq 1 - 3 \sup_{t \in [e^{-l}, 1]} P(|X(t) - X(e^{-l})| > \varepsilon r/6) \\ &\geq 1 - 3c_2 P(|X(1) - X(e^{-l})| > (1 - \varepsilon)\varepsilon r/6). \end{aligned}$$

Taking sufficiently large $R > 0$ such that

$$3c_2 P(|X(1) - X(e^{-l})| > (1 - \varepsilon)\varepsilon r/6) < 1/2 \quad \text{for } r > R,$$

we have (4.24). \square

Proof of Proposition 4.1: Assertion (i) follows from (i) of Lemma 4.6. Next we prove (ii). Suppose that (4.3) holds. Let $M > 0$ be a sufficiently large positive integer. Take a positive constant b such that $e^l \geq b > 1$ and $b^M = e$. We see from Lemma 4.11 that, for any $\varepsilon \in (0, 1)$, there is $R > 0$ such that, for $r > R$,

$$P\left(\sup_{t \in [b^{-1}, 1]} |X(t) - X(e^{-l})| > r\right) \leq 2P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r).$$

Thus, by using the selfsimilarity, we have

$$\begin{aligned} &P\left(\sup_{t \in [b^{-m-1}e^n, b^{-m}e^n]} |X(t) - X(b^{-m}e^{n-l})| > (b^{-m}e^n)^H r\right) \\ &\leq 2P(|X(1) - X(e^{-l})| > (1 - \varepsilon)r) \end{aligned}$$

for $0 \leq m \leq M - 1$ and $n \in \mathbb{Z}$. Let $r_n := g(n - 1)/(1 - \varepsilon)$. We obtain from (4.3) that

$$\sum_{n=1}^{\infty} P\left(\sup_{t \in [b^{-m-1}e^n, b^{-m}e^n]} |X(t) - X(b^{-m}e^{n-l})| > (b^{-m}e^n)^H r_n\right) < \infty.$$

By virtue of the Borel-Cantelli lemma, there is a random number N_m such that, for any $n \geq N_m$, almost surely

$$\sup_{t \in [b^{-m-1}e^n, b^{-m}e^n]} |X(t)| - |X(b^{-m}e^{n-l})| \leq \frac{(b^{-m}e^n)^H g(n-1)}{1-\varepsilon}. \quad (4.25)$$

Let $k := k_{(m,n)}$ be the maximum integer satisfying $n - k_{(m,n)}l \geq N_m$. Let $1 \leq j \leq k$. Then we substitute $n - jl$ for n in (4.25) and see that, for $1 \leq j \leq k$, almost surely

$$|X(b^{-m}e^{n-jl})| - |X(b^{-m}e^{n-(j+1)l})| \leq \frac{(b^{-m}e^{n-jl})^H g(n-jl-1)}{1-\varepsilon}. \quad (4.26)$$

Recall that $e = b^M$. Adding up (4.25) and (4.26) for $1 \leq j \leq k$, we have

$$\begin{aligned} \sup_{t \in [b^{Mn-m-1}, b^{Mn-m}]} |X(t)| - |X(b^{-m}e^{n-(k+1)l})| &\leq \sum_{j=0}^k \frac{(b^{-m}e^{n-jl})^H g(n-jl-1)}{1-\varepsilon} \\ &\leq \frac{(b^{Mn-m})^H g(n-1)}{(1-\varepsilon)(1-e^{-lH})}. \end{aligned}$$

Notice that $N_m - l \leq n - (k+1)l < N_m$. Thus,

$$\begin{aligned} \limsup_{t \rightarrow \infty} \frac{|X(t)|}{t^H g(\log t)} &\leq \limsup_{n \rightarrow \infty} \sup_{t \in [b^{M(n-1)}, b^{Mn}]} \frac{|X(t)|}{t^H g(\log t)} \\ &\leq \limsup_{n \rightarrow \infty} \sup_{0 \leq m \leq M-1} \sup_{t \in [b^{Mn-m-1}, b^{Mn-m}]} \frac{b^H |X(t)|}{(b^{Mn-m})^H g(n-1)} \\ &\leq \frac{b^H}{(1-\varepsilon)(1-e^{-lH})} + \limsup_{n \rightarrow \infty} \max_{N_m-l \leq i < N_m} \frac{b^H |X(b^{-m}e^i)|}{(b^{Mn-m})^H g(n-1)} \\ &= \frac{b^H}{(1-\varepsilon)(1-e^{-lH})}. \end{aligned}$$

Letting $\varepsilon \downarrow 0$ and $b \downarrow 1$, we have (4.4). \square

Proof of Proposition 4.2: Assertion (i) follows from (ii) of Lemma 4.6. Next we prove (ii). Suppose that (4.7) holds. In the same way as in the proof of Proposition 4.1, there is a random number N such that, for any $n \geq N$, almost surely

$$\sup_{t \in [e^{-l(n+1)}, e^{-ln}]} |X(t) - X(e^{-(n+1)l})| \leq \frac{e^{-lnH} g((n+1)l)}{1-\varepsilon}.$$

Let $0 < H' < H$. Then we have $g(n+1)/g(n) < e^{H'}$ for all sufficiently large n . Put $H_1 = H - H'$. Note from Theorem 2.1 of Watanabe (2002a) that $\lim_{n \rightarrow \infty} X(e^{-ln}) = 0$ a.s. Hence we have

$$\begin{aligned} \sup_{t \in [e^{-l(n+1)}, e^{-ln}]} |X(t)| &\leq (1-\varepsilon)^{-1} \sum_{j=0}^{\infty} e^{-l(n+j)H} g(l(n+j+1)) \\ &\leq (1-\varepsilon)^{-1} e^{-lnH} \sum_{j=0}^{\infty} e^{-ljH_1} g(l(n+1)) \\ &= \frac{e^{-lnH} g(l(n+1))}{1-\varepsilon} \cdot (1-e^{-lH_1})^{-1} \end{aligned}$$

for all sufficiently large n . Here we have

$$\frac{e^{-lnH}g(l(n+1))}{t^H g(|\log t|)} \leq \frac{g(l(n+1))}{e^{-lH}g(nl)} < 1$$

for $t \in [e^{-l(n+1)}, e^{-ln}]$. Therefore,

$$\limsup_{t \rightarrow 0} \frac{|X(t)|}{t^H g(|\log t|)} \leq (1 - \varepsilon)^{-1} \cdot (1 - e^{-lH_1})^{-1}.$$

Letting $\varepsilon \rightarrow 0$, we obtain (4.8). \square

Proof of Theorem 2.11: Note that, for $\delta > 0$ and $l \in \mathbb{N}$,

$$\begin{aligned} \int_0^\infty P(|X(1) - X(e^{-l})| > \delta g(x)) dx &= \int_0^\infty G_l(\delta g(x)) dx \\ &= \int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx). \end{aligned} \quad (4.27)$$

Thus the proof of (ii) is clear from Proposition 4.1. Assertion (i) follows from (ii). Suppose that $g \in \mathcal{G}_1$ and $g^{-1}(|x|)$ is submultiplicative on \mathbb{R}^d . We prove the equivalence of (2.13) and (2.14). Recall the Lévy measure η_1 of ρ_1 , that is, (2.11). We find from Lemma 3.5 that

$$\int_1^\infty \frac{K(r) - K(e^H r)}{r} g^{-1}(\delta^{-1}r) dr < \infty \quad (4.28)$$

if and only if $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_1(dx) < \infty$. For some $c_1 > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) &= \int_{(\mathbb{R}^d)^l} g^{-1} \left(\delta^{-1} \left| \sum_{k=1}^l e^{(-k+1)H} x_k \right| \right) \prod_{k=1}^l \rho_1(dx_k) \\ &\leq c_1^l \int_{(\mathbb{R}^d)^l} \prod_{k=1}^l g^{-1}(\delta^{-1} e^{(-k+1)H} |x_k|) \rho_1(dx_k) \\ &\leq c_1^l \left(\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_1(dx) \right)^l. \end{aligned}$$

Thus (4.28) is equivalent to that $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) < \infty$ for all l . Suppose that (2.14) holds. Let $\delta > C$. Then (4.28) holds, and thus $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) < \infty$ for all l . Let $\delta < C$. Then we have

$$\infty = \int_1^\infty \frac{K(r) - K(e^H r)}{r} g^{-1}(\delta^{-1}r) dr \leq \int_1^\infty \frac{K(r) - K(e^l H r)}{r} g^{-1}(\delta^{-1}r) dr.$$

By Lemma 3.5, $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) = \infty$ for all l , and thereby (2.13) holds. Conversely, suppose that (2.13) holds. Let $\delta > C$. Then $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) < \infty$ for all l , and hence (4.28) holds. Let $\delta < C$. Since $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_l(dx) = \infty$ for

some l , we see that

$$\begin{aligned}
\infty &= \int_1^\infty \frac{K(r) - K(e^l H r)}{r} g^{-1}(\delta^{-1} r) dr \\
&= \sum_{k=1}^l \int_1^\infty \frac{K(e^{(k-1)H} r) - K(e^{kH} r)}{r} g^{-1}(\delta^{-1} r) dr \\
&= \sum_{k=1}^l \int_{e^{(k-1)H}}^\infty \frac{K(r) - K(e^H r)}{r} g^{-1}(\delta^{-1} e^{-(k-1)H} r) dr \\
&\leq l \int_1^\infty \frac{K(r) - K(e^H r)}{r} g^{-1}(\delta^{-1} r) dr.
\end{aligned}$$

Hence (2.14) holds. We have proved the equivalence of (2.13) and (2.14). Thus assertion (iii) follows from (ii). The second assertion of (iii) is trivial. \square

Proof of Theorem 2.13: We see from Theorem 3.13 that $F(r) \in OR$ is equivalent to $G_1(r) \in OR$. Suppose that $G_1(r) \in OR$ and there is $g(x) \in \mathcal{G}_1$ such that (1.2) holds with $C = 1$. Then we see from Proposition 4.1 that

$$\begin{aligned}
&\int_0^\infty G_1(2^{-1}(1 - e^{-H})g(x))dx \\
&= \int_0^\infty P(|X(1) - X(e^{-1})| > 2^{-1}(1 - e^{-H})g(x))dx = \infty
\end{aligned}$$

and

$$\int_0^\infty G_1(2g(x))dx = \int_0^\infty P(|X(1) - X(e^{-1})| > 2g(x))dx < \infty.$$

As $G_1(r) \in OR$, this is a contradiction. Hence if $G_1(r) \in OR$, then there is no $g(x) \in \mathcal{G}_1$ such that (1.2) holds with $C = 1$. Conversely, suppose that $G_1(r) \notin OR$. Then there is a positive sequence $y_n \uparrow \infty$ for $n \in \mathbb{Z}_+$ such that $2^{-n}G_1(y_n) \geq G_1(2y_n)$ for $n \in \mathbb{Z}_+$. Take $x_n \uparrow \infty$ satisfying $x_0 = 0$ and

$$1 \leq G_1(y_n)(x_{n+1} - x_n) \leq 2 \quad \text{for } n \in \mathbb{Z}_+.$$

Furthermore, we define $g(x) \in \mathcal{G}_1$ by $g(x) = y_n$ on $[x_n, x_{n+1})$. Then we obtain that

$$\int_0^\infty P(|X(1) - X(e^{-1})| > g(x))dx = \sum_{n=0}^\infty G_1(y_n)(x_{n+1} - x_n) = \infty$$

and

$$\int_0^\infty P(|X(1) - X(e^{-1})| > 2g(x))dx \leq \sum_{n=0}^\infty 2^{-n}G_1(y_n)(x_{n+1} - x_n) < \infty.$$

It follows from Proposition 4.1 and Theorem 2.11 that there is $C_0 \in [1, 2(1 - e^{-H})^{-1}]$ such that (1.2) holds with $C = C_0$. Thus (1.2) holds with $C = 1$ by replacing $g(x)$ with $C_0g(x)$. \square

Proof of Theorem 2.14: By Theorem 3.13, we have $F(r) \asymp G_1(r)$. As $G_1(r) \in OR$, the theorem holds from Proposition 4.1. \square

Proof of Corollary 2.15: The proof is obvious from Theorem 2.14. \square

Proof of Corollary 2.16: By using Proposition 3.14, we can obtain the corollary from Theorems 2.13 and 2.14. \square

Proof of Example 2.17: Use Corollary 2.16 (ii) for

$$M(r) = \Gamma\left(\frac{m+1}{2}\right) \left(\sqrt{\pi}\Gamma\left(\frac{m}{2}\right)\right)^{-1} (1+r^2)^{-(m+1)/2}.$$

Remaining assertion is clear from the first assertion. \square

Proof of Theorem 2.18: By (4.27), we have

$$\int_0^\infty G_1(\delta g(x))dx = \int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|)\rho_1(dx) \quad \text{for } \delta > 0. \quad (4.29)$$

Suppose that $g^{-1}(x) + \log x \in OR$ and there is $\{X(t)\}$ such that (1.2) holds with $C = 1$. Note that $\int_{|x|>1} \log|x|\rho_1(dx) < \infty$. By the same way as in the proof of Theorem 2.13, we see from (4.29) that absurdity occurs. Thus if $g^{-1}(x) + \log x \in OR$, then there is no $\{X(t)\}$ such that (1.2) holds with $C = 1$.

Conversely, we suppose that $g^{-1}(|x|)$ is submultiplicative on \mathbb{R}^d and $g^{-1}(x) + \log x \notin OR$. There is $x_n \uparrow \infty$ for $n \in \mathbb{Z}_+$ such that $x_0 = 1$, $e^{2H}x_n < x_{n+1}$, and $2^{-n}(g^{-1}(x_n) + \log x_n) \geq g^{-1}(e^{-H}x_n) + \log(e^{-H}x_n)$ for $n \in \mathbb{Z}_+$ and that, for $C_n := \int_{x_n}^{e^H x_n} (g^{-1}(x) + \log x)x^{-1}dx$ with $n \in \mathbb{Z}_+$, it holds that $\sum_{n=0}^\infty C_n^{-1} < \infty$. Define $K(r)$ as

$$K(r) = \begin{cases} \sum_{j=0}^\infty C_j^{-1} & \text{for } 0 < r < e^H x_0, \\ \sum_{j=n}^\infty C_j^{-1} & \text{for } e^H x_{n-1} \leq r < e^H x_n \text{ with } n \in \mathbb{N}. \end{cases}$$

Then we have

$$K(r) - K(e^H r) = \begin{cases} C_n^{-1} & \text{for } x_n \leq r < e^H x_n \text{ with } n \in \mathbb{Z}_+, \\ 0 & \text{for } e^H x_n \leq r < x_{n+1} \text{ with } n \in \mathbb{Z}_+. \end{cases}$$

Then we obtain that

$$\int_1^\infty (g^{-1}(r) + \log r) \frac{K(r) - K(e^H r)}{r} dr = \sum_{n=0}^\infty C_n C_n^{-1} = \infty$$

and

$$\begin{aligned} & \int_1^\infty (g^{-1}(e^{-2H}r) + \log(e^{-2H}r)) \frac{K(r) - K(e^H r)}{r} dr \\ & \leq \sum_{n=0}^\infty (g^{-1}(e^{-H}x_n) + \log(e^{-H}x_n)) \int_{x_n}^{e^H x_n} \frac{K(r) - K(e^H r)}{r} dr \\ & \leq \sum_{n=0}^\infty 2^{-n} \int_{x_n}^{e^H x_n} (g^{-1}(r) + \log r) \frac{K(r) - K(e^H r)}{r} dr \\ & = \sum_{n=0}^\infty 2^{-n} C_n C_n^{-1} < \infty. \end{aligned}$$

Notice that $\int_1^\infty (\log r)(K(r) - K(e^H r))r^{-1}dr < \infty$ and $\int_1^\infty K(r)r^{-1}dr < \infty$. It follows from Theorem 2.11 (iii) that (1.2) holds with $C = C_a$ for some $C_a \in [1, e^{2H}]$. Hence (1.2) holds with $C = 1$ by replacing $\{X(t)\}$ with $\{C_a^{-1}X(t)\}$. \square

Proof of Theorem 2.19: Note that $\int_{|x|>1} \log|x|\rho_1(dx) < \infty$. By using (4.29), the proof is clear from Proposition 4.1. \square

Proof of Proposition 2.20: Let $C \in [0, \infty)$. As mentioned in Remark 3.3 (ii), $g^{-1}(|x|)$ is quasi-submultiplicative on \mathbb{R}^d . Thus we see from Proposition 3.6 that (2.23) holds

if and only if (2.13) holds. Here we assumed that $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|)\mu(dx) < \infty$ for some $\delta \in (0, \infty)$ in the “if” part. Hence assertion (i) holds from Theorem 2.11. Now we see from Lemma 3.5 that (2.15) is equivalent to (2.23). Furthermore, we see that $\int_1^\infty K(r)g^{-1}(\delta^{-1}r)r^{-1}dr < \infty$ if and only if $\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|)\mu(dx) < \infty$. Thus assertion (ii) holds from (i). \square

Proof of Proposition 2.22: Define ζ_1 on \mathbb{R}_+ by $\zeta_1(x > r) := \rho_1(|x| > r)$. Suppose that $\rho_1 \in \mathcal{OS}$ on \mathbb{R}^d , that is, $\zeta_1 \in \mathcal{OS}$ on \mathbb{R}_+ . Now we have, for $g \in \mathcal{G}_1$ and $l \in \mathbb{N}$,

$$\begin{aligned} \int_{\mathbb{R}^d} g^{-1}(|x|)\rho_l(dx) &= \int_{(\mathbb{R}^d)^l} g^{-1}\left(\left|\sum_{k=1}^l e^{(-k+1)H}x_k\right|\right) \prod_{k=1}^l \rho_1(dx_k) \\ &\leq \int_{(\mathbb{R}^d)^l} g^{-1}\left(\sum_{k=1}^l |x_k|\right) \prod_{k=1}^l \rho_1(dx_k) \\ &= \int_{\mathbb{R}_+} g^{-1}(r)\zeta_1^{l*}(dr) \\ &= \int_0^\infty \zeta_1^{l*}(x > g(r))dr. \end{aligned}$$

By Lemma 5.5 (iii) in Sect.5, $\zeta_1^{l*}(x > r) \asymp \zeta_1(x > r)$. Thus, if $\int_{\mathbb{R}^d} g^{-1}(|x|)\rho_1(dx) < \infty$, then $\int_{\mathbb{R}^d} g^{-1}(|x|)\rho_l(dx) < \infty$ for all $l \in \mathbb{N}$. Therefore we see that (2.13) holds if and only if (2.25) holds. By virtue of Theorem 3.9 (iv), assertion (ii) holds from (i). The proof of (iii) is clear from Proposition 2.20 (i) and Theorem 3.9 (iv). \square

Proof of Example 2.23: The example of Sect.6 in Embrechts et al. (1979) shows that $\mu \in \mathcal{S}$ on \mathbb{R}_+ and

$$\mu(x > r) \sim (2\pi)^{-1/2}(\log r)^{-1} \exp(-(\log r)^2/2). \quad (4.30)$$

Hence we see from Theorem 1 of Embrechts et al. (1979) that

$$\int_r^\infty \frac{K(x)}{x} dx \sim \mu(x > r).$$

By (2.11) and (4.30), we have

$$\eta_1(x > r) = \int_r^\infty \frac{K(x)}{x} dx - \int_{eHr}^\infty \frac{K(x)}{x} dx \sim \mu(x > r). \quad (4.31)$$

It follows from Lemma A3.15 of Embrechts et al. (1997) that $\tilde{\eta}_1 \in \mathcal{S}$, and thereby $\tilde{\eta}_1 \in \mathcal{OS}$. For $\delta > 0$ and $g \in \mathcal{G}_1$, we find from (4.31) that

$$\int_{\mathbb{R}_+} \eta_1(x > \delta g(r))dr = \int_{\mathbb{R}_+} g^{-1}(\delta^{-1}x)\eta_1(dx) < \infty$$

if and only if

$$\int_{\mathbb{R}_+} \mu(x > \delta g(r))dr = \int_{\mathbb{R}_+} g^{-1}(\delta^{-1}x)\mu(dx) < \infty.$$

Thus we obtain the first assertion from Proposition 2.22 (ii). Setting $g^{-1}(x) = \exp((\log x)^2/2)$ for $x > 1$, we obtain the second assertion from the first one. The second assertion can be proved also by employing Proposition 2.20 (i). \square

Lemma 4.12. *Let $C \in [0, \infty]$. Let $\phi(r)$ be a positive, increasing, and regularly varying function with positive index. Suppose that $g \in \mathcal{G}_1$ satisfies that*

$$g^{-1}(r) = \exp(\phi(r)). \quad (4.32)$$

Then (2.13) holds if and only if

$$\int_1^\infty \rho_l(|x| > r) g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C \text{ and all } l \in \mathbb{N}, \\ = \infty & \text{for } \delta < C \text{ and some } l = l(\delta) \in \mathbb{N}. \end{cases} \quad (4.33)$$

Moreover, if $g^{-1}(r)$ is submultiplicative on \mathbb{R}_+ , then (2.13) is also equivalent to (2.23), and to

$$\int_1^\infty K(r) g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (4.34)$$

Proof: By Theorem 4.12.10 (ii) of Bingham et al. (1987), (2.13) holds if and only if

$$\int_{\mathbb{R}^d} \rho_l(dx) \int_0^{|x|} g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C \text{ and all } l \in \mathbb{N}, \\ = \infty & \text{for } 0 < \delta < C \text{ and some } l = l(\delta) \in \mathbb{N}. \end{cases} \quad (4.35)$$

Here we used the assumption that $\phi(r)$ is regularly varying. The integral of (4.35) is equal to that of (4.33).

Suppose that $g^{-1}(r)$ is submultiplicative on \mathbb{R}_+ . Then $\int_0^{|x| \vee 1} g^{-1}(r) dr$ is submultiplicative on \mathbb{R}^d and thus, by Lemma 3.5, (4.35) is equivalent to

$$\int_{\mathbb{R}^d} \eta_l(dx) \int_0^{|x|} g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C \text{ and all } l \in \mathbb{N}, \\ = \infty & \text{for } 0 < \delta < C \text{ and some } l = l(\delta) \in \mathbb{N}, \end{cases}$$

equivalently,

$$\int_1^\infty \eta_l(|x| > r) g^{-1}(\delta^{-1}r) dr \begin{cases} < \infty & \text{for } \delta > C \text{ and all } l \in \mathbb{N}, \\ = \infty & \text{for } 0 < \delta < C \text{ and some } l = l(\delta) \in \mathbb{N}. \end{cases} \quad (4.36)$$

By (2.11), we have

$$lHK(r) \geq \eta_l(|x| > r) \geq (\log(1 + \varepsilon))K((1 + \varepsilon)r) \quad (4.37)$$

for sufficiently small $\varepsilon > 0$. Thus (4.36) is equivalent to (4.34). Moreover, by Lemma 3.5, (2.23) is equivalent to (2.15). As $\phi(x)$ is regularly varying with positive index, (2.15) is equivalent to (4.34). Thus the second assertion is true. \square

Proof of Proposition 2.24: Let

$$g^{-1}(r) = \exp(r^\alpha f(\log(r \vee 1))).$$

Since $f(r)$ is regularly varying, $f(\log(r \vee 1))$ is slowly varying. Thus we have

$$g(r) \sim \frac{(\log r)^{1/\alpha}}{(f(\alpha^{-1} \log_{(2)} r))^{1/\alpha}}.$$

Notice that (2.27) and (2.28) are equivalent to (4.34) and (2.23), respectively. Since $g^{-1}(r)$ is submultiplicative on \mathbb{R}_+ , we see from Lemma 4.12 that (2.26) is equivalent to (2.27) and to (2.28). \square

Proof of Proposition 2.25: Let

$$g^{-1}(r) = \exp(r f^{-1}(\log(r \vee 1))).$$

We see from Definition 2.7 that, for all sufficiently small $\varepsilon > 0$,

$$\limsup_{x \rightarrow \infty} \frac{f^{-1}((1 + \varepsilon)x)}{f^{-1}(x)} \leq 1 + \varepsilon.$$

Thus $f^{-1}(\phi(t)) \sim f^{-1}(t)$ provided that $\phi(t) \sim t$ as $t \rightarrow \infty$. This implies that $r f^{-1}(\log(r + 1))$ is regularly varying with index 1, and we have

$$g(r) \sim \frac{\log r}{f^{-1}(\log_{(2)} r)}.$$

By Theorem 2.11 and Lemma 4.12, (2.29) is equivalent to (4.33), that is,

$$\int_1^\infty \exp(\delta^{-1} r f^{-1}(\log(r + 1))) \rho_l(|x| > r) dr \begin{cases} < \infty & \forall l \in \mathbb{N}, \text{ for } \delta > C, \\ = \infty & \exists l = l(\delta) \in \mathbb{N}, \text{ for } 0 < \delta < C. \end{cases}$$

By Lemma 3.8, this is equivalent to

$$\int_1^\infty \exp(h(\delta^{-1} r)) \eta_l(|x| > r) dr \begin{cases} < \infty & \forall l \in \mathbb{N}, \text{ for } \delta > C, \\ = \infty & \exists l = l(\delta) \in \mathbb{N}, \text{ for } 0 < \delta < C. \end{cases} \quad (4.38)$$

Using (4.37) for sufficiently small $\varepsilon > 0$, we see that (4.38) is equivalent to

$$\int_1^\infty \exp(h(\delta^{-1} r)) K(r) dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (4.39)$$

Using (4.37) again, we see that this is equivalent to

$$\int_1^\infty \exp(h(\delta^{-1} r)) \eta_1(|x| > r) dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (4.40)$$

Further, (4.39) is equivalent to (2.30). We see from Lemma 3.8 that (4.40) is equivalent to (2.31).

Lastly, we prove the equivalence of (2.30) and (2.32) provided that $h(r)$ is regularly varying with positive index. Now (2.30) is equivalent to (4.39), equivalently,

$$\int_1^\infty \exp(h(\delta^{-1} r)) K(r) r^{-1} dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases}$$

Here we used the regular variation. By virtue of Theorem 4.12.10 (ii) of Bingham et al. (1987), this is equivalent to

$$\int_1^\infty \left(\int_0^r \exp(h(\delta^{-1} s)) ds \right) K(r) r^{-1} dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C, \end{cases}$$

equivalently,

$$\int_1^\infty \left(\int_r^\infty K(s) s^{-1} ds \right) \exp(h(\delta^{-1} r)) dr \begin{cases} < \infty & \text{for } \delta > C, \\ = \infty & \text{for } 0 < \delta < C. \end{cases} \quad (4.41)$$

Notice that $\nu(|x| > r) = \int_r^\infty K(s)s^{-1}ds$. It follows that (4.41) is equivalent to

$$\liminf_{r \rightarrow \infty} \frac{h^{-1}(-\log \nu(|x| > r))}{r} = C.$$

By Lemma 3.8, this is equivalent to (2.32). \square

Proof of Proposition 2.26: Let $g^{-1}(r) = \exp(r \log(r \vee 1))$. The proof of Proposition 2.26 is similar to that of Proposition 2.25 by using Lemma 3.7 in place of Lemma 3.8. It is omitted. \square

Proof of Theorem 2.28: Note from Theorem 2.11 (ii) that if $g_1^{-1}(r) \asymp g_2^{-1}(r)$ for $g_1, g_2 \in \mathcal{G}_1$, then (1.2) holds for $g = g_1$ and for $g = g_2$ with the same constant C . Thus in assertion (i), (1), (2), and (3) are from Remark 2.27, Proposition 2.26 and Proposition 2.25, respectively.

Next we prove assertion (ii). Suppose that (2.38) holds. Since ρ_1 is infinitely divisible, we see from Lemma 3.7 that

$$\int_{\mathbb{R}^d} e^{c|x|^2} \rho_1(dx) = \infty \quad \text{for some } c > 0.$$

Hence, by (2.38), we have

$$\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \rho_1(dx) = \infty \quad \text{for } 0 < \delta < \infty. \quad (4.42)$$

By Theorem 2.11 (ii), (1.2) holds with $C = \infty$. Hence assertion (1) is true. Suppose that (2.39) and (2.40) hold. In the case where $\nu \neq 0$, we find from Lemma 3.7 that

$$\int_{|x|>1} e^{c|x| \log |x|} \rho_1(dx) = \infty \quad \text{for some } c > 0.$$

Thus we have (4.42) by (2.39), and thereby (1.2) holds with $C = \infty$. In the case where $\nu = 0$, we see that

$$\int_{\mathbb{R}^d} e^{c|x|^2} \mu(dx) < \infty \quad \text{for some } c > 0.$$

Hence, by (2.40), we have

$$\int_{\mathbb{R}^d} g^{-1}(\delta^{-1}|x|) \mu(dx) < \infty \quad \text{for } 0 < \delta < \infty.$$

Hence we obtain from Proposition 2.20 that (1.2) holds with $C = 0$. Hence assertion (2) is true. \square

Proof of Example 2.30: (i) Use Proposition 2.24 with $f(x) = x^\beta$.

(ii) Use Proposition 2.25 with $C = c^{-1/\alpha}$ and $h(x) = x^\alpha (\log x)^\beta$ for sufficiently large x . Then it suffices to prove that

$$f^{-1}(y) \sim \begin{cases} (\log y)^\beta & \text{for } \alpha = 1, \\ \alpha^{-(\beta-\alpha)/\alpha} (\alpha-1)^{-(\alpha-1)/\alpha} y^{(\alpha-1)/\alpha} (\log y)^{\beta/\alpha} & \text{for } \alpha > 1. \end{cases}$$

Suppose that $\alpha > 1$. As $h(x) = \int_0^x q(t)dt$, it follows that $q(t) \sim \alpha t^{\alpha-1} (\log t)^\beta$ as $t \rightarrow \infty$. Put $y = q(t)$. Then $y \sim \alpha t^{\alpha-1} (\log t)^\beta$ and $\log y \sim (\alpha-1) \log t$ as $t \rightarrow \infty$. Hence

$$t \sim \left(\frac{y}{\alpha (\log t)^\beta} \right)^{1/(\alpha-1)} \sim \left(\frac{y}{\alpha ((\alpha-1)^{-1} \log y)^\beta} \right)^{1/(\alpha-1)}$$

as $t \rightarrow \infty$. This implies that

$$q^{-1}(y) \sim \alpha^{-1/(\alpha-1)}(\alpha-1)^{\beta/(\alpha-1)}y^{1/(\alpha-1)}(\log y)^{-\beta/(\alpha-1)}.$$

Hence,

$$\begin{aligned} f(x) &= \int_0^x q^{-1}(y)dy \sim \alpha^{-1/(\alpha-1)}(\alpha-1)^{\beta/(\alpha-1)} \frac{x^{(\alpha-1)^{-1}+1}}{(\alpha-1)^{-1}+1} (\log x)^{-\beta/(\alpha-1)} \\ &= \alpha^{-\alpha/(\alpha-1)}(\alpha-1)^{(\alpha+\beta-1)/(\alpha-1)} x^{\alpha/(\alpha-1)} (\log x)^{-\beta/(\alpha-1)}. \end{aligned}$$

Moreover, put $y = f(x)$. Then we have $\log y \sim \alpha(\alpha-1)^{-1} \log x$ as $x \rightarrow \infty$. Hence,

$$\begin{aligned} x &\sim \left(\frac{\alpha^{\alpha/(\alpha-1)}(\alpha-1)^{-(\alpha+\beta-1)/(\alpha-1)}y}{(\log x)^{-\beta/(\alpha-1)}} \right)^{(\alpha-1)/\alpha} \\ &\sim \left(\frac{\alpha^{\alpha/(\alpha-1)}(\alpha-1)^{-(\alpha+\beta-1)/(\alpha-1)}y}{((\alpha-1)\alpha^{-1} \log y)^{-\beta/(\alpha-1)}} \right)^{(\alpha-1)/\alpha} \\ &= \alpha^{-(\beta-\alpha)/\alpha}(\alpha-1)^{-(\alpha-1)/\alpha} y^{(\alpha-1)/\alpha} (\log y)^{\beta/\alpha}. \end{aligned}$$

This implies that $f^{-1}(y) \sim \alpha^{-(\beta-\alpha)/\alpha}(\alpha-1)^{-(\alpha-1)/\alpha} y^{(\alpha-1)/\alpha} (\log y)^{\beta/\alpha}$. Next suppose that $\alpha = 1$. Then $y = q(t) = (\log t)^\beta + \beta(\log t)^{\beta-1}$ for all sufficiently large t . Hence,

$$\exp(y^{\beta-1}) = \exp((\log t)(1 + \beta(\log t)^{-1})^{\beta-1}) \sim e \cdot t \quad \text{as } t \rightarrow \infty.$$

This yields that $q^{-1}(y) \sim e^{-1} \exp(y^{\beta-1})$. Hence we obtain that

$$f(x) = \int_0^x q^{-1}(y)dy \sim e^{-1} \exp(x^{\beta-1}) \cdot \beta x^{1-\beta^{-1}}.$$

Put $y = f(x)$. Then $\log y \sim x^{\beta-1}$. Hence we obtain that $f^{-1}(y) \sim (\log y)^\beta$. Thus all assertions are true. \square

5. Proof of Theorem 3.9.

In this section, we prove Theorem 3.9 mentioned in Sect.3. The proof of Theorem 3.9 is similar to that of Theorem 1.1 of Shimura and Watanabe (2005), but there is a difficulty peculiar to the multi-dimensional case.

Lemma 5.1. *Let ρ_j for $j = 1, 2$ be distributions on \mathbb{R}^d . For $j = 1, 2$, we define a distribution ζ_j on \mathbb{R}_+ by $\zeta_j(x > r) := \rho_j(|x| > r)$ for $r \geq 0$.*

(i) *We have*

$$\rho_1 * \rho_2(|x| > r) \leq \zeta_1 * \zeta_2(x > r) \quad \text{for } r \geq 0. \quad (5.1)$$

In particular, suppose $n \in \mathbb{N}$, then we have

$$\rho_1^{n*}(|x| > r) \leq \zeta_1^{n*}(x > r) \quad \text{for } r \geq 0. \quad (5.2)$$

(ii) *There are $s > 0$ and $c_1 > 1$ both independent of ρ_1 such that*

$$\rho_1(|x| > r) \leq c_1 \rho_1 * \rho_2(|x| > r - s) \quad \text{for } r \geq 0. \quad (5.3)$$

Proof: Let $\{X_j\}$ be independent \mathbb{R}^d -valued random variables such that the distribution of X_j is ρ_j for $j = 1, 2$. Then ζ_j is the distribution of $|X_j|$ for $j = 1, 2$. Thus we have, for $r \geq 0$,

$$\begin{aligned} \rho_1 * \rho_2(|x| > r) &= P(|X_1 + X_2| > r) \\ &\leq P(|X_1| + |X_2| > r) = \zeta_1 * \zeta_2(x > r). \end{aligned}$$

The second assertion of (i) is trivial.

Choose $s > 0$ such that $c_1^{-1} := P(|X_2| \leq s) > 0$. Then we see that for $r > s$,

$$\begin{aligned} \rho_1(|x| > r) &= c_1 P(|X_2| \leq s) P(|X_1| > r) \\ &= c_1 P(|X_1| > r, |X_2| \leq s) \\ &\leq c_1 P(|X_1 + X_2| > r - s) = c_1 \rho_1 * \rho_2(|x| > r - s). \end{aligned}$$

We have proved (ii). \square

Lemma 5.2. *Let ζ_j for $j = 1, 2, 3$ be distributions on \mathbb{R}_+ . If $\zeta_1(x > r) \leq c_1 \zeta_2(x > r)$ for some $c_1 > 1$, then*

$$\zeta_1 * \zeta_3(x > r) \leq c_1 \zeta_2 * \zeta_3(x > r) \quad \text{for } r \geq 0, \quad (5.4)$$

and, for any $n \in \mathbb{N}$,

$$\zeta_1^{n*}(x > r) \leq c_1^n \zeta_2^{n*}(x > r) \quad \text{for } r \geq 0. \quad (5.5)$$

Proof: Suppose that $\zeta_1(x > r) \leq c_1 \zeta_2(x > r)$ for some $c_1 > 1$. Then we see

$$\begin{aligned} \zeta_1 * \zeta_3(x > r) &= \int_{0-}^{r+} \zeta_1(x > r - u) \zeta_3(du) + \zeta_3(x > r) \\ &\leq c_1 \int_{0-}^{r+} \zeta_2(x > r - u) \zeta_3(du) + c_1 \zeta_3(x > r) \\ &= c_1 \zeta_2 * \zeta_3(x > r). \end{aligned}$$

The inequality (5.5) is trivial from (5.4). \square

Lemma 5.3. *Let ζ_j for $j = 1, 2$ be distributions on \mathbb{R}_+ .*

- (i) *If $\zeta_1 \in \mathcal{OS}$ and $\zeta_1(x > r) \asymp \zeta_2(x > r)$, then $\zeta_2 \in \mathcal{OS}$.*
- (ii) *If $\zeta_1 \in \mathcal{D}$ and $\zeta_1(x > r) \asymp \zeta_2(x > r)$, then $\zeta_2 \in \mathcal{D}$.*

Proof: Assertion (i) is from Theorem 2.3 of Klüppelberg (1990). Assertion (ii) is clear from the definition. \square

Lemma 5.4. *Let ζ be distribution on \mathbb{R}_+ . Then $\zeta^{n*} \in \mathcal{D}$ for some $n \in \mathbb{N}$ if and only if $\zeta \in \mathcal{D}$.*

Proof: This is from Proposition 1.1 (iii) and Proposition 2.5 (iii) of Shimura and Watanabe (2005). \square

For $\zeta \in \mathcal{OS}$ on \mathbb{R}_+ , we define

$$\ell^*(\zeta) := \limsup_{r \rightarrow \infty} \frac{\zeta^{2*}(x > r)}{\zeta(x > r)} < \infty. \quad (5.6)$$

Lemma 5.5. *Suppose that $\zeta \in \mathcal{OS}$ on \mathbb{R}_+ .*

- (i) *For any $s > 0$, there is $c_1 > 0$ such that $\zeta(x > r) \leq c_1 \zeta(x > r + s)$. Thus $\zeta(x > \log(r + 1)) \in \mathcal{D}$ and there are $c_2, c_3 > 0$ such that*

$$\zeta(x > r) \geq c_2 \exp(-c_3 r) \quad \text{for } r \geq 0. \quad (5.7)$$

- (ii) *For every $\varepsilon > 0$ there is $c_4 > 0$ such that*

$$\zeta^{k*}(x > r) \leq c_4 (\ell^*(\zeta) - 1 + \varepsilon)^k \zeta(x > r) \quad \text{for every } r \geq 0 \text{ and } k \geq 1. \quad (5.8)$$

- (iii) *For every $k \in \mathbb{N}$, we have*

$$\zeta^{k*}(x > r) \asymp \zeta(x > r). \quad (5.9)$$

Proof: Assertion (i) is from Proposition 2.2 (ii) of Shimura and Watanabe (2005). Assertion (ii) is from Proposition 2.4 of Shimura and Watanabe (2005). As $\zeta^{k*}(x > r) \geq \zeta(x > r)$ for every $k \in \mathbb{N}$, assertion (iii) is clear from (ii). \square

Lemma 5.6. *Let ρ be an infinitely divisible distribution on \mathbb{R}^d with Lévy measure η . Take $c > 0$ such that $\eta(|x| > c) > 0$. Let ρ_j for $j = 1, 2$ be the infinitely divisible distributions on \mathbb{R}^d such that ρ_2 is a compound Poisson distribution on \mathbb{R}^d with Lévy measure $1_{\{|x|>c\}}\eta(dx)$ and $\rho = \rho_1 * \rho_2$. Then the following hold:*

- (i) $\rho \in \mathcal{OS}$ if and only if $\rho_2 \in \mathcal{OS}$.
- (ii) If $\rho \in \mathcal{OS}$, then $\rho(|x| > r) \asymp \rho_2(|x| > r)$.

Proof: Define the distributions ζ_j for $j = 1, 2$ and ζ_ρ by $\zeta_j(x > r) = \rho_j(|x| > r)$ and $\zeta_\rho(x > r) = \rho(|x| > r)$ for $r \geq 0$. Note from Lemma 3.7 that $\zeta_1(x > r) \leq c_1 \zeta_2(x > r)$ for some $c_1 > 1$. On the one hand, we see from Lemmas 5.1 (i) and 5.2 that

$$\rho(|x| > r) = \rho_1 * \rho_2(|x| > r) \leq \zeta_1 * \zeta_2(x > r) \leq c_1 \zeta_2^{2*}(x > r). \quad (5.10)$$

On the other hand, we see from Lemma 5.1 (ii) that, for some $c_2 \in (0, 1)$ and $s > 0$,

$$\rho(|x| > r) = \rho_1 * \rho_2(|x| > r) \geq c_2 \zeta_2(x > r + s). \quad (5.11)$$

Suppose that $\rho_2 \in \mathcal{OS}$, that is, $\zeta_2 \in \mathcal{OS}$. Thus $\zeta_2^{2*}(x > r) \asymp \zeta_2(x > r)$. Then, by using Lemma 5.5 (i) for ζ_2 , we obtain from (5.10) and (5.11) that $\rho(|x| > r) \asymp \zeta_2(x > r)$, and, by Lemma 5.3 (i), $\rho \in \mathcal{OS}$. Conversely, suppose that $\rho \in \mathcal{OS}$. Then $\zeta_\rho^{2*}(x > r) \leq c_3 \zeta_\rho(x > r)$ for some $c_3 > 0$. By using Lemma 5.5 (i) for ζ_ρ , we have, for some $c_4 > 0$,

$$\rho(|x| > r) \geq c_4 \rho(|x| > r - s) \geq c_2 c_4 \zeta_2(x > r). \quad (5.12)$$

This implies that there is $c_5 > 1$ such that $\zeta_2(x > r) \leq c_5 \zeta_\rho(x > r)$ for $r \geq 0$. From Lemma 3.7, we can take $\varepsilon > 0$ and $A > 0$ such that $\varepsilon c_3 c_5 < 1$ and $\zeta_1(x > r) < \varepsilon \zeta_\rho(x > r)$ for $r \geq A$. Hence we see that

$$\begin{aligned} \rho(|x| > r + A) &\leq \zeta_1 * \zeta_2(x > r + A) \\ &\leq \int_{0-}^{r+} \zeta_1(x > r + A - u) \zeta_2(du) + \zeta_2(x > r) \\ &\leq \varepsilon \int_{0-}^{r+} \zeta_\rho(x > r + A - u) \zeta_2(du) + \zeta_2(x > r) \\ &\leq \varepsilon \zeta_\rho * \zeta_2(x > r + A) + \zeta_2(x > r) \\ &\leq \varepsilon c_5 \zeta_\rho * \zeta_\rho(x > r + A) + \zeta_2(x > r) \\ &\leq \varepsilon c_3 c_5 \zeta_\rho(x > r + A) + \zeta_2(x > r). \end{aligned}$$

This yields

$$(1 - \varepsilon c_3 c_5) \rho(|x| > r + A) \leq \zeta_2(x > r).$$

We obtain from Lemma 5.5 (i) and (5.12) that $\rho(|x| > r) \asymp \rho_2(|x| > r)$, and consequently $\rho_2 \in \mathcal{OS}$. Here also assertion (ii) has been proved. \square

Proposition 5.7. *Let ρ be a compound Poisson distribution on \mathbb{R}^d with Lévy measure η . Define a distribution ζ on \mathbb{R}_+ satisfying $\zeta(x > r) = \tilde{\eta}(|x| > r)$ for $r \geq 0$.*

- (i) $\rho \in \mathcal{OS}$ on \mathbb{R}^d if and only if there is a positive integer n such that $\tilde{\eta}^{n*} \in \mathcal{OS}$ on \mathbb{R}^d . Moreover, if $\rho \in \mathcal{OS}$, then (3.12) holds.

(ii) If $\tilde{\eta} \in \mathcal{OS}$, then (3.13) holds.

(iii) $\rho \in \mathcal{OS}$ if and only if there is a positive integer m such that $\zeta^{m*} \in \mathcal{OS}$ on \mathbb{R}_+ and (3.14) holds.

Proof: We define distributions ζ_ρ and ζ_n on \mathbb{R}_+ as follows: For $r \geq 0$,

$$\zeta_\rho(x > r) := \rho(|x| > r) \quad \text{and} \quad \zeta_n(x > r) := \tilde{\eta}^{n*}(|x| > r).$$

Let $\delta := \eta(\mathbb{R}^d) < \infty$. Suppose that there is $n \in \mathbb{N}$ such that $\tilde{\eta}^{n*} \in \mathcal{OS}$ on \mathbb{R}^d , namely, $\zeta_n \in \mathcal{OS}$ on \mathbb{R}_+ . Then we see from Lemma 5.5 (ii) that there is $c_1 > 0$ such that

$$\zeta_n^{k*}(x > r) \leq c_1 \ell^*(\zeta_n)^k \zeta_n(x > r) \quad \text{for every } r \geq 0 \text{ and } k \geq 1.$$

By using Lemma 5.1 (ii) for $\tilde{\eta}^{(nk+j)*}$ with $0 \leq j \leq n-1$ and $s > 0$, we obtain from Lemmas 5.1 (i) and 5.5 (i) that, for some $c_2, c_3 > 0$,

$$\begin{aligned} \rho(|x| > r) &= e^{-\delta} \sum_{k=0}^{\infty} \frac{\delta^k}{k!} \tilde{\eta}^{k*}(|x| > r) \\ &= e^{-\delta} \sum_{j=0}^{n-1} \sum_{k=0}^{\infty} \frac{\delta^{nk+j}}{(nk+j)!} \tilde{\eta}^{(nk+j)*}(|x| > r) \\ &\leq c_2 \sum_{k=0}^{\infty} \frac{\delta^{nk}}{(nk)!} \tilde{\eta}^{n(k+1)*}(|x| > r-s) \\ &\leq c_2 \sum_{k=0}^{\infty} \frac{\delta^{nk}}{(nk)!} \zeta_n^{(k+1)*}(x > r-s) \\ &\leq c_2 c_1 \sum_{k=0}^{\infty} \frac{\ell^*(\zeta_n)^{k+1} \delta^{nk}}{(nk)!} \zeta_n(x > r-s) \\ &\leq c_3 \zeta_n(x > r). \end{aligned}$$

Obviously, it follows that

$$\rho(|x| > r) \geq e^{-\delta} \frac{\delta^n}{n!} \tilde{\eta}^{n*}(|x| > r) = e^{-\delta} \frac{\delta^n}{n!} \zeta_n(x > r). \quad (5.13)$$

Thus we have

$$\rho(|x| > r) \asymp \zeta_n(x > r) = \tilde{\eta}^{n*}(|x| > r). \quad (5.14)$$

Hence $\rho \in \mathcal{OS}$ by Lemma 5.3 (i).

Conversely, we suppose that $\rho \in \mathcal{OS}$ on \mathbb{R}^d . By using the method in the proof of Theorem 1.5 of Watanabe (2008), we shall prove that there is $n \in \mathbb{N}$ such that

$$\liminf_{r \rightarrow \infty} \frac{\tilde{\eta}^{n*}(|x| > r)}{\rho(|x| > r)} > 0. \quad (5.15)$$

Suppose that, for all $n \in \mathbb{N}$,

$$\liminf_{r \rightarrow \infty} \frac{\tilde{\eta}^{n*}(|x| > r)}{\rho(|x| > r)} = 0.$$

Then we can choose $N \in \mathbb{N}$ and positive increasing sequence $\{r_n\}$ with $\lim_{n \rightarrow \infty} r_n = \infty$ such that

$$\limsup_{r \rightarrow \infty} \frac{\zeta_\rho^{2*}(x > r)}{\zeta_\rho(x > r)} < e^{-\delta} 2^{N+1}, \quad (5.16)$$

and

$$\lim_{n \rightarrow \infty} \frac{\tilde{\eta}^{N*}(|x| > r_n)}{\rho(|x| > r_n)} = 0.$$

Define $I_j(r)$ and $J_j(r)$ for $j = 1, 2$ as

$$I_j(r) := e^{-j\delta} \sum_{k=1}^N \frac{(j\delta)^k \tilde{\eta}^{k*}(|x| > r)}{k!} \quad \text{and} \quad J_j(r) := e^{-j\delta} \sum_{k=N+1}^{\infty} \frac{(j\delta)^k \tilde{\eta}^{k*}(|x| > r)}{k!}.$$

By Lemma 5.1 (ii), there are $s_1 > 0$ and $c_4 > 0$ such that, for $1 \leq k \leq N$,

$$\tilde{\eta}^{k*}(|x| > r + s_1) \leq c_4 \tilde{\eta}^{N*}(|x| > r) \quad \text{for } r \geq 0.$$

Since $\rho \in \mathcal{OS}$, we find from Lemma 5.5 (i) that there is $c_5 > 0$ such that

$$\rho(|x| > r) \leq c_5 \rho(|x| > r + s_1) \quad \text{for } r \geq 0.$$

Thus we have, for $1 \leq k \leq N$,

$$\limsup_{n \rightarrow \infty} \frac{\tilde{\eta}^{k*}(|x| > r_n + s_1)}{\rho(|x| > r_n + s_1)} \leq c_4 c_5 \lim_{n \rightarrow \infty} \frac{\tilde{\eta}^{N*}(|x| > r_n)}{\rho(|x| > r_n)} = 0.$$

Here, as we have $I_1(r) + J_1(r) = \rho(|x| > r)$ and $I_1(r_n + s_1)/\rho(|x| > r_n + s_1) \rightarrow 0$ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \frac{J_1(r_n + s_1)}{\rho(|x| > r_n + s_1)} = 1.$$

Hence we establish that

$$\begin{aligned} \limsup_{r \rightarrow \infty} \frac{\zeta_\rho^{2*}(x > r)}{\zeta_\rho(x > r)} &\geq \limsup_{r \rightarrow \infty} \frac{\rho^{2*}(|x| > r)}{\rho(|x| > r)} \\ &\geq \limsup_{n \rightarrow \infty} \frac{I_2(r_n + s_1) + J_2(r_n + s_1)}{\rho(|x| > r_n + s_1)} \\ &\geq \liminf_{r \rightarrow \infty} \frac{J_2(r)}{J_1(r)} \geq e^{-\delta} 2^{N+1}. \end{aligned}$$

This contradicts (5.16). Thus (5.15) holds and it follows from (5.13) that there is $n \in \mathbb{N}$ such that (5.14) holds. We find from Lemma 5.3 (i) that $\tilde{\eta}^{n*} \in \mathcal{OS}$.

Assertion (ii) is clear from (i). Next we prove (iii). Suppose that $\rho \in \mathcal{OS}$. Define a compound Poisson distribution τ on \mathbb{R}_+ such that its Lévy measure is $\delta\zeta$. Recall that $\delta = \eta(\mathbb{R}^d)$. By Lemma 5.1 (i), we have

$$\zeta_\rho(x > r) = e^{-\delta} \sum_{k=1}^{\infty} \frac{\delta^k \tilde{\eta}^{k*}(|x| > r)}{k!} \leq \tau(x > r) \quad \text{for } r \geq 0.$$

Note that $\zeta_\rho(x > r) \geq c^{-1} \zeta(x > r)$ with $c := e^\delta \delta^{-1}$. By using (5.8), we see from Lemma 5.2 that, for $r \geq 0$,

$$\begin{aligned} \tau(x > r) &= e^{-\delta} \sum_{k=1}^{\infty} \frac{\delta^k}{k!} \zeta^{k*}(x > r) \\ &\leq e^{-\delta} \sum_{k=1}^{\infty} \frac{(c\delta)^k}{k!} \zeta_\rho^{k*}(x > r) \\ &\leq c_1 e^{-\delta} \sum_{k=1}^{\infty} \frac{(c\delta)^k}{k!} (\ell^*(\zeta_\rho) - 1 + \varepsilon)^k \zeta_\rho(x > r). \end{aligned}$$

Thus we find that

$$\tau(x > r) \asymp \zeta_\rho(x > r) = \rho(|x| > r)$$

and, by Lemma 5.3 (i), $\tau \in \mathcal{OS}$ on \mathbb{R}_+ . Use (i) for the compound Poisson distribution τ on \mathbb{R}_+ . There is a positive integer m such that $\zeta^{m*} \in \mathcal{OS}$ on \mathbb{R}_+ and (3.14) holds. By Lemma 5.3, the converse assertion is trivial. \square

Now we prove Theorem 3.9.

Proof of Theorem 3.9: The proofs of (i), (ii), and (iii) are clear from Lemma 5.6 and Proposition 5.7. Assertion (iv) is proved in the same way of Corollary 1.1 (ii) of Shimura and Watanabe (2005). \square

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