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Blow-up for a system with time-dependent generators

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Abstract. Sufficient conditions, for finite time blow-up of nonnegative solutions, are given for the weakly coupled system:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= k_1(t)\,\Delta_{\alpha_1}u_1 + u_1u_2^{\beta_1},\ u_1(0)=\varphi_1,\\ \frac{\partial u_2}{\partial t} &= k_2(t)\,\Delta_{\alpha_2}u_2 + u_2u_1^{\beta_2},\ u_2(0)=\varphi_2,\end{aligned}$$

where Δ_{α_i} is the fractional Laplacian, $0<\alpha_i\le 2$, $\beta_i\ge 1$ are constants, φ_i are positive, bounded, continuous and not identically zero, and $k_i:[0,\infty)\to[0,\infty)$, are continuous with $\int_0^t k_i(r)\,dr = O(t^{\rho_i})$, $\rho_i>0$, for $i=1,2$.

1. Introduction: statement of the result and overview

In what follows $i\in\{1,2\}$ and $i'=3-i$. The purpose of this paper is to extend some results obtained in the literature about blow up in finite time of the nonnegative solutions for the following system of equations,

$$\begin{aligned}\frac{\partial u_i(t,x)}{\partial t} &= k_i(t)\,\Delta_{\alpha_i}u_i(t,x) + u_i(t,x)\,u_{i'}^{\beta_i}(t,x),\ t>0,\ x\in\mathbb{R}^d, \qquad (1.1)\\ u_i(0,x) &= \varphi_i(x)\ge 0,\quad x\in\mathbb{R}^d.\end{aligned}$$

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For $\Delta_{\alpha_i} = -(-\Delta)^{\alpha_i/2}$ we denote the α_i-Laplacian, $0 < \alpha_i \leq 2$, $\beta_i \geq 1$ are constants, φ_i is bounded, continuous and not identically zero, and $k_i : [0,\infty) \to [0,\infty)$ is continuous. Let us define

$$K_i(t,s) = \int_s^t k_i\,(r)\,dr, \quad t \geq s \geq 0. \tag{1.2}$$

We also assume

$$\varepsilon_{1,i} t^{\rho_i} \leq K_i\,(t,0) \leq \varepsilon_{2,i} t^{\rho_i}, \tag{1.3}$$

for all t large enough, where $\varepsilon_{1,i}$, $\varepsilon_{2,i}$ and ρ_i are positive constants.

The integral version of (1.1) is given by

$$u_i(t,x) \quad = \quad U_i\,(t,0)\,\varphi_i\,(x) + \int_0^t U_i\,(t,r)\,u_i\,(r,x)\,u_{i'}^{\beta_i}\,(r,x)\,dr, \tag{1.4}$$

where $\{U_i\,(t,s)\}_{t\geq s\geq 0}$ is the evolution family, on the space of bounded Borel measurable functions on $\mathbb{R}^d$, that solves the homogeneous Cauchy problem, for the family of generators $\{k_i\,(t)\,\Delta_{\alpha_i}\}_{t\geq 0}$. We have

$$U_i\,(t,s) = S_i\,(K_i\,(t,s))\,, \quad t \geq s \geq 0,$$

where $\{S_i\,(t)\}_{t\geq 0}$ is the semigroup of a α_i-stable Lévy process with infinitesimal generator Δ_{α_i}.

A solution of (1.4) is called a mild solution of (1.1). By a solution of (1.1) we always mean a mild solution. If there exist a solution (u_1,u_2) of (1.1) in $[0,\infty)\times\mathbb{R}^d$ such that $\|u_1\,(t,\cdot)\|_\infty + \|u_2\,(t,\cdot)\|_\infty < \infty$ for any $t \geq 0$, we say that (u_1,u_2) is a global solution. If there exist a number $0 < T_e < \infty$ such that (1.1) has a bounded solution (u_1,u_2) in $[0,T]\times\mathbb{R}^d$, for all $T < T_e$, with $\lim_{t\uparrow T_e}\left(\|u_1\,(t,\cdot)\|_\infty + \|u_2\,(t,\cdot)\|_\infty\right) = \infty$, then we say that (u_1,u_2) is non-global or that blows up in finite time.

From (1.4) we see that if u_i blows up in finite time then $u_{i'}$ also does. In this paper we will see that all nonnegative solutions (u_1,u_2) of (1.1) blow up in finite time. More precisely, we prove in Section 2 the following result.

Theorem 1.1. *Let $a = \min\{\rho_1/\alpha_1, \rho_2/\alpha_2\}$. If*

$$d < \max\left\{\left(\frac{\rho_{i'}\beta_i}{\alpha_{i'}} + \frac{\rho_i}{\alpha_i} - a\right)^{-1} : i = 1,2\right\}, \tag{1.5}$$

then the solution (u_1,u_2) of (1.1) blows up in finite time.

The study of systems like (1.1) is of interest due to their applications. Such models arise, for example, in chemical reaction processes, combustion theory, heat conduction, physic and engineering; see Bebernes and Eberly (1989) and Samarskii et al. (1995). Generators of the form $k_i\,(t)\,\Delta_{\alpha_i}$ are used for example in models of anomalous growth of certain fractal interfaces and in hydrodynamic models with modified diffusivity; see, for example, Bardos et al. (1979) and Jr. and Woyczynski (2001). From a probabilistic perspective, operators Δ_{α_i} correspond to stable Lévy processes; see Sato (1999) and Bogdan et al. (2009).

Using a probabilistic approach Birkner et al. (2002) gave sufficient conditions for the blow up of solutions of (1.1). In fact, when $k_1 \equiv k_2 \equiv 1$, Birkner et al. (2002) used the Feynman-Kac formula to prove that if $a = \min\{1/\alpha_1, 1/\alpha_2\}$ and

$$d < \max\left\{\left(\frac{\beta_i}{\alpha_{i'}} + \frac{\rho_i}{\alpha_i} - a\right)^{-1} : i = 1,2\right\} \tag{1.6}$$

then all nontrivial nonnegative solution of (1.1) blow up in finite time. Observe that when $\rho_1 = \rho_2 = 1$, then the condition (1.5) coincides with (1.6). Kolkovska et al. (2008) proved that the nonnegative solutions of (1.1) blow up in finite time for a single equation with $0 < \alpha \leq 2$, where $k : [0, \infty) \to [0, \infty)$ is a locally integrable function satisfying (1.3), and $d < \alpha/(\rho\beta)$. The case $d = \alpha/(\rho\beta)$ also implies blow up in finite time, and it was studied in Pérez and Villa (2010) by analytical methods. We like to note that the subject was initiated by Fujita (1966) and it is very active even now, see for example Escobedo and Levine (1995), Lu (1995), Zheng (1999), Guedda and Kirane (1999), Guedda and Kirane (2001), Wang (2001), Wang (2000), Kirane and Qafsaoui (2002), Yamauchi (2006), Pérez-Pérez (2006) and references therein.

When $\varphi_i \equiv c_i > 0$, a constant, then the solution $(v_1(t), v_2(t))$ of the system of equations

$$v_i(t) \quad = \quad c_i + \int_0^t v_i(s) v_{i'}^{\beta_i}(s) ds,$$

is a solution of (1.4). This system induce a Bernoulli type equation for v_i, and thus it blows up in finite time (see p. 211). Here the blow up does not depend on the parameters α, d, β and ρ. We have a completely different behavior when φ_i is not a constant. Assume, without loss of generality, that 0 is in the support of φ_i and denote by B_1 the unitary open ball with center 0. Then (see Lemma 4.1),

$$u_i(t,x) \geq c t^{-d/(\alpha_i/\rho_i)} 1_{B_1}(c t^{-1/(\alpha_i/\rho_i)} x). \tag{1.7}$$

In the sequel by c or c_i we denote positive constants whose specific values are unimportant and in general are different form place to place. The lower estimation in (1.7) of u_i is very rough, but using it as initial condition in (1.1), by the Feynman-Kac formula we find a better lower estimation for u_i. In particular, we can see that u_i is not bounded.

Proposition 1.2. *If d satisfies (1.5), then $\inf_{x \in B_1} u_i(t,x) \to \infty$, as $t \to \infty$, for each $i \in \{1,2\}$.*

In fact, we will see that $u_i(t,x)$ can be bounded from below by a solution of a Bernoulli type equation whose explosion time depends on the fact that we can take $u_i(t,x)$ large enough. The latter is ensured by Proposition 1.2, whose proof is given in Section 4.

Heuristically, from (1.7) we see that the behavior of $k_i(t)\Delta_{\alpha_i}$ is similar to that of Δ_{α_i/ρ_i}. If for example the initial condition is sufficiently small and α_i/ρ_i is also small, then the related α_i/ρ_i-stable process has long jumps and can reach those regions where the initial values are close to 0, hence bigger values of β_i tend to decrease the values of d. In this way $0 < \rho_i < 1$ diminishes the influence of jumps of the process and $\rho_i > 1$ enhances it, influencing dimensions for wich we obtain explosion. As an example, observe that if $\rho_i = 1$, $\beta_i = 1$ and $\alpha_i > 1$, then our criterion yields the blow-up of all the non-negative solutions in finite time only for $d = 1$, and if $\alpha_i = 1$, $\beta_i = 2$, $\rho_i = 1/8$, then we certainly have explosion up to $d < 4$.

2. Local existence and Proof of Theorem 1.1

The proof of the local existence is an adaptation, to our case, of the proof given by Uda (1995).

Let $\tau > 0$ and

$$E_\tau \equiv \{(u_1, u_2) : [0, \tau] \to C_b(\mathbb{R}^d) \times C_b(\mathbb{R}^d), |||(u_1, u_2)||| < \infty\},$$

where $C_b(\mathbb{R}^d)$ is the space of real-valued continuous and bounded functions defined on $\mathbb{R}^d$, and

$$|||(u_1, u_2)||| \equiv \sup_{0 \le t \le \tau} \{||u_1(t, \cdot)||_\infty + ||u_2(t, \cdot)||_\infty\}.$$

The set E_τ is a Banach space and $P_\tau \equiv \{(u_1, u_2) \in E_\tau, u_1 \ge 0, u_2 \ge 0\}$ and $C_R \equiv \{(u_1, u_2) \in E_\tau, |||(u_1, u_2)||| \le R\}$, $R > 0$, are closed subsets of E_τ.

Theorem 2.1. *There exists a constant $\tau = \tau(\varphi_1, \varphi_2) > 0$ such that the integral system (1.4) has a local solution in $C_b([0, \tau] \times \mathbb{R}^d) \times C_b([0, \tau] \times \mathbb{R}^d)$.*

Proof: Define the operator Ψ on $C_b([0, \tau] \times \mathbb{R}^d) \times C_b([0, \tau] \times \mathbb{R}^d)$ as

$$\begin{aligned}\Psi(u_1, u_2) &= (U_1(t, 0)\varphi_1(x), U_2(t, 0)\varphi_2(x)) \\ &+ \left(\int_0^t U_1(t, r)\, u_1(r, x)\, u_2^{\beta_1}(r, x)\, dr, \int_0^t U_2(t, r)\, u_2(r, x)\, u_1^{\beta_2}(r, x)\, dr\right).\end{aligned}$$

Then, by taking $\tau > 0$ small enough and $R > 0$ sufficiently large, we see that Ψ is a contraction mapping on $C_R \cap P_\tau$, hence the result follows by the fixed point theorem. □

Now let us handle the proof of the main result. We begin by introducing some concepts. Let $p_i(t, x)$, $t \ge 0$, $x \in \mathbb{R}^d$, be the transition density of the d-dimensional isotropic α_i-stable process $Z_i \equiv \{Z_i(t)\}_{t \ge 0}$. We use P_x to denote the distribution of Z_i, where $P_x[Z_i(0) = x] = 1$.

Lemma 2.2. *Let*

$$\xi_i := \min_{x \in B_1} \min_{0 \le r \le 1} P_x[Z_i(r) \in B_1].$$

Then $\xi_i > 0$ and for any $0 \le s \le t$, $x \in B_1$ and $t_0 \ge 0$,

$$\int_{B_1} p_i(K_i(t + t_0, s + t_0), y - x) dy \ge \xi_i^{\lfloor K_i(t+t_0, s+t_0) \rfloor}, \tag{2.1}$$

where $\lfloor z \rfloor$ denotes the least integer no smaller than $z \in [0, \infty)$.

Proof: See Lemma 4.2 in Kolkovska et al. (2008). □

Proof of Theorem 1.1: Let $t_1 > 0$ such that $\|u_1(t_1, \cdot)\|_\infty + \|u_2(t_1, \cdot)\|_\infty < \infty$. Then

$$\begin{aligned}u_i(t + t_1, x) &\ge \int_{B_1} p_i(K_i(t + t_1, t_1), y - x)\, u_i(t_1, y)\, dy \\ &+ \int_0^t \int_{B_1} p_i(K_i(t + t_1, s + t_1), y - x)\, u_i(s + t_1, y)\, u_{i'}^{\beta_i}(s + t_1, y)\, dyds.\end{aligned}$$

Let $w_i(t, \cdot) := u_i(t_1 + t, \cdot)$, then

$$\begin{aligned}w_i(t, x) &\ge \left(\min_{y \in B_1} u_i(t_1, y)\right) \int_{B_1} p_i(K_1(t + t_1, t_1), y - x)\, dy \\ &+ \int_0^t \int_{B_1} p_i(K_1(t + t_1, s + t_1), y - x) \left(\min_{z \in B_1} w_i(s, z)\right) \left(\min_{z \in B_1} w_{i'}(s, z)\right)^{\beta_i} dyds.\end{aligned}$$

Let $t \in [0,1]$. From Lemma 2.2 and observing that

$$K_i(t+t_1, s+t_1) \le K_i(t+t_1, t_1) \le K_i(1+t_1, t_1),$$

we get

$$\begin{aligned}
\min_{x\in B_1} w_i(t,x) &\ge \left(\min_{y\in B_1} u_i(t_1,y)\right) \xi_i^{\lfloor K_i(t+t_1,t_1)\rfloor} \\
&\quad + \int_0^t \xi_i^{\lfloor K_i(t+t_1,s+t_1)\rfloor} \left(\min_{z\in B_1} w_i(s,z)\right)\left(\min_{z\in B_1} w_{i'}(s,z)\right)^{\beta_i} dyds \\
&\ge C_i(t_1)\xi_i(t_1) \\
&\quad + \xi_i(t_1)\int_0^t \left(\min_{z\in B_1} w_i(s,z)\right)\left(\min_{z\in B_1} w_{i'}(s,z)\right)^{\beta_i} dyds,
\end{aligned}$$

where

$$C_i(t_1) = \min_{y\in B_1} u_i(t_1,y) \quad \text{and} \quad \xi_i(t_1) = \xi_i^{\lfloor K_i(1+t_1,t_1)\rfloor}.$$

Consider the integral system

$$v_i(t) \;=\; C_i(t_1) + \int_0^t v_i(s) v_{i'}^{\beta_i}(s) ds.$$

From this we see that v_i blows up in finite time if and only if $v_{i'}$ does. Moreover, by the comparison theorem we have $\min_{z\in B_1} w_i(t,z)/\xi_i(t_1) \ge v_i(t)$. In this manner, it is enough to deal with v_i. Since

$$v_i^{\beta_{i'}-1}(t)\frac{dv_i}{dt}(t) = v_{i'}^{\beta_i-1}(t)\frac{dv_{i'}}{dt}(t),$$

then we get a Bernoulli type equation for v_i, namely,

$$\frac{dv_i}{dt}(t) + \left(\frac{\beta_i}{\beta_{i'}} v_i^{\beta_{i'}}(0) - v_{i'}^{\beta_i}(0)\right) v_i(t) = \frac{\beta_i}{\beta_{i'}} v_i^{\beta_{i'}+1}(t). \tag{2.2}$$

Let $C_i^*(t_1) = \beta_i C_i(t_1)^{\beta_{i'}}$ and $C_{i'}^*(t_1) = \beta_{i'} C_{i'}(t_1)^{\beta_i}$. Without loss of generality, we can assume that $C_{i'}^*(t_1) \ge C_i^*(t_1)$. The solution of (2.2), when $C_{i'}^*(t_1) > C_i^*(t_1)$ is

$$v_i^{\beta_{i'}}(t) = \frac{C_i^*(t_1)\left(C_{i'}^*(t_1) - C_i^*(t_1)\right)\beta_i^{-1} C_{i'}^*(t_1)^{-1} \exp\left[\left(C_{i'}^*(t_1) - C_i^*(t_1)\right)t\right]}{1 - \frac{C_i^*(t_1)}{C_{i'}^*(t_1)} \exp\left[\left(C_{i'}^*(t_1) - C_i^*(t_1)\right)t\right]}$$

and when $C_{i'}^*(t_1) = C_i^*(t_1)$,

$$v_i^{\beta_{i'}}(t) = \frac{1}{\beta_i\left(\frac{1}{C_{i'}^*(t_1)} - t\right)}.$$

Thus, the explosion time for v_i, the solution of (2.2), is

$$\tau = \begin{cases} \frac{\ln C_{i'}^*(t_1) - \ln C_i^*(t_1)}{C_{i'}^*(t_1) - C_i^*(t_1)}, & \text{if } C_{i'}^*(t_1) > C_i^*(t_1), \\ \frac{1}{C_i^*(t_1)}, & \text{if } C_{i'}^*(t_1) = C_i^*(t_1). \end{cases}$$

By Proposition 1.2 we can choose t_1 such that $C_i^*(t_1) > 1$. Since $f(x) = x - \ln(x)$ is strictly increasing for $x > 1$, we have $\tau < 1$, and thus

$$\frac{\min_{z\in B_1} w_i(1,z)}{\xi_i(t_1)} \ge v_i(1) = \infty,$$

which implies that (u_1, u_2) blows up at time $t_1 + 1$. $\square$

3. Feynman-Kac formula and subsolutions

Let $X_i \equiv \{X_i(t)\}_{t\geq 0}$ be the (time-inhomogeneous) càdlàg Feller process corresponding to the family of generators $\{k_i(t)\Delta_{\alpha_i}\}$. Note that $X_i(t) = Z_i(K_i(t,0))$. We use P_x to denote the distribution of X_i, where $P_x[X_i(0) = x] = 1$ and we write E_x for the expectation with respect to P_x, $x \in \mathbb{R}^d$.

It is known (see e.g. Kolkovska et al., 2008) that for any $0 < T < T_e$ the solution (u_1, u_2) of (1.1) admits the Feynman-Kac representation

$$u_i(t,x) \quad = \quad E_x\left[\varphi_i(X_i(t))\exp\left\{\int_0^t u_{i'}^{\beta_i}(t-s, X_i(s))\,ds\right\}\right], \tag{3.1}$$

for each $(t,x) \in [0,T]\times\mathbb{R}^d$. We say that $(\overline{u}_1, \overline{u}_2)$ is a subsolution of (u_1,u_2), the solution of (1.1), if $\overline{u}_i \leq u_i$. Such formula implies that (u_1^*, u_2^*), where

$$u_i^*(t,x) := E_x[\varphi_i(X_i(t))],$$

is a subsolution of (1.1). Also, the next lemma, which we will need in the following section, is a direct consequence of the Feynman-Kac representation.

Lemma 3.1. *If $(\overline{u}_1, \overline{u}_2)$ is a subsolution of (1.1), then any solution of*

$$\begin{aligned}
\frac{\partial u_i(t,x)}{\partial t} \quad &= \quad k_i(t)\Delta_{\alpha_i}u_i(t,x) + u_i(t,x)\,\overline{u}_{i'}^{\beta_i}(t,x), \quad (t,x)\in(0,T_e)\times\mathbb{R}^d,\\
u_i(0,x) \quad &= \quad \varphi_i(x), \qquad x\in\mathbb{R}^d,
\end{aligned}$$

remains a subsolution of (1.1).

4. Unboundness of positive solutions

We recall that the transition densities, $p_i(t,x)$, of the d-dimensional symmetric α_i-stable process Z_i, are strictly positive, radially symmetric, continuous. They also have the following properties.

Lemma 4.1. *Let $s,t > 0$ and $x,y \in \mathbb{R}^d$. Then*
(i) $p_i(ts,x) = t^{-d/\alpha_i}p_i(s, t^{-1/\alpha_i}x)$ (self similarity),
(ii) $p_i(t,x) \leq p_i(t,y)$, when $|x| \geq |y|$ (radially decreasing).

Proof: See Sugitani (1975) or Guedda and Kirane (1999), for example. □

Fix $t_0 > 1$ such that $K_i(t_0, 0) \geq 1$ and (1.3) holds for all $t \geq t_0$. Define

$$\delta_0 = \min\left\{\left(\frac{\varepsilon_{1,i}}{2\varepsilon_{2,i}}\right)^{1/\rho_i} : i = 1,2\right\}.$$

Lemma 4.2. *There exists $c > 0$ such that for all $x,y \in B_1$ and $t \geq (t_0/\delta_0)$,*

$$P_x\left[X_i(s) \in B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)}\,|X_i(t) = y\right] \geq cs^{(a-\rho_i/\alpha_i)d}$$

for all $s \in [t_0, \delta_0 t]$, where $a = \min\{\rho_j/\alpha_j : j \in \{1,2\}\}$.

Proof: Using Lemma 4.1 (self similarity) we get

$$\begin{aligned}
&P_x\left[X_i(s)\in B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)}\,|X_i(t)=y\right]\\
&=\frac{K_i^{-d/\alpha_i}(s,0)\,K_i^{-d/\alpha_i}(t,s)}{K_i^{-d/\alpha_i}(t,0)\,p_i(1,K_i^{-1/\alpha_i}(t,0)\,(x-y))}\\
&\times\int_{B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)}} p_i(1,K_i^{-1/\alpha_i}(s,0)\,(x-z))p_i(1,K_i^{-1/\alpha_i}(t,s)\,(z-y))dz.
\end{aligned}$$

The unimodality of $p_i(1,\cdot)$, $K_i\,(t,0)\geq K_i\,(t,s)$ and (1.3) implies

$$\frac{K_i^{-d/\alpha_i}(s,0)\,K_i^{-d/\alpha_i}(t,s)}{K_i^{-d/\alpha_i}(t,0)\,p_i(1,K_i^{-1/\alpha_i}(t,0)\,(x-y))}\geq\frac{c}{p_i(1,0)}s^{-\rho_i d/\alpha_i}. \tag{4.1}$$

Since $t_0\leq s\leq\delta_0 t$, then (1.3) gives

$$K_{i'}(t-s,0)\geq\varepsilon_{1,i'}(t-s)^{\rho_{i'}}\geq\varepsilon_{1,i'}\left(\frac{1}{\delta_0}-1\right)^{\rho_{i'}}s^{\rho_{i'}},$$

therefore $B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)}\supset B_{cs^a}$. Now, let us estimate

$$\int_{B_{cs^a}} p_i(1,K_i^{-1/\alpha_i}(s,0)\,(x-z))p_i(1,K_i^{-1/\alpha_i}(t,s)\,(z-y))dz. \tag{4.2}$$

Observe that

$$\begin{aligned}
|K_i^{-1/\alpha_i}(s,0)\,(x-z)\,| &\leq K_i^{-1/\alpha_i}(s,0)+K_i^{-1/\alpha_i}(s,0)\,cs^a\\
&\leq 1+cs^{a-\rho_i/\alpha_i}\leq c_1,
\end{aligned}$$

and

$$K_i\,(t,s)=K_i\,(t,0)-K_i\,(s,0)\geq\varepsilon_{1,i}t^{\rho i}-\varepsilon_{2,i}\delta_0^{\rho_i}t^{\rho_i}\geq ct^{\rho_i}$$

implies

$$|K_i^{-1/\alpha_i}(t,s)\,(z-y)\,|\leq c_2.$$

Thus, p_i radialy decreasing implies that (4.2) is bounded below by

$$\int_{B_{cs^a}}\inf\{p_i(1,\varsigma)p_i(1,\zeta):\varsigma,\zeta\in B_{c_1+c_2}\}dz=cs^{da}.$$

The result follows from (4.1) and (4.2). □

Lemma 4.3. *There exists a constant $c>0$ such that,*

$$u_i^*(t,x)\geq cK_i^{-d/\alpha_i}(t,0)1_{B_1}(K_i^{-1/\alpha_i}(t,0)x),$$

for all $x\in\mathbb{R}^d$ and all $t>0$ satisfying $K_i(t,0)\geq 1$.

Proof: See page 6 in Kolkovska et al. (2008). □

Proof of Proposition 1.2: By the law of total probability, the Feynman-Kac representation (3.1) of u_i can be written as

$$\begin{aligned}
u_i(t,x) &= \int\varphi_i(y)p_i\,(K_i(t,0),y-x)\\
&\quad\times E_x\left[\exp\int_0^t u_{i'}^{\beta_i}(t-s,X_i(s))ds\ \mid\ X_i(t)=y\right]dy.
\end{aligned}$$

By Lemmas 4.1, 4.3 and Jensen's inequality we get

$$\begin{aligned} u_i(t,x) &\geq \int_{B_1} \varphi_i(y) K_i^{-d/\alpha_i}(t,0) p_i(1, K_i^{-1/\alpha_i}(t,0)(y-x)) \\ &\quad \times \exp\Big(\int_\theta^{\delta_0 t} c K_{i'}^{-d\beta_i/\alpha_{i'}}(t-s,0) \\ &\quad \times P_x \Big[X_i(s) \in B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)} \mid X_i(t) = y \Big] ds \Big) dy. \end{aligned}$$

If $x, y \in B_1$, then

$$p_i(1, K_i^{-1/\alpha_i}(t,0)(y-x)) \geq \inf\{p_i(1,z) : z \in B_2\}.$$

This, together with $K_{i'}(t-s,0) \leq K_{i'}(t,0)$ and (1.3), yields

$$\begin{aligned} u_i(t,x) &\geq c K_i^{-d/\alpha_i}(t,0) \int_{B_1} \varphi_i(y) \exp\Big(c \int_\theta^{\delta_0 t} K_{i'}^{-d\beta_i/\alpha_{i'}}(t-s,0) \\ &\quad \times P_x \Big[X_i(s) \in B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)} \mid X_i(t) = y \Big] ds \Big) dy \\ &\geq c t^{-\rho_i d/\alpha i} \exp\Big(c t^{-d\rho_{i'}\beta_i/\alpha_{i'}} \\ &\quad \times \int_\theta^{\delta_0 t} \min_{y \in B_1} P_x \Big[X_i(s) \in B_{K_{i'}^{1/\alpha_{i'}}(t-s,0)} \mid X_i(t) = y \Big] ds \Big). \end{aligned}$$

We are done by Lemma 4.2. □

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