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Diagonal approximations by martingales

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Abstract. In the paper, we study and compare several types of approximations of stationary processes by martingales.

1. Introduction

Let $(\Omega, \mathcal{A}, \mu)$ be a probability space with a bijective, bimeasurable and measure preserving transformation T. For a measurable function f on Ω, $(f \circ T^i)_i$ is a (strictly) stationary process and reciprocally, any (strictly) stationary process can be represented in this way. We shall denote $Uf = f \circ T$; U is a unitary operator in L^2. The function f will be supposed to have a zero mean. $(\mathcal{F}_i)_{i \in \mathbb{Z}}$, where $T^{-1}\mathcal{F}_i = \mathcal{F}_{i+1}$, is an increasing filtration of sub-σ-fields of $\mathcal{A}$. H_i denotes the space of $\mathcal{F}_i$-measurable and square integrable random variables. $P_i(f)$ is the projection of f onto the space $H_i \ominus H_{i-1}$, i.e. $P_i(f) = \mathrm{E}(f|T^{-i}(\mathcal{F}_0)) - \mathrm{E}(f|T^{-i+1}(\mathcal{F}_0))$. We suppose the function f to be regular, i.e. $f = \sum_{i=-\infty}^{+\infty} P_i(f)$. Let $\sigma_n = ||S_n(f)||$, where $||\cdot||$ is the L_2-norm and $S_n(f) = \sum_{i=1}^{n} f \circ T^i$. For simplicity of notation we write $Q_0(S_n(f))$ for $\mathrm{E}(S_n(f)|\mathcal{F}_0)$ and $R_n(S_n(f))$ for $S_n(f) - \mathrm{E}(S_n(f)|\mathcal{F}_n)$.

Definition 1.1. We say that a function f has **a martingale approximation** if there exists a martingale difference sequence $(m \circ T^i)_{i \in \mathbb{Z}}$ such that $||S_n(f - m)||^2 = o(n)$.

It is a classical result that the martingale approximation implies the central limit theorem Gordin (1969). In his article, Gordin noticed that if $f = \sum_{i=j}^{k} P_i f$, $j \leq k$, then for $m = P_0 \sum_{i=-k}^{-j} U^i f$ we have $f = m + g - Ug$ with $g \in L^2$ hence we get a martingale approximation.

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Definition 1.2. We say that for a function f there exists **a weak diagonal approximation** if there is a sequence $(k_n)_{n\in\mathsf{N}}$, $k_n = o(n)$ of integers such that

(i) for $f_n = \sum_{i=-k_n}^{k_n} P_i f$ we have

$$\frac{1}{\sigma_n}||S_n(f - f_n)|| \to 0,$$

(ii) for $f_n = m_n + g_n - Ug_n$ with

$$\begin{aligned} m_n &= \sum_{i=-k_n}^{k_n} P_0 U^i f \\ g_n &= \sum_{i=0}^{k_n-1}\sum_{j=1}^{k_n-i} P_{-i}U^j f_n - \sum_{i=1}^{k_n}\sum_{j=0}^{k_n-i} P_i U^{-j} f_n \end{aligned}$$

we have $||g_n|| = o(\sigma_n)$.

If for the array $(U^i m_n)_{i=0,k_n-1,n\geq 1}$ of martingale differences a CLT holds, we get it for $S_n(f)/\sigma_n$ as well (cf. Volný, 2006); this generalises Gordin's result to processes with nonlinear growth of variances (of partial sums).

Hannan (1973) has noticed that if the series of $||P_i f|| = ||P_0 U^i f||$ is summable then the series of $m = \sum_{i\in\mathsf{Z}} P_0 U^i f$ converges and gives a martingale approximation. We, moreover, get that for any $\epsilon > 0$ there is an n such that for $m_n = \sum_{i=-n}^{n} P_0 U^i f$, $\limsup_{k\to\infty} ||S_k(f-m)||^2/k \leq \epsilon$; this holds if we replace m_n by $f_n = \sum_{i=-n}^{n} P_i f$ as well.

Definition 1.3. We say that for a function f there exists **a strong diagonal approximation** if there is an integer sequence of $(d(n)))_{n\in\mathsf{N}}$, $d(n) = o(n)$, such that for all integer sequences $(k_n)_{n\in\mathsf{N}}$ which satisfy $k_n \geq d(n)$ for all n and $k_n = o(n)$:

(i) for $f_n = \sum_{i=-k_n}^{k_n} P_i f$ we have

$$\frac{1}{\sigma_n}||S_n(f - f_n)|| \to 0,$$

(ii) for $f_n = m_n + g_n - Ug_n$ with

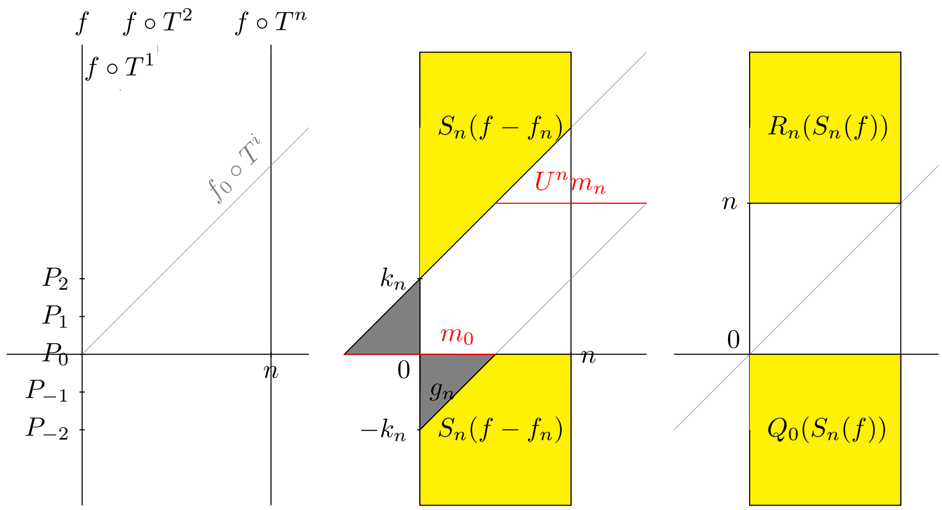
$$\begin{aligned} m_n &= \sum_{i=-k_n}^{k_n} P_0 U^i f \\ g_n &= \sum_{i=0}^{k_n-1}\sum_{j=1}^{k_n-i} P_{-i}U^j f_n - \sum_{i=1}^{k_n}\sum_{j=0}^{k_n-i} P_i U^{-j} f_n \end{aligned}$$

we have $||g_n|| = o(\sigma_n)$.

The Hannan's condition thus implies the strong diagonal approximation.

The Hannan's condition implies the weak invarinace principle; in full generality the result was proved in Dedecker et al. (2007), the martingale approximation for non adapted processes can be found in Volný (1993). Heyde has shown that if the series $m = \sum_{i\in\mathsf{Z}} P_0 U^i f$ converges in L^2 and $\|m\|_2 \geq \liminf_{n\to\infty} \|S_n(f)\|_2$ then a martingale approximation exists (cf. Hall and Heyde, 1980). Volný (1993) has shown that the convergence in L^2 is not sufficient for the martingale approximation and that an unconditional convergence is a sufficient condition.

FIGURE 1.1. Diagonal and Wu-Woodroofe approximation



Definition 1.4. We say that a function f has **Wu-Woodroofe approximation** if $||Q_0(S_n(f))|| = o(\sigma_n)$ and $||R_n(S_n(f))|| = o(\sigma_n)$.

This is equivalent to the existence of an array $(U^i m_n)_{i=0}^{n-1}$, $n \geq 1$ of martingale differences such that $||S_n(f - m_n)|| = o(\sigma_n)$; for m_n we can, in such a case, choose $m_n = \sum_{i=0}^{n-1} \frac{n-i}{n} P_0 U^i f + \sum_{i=1}^{n-1} \frac{n-i}{n} P_0 U^{-i} f = \frac{1}{n} \sum_{k=0}^{n-1} \sum_{i=-k}^{k} P_0 U^i f$.

The Wu-Woodroofe approximation was introduced in Wu and Woodroofe (2004) for adapted processes. The nonadapted case can been found in Volný (2006).

The Wu-Woodroofe approximation together with linear growth of variances – $||S_n(f)||^2$ – are necessary (but not sufficient, cf. Klicnarová and Volný, 2009) conditions for the martingale approximation. A condition which, for processes with linear growth of variance, is necessary and sufficient for the existence of a martingale approximation, has been found by Zhao and Woodroofe (2008). This condition, valid for adapted processes, can be formulated in the following way: For $f_n = \sum_{i=0}^{n} P_{-i} f$

$$\frac{1}{n} \sum_{k=1}^{n} \limsup_{j \to \infty} \frac{1}{\sqrt{j}} ||S_j(f - f_k)|| \to 0.$$

2. Results

Theorem 2.1. *The weak diagonal approximation and the Wu-Woodroofe approximation are equivalent and both are implied by the strong diagonal approximation.*

Now, we shall deal with adapted processes, i.e. f will be supposed $\mathcal{F}_0$ measurable.

Theorem 2.2. *There exists a stationary linear process $(f \circ T^i)_i$ such that f has a martingale approximation but no strong diagonal approximation. More precisely, for any sequence of $d(n) = o(n)$ there exists a sequence (k_n) such that $k_n \geq d(n)$, $k_n = o(n)$ and for $f_n = \sum_{i=-k_n}^{k_n} P_i f$, $\frac{1}{\sqrt{n}} ||S_n(f - f_n)|| \not\to 0$.*

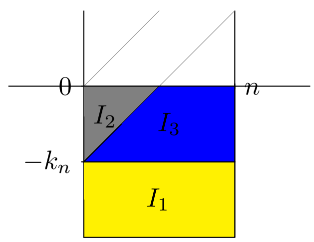
The Proposition shows that in the Zhao-Woodroofe's condition, the limit of averages cannot be replaced by a limit of $||f||_{k+} = \limsup_{j\to\infty} \frac{1}{\sqrt{j}}||S_j(f - f_k)||$.

3. Proofs

3.1. *Proof of Theorem 2.1.* Let us prove the first part of the theorem.

At first we prove that the weak diagonal approximation implies the Wu-Woodroofe approximation. We have

$$\begin{aligned} ||Q_0(S_n(f))|| &= ||\sum_{i=-\infty}^{0}(\sum_{j=1}^{n} P_i U^j f)|| \\ &\le ||\sum_{i=-\infty}^{-k_n}(\sum_{j=1}^{n} P_i U^j f)|| + ||\sum_{i=-k_n+1}^{0}(\sum_{j=1}^{i+k_n} P_i U^j f)|| + \\ &\quad +||\sum_{i=-k_n+1}^{0}(\sum_{j=i+k_n+1}^{n} P_i U^j f)|| \\ &= I_1 + I_2 + I_3 \end{aligned}$$



and

$$\begin{aligned} I_1 + I_3 &\le ||S_n(f - f_n)|| \\ I_2 &\le ||Ug_n|| = ||g_n||. \end{aligned}$$

Therefore, under the assumptions of the weak diagonal approximation,

$$||Q_0(S_n(f))|| = o(\sigma_n).$$

The proof for $||R_n(S_n(f))||$ is similar.

Now, we show that the Wu-Woodroofe approximation implies the weak diagonal approximation.

We can suppose that f is adapted to the filtration and regular, i.e. $f = \sum_{i=0}^{\infty} P_{-i} f$; the non adapted case can be treated in the same manner (cf. Volný, 2006) and the non regular part of f does not affect the limit behaviour.

We have $\sigma_n^2 = ||S_n(f)||_2^2 = nh_n$ where h_n is a slowly varying function in the sense of Karamata (cf. Kallenberg, 1997) and $||E(S_n(f)|\mathcal{F}_0)||_2 = o(\sigma_n)$.

Recall that the Wu-Woodroofe condition implies that there is a martingale difference array $(U^i m_n)$ such that $||S_n(f - m_n)||_2 = o(\sigma_n)$; we then have $||m_n||_2 = \sigma_n/\sqrt{n} + o(\sigma_n/\sqrt{n})$.

Note that

$$||S_n(f-m_n)||_2^2 = ||E(S_n(f)|\mathcal{F}_0)||_2^2 + \sum_{j=1}^n ||U^j m_n - P_j \sum_{i=j}^n U^i f||_2^2 =$$
$$= ||E(S_n(f)|\mathcal{F}_0)||_2^2 + \sum_{j=1}^n ||m_n - P_0 \sum_{i=0}^{n-j} U^i f||_2^2$$

(by the assumptions, f is adapted hence $P_j U^i f = 0$ for $j > i$).

We thus have that for any $\epsilon > 0$ and n big enough, for all but ϵn $j \in \{0, \ldots, n\}$, $||m_n - P_0 \sum_{i=0}^{n-j} U^i f||_2^2 < \epsilon h_n$. Let $k_n \le \epsilon n$ be one of such j. We then have

$$\sum_{j=1}^n ||P_0 \sum_{i=j}^n U^i f - P_0 \sum_{i=0}^{k_n} U^i f||_2^2 < 2\epsilon h_n$$

hence for $m_n' = P_0 \sum_{i=0}^{k_n} U^i f$,

$$||S_n(f-m_n')||_2^2 = o(\sigma_n^2).$$

Denote $f_n = \sum_{i=0}^{k_n} P_{-i} f$. To show that $||S_n(f-f_n)||_2^2 = o(\sigma_n^2)$ it suffices to prove

$$||E(S_{k_n}(f_n)|\mathcal{F}_0)||_2 = o(\sigma_n).$$

By the assumptions

$$||\sum_{j=1}^n U^j m_n - P_j \sum_{i=j}^n U^i f||_2 = o(\sigma_n)$$

hence (recall that $P_j U^j m_n = U^j m_n$)

$$\sum_{j=n-k_n+1}^n ||m_n - P_0 \sum_{i=0}^{n-j} U^i f||_2 = o(\sigma_n)$$

so that

$$\sum_{j=0}^{k_n-1} ||m_n - P_0 \sum_{i=0}^{j} U^i f||_2 = o(\sigma_n).$$

Because $||\sum_{j=1}^{k_n} U^j m_n||_2^2 \approx \frac{k_n}{n}\sigma_n^2 = o(\sigma_n^2)$, we thus get

$$\sum_{j=0}^{k_n-1} ||P_0 \sum_{i=0}^{j} U^i f||_2 = o(\sigma_n)$$

hence $f_n = m_n' + g_n - Ug_n$ where $||g_n||_2 = o(\sigma_n)$ and $||S_n(f-f_n)||_2 = o(\sigma_n)$.

So, the f has a weak diagonal approximation.

It remains to prove the second part of the theorem. This part follows immediately from the definitions of the strong diagonal approximation and the weak diagonal approximation.

△

3.2. *Proof of Theorem 2.2.*

3.2.1. *Construction of the process.* Let $e \in H_0 \ominus H_{-1}$ and $||e|| = 1$. $(U^i e)_i$ is thus a martingale difference sequence. For $i, j \in \mathsf{N}$ (N denotes the set of all positive integers) we denote

$$\begin{aligned} K_j &= \lfloor \exp j^2 \rfloor \\ N_j &= 2\sum_{i=0}^{j-1} K_i + 1, \\ b_j &= \frac{1}{j^2} \end{aligned}$$

and put $K_0, N_0 = 0,\ b_0 = 1$.

We define functions $f_i, \bar{f}_i$ for $i \in \mathsf{N}$:

$$\begin{aligned} f_i &:= \sum_{k=1}^{K_i} \left(-\frac{1}{k}\right) e \circ T^k + \sum_{k=0}^{K_i - 1} \frac{1}{k+1} e \circ T^{-k}, \\ \bar{f}_i &:= f_i \circ T^{-N_i - K_i} \end{aligned}$$

and put

$$f = \sum_{i=1}^{+\infty} b_i \bar{f}_i + f_0, \tag{3.1}$$

where $f_0 = b_0 e$.

3.2.2. *Main idea of the proof.* We will prove that $\frac{||Q_0(S_n(f))||}{||S_n(f)||} \to 0$. Remark that this is a necessary and sufficient condition for the Wu-Woodroofe approximation (cf. Wu and Woodroofe, 2004). Then we will deduce that there is a martingale approximation. In the second part of the proof we will show that for any sequence of $d(n) = o(n)$ there exists a sequence (k_n) such that $k_n \geq d(n)$, $k_n = o(n)$ and for $f_n = \sum_{i=-k_n}^{k_n} P_i f$, $\frac{1}{\sqrt{n}}||S_n(f - f_n)|| \not\to 0$.

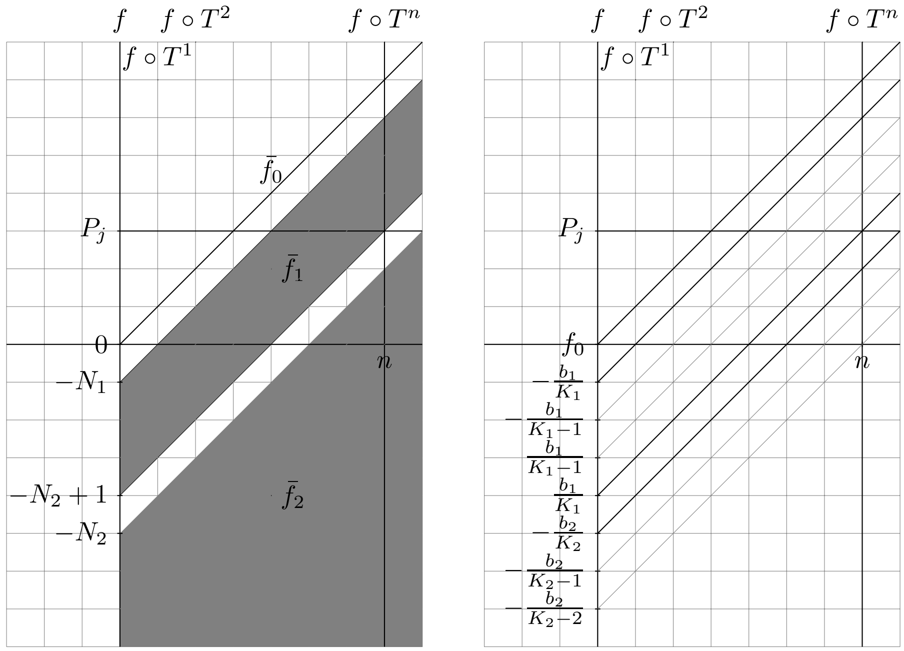
In the proof, we shall need both – upper and lower – estimations of the term $||S_n(f)||$, an upper estimation of $||Q_0(S_n(f)||$ and a lower estimation of the term $||S_n(f - f_n)||$.

We have

$$\begin{aligned} ||S_n(f)|| &\geq ||S_n(f) - Q_0(S_n(f))||, \\ ||S_n(f)|| &\leq ||S_n(f) - Q_0(S_n(f))|| + ||Q_0(S_n(f))||, \end{aligned}$$

hence for the estimation of the term $||S_n(f)||$ it suffices to estimate the terms $||Q_0(S_n(f))||$ and $||S_n(f) - Q_0(S_n(f))||$.

FIGURE 3.2. The construction of the process



3.2.3. *Projections* $P_j(S_n(f))$. Since $Q_0(S_n(f)) = \sum_{j=-\infty}^{0} P_j(S_n(f))$ and $S_n(f) - Q_0(S_n(f)) = \sum_{j=1}^{+\infty} P_j(S_n(f))$, in order to estimate the terms $||Q_0(S_n(f))||$ and $||S_n(f) - Q_0(S_n(f))||$ we need to know estimations of $P_j(S_n(f))$. From the construction (3.1) of the function f we have

$$P_j(S_n(f)) = P_j\left(S_n\left(\sum_{i=1}^{+\infty} b_i\bar{f}_i + f_0\right)\right). \tag{3.2}$$

Let us study the functions $\bar{f}_i$. From the construction of $\bar{f}_i$ we can see that

$$\bar{f}_i \in H_{-N_i} \ominus H_{-N_i-2K_i}.$$

It follows that

$$\bar{f}_i \circ T^k \in H_{-N_i+k} \ominus H_{-N_i-2K_i+k}$$

and

$$S_n(\bar{f}_i) \in H_{-N_i+n} \ominus H_{-N_i-2K_i+1}. \tag{3.3}$$

Now, let us fix i and study $P_j(S_n(b_i\bar{f}_i))$. We can write

$$P_j(S_n(b_i\bar{f}_i)) = y_{n,j}e_j.$$

The part $y_{n,j}$ on the right hand side of the equation depends in fact on i, n and j. Let us split all possibilities into the following three cases. (Recall that $N_{i+1} = N_i + 2K_i$.)

The case of $n > N_{i+1}$. If we suppose $n \geq N_{i+1}$, then the term $y_{n,j}$ is as follows:

1.1 $\mathbf{j}:\ \mathbf{j} > \mathbf{n} - \mathbf{N_i}$ or $\mathbf{j} \leq -\mathbf{N_{i+1}} + \mathbf{1}$. In such case (cf. (3.3)) $y_{n,j} = 0$.

1.2 $\mathbf{j}:\ \mathbf{n} - \mathbf{N_i} - \mathbf{K_i} + \mathbf{1} \leq \mathbf{j} \leq \mathbf{n} - \mathbf{N_i}$

$$y_{n,j} = b_i \sum_{k=j-n+N_i+K_i}^{K_i} \left(-\frac{1}{k}\right).$$

1.3 $\mathbf{j}:\ \mathbf{n} - \mathbf{N_i} - \mathbf{2K_i} + \mathbf{2} \leq \mathbf{j} \leq \mathbf{n} - \mathbf{N_i} - \mathbf{K_i}$

$$y_{n,j} = b_i \sum_{k=n+2-N_i-K_i-j}^{K_i} \left(-\frac{1}{k}\right).$$

1.4 $\mathbf{j}:\ -\mathbf{N_i} + \mathbf{1} \leq \mathbf{j} \leq \mathbf{n} - \mathbf{N_i} - \mathbf{2K_i} + \mathbf{1}$. It is easily seen (from the construction of the function $\bar{f}_i$) that

$$P_j(S_n(b_i \bar{f}_i)) = b_i \left(\sum_{k=1}^{K_i} \left(-\frac{1}{k}\right) + \sum_{k=1}^{K_i} \left(\frac{1}{k}\right) \right) = 0,$$

so $y_{n,j} = 0$.

1.5 $\mathbf{j}:\ -\mathbf{N_i} - \mathbf{K_i} + \mathbf{2} \leq \mathbf{j} \leq -\mathbf{N_i}$. Alike as in (3.)

$$y_{n,j} = b_i \sum_{k=N_i+K_i+j}^{K_i} \left(-\frac{1}{k}\right).$$

1.6 $\mathbf{j}:\ -\mathbf{N_i} - \mathbf{2K_i} + \mathbf{2} \leq \mathbf{j} \leq -\mathbf{N_i} - \mathbf{K_i} + \mathbf{1}$. It is similar case to (2.), we have

$$y_{n,j} = b_i \sum_{k=-N_i-K_i-j+2}^{K_i} \left(-\frac{1}{k}\right).$$

The case of $n \leq N_i$. Remark that $P_j(S_n(f)) = 0$ for $j > n$ (since f is adapted). So, if have the case of $n < N_i$, then we need to express the projection $P_j(S_n(b_i \bar{f}_i))$ for negative j only, especially for $j \in [n - N_I, -N_I - 2K_I]$.

Now, let us have $n \leq N_i$. Since $S_n(\bar{f}_i) \in H_{-N_i+n} \ominus H_{-N_i-2K_i+1}$, we are interested only in projections P_j for negative j. The expression of the term $y_{n,j}$ is as follows.

2.1 $\mathbf{j}:\ \mathbf{j} > \mathbf{n} - \mathbf{N_i}$ or $\mathbf{j} \leq -\mathbf{N_{i+1}} + \mathbf{1}$. In such case (cf. (3.3)) $y_{n,j} = 0$.

2.2 $\mathbf{j}:\ -\mathbf{N_i} + \mathbf{1} \leq \mathbf{j} \leq \mathbf{n} - \mathbf{N_i}$

$$y_{n,j} = b_i \sum_{k=j-n+N_i+K_i}^{K_i} \left(-\frac{1}{k}\right).$$

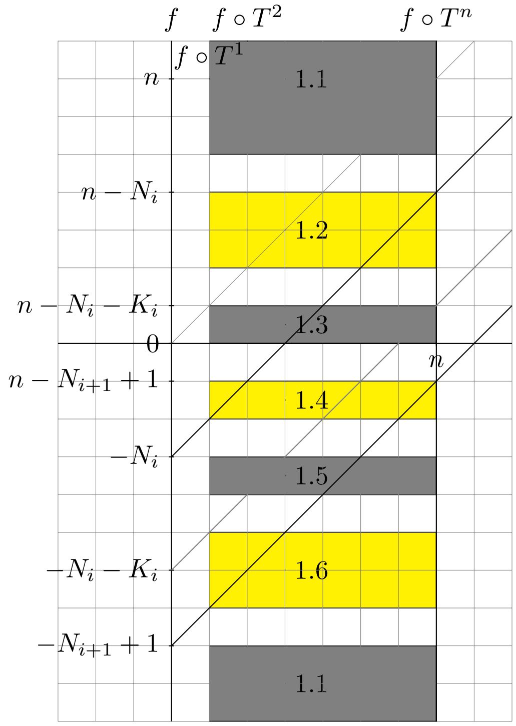
2.3 $\mathbf{j}:\ \mathbf{n} - \mathbf{N_i} - \mathbf{K_i} + \mathbf{1} \leq \mathbf{j} \leq -\mathbf{N_i}$

$$y_{n,j} = b_i \sum_{k=j-n+N_i+K_i}^{K_i+j+N_i-1} \left(-\frac{1}{k}\right).$$

2.4 $\mathbf{j}:\ \left\lceil \frac{\mathbf{n}+\mathbf{1}}{\mathbf{2}} \right\rceil - \mathbf{N_i} - \mathbf{K_i} + \mathbf{1} \leq \mathbf{j} \leq \mathbf{n} - \mathbf{N_i} - \mathbf{K_i}$

$$y_{n,j} = b_i \sum_{k=-j+n-N_i-K_i+1}^{K_i+j+N_i-1} \left(-\frac{1}{k}\right).$$

FIGURE 3.3. The case of $n \geq 2K_i$.



2.5 $\mathbf{j:\ -N_i - K_i + 1 \leq j \leq \left\lfloor \frac{n-1}{2} \right\rfloor - N_i - K_i}$

$$y_{n,j} = b_i \sum_{k=j+N_i+K_i+1}^{-K_i-j-N_i+n} \left(\frac{1}{k}\right).$$

2.6 $\mathbf{j:\ n - N_i - 2K_i + 1 \leq j \leq -N_i - K_i}$

$$y_{n,j} = b_i \sum_{k=-j-N_i-K_i+1}^{-K_i-j-N_i+n} \left(\frac{1}{k}\right).$$

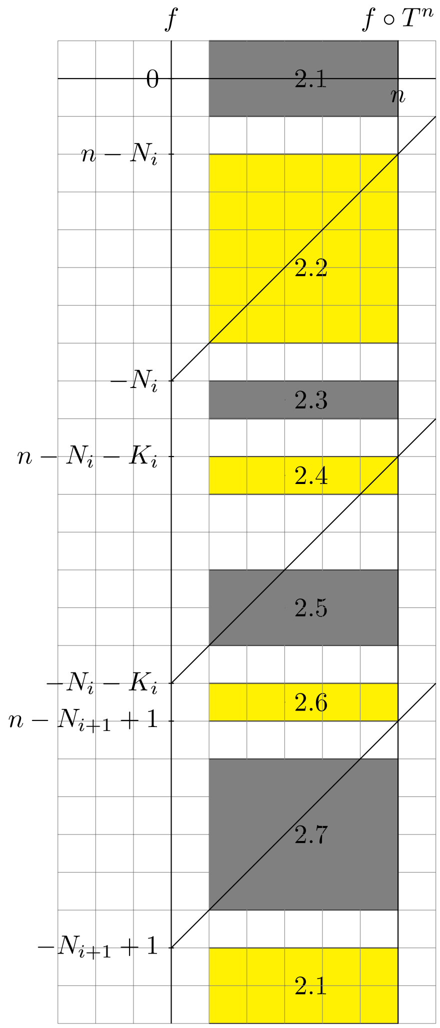
2.7 $\mathbf{j:\ -N_i - 2K_i + 1 \leq j \leq n - N_i - 2K_i}$

$$y_{n,j} = b_i \sum_{k=-j-N_i-K_i+1}^{K_i} \left(\frac{1}{k}\right).$$

Observe that if n is even, then the term $P_{-N_i-K_i+1+\frac{n}{2}}(S_n(\bar{f}_i))$ is equal to zero.

The case of $N_i < n < N_{i+1}$. We can split this case into the following three parts. If $n \geq 2K_i$, then we can apply formulas (1.1)–(1.6). In the case of $n \leq K_i$, the formulas (2.1)–(2.7) can be used. When n is such that $K_i < n < 2K_i$ then we can use the formulas (2.1)–(2.5) and (2.7), where the upper bound in (2.5) is $-N_i - K_i$. (In such a case the set of j in the formula (2.6) is empty.)

FIGURE 3.4. The case of $n \le N_i$



General expression for positive j. As we saw above, the projection – for fixed i – $P_j(S_n(b_i\bar{f}_i))$ depends on n, j. Let us denote

$$P_j(S_n(b_i\bar{f}_i)) = y_{n,j}e_j.$$

For general $j, n \in \mathsf{N}$, we can thus express the term $y_{n,j}$ ($i_{n,j}$ is such i that $n - N_{i+1} + 2 \le j \le n - N_i$; recall that $i_{n,j}$ is the only i for which the projection

$P_j(S_n(\bar{f}_i))$ is nonzero) as

$$\begin{aligned} y_{n,j} &= b_{i_{n,j}} \sum_{k=j-n+N_{i_{n,j}}+K_{i_{n,j}}}^{K_{i_{n,j}}} \left(-\frac{1}{k}\right) \\ &\qquad \text{for } j: \ n-N_{i_{n,j}}-K_{i_{n,j}}+1 \le j \le n-N_{i_{n,j}} \\ &= b_{i_{n,j}} \sum_{k=n+2-N_{i_{n,j}}-K_{i_{n,j}}-j}^{K_{i_{n,j}}} \left(-\frac{1}{k}\right) \\ &\qquad \text{for } j: \ n-N_{i_{n,j}+1}+2 \le j \le n-N_{i_{n,j}}-K_{i_{n,j}} \\ &= 0 \text{ in other case.} \end{aligned} \tag{3.4}$$

From the previous calculations we can deduce that (still for positive j)

$$||P_j(S_n(f))||^2 = ||P_j(S_n(f_0 + b_{i_{n,j}}\bar{f}_{i_{n,j}}))||^2 = (1+y_{n,j})^2 \tag{3.5}$$

where $i_{n,j}$ is such that j belongs to the interval $[n-N_{i_{n,j}+1}+2, n-N_{i_{n,j}}]$ and $y_{n,j}$ is as above. Remark that $y_{n,j} \in [-1,0]$.

The upper estimation of the term $||S_n(f) - Q_0(S_n(f))||$. Now, let us estimate the term $||S_n(f)-Q_0(S_n(f))||$. An upper estimation of this term can, using the previous remarks, be easily seen (the norm of each term $P_j(S_n(f))$ is less or equal to one, cf. (3.5)) to be

$$||S_n(f) - Q_0(S_n(f))||^2 = \sum_{j=1}^{n} ||P_j(S_n(f))||^2 \le n. \tag{3.6}$$

The lower estimation of the term $||S_n(f) - Q_0(S_n(f))||$. Since (cf. (3.5))

$$||P_j(S_n(f))||^2 = ||P_j(S_n(f_0 + b_{i_{n,j}}\bar{f}_{i_{n,j}}))||^2,$$

for the lower estimation of $||S_n(f) - Q_0(S_n(f))||^2$ we need to estimate

$$||P_j(S_n(f_0 + b_{i_{n,j}}\bar{f}_{i_{n,j}}))||^2 \text{ for } 0 \le j \le n.$$

Let us fix $n \in \mathsf{N}$. Then, there exists an integer I such that $N_I < n \le N_{I+1}$. Using (3.5) we have

$$||S_n(f) - Q_0(S_n(f))||^2 = \sum_{j=1}^{n} ||P_j(S_n(f))||^2 = \sum_{j=1}^{n} ||P_j(S_n(f_0 + b_{i_{n,j}}\bar{f}_{i_{n,j}}))||^2.$$

We can write the term on the right hand side as

$$\sum_{j=1}^{n} ||P_j(S_n(f))||^2 = \sum_{i=0}^{I} \sum_{j=\max(1,n-N_{i+1}+1)}^{n-N_i} ||P_j(S_n(f_0 + b_i\bar{f}_i))||^2.$$

Recall that for positive j, the projections $P_j(S_n(b_i\bar{f}_i))$ are equal to zero for every $i \ge 1$ except one of them ($i_{n,j}$). We get

$$||S_n(f) - Q_0(S_n(f))||^2 = \sum_{i=0}^{I} A_i$$

where for $i \leq I-1$

$$A_i = \sum_{j=n-N_{i+1}+1}^{n-N_i} ||P_j(S_n(f))||^2 = \sum_{j=n-N_i-2K_i+1}^{n-N_i} ||P_j(S_n(f_0 + b_{i_{n,j}} \bar{f}_{i_{n,j}}))||^2$$

and

$$A_I = \sum_{j=1}^{n-N_I} ||P_j(S_n(f_0 + b_I \bar{f}_I))||^2.$$

Remark that all functions $\bar{f}_{i_{n,j}}$ contained in A_i are the same.

Let us study the case of $i \leq I-1$, using ((1.3),(1.4)) (recall that $N_{i+1} = N_i + 2K_i$):

$$\begin{aligned} A_i &= \sum_{j=n-N_i-2K_i+1}^{n-N_i} ||P_j(S_n(f_0 + b_{i_{n,j}} \bar{f}_{i_{n,j}}))||^2 \\ &= \sum_{j=n-N_i-2K_i+2}^{n-N_i} ||P_j(S_n(f_0 + b_{i_{n,j}} \bar{f}_{i_{n,j}}))||^2 + ||P_{n-N_i-2K_i+1}(S_n(f_0))||^2 \\ &= 2 \sum_{j=n-N_i-K_i+2}^{n-N_i} (1 + y_{n,j})^2 + (1 + y_{n,n-N_i-K_i+1})^2 + 1. \end{aligned}$$

Since

$$(1 + y_{n,j})^2 \geq 1 + 2y_{n,j}$$

we have

$$A_i \geq 2 \sum_{j=n-N_i-K_i+2}^{n-N_i} (1 + 2y_{n,j}) + (1 + 2y_{n,n-N_i-K_i+1}) + 1.$$

Using the expression of $y_{n,j}$ as a sum of the fractions, see (3.4), putting $l := j-n+N_i+K_i$, we obtain (recall that $\sum_{k=l}^{m}\left(\frac{1}{k}\right) \leq \log\frac{m}{l-1}$ and $\sum_{k=l}^{m} \log k \leq \int_{l-1}^{m} \log x \mathrm{d}x$):

$$\begin{aligned} A_i &\geq 2\sum_{l=2}^{K_i}\left(1 - \frac{2}{i^2}\sum_{k=l}^{K_i}\left(\frac{1}{k}\right)\right) + \left(1 - \frac{2}{i^2}\sum_{k=1}^{K_i}\left(\frac{1}{k}\right)\right) + 1 \\ &\geq 2\left(K_i - 1 - \frac{2}{i^2}\sum_{l=2}^{K_i}\log\frac{K_i}{l-1}\right) + 2 - \frac{2}{i^2}(1 + \log(K_i)) \\ &= 2K_i - \frac{2}{i^2}\left((2K_i - 1)\log(K_i) - 2\sum_{l=2}^{K_i}\log(l-1) + 1\right) \\ &\geq 2K_i - \frac{2}{i^2}\left((2K_i - 1)\log(K_i) - 2[x\log x - x]_1^{K_i-1} + 1\right) \\ &= 2K_i - \frac{2}{i^2}\left((2K_i - 2)(\log(K_i) - \log(K_i - 1)) + \log(K_i) + (2K_i - 2) - 1\right) \\ &= 2K_i - \frac{2}{i^2}\left((2K_i - 2)\left(\log\frac{K_i}{K_i - 1} + 1\right) + \log(K_i) - 1\right) \end{aligned}$$

Using $\log\left(\frac{x}{x-1}\right) = \log\left(1 + \frac{1}{x-1}\right) \leq \frac{1}{x-1}$ for $x > 1$, we have

$$\begin{aligned} A_i &\geq 2K_i - \frac{2}{i^2}\left((2K_i-2)\left(\frac{1}{K_i-1}+1\right)+\log(K_i)-1\right) \\ &= 2K_i - \frac{2}{i^2}(2K_i+\log(K_i)-1) \\ &= 2K_i - \frac{4K_i+2\log K_i-2}{i^2}. \end{aligned}$$

We have derived (recall that $K_i = \lfloor \exp i^2 \rfloor$)

$$A_i \sim 2K_i + O\left(\frac{K_i}{\log K_i}\right).$$

Denote $C_1 = \sum_{i=1}^{I-1} A_i$. We have

$$\begin{aligned} C_1 = \sum_{i=1}^{I-1} A_i &= \sum_{i=1}^{I-1}\left(2K_i - \frac{4K_i+2\log K_i-2}{i^2}\right) \\ &\geq N_I - 1 - \sum_{i=1}^{I-1}\frac{4K_i+2\log K_i-2}{i^2} \\ &\sim N_I + O\left(\frac{K_{I-1}}{\log K_{I-1}}\right). \end{aligned}$$

Now, for finishing of the estimation of the term $||S_n(f) - Q_0(S_n(f))||$, we need to estimate the term A_I. The estimation can be divided into following three parts.

(1) Let $n = N_I + K_I$. Then, (cf. (3.4))

$$||P_j(S_n(f))||^2 = ||P_j(S_n(f_0 + b_I\bar{f}_I)||^2, \text{for } j \in [1, n - N_I].$$

Hence, in this case, we estimate the terms $||P_j(S_n(f_0 + b_I\bar{f}_I))||^2$ using the same methods as above and we obtain (using $n - N_I - K_I + 1 = 1$ and $n - N_I = K_I$)

$$A_I^{(1)} = \sum_{j=1}^{K_I} ||P_j(S_n(f_0 + b_I\bar{f}_I))||^2 \geq K_I - \frac{2K_I + \log K_I - 1}{I^2}.$$

For $||S_n(f) - Q_0(S_n(f))||^2$ we have deduced that (recall that in such a case $n \sim K_I$ and $\log K_I \sim I^2$)

$$\begin{aligned} ||S_n(f) - Q_0(S_n(f))||^2 &\geq C_1 + K_I - \frac{2K_I + \log K_I - 1}{I^2} \\ &\geq n + O\left(\frac{K_{I-1}}{\log K_{I-1}}\right) - \frac{2K_I + \log K_I - 1}{I^2} \\ &= n + o(n). \end{aligned}$$

(2) Let $N_I < n < N_I + K_I$.

We need to estimate the term

$$A_I^{(2)} = \sum_{j=1}^{n-N_I} ||P_j(S_n(f_0 + b_I\bar{f}_I))||^2.$$

Using similar arguments as before ($n - N_I < K_I$) we get

$$\begin{aligned}
A_I^{(2)} &\geq n - N_I - \frac{2}{I^2} \sum_{j=N_I-n+K_I+1}^{K_I} \sum_{k=j}^{K_I} \left(\frac{1}{k}\right) \\
&\geq n - N_I - \frac{2}{I^2} \sum_{j=N_I-n+K_I+1}^{K_I} \log \frac{K_I}{j-1} \\
&\geq n - N_I - \frac{2}{I^2} \Big((n - N_I)\log(K_I) - [x \log x - x]_{K_I-n+N_I-1}^{K_I-1} \Big) \\
&= n - N_I - \\
&\quad -\frac{2}{I^2} \left((n-N_I) \log \frac{K_I}{K_I-n+N_I-1} - (K_I-1) \log \frac{K_I-1}{K_I-n+N_I-1} + n - N_I \right) \\
&\geq n - N_I - \\
&\quad -\frac{2}{I^2} \left((n-N_I-K_I+1) \log \frac{K_I-1}{K_I-n+N_I-1} + (n-N_I) \log \frac{K_I}{K_I-1} + n - N_I \right) \\
&\geq n - N_I - \frac{3}{I^2}(n - N_I).
\end{aligned}$$

We thus have got

$$A_I^{(2)} \geq n - N_I - \frac{3}{I^2}(n - N_I).$$

So, for $||S_n(f) - Q_0(S_n(f))||^2$, we have

$$\begin{aligned}
||S_n(f) - Q_0(S_n(f))||^2 &\geq C_1 + n - N_I - \frac{3}{I^2}(n - N_I) \\
&\geq n + O\left(\frac{K_{I-1}}{\log K_{I-1}}\right) - \frac{3}{I^2}(n - N_I) \\
&= n + o(n).
\end{aligned}$$

(3) Let $N_I + K_I < n < N_I + 2K_I$. In such a case we need to estimate the term

$$A_I^{(3)} = \sum_{j=1}^{n-N_I} ||P_j(S_n(f_0 + b_I \bar{f}_I))||^2.$$

Since $n - N_I > K_I$ we divide the term $A_I^{(3)}$ into two parts as follows

$$\begin{aligned}
\sum_{j=1}^{n-N_I} ||P_j(S_n(f_0 + b_I \bar{f}_I))||^2 &= \sum_{j=n-N_I-K_I+1}^{n-N_I} ||P_j(S_n(f_0 + b_I \bar{f}_I))||^2 + \\
&\qquad + \sum_{j=1}^{n-N_I-K_I} ||P_j(S_n(f_0 + b_I \bar{f}_I))||^2
\end{aligned}$$

where we know the estimation of the first term on the right-hand side. It is the same estimation as for the term $A_I^{(1)}$. So, let us denote by $A_I^{(3_2)}$ the second term on the right-hand side. This term can be expressed as

$$A_I^{(3_2)} = \sum_{j=1}^{n-N_I-K_I} \left(1 - \frac{1}{I^2} \sum_{k=j+1}^{K_I} \left(\frac{1}{k}\right)\right)^2.$$

$$\begin{aligned}
A_I^{(3_2)} &\geq \sum_{j=1}^{n-N_I-K_I} \left(1 - \frac{2}{I^2} \sum_{k=j+1}^{K_I} \left(\frac{1}{k}\right)\right) \\
&\geq (n - N_I - K_I) - \frac{2}{I^2} \left(\sum_{j=1}^{n-N_I-K_I} \log \frac{K_I}{j}\right) \\
&\geq (n - N_I - K_I) - \frac{2}{I^2} \left((n - N_I - K_I) \log(K_I) - [x \log x - x]_1^{n-N_I-K_I}\right) \\
&= (n - N_I - K_I) - \frac{2}{I^2} \left((n - N_I - K_I) \left(\log \frac{K_I}{n - N_I - K_I} + 1\right) - 1\right)
\end{aligned}$$

Now, let us take a constant c such that $0 < c < 1$ and $n - N_I - K_I = cK_I$; we thus have

$$\begin{aligned}
A_I^{(3_2)} &\geq cK_I - \frac{2}{I^2} \left(cK_I \left(\log \frac{K_I}{cK_I} + 1\right) - 1\right) \\
&\geq cK_I - \frac{2}{I^2} \left(cK_I (1 - \log c)\right)
\end{aligned}$$

So, for the term $A_I^{(3)}$ we have derived ($\frac{\log K_I - 1}{I^2} \leq 1$):

$$\begin{aligned}
A_I^{(3)} &\geq (c+1)K_I - \frac{2}{I^2} \left(cK_I - cK_I \log c + K_I\right) - 1 \\
&= (c+1)K_I \left(1 - \frac{2}{I^2}\right) + \frac{2K_I c \log c}{I^2} - 1
\end{aligned}$$

Using the inequality $-\log c \leq \frac{1}{c} - 1$, we get

$$A_I^{(3)} \geq (c+1)K_I - \frac{4K_I}{I^2} - 1.$$

We have obtained for the term $||S_n(f) - Q_0(S_n(f))||^2$ (recall that $C_1 = N_I + O\left(\frac{K_{I-1}}{\log K_I - 1}\right)$ and $n = N_I + K_I(c+1)$):

$$\begin{aligned}
||S_n(f) - Q_0(S_n(f))||^2 &\geq C_1 + K_I(c+1) - \frac{4}{I^2} K_I - 1 \\
&\geq n + O\left(\frac{K_{I-1}}{\log K_{I-1}}\right) - \frac{4}{I^2} K_I - 1 \\
&= n + o(n).
\end{aligned}$$

We thus have deduced (recall the upper estimate) that for each $n \in \mathsf{N}$,

$$n \geq ||S_n(f) - Q_0(S_n(f))||^2 = n + o(n). \tag{3.7}$$

Estimation of the term $||Q_0(S_n(f))||$. Now, we will study the term $||Q_0(S_n(f))||$. Recall that

$$||Q_0(S_n(f))||^2 = \sum_{j=-\infty}^{0} ||P_j(S_n(f))||^2 = \sum_{j=-\infty}^{0} \left|\left| P_j \left(S_n \left(f_0 + \sum_{i=1}^{+\infty} b_i \bar{f}_i \right)\right)\right|\right|^2 .$$

Using (3.5) and an inequality $(x-y)^2 \le (x^2+y^2)$ for $x, y \ge 0$. We thus get

$$||Q_0(S_n(f))||^2 \le \sum_{i=0}^{+\infty} I_i^n,$$

where $I_i^n = \sum_{j=-\infty}^{0} ||P_j(S_n(b_i\bar{f}_i))||^2$. The estimation of the terms (I_i^n) can be divided into three parts. It depends on the relation between i and n.

(1) Let i be such that $i \le I-1$ (we still suppose $n : N_I \le n < N_{I+1}$; this means that $n \ge N_i + 2K_i$. In such a case we can use ((1.5),(1.6)) (some negative j are also in the part (1.4) but the projections in (1.4) are equal to zero) and putting $l := j + N_i + K_i$ in (1.5) and $l := j + N_i + K_i - 2$ in (1.6) we obtain

$$I_i^n \le 2 \sum_{l=1}^{K_i} \left(\frac{1}{i^2} \sum_{k=l}^{K_i} \left(\frac{1}{k} \right) \right)^2 . \tag{3.8}$$

$$\begin{aligned} I_i^n &\le \frac{2}{i^4} \left((1+\log K_i)^2 + \sum_{j=2}^{K_i} \log^2 \frac{K_i}{j-1} \right) \\ &\le \frac{2}{i^4} \left(1 + 2\log K_i + \log^2 K_i + (K_i - 1)\log^2 K_i - \sum_{j=1}^{K_i-1} \log^2 j \right) \\ &\le \frac{2}{i^4} \left(1 + 2\log K_i + K_i \log^2 K_i - [x\log^2 x - 2x\log x + 2x]_1^{K_i} \right) \\ &\le \frac{2}{i^4} (2 + 2(K_i+1)\log K_i - 2K_i) \\ &\le \frac{5K_i}{i^2}. \end{aligned}$$

The sum of such I_i^n can be estimated by (recall that $K_j = \lfloor \exp j^2 \rfloor$)

$$\sum_{i=1}^{I-1} \frac{5K_i}{i^2} \le \frac{5K_{I-1}}{(I-1)^2} + \sum_{i=1}^{I-2} \frac{5K_i}{i^2} \le \frac{6K_{I-1}}{(I-1)^2} \le \frac{6n}{(I-1)^2} \tag{3.9}$$

(2) Let $i \ge I+1$, i.e. $n \le N_i$. Then, using expressions (2.1) – (2.7) we get

$$\begin{aligned} I_i^n &= \sum_{j=-N_i-2K_i}^{-N_i-K_i} ||P_j(S_n(b_i\bar{f}_i))||^2 + \sum_{j=-N_i-K_i+1}^{-N_i} ||P_j(S_n(b_i\bar{f}_i))||^2 + \\ &\quad + \sum_{j=-N_i+1}^{0} ||P_j(S_n(b_i\bar{f}_i))||^2 \end{aligned}$$

hence

$$I_i^n \leq \frac{3}{i^4}\sum_{j=1}^{K_i}\left(\sum_{k=j}^{j+n-1}\left(\frac{1}{k}\right)\right)^2.$$

Using the inequality $(\sum_{i=1}^n a_i)^2 \leq n(\sum_{i=1}^n a_i^2)$ we get

$$\begin{aligned}
\sum_{j=1}^{K_i}\left(\sum_{k=j}^{j+n-1}\left(\frac{1}{k}\right)\right)^2 &\leq \sum_{j=1}^{K_i} n \sum_{k=j}^{j+n-1}\frac{1}{k^2} \\
&\leq n\sum_{j=2}^{K_i}\left(\frac{1}{j-1}-\frac{1}{j+n-1}\right)+n\left(2-\frac{1}{n}\right) \\
&= n\left(\sum_{j=1}^{n}\frac{1}{j}-\sum_{j=K_i}^{K_i+n-1}\frac{1}{j}\right)+n\left(2-\frac{1}{n}\right) \\
&\leq n(3+\log n)
\end{aligned}$$

hence

$$I_i^n \leq \frac{3}{i^4}n(3+\log n).$$

Now, we can sum the terms I_i^n and using the inequality $\log n \leq i^2$ we get

$$\sum_{i=I+1}^{+\infty} I_i^n \leq \sum_{i=I+1}^{+\infty}\frac{3}{i^4}n(3+\log n) \leq 3n\sum_{i=I+1}^{+\infty}\frac{3+i^2}{i^4} \leq \frac{4n}{I}.$$

$$\begin{aligned}
I_i^n &= b_i^2\left(\sum_{j=1}^{n}\sum_{k=K_i-n+j}^{K_i}\left(-\frac{1}{k}\right)\right)^2 + b_i^2\left(\sum_{j=1}^{K_i-n}\sum_{k=j+1}^{n+j}\left(-\frac{1}{k}\right)\right)^2 \\
&\quad + b_i^2\left(\sum_{j=\lceil\frac{n+1}{2}\rceil}^{n}\sum_{k=n-j+2}^{j+1}\left(-\frac{1}{k}\right)\right)^2 \\
&\quad + b_i^2\left(\sum_{j=1}^{n}\sum_{k=K_i-j+1}^{K_i}\left(\frac{1}{k}\right)\right)^2 + b_i^2\left(\sum_{j=1}^{K_i-n}\sum_{k=-n+K_i+1-j}^{K_i-j}\left(\frac{1}{k}\right)\right)^2 \\
&\quad + b_i^2\left(\sum_{j=0}^{\lceil\frac{n-1}{2}\rceil}\sum_{k=j+1}^{-j+n}\left(\frac{1}{k}\right)\right)^2 \\
&\leq 2b_i^2\sum_{j=1}^{K_i}\left(\sum_{k=j+1}^{j+n}\frac{1}{k}\right)^2 + 2b_i^2\sum_{j=0}^{\lceil\frac{n-1}{2}\rceil}\left(\sum_{k=j+1}^{-j+n}\frac{1}{k}\right)^2,
\end{aligned}$$

$$\begin{aligned} I_i^n &\leq \frac{2}{i^4}\sum_{j=1}^{K_i}\left(\sum_{k=j+1}^{j+n}\frac{1}{k}\right)^2 + \frac{2}{i^4}\sum_{j=0}^{\lceil\frac{n-1}{2}\rceil}\left(\sum_{k=j+1}^{-j+n}\frac{1}{k}\right)^2 \\ &\leq \frac{4}{i^4}\sum_{j=1}^{K_i}\left(\sum_{k=j+1}^{j+n}\frac{1}{k}\right)^2. \end{aligned}$$

Using the inequality $(\sum_{i=1}^n a_i)^2 \leq n(\sum_{i=1}^n a_i^2)$ we get

$$\begin{aligned} I_i^n &\leq \frac{4n}{i^4}\sum_{j=1}^{K_i}\sum_{k=j+1}^{j+n}\frac{1}{k} \\ &\leq \frac{4n}{i^4}\sum_{j=1}^{K_i}\left(-\frac{1}{j+n}+\frac{1}{j}\right) \\ &\leq \frac{4n}{i^4}\sum_{j=1}^{n}\frac{1}{j} \\ &\leq \frac{4n}{i^4}(1+\log n). \end{aligned}$$

(3) It remains the case of $i = I$. According to paragraph **The case of** n **:** $N_i < n < N_{i+1}$ in the subsection **Projections** $P_j(S_n(f))$, this estimation can be split into three parts. In the case of $n \geq 2K_I$ we have

$$I_I^n \leq \frac{5K_I}{I^2}.$$

For $n \leq K_I$ we have

$$I_I^n \leq \frac{3n(3+\log n)}{I^4}.$$

For $K_I \leq n < N_I + 2K_I$ we have, similarly as in the previous case,

$$||Q_0(S_n(b_I\bar{f}_I)||^2 \leq 3\sum_{j=1}^{K_I}\left(\frac{1}{I^2}\sum_{k=j}^{K_I}\left(\frac{1}{k}\right)\right)^2$$

In (3.8) we get

$$||Q_0(S_n(b_I\bar{f}_I)||^2 \leq \frac{10K_I}{I^2}.$$

It remains to estimate $||Q_0(S_n(\bar{f}_I)||^2$ for $N_I < n < K_I$. We have

$$\begin{aligned}||Q_0(S_n(b_I\bar{f}_I)||^2 \quad &\leq \sum_{j=-N_I-2K_I}^{-N_I-2K_I+n-1} ||P_j(S_n(b_I\bar{f}_I))||^2 + \sum_{j=-N_I-n+1}^{-N_I} ||P_j(S_n(b_I\bar{f}_I))||^2 + \\ &+ \sum_{j=-N_I+1}^{0} ||P_j(S_n(b_I\bar{f}_I))||^2 + \sum_{j=-N_I-2K_I+n}^{-N_I-K_I} ||P_j(S_n(b_I\bar{f}_I))||^2 + \\ &+ \sum_{j=-N_I-K_I+1}^{-N_I-n} ||P_j(S_n(b_I\bar{f}_I))||^2 \leq \\ &\leq \frac{1}{I^4}\left[3\sum_{j=1}^{n}\left(\sum_{k=K_I-j}^{K_I}\left(\frac{1}{k}\right)\right)^2 + 2\sum_{j=1}^{K_I-n}\left(\sum_{k=K_I-j-n+2}^{K_I-j+1}\left(\frac{1}{k}\right)\right)^2\right]\end{aligned}$$

which, like in (3.8), can be estimated by

$$\frac{25K_I}{I^2}.$$

For all n it is thus

$$||Q_0(S_n(f))||^2 \leq \frac{25n}{I-1}.$$

Wu–Woodroofe approximation of the function f. Using the estimations for terms $||S_n(f) - Q_0(S_n(f))||$ and $||Q_0(S_n(f))||$ we obtain

$$\frac{||Q_0(S_n(f))||^2}{||S_n(f)||^2} \leq \frac{\frac{25n}{I-1}}{n+o(n)} \sim O\left(\frac{1}{I-1}\right).$$

Since $I \to +\infty$ (recall that $I - 1 \leq \log n \leq I + 1$) as $n \to +\infty$, we have proved Wu-Woodrofe approximation.

In the Wu-Woodroofe approximation, we can take $D_{n,0} = \sum_{i=0}^{n-1} \frac{n-i}{n} P_0U^i f + \sum_{i=1}^{n-1} \frac{n-i}{n} P_0U^{-i} f$. In the case of a stationary linear process $P_0U^i f$ is a multiple of e hence we get $D_{n,0} = c_n e$. From $||S_n(f)||^2/n \to const.$ (cf. 3.7) it follows that the c_n converge to a constant c and we get a martingale approximation with $m = ce$.

Strong diagonal approximation of the function f. Now, we show that the function f does not satisfy the strong diagonal approximation. Take a sequence $d(n)$ such that $d(n) = o(n)$. Then, there exists a sequence of

$$n_j = l_j(N_j + K_j),$$

where $N_j + K_j \geq d(n_j)$, $N_j + K_j = o(n)$ and l_j tends to infinity. Define

$$h_j = \sum_{i=-N_j-K_j}^{0} P_i f.$$

Then $||S_{n_j}(h_j)||^2 = o(n_j)$. As we proved above $||S_{n_j}(f)||^2 = n_j + o(n_j)$, hence $||S_{n_j}(f - h_j)||^2 = n_j - o(n_j)$. This shows that for arbitrary $d(n) = o(n)$ the condition (i) of Definition 1.3 is not satisfied.

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