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TASEP with discontinuous jump rates

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Abstract. We prove a hydrodynamic limit for the totally asymmetric simple exclusion process with spatially inhomogeneous jump rates given by a speed function that may admit discontinuities. The limiting density profiles are described with a variational formula. This formula enables us to compute explicit density profiles even though we have no information about the invariant distributions of the process. In the case of a two-phase flux for which a suitable p.d.e. theory has been developed we also observe that the limit profiles are entropy solutions of the corresponding scalar conservation law with a discontinuous speed function.

1. Introduction

This paper studies hydrodynamic limits of totally asymmetric simple exclusion processes (TASEPs) with spatially inhomogeneous jump rates given by functions that are allowed to have discontinuities. We prove a general hydrodynamic limit and compute some explicit solutions, even though information about invariant distributions is not available. The results come through a variational formula that takes advantage of the known behavior of the homogeneous TASEP. This way we

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are able to get explicit formulas, even though the usual scenario in hydrodynamic limits is that explicit equations and solutions require explicit computations of expectations under invariant distributions. Together with explicit hydrodynamic profiles we can present explicit limit shapes for the related last-passage growth models with spatially inhomogeneous rates.

The class of particle processes we consider are defined by a positive speed function $c(x)$ defined for $x \in \mathbb{R}$, lower semicontinuous and assumed to have a discrete set of discontinuities. Particles reside at sites of $\mathbb{Z}$, subject to the exclusion rule that admits at most one particle at each site. The dynamical rule is that a particle jumps from site i to site $i+1$ at rate $c(i/n)$ provided site $i+1$ is vacant. Space and time are both scaled by the factor n and then we let $n \to \infty$. We prove the almost sure vague convergence of the empirical measure to a density $\rho(x,t)$, assuming that the initial particle configurations have a well-defined macroscopic density profile ρ_0.

From known behavior of driven conservative particle systems a natural expectation would be that the macroscopic density $\rho(x,t)$ of this discontinuous TASEP ought to be, in some sense, the unique entropy solution of an initial value problem of the type

$$\rho_t + (c(x)f(\rho))_x = 0, \quad \rho(x,0) = \rho_0(x). \tag{1.1}$$

Our proof of the hydrodynamic limit does not lead directly to this scalar conservation law. We can make the connection through some recent PDE theory in the special case of the two-phase flow where the speed function is piecewise constant with a single discontinuity. In this case the discontinuous TASEP chooses the unique entropy solution. We would naturally expect TASEP to choose the correct entropy solution in general, but we have not investigated the PDE side of things further to justify such a claim.

The remainder of this introduction reviews briefly some relevant literature and then closes with an overview of the contents of the paper. The model and the results are presented in Section 2.

Discontinuous scalar conservation laws. The study of scalar conservation laws

$$\rho_t + F(x,\rho)_x = 0 \tag{1.2}$$

whose flux F may admit discontinuities in x has taken off in the last decade. As with the multiple weak solutions of even the simplest spatially homogeneous case, a key issue is the identification of the unique physically relevant solution by means of a suitable *entropy condition*. (See Sect. 3.4 of Evans, 1998 for textbook theory.) Several different entropy conditions for the discontinuous case have been proposed, motivated by particular physical problems. See for example Adimurthi and Gowda (2003), Adimurthi et al. (2007), Audusse and Perthame (2005), Chen et al. (2008), Diehl (1995), Klingenberg and Risebro (1995), Ostrov (2002). Adimurthi et al. (2007) discuss how different theories lead to different choices of relevant solution. An interesting phenomenon is that limits of vanishing higher order effects can lead to distinct choices (such as vanishing viscosity vs. vanishing capillarity).

However, the model we study does not offer more than one choice. In our case the graphs of the different fluxes do not intersect as they are all multiples of $f(\rho) = \rho(1-\rho)$. In such cases it is expected that all the entropy criteria single out the same solution (Remark 4.4 on p. 811 of Adimurthi et al., 2005). By appeal to the theory developed by Adimurthi and Gowda (2003) we show that the discontinuous

TASEP chooses entropy solutions of equation (1.1) in the case where $c(x)$ takes two values separated by a single discontinuity.

Our approach to the hydrodynamic limit goes via the interface process whose limit is a Hamilton-Jacobi equation. Hamilton-Jacobi equations with discontinuous spatial dependence have been studied by Ostrov (2002) via mollification.

Hydrodynamic limits for spatially inhomogeneous, driven conservative particle systems. Hydrodynamic limits for the case where the speed function possesses some degree of smoothness were proved over a decade ago by Covert and Rezakhanlou (1997) and Bahadoran (1998). For the case where the speed function is continuous, a hydrodynamic limit was proved by Rezakhanlou (2002) by the method of Seppäläinen (1999).

The most relevant and interesting predecessor to our work is the study of Chen et al. (2008). They combine an existence proof of entropy solutions for (1.2) under certain technical hypotheses on F with a hydrodynamic limit for an attractive zero-range process (ZRP) with discontinuous speed function. The hydrodynamic limit is proved through a compactness argument for approximate solutions that utilizes measure-valued solutions. The approach follows Bahadoran (1998) and Covert and Rezakhanlou (1997) by establishing a microscopic entropy inequality which under the limit turns into a macroscopic entropy inequality.

The scope of Chen et al. (2008) and our work are significantly different. Our flux $F(x, \rho) = c(x)\rho(1 - \rho)$ does not satisfy the hypotheses of Chen et al. (2008). Even with spatial inhomogeneities, a ZRP has product-form invariant distributions that can be readily written down and computed with. This is a key distinction in comparison with exclusion processes. The microscopic entropy inequality in Chen et al. (2008) is derived by a coupling with a stationary process.

Finally, let us emphasize the distinction between the present work and some other hydrodynamic limits that feature spatial inhomogeneities. Random rates (as for example in Seppäläinen, 1999) lead to homogenization (averaging) and the macroscopic flux does not depend on the spatial variable. Somewhat similar but still fundamentally different is TASEP with a slow bond. In this model jumps across bond $(0, 1)$ occur at rate $c < 1$ while all other jump rates are 1. The deep question is whether the slow bond disturbs the hydrodynamic profile for all $c < 1$. V. Beffara, V. Sidoravicius and M. E. Vares have announced a resolution of this question in the affirmative. Then the hydrodynamic limit can be derived in the same way as in the main theorem of the present paper. The solution is not entirely explicit, however: one unknown constant remains that quantifies the effect of the slow bond (see Seppäläinen, 2001). Bahadoran (2004) generalizes the hydrodynamic limit of Seppäläinen (2001) to a broad class of driven particle systems with a microscopic blockage.

Organization of this paper. Section 2 contains the main results for the inhomogeneous corner growth model and TASEP. Sections 3 and 4 prove the limits. Section 5 outlines the explicit computation of density profiles for the two-phase TASEP. Section 6 discusses the connection with PDE theory.

Notational conventions. $\mathbb{N} = \{1, 2, 3, \dots\}$ and $\mathbb{Z}_+ = \{0, 1, 2, \dots\}$. The Exp($c$) distribution has density $f(x) = ce^{-cx}$ for $0 < x < \infty$. Two last passage time models appear in our paper: the corner growth model whose last-passage times are denoted by G, and the equivalent wedge growth model with last-passage times T. $H(x) = \mathbf{1}_{[0,\infty)}(x)$ is the Heavyside function. C is a constant that may change from line to line.

2. Results

The corner growth model connected with TASEP has been a central object of study in this area since the seminal paper of Rost (1981). So let us begin with an explicit description of the limit shape for a two-phase corner growth model with a boundary along the diagonal. Put independent exponential random variables $\{Y_v\}_{v\in\mathbb{N}^2}$ on the points of the lattice with distributions

$$Y_{(i,j)} \sim \begin{cases} \text{Exp}(c_1), & \text{if } i < j \\ \text{Exp}(c_1 \wedge c_2), & \text{if } i = j \\ \text{Exp}(c_2), & \text{if } i > j. \end{cases} \tag{2.1}$$

We assume that the rates satisfy $c_1 \geq c_2$.

Define the last passage time

$$G(m,n) = \max_{\pi\in\Pi(m,n)} \sum_{v\in\pi} Y_v, \quad (m,n) \in \mathbb{N}^2, \tag{2.2}$$

where $\Pi(m,n)$ is the collection of weakly increasing nearest-neighbor paths in the rectangle $[m] \times [n]$ that start from $(1,1)$ and go up to (m,n). That is, elements of $\Pi(m,n)$ are sequences $\{(1,1) = v_1, v_2, \ldots, v_{m+n-1} = (m,n)\}$ such that $v_{i+1} - v_i = (1,0)$ or $(0,1)$.

Theorem 2.1. *Let the rates $c_1 \geq c_2 > 0$. Define $c = c_1/c_2 \geq 1$ and $b = 2c - 1 - 2\sqrt{c(c-1)}$. Then the a.s. limit*

$$\Phi(x,y) = \lim_{n\to\infty} n^{-1} G(\lfloor nx \rfloor, \lfloor ny \rfloor)$$

exists for all $(x,y) \in (0,\infty)^2$ and is given by

$$\Phi(x,y) = \begin{cases} c_1^{-1}\left(\sqrt{x} + \sqrt{y}\right)^2, & \text{if } 0 < x \leq b^2 y \\ x\dfrac{4c-(1+b)^2}{c_1(1-b^2)} + y\dfrac{(1+b)^2 - 4cb^2}{c_1(1-b^2)}, & \text{if } b^2 y < x < y \\ c_2^{-1}\left(\sqrt{x} + \sqrt{y}\right)^2, & \text{if } y \leq x < +\infty. \end{cases}$$

This theorem will be obtained as a side result of the development in Section 3.

We turn to the general hydrodynamic limit. The variational description needs the following ingredients. Define the wedge

$$\mathcal{W} = \{(x,y) \in \mathbb{R}^2 : y \geq 0,\ x \geq -y\}$$

and on $\mathcal{W}$ the last-passage function of homogeneous TASEP by

$$\gamma(x,y) = (\sqrt{x+y} + \sqrt{y})^2. \tag{2.3}$$

Let $\mathbf{x}(s) = (x_1(s), x_2(s))$ denote a path in $\mathbb{R}^2$ and set

$$\mathcal{H}(x,y) = \{\mathbf{x} \in C([0,1],\mathcal{W}) : \mathbf{x} \text{ is piecewise } C^1, \mathbf{x}(0) = (0,0), \\ \mathbf{x}(1) = (x,y),\ \mathbf{x}'(s) \in \mathcal{W} \text{ wherever the derivative is defined}\}.$$

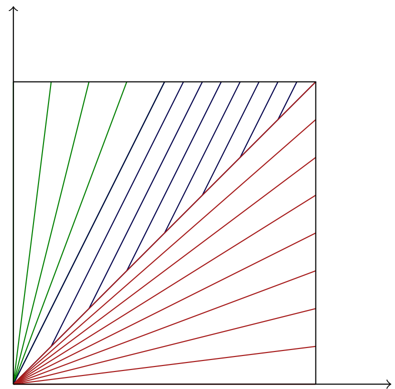

FIGURE 2.1. Optimal macroscopic paths that give the last passage time constant described in Theorem 2.1.

The *speed function* c of our system is by assumption a positive lower semicontinuous function on $\mathbb{R}$. We assume that at each $x \in \mathbb{R}$

$$c(x) = \min\Big\{ \lim_{y \nearrow x} c(y), \lim_{y \searrow x} c(y) \Big\}. \tag{2.4}$$

In particular we assume that the limits in (2.4) exist. We also assume that $c(x)$ has only finitely many discontinuities in any compact set, hence it is bounded away from 0 in any compact set.

For the hydrodynamic limit consider a sequence of exclusion processes $\eta^n = (\eta^n_i(t) : i \in \mathbb{Z}, t \in \mathbb{R}_+)$ indexed by $n \in \mathbb{N}$. These processes are constructed on a common probability space that supports the initial configurations $\{\eta^n(0)\}$ and the Poisson clocks of each process. As always, the clocks of process η^n are independent of its initial state $\eta^n(0)$. The joint distributions across the index n are immaterial, except for the assumed initial law of large numbers (2.10) below. In the process η^n a particle at site i attempts a jump to $i+1$ with rate $c(i/n)$. Thus the generator of η^n is

$$L_n f(\eta) = \sum_{x \in \mathbb{Z}} c(xn^{-1})\eta(x)(1 - \eta(x+1))(f(\eta^{x,x+1}) - f(\eta)) \tag{2.5}$$

for cylinder functions f on the state space $\{0,1\}^{\mathbb{Z}}$. The usual notation is that particle configurations are denoted by $\eta = (\eta(i) : i \in \mathbb{Z}) \in \{0,1\}^{\mathbb{Z}}$ and

$$\eta^{x,x+1}(i) = \begin{cases} 0 & \text{when } i = x \\ 1 & \text{when } i = x+1 \\ \eta(i) & \text{when } i \neq x, x+1 \end{cases}$$

is the configuration that results from moving a particle from x to $x+1$. Let $J^n_i(t)$ denote the number of particles that have made the jump from site i to site $i+1$ in time interval $[0,t]$ in the process η^n.

An initial macroscopic profile ρ_0 is a measurable function on $\mathbb{R}$ such that $0 \le \rho_0(x) \le 1$ for all real x, with antiderivative v_0 satisfying

$$v_0(0) = 0, \quad v_0(b) - v_0(a) = \int_a^b \rho_0(x)\, dx. \tag{2.6}$$

The macroscopic flux function of the constant rate 1 TASEP is

$$f(\rho) = \begin{cases} \rho(1-\rho), & \text{if } 0 \le \rho \le 1 \\ -\infty, & \text{otherwise.} \end{cases} \tag{2.7}$$

Its Legendre conjugate

$$f^*(y) = \inf_{0 \le \rho \le 1} \{y\rho - f(\rho)\}$$

represents the limit shape in the wedge. We orient our model so that growth in the wedge proceeds upward, and so we use $g(y) = -f^*(y)$. It is explicitly given by

$$g(y) = \sup_{0 \le \rho \le 1} \{f(\rho) - y\rho\} = \begin{cases} -y, & \text{if } y \le -1 \\ \frac{1}{4}(1-y)^2, & \text{if } -1 \le y \le 1 \\ 0, & \text{if } y \ge 1. \end{cases} \tag{2.8}$$

For $x \in \mathbb{R}$ define $v(x, 0) = v_0(x)$, and for $t > 0$,

$$v(x,t) = \sup_{w(\cdot)} \left\{ v_0(w(0)) - \int_0^t c(w(s))\, g\left(\frac{w'(s)}{c(w(s))}\right) ds \right\} \tag{2.9}$$

where the supremum is taken over continuous piecewise C^1 paths $w : [0, t] \longrightarrow \mathbb{R}$ that satisfy $w(t) = x$. The function $v(x, t)$ is Lipschitz continuous jointly in (x, t) (see Section 4) and it has a derivative almost everywhere. The macroscopic density is defined by $\rho(x, t) = v_x(x, t)$.

The initial distributions of the processes η^n are arbitrary subject to the condition that the following strong law of large numbers holds at time $t = 0$: for all real $a < b$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=\lfloor na \rfloor+1}^{\lfloor nb \rfloor} \eta_i^n(0) = \int_a^b \rho_0(x)\, dx \quad \text{a.s.} \tag{2.10}$$

The second theorem gives the hydrodynamic limit of current and particle density for TASEP with discontinuous jump rates.

Theorem 2.2. *Let $c(x)$ be a lower semicontinuous positive function satisfying (2.4), with finitely many discontinuities in any compact set. Under assumption (2.10), these strong laws of large numbers hold at each $t > 0$: for all real numbers $a < b$*

$$\lim_{n\to\infty} n^{-1} J^n_{\lfloor na \rfloor}(nt) = v_0(a) - v(a,t) \quad a.s. \tag{2.11}$$

and

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=\lfloor na \rfloor+1}^{\lfloor nb \rfloor} \eta_i^n(nt) = \int_a^b \rho(x,t)\, dx \quad a.s. \tag{2.12}$$

where $v(x, t)$ is defined by (2.9) and $\rho(x, t) = v_x(x, t)$.

Remark 2.3. In a totally asymmetric K-exclusion with speed function c the state space would be $\{0, 1, \ldots, K\}^{\mathbb{Z}}$ with K particles allowed at each site, and one particle moved from site x to $x+1$ at rate $c(x/n)$ whenever such a move can be legitimately completed. Theorem 2.2 can be proved for this process with the same method of proof. The definition of the limit (2.9) would be the same, except that the explicit flux f and wedge shape g would be replaced by the unknown functions f and g whose existence was proved in Seppäläinen (1999).

To illustrate Theorem 2.2 we compute the macroscopic density profiles $\rho(x,t)$ from constant initial conditions in the two-phase model with speed function

$$c(x) = c_1(1 - H(x)) + c_2 H(x) \tag{2.13}$$

where $H(x) = \mathbf{1}_{[0,\infty)}(x)$ is the Heavyside function and $c_1 \geq c_2$. (The case $c_1 < c_2$ can then be deduced from particle-hole duality.) The particles hit the region of lower speed as they pass the origin from left to right. Depending on the initial density ρ, we see the system adjust to this discontinuity in different ways to match the actual throughput of particles on either side of the origin. The maximal flux on the right is $c_2/4$ which is realized on the left at densities ρ^* and $1-\rho^*$ with

$$\rho^* = \tfrac{1}{2} - \tfrac{1}{2}\sqrt{1 - c_2/c_1}.$$

Corollary 2.4. *Let $c_1 \geq c_2$ and the speed function as in (2.13). Then the macroscopic density profiles with initial conditions $\rho_0(x,0) = \rho$ are given as follows.*

(i) *Suppose $0 < \rho < \rho^*$. Define $r^* = r^*(\rho) = \frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\rho(1-\rho)c_1/c_2}$. Then*

$$\rho(x,t) = \begin{cases} \rho & \text{if } -\infty \leq x \leq 0 \\ r^* & \text{if } 0 \leq x \leq c_2(1-2r^*)t \\ \frac{1}{2}\left(1 - \frac{x}{tc_2}\right) & \text{if } c_2(1-2r^*)t \leq x \leq c_2(1-2\rho)t \\ \rho & \text{if } (1-2\rho)tc_2 \leq x < +\infty \end{cases} \tag{2.14}$$

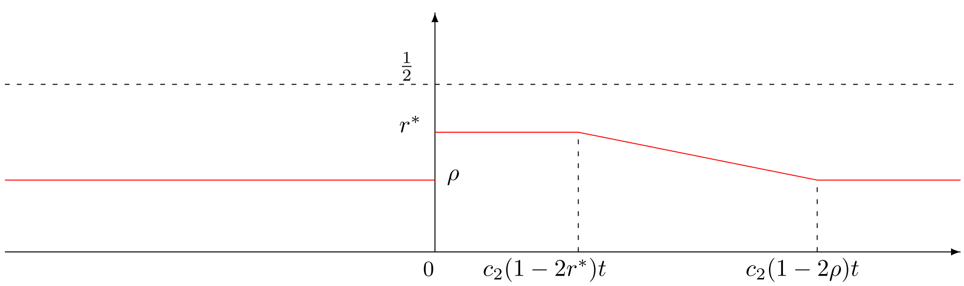

FIGURE 2.2. Density profile $\rho(x,t)$ in the two-phase ($c_1 > c_2$) TASEP when we start from constant initial configurations $\rho_0(x) \equiv \rho \in (0, \rho^*)$.

(ii) *Suppose $\rho^* \leq \rho \leq \frac{1}{2}$. Then*

$$\rho(x,t) = \begin{cases} \rho & \text{if } -\infty \leq x \leq -tc_1(\rho - \rho^*) \\ 1-\rho^* & \text{if } -tc_1(\rho-\rho^*) \leq x \leq 0 \\ \frac{1}{2}\left(1 - \frac{x}{tc_2}\right) & \text{if } 0 \leq x \leq (1-2\rho)tc_2 \\ \rho & \text{if } (1-2\rho)tc_2 \leq x < +\infty \end{cases} \tag{2.15}$$

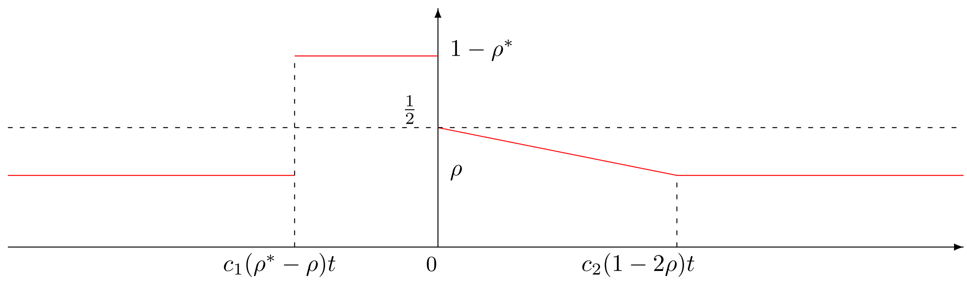

FIGURE 2.3. Density profile $\rho(x,t)$ in the two-phase ($c_1 > c_2$) TASEP when we start from constant initial configurations $\rho_0(x) \equiv \rho \in [\rho^*, \frac{1}{2}]$.

(iii) *Suppose* $\rho \geq \frac{1}{2}$. *Define* $r^* = r^*(\rho) = \frac{1}{2} - \frac{1}{2}\sqrt{1-4\rho(1-\rho)c_2/c_1}$. *Then*

$$\rho(x,t) = \begin{cases} \rho & \textit{if } -\infty \leq x \leq -tc_1(\rho - r^*) \\ 1-r^* & \textit{if } -tc_1(\rho-r^*) \leq x \leq 0 \\ \rho & \textit{if } 0 < x < +\infty \end{cases} \tag{2.16}$$

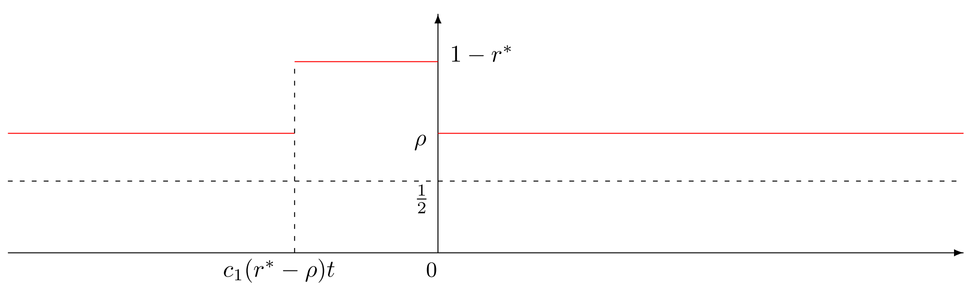

FIGURE 2.4. Density profile $\rho(x,t)$ in the two-phase ($c_1 > c_2$) TASEP when we start from constant initial configurations $\rho_0(x) \equiv \rho \in (\frac{1}{2}, 1)$.

Remark 2.5. Taking $t \to \infty$ in the three cases of Corollary 2.4 gives a family of macroscopic profiles that are fixed by the time evolution. A natural question to investigate would be the existence and uniqueness of invariant distributions that correspond to these macroscopic profiles.

Next we relate the density profiles picked by the discontinuous TASEP to entropy conditions for scalar conservation laws with discontinuous fluxes. The entropy conditions defined by Adimurthi and Gowda (2003) are particularly suited to our needs. Their results give uniqueness of the solution for the scalar conservation law

$$\begin{cases} \rho_t + (F(x,\rho))_x = 0, & x \in \mathbb{R}, t > 0 \\ \rho(x,0) = \rho_0(x), & x \in \mathbb{R} \end{cases} \tag{2.17}$$

with distinct fluxes on the half-lines:

$$F(x,\rho) = H(x) f_r(\rho) + (1 - H(x)) f_\ell(\rho) \tag{2.18}$$

where $f_r, f_\ell \in C^1(\mathbb{R})$ are strictly concave with superlinear decay to $-\infty$ as $|x| \to \infty$. A solution of (2.17) means a weak solution, that is, $\rho \in L^\infty_{\text{loc}}(\mathbb{R}\times\mathbb{R}_+)$ such that for all continuously differentiable, compactly supported test functions $\phi \in C^1_c(\mathbb{R}\times\mathbb{R}_+)$,

$$\int_{-\infty}^{+\infty}\int_0^{+\infty}\left(\rho\frac{\partial\phi}{\partial t} + F(x,\rho)\frac{\partial\phi}{\partial x}\right) dt\,dx + \int_{-\infty}^{+\infty}\rho(x,0)\phi(x,0)\,dx = 0. \tag{2.19}$$

(2.19) is the weak formulation of the problem

$$\begin{cases} \rho_t + f_r(\rho)_x = 0, & \text{for } x > 0, t > 0 \\ \rho_t + f_\ell(\rho)_x = 0, & \text{for } x < 0, t > 0 \\ f_r(\rho(0+,t)) = f_\ell(\rho(0-,t)) & \text{for a.e. } t > 0 \\ \rho(x,0) = \rho_0(x). \end{cases} \tag{2.20}$$

The entropy conditions used in Adimurthi and Gowda (2003) come in two sets and assume the existence of certain one-sided limits:

(E_i) *Interior entropy condition, or Lax-Oleinik entropy condition:*

$$\rho(x+,t) \geq \rho(x-,t) \quad \text{for } x \neq 0 \text{ and for all } t > 0. \tag{2.21}$$

(E_b) *Boundary entropy condition at $x = 0$:* for almost every t, the limits $\rho(0\pm,t)$ exist and one of the following holds:

$$f_r'(\rho(0+,t)) \geq 0 \quad \text{and} \quad f_\ell'(\rho(0-,t)) \geq 0, \tag{2.22}$$

$$f_r'(\rho(0+,t)) \leq 0 \quad \text{and} \quad f_\ell'(\rho(0-,t)) \leq 0, \tag{2.23}$$

$$f_r'(\rho(0+,t)) \leq 0 \quad \text{and} \quad f_\ell'(\rho(0-,t)) \geq 0. \tag{2.24}$$

Define

$$G_x(p) = \mathbf{1}\{x>0\} f_r^*(p) + \mathbf{1}\{x<0\} f_\ell^*(p) + \mathbf{1}\{x=0\}\min\bigl(f_r^*(0), f_\ell^*(0)\bigr),$$

where f_r^* and f_ℓ^* are the convex duals of f_r and f_ℓ. Set $V_0(x) = \int_0^x \rho_0(\theta)\,d\theta$ and define

$$V(x,t) = \sup_{w(\cdot)}\left\{V_0(w(0)) + \int_0^t G_{w(s)}\bigl(w'(s)\bigr)\,ds\right\} \tag{2.25}$$

where the supremum is over continuous, piecewise linear paths $w : [0,t] \longrightarrow \mathbb{R}$ with $w(t) = x$.

Theorem 2.6. *Adimurthi and Gowda (2003). Let $\rho_0 \in L^\infty(\mathbb{R})$ and define V by (2.25). Then V is a uniformly Lipschitz continuous function and $\rho(x,t) = V_x(x,t)$ is the unique weak solution of (2.20) that satisfies the entropy assumptions (E_i) and (E_b) in the class $L^\infty \cap BV_{\text{loc}}$ and with discontinuities given by a discrete set of Lipschitz curves.*

It is easy to check that the two-phase density profile $\rho(x,t)$ in Corollary 2.4 is a weak solution (in the sense of (2.19)) to the scalar conservation law (2.17) with flux function $F(x,\rho) = c(x)\rho(1-\rho)$. However we cannot immediately apply this theorem in our case since the two-phase flux function $\widetilde{F}(x,\rho) = (1-H(x))c_1 f(\rho) + H(x)c_2 f(\rho)$ is finite only for $\rho \in [0,1]$ and in particular is not C^1. We show how we can replace $F(x,\rho)$ with $\widetilde{F}(x,\rho)$ in the above theorems in Section 6. In particular, we prove the following.

Theorem 2.7. *For $\rho \in \mathbb{R}$ define $f_r(\rho) = c_2(1-\rho)\rho$ and $f_\ell(\rho) = c_1(1-\rho)\rho$ to be the flux functions for the scalar conservation law (2.20). Let the initial macroscopic profile for the hydrodynamic limit be a measurable function $0 \le \rho_0(x) \le 1$. Then the macroscopic density profile $\rho(x,t)$ from the hydrodynamic limit in Theorem 2.2 is the unique solution described in Theorem 2.6.*

3. Wedge last passage time

The strategy of the proof of the hydrodynamic limit is the one from Seppäläinen (1999) and Seppäläinen (2001). Instead of the particle process we work with the height process. The limit is first proved for the jam initial condition of TASEP (also called step initial condition) which for the height process is an initial wedge shape. This process can be equivalently represented by the wedge last-passage model. Subadditivity gives the limit. The general case then follows from an envelope property that also leads to the variational representation of the limiting height profile. In this section we treat the wedge case, and the next section puts it all together.

Recall the notation and conventions introduced in the previous section. In particular, $c(x)$ is a positive, lower semicontinuous speed function with only finitely many discontinuities in any compact set. Define a lattice analogue of the wedge $\mathcal{W}$ by

$$\mathcal{L} = \{(i,j) \in \mathbb{Z}^2 : j \ge 1, i \ge -j+1\} \tag{3.1}$$

with boundary $\partial\mathcal{L} = \{(i,0) : i \ge 0\} \cup \{(i,-i) : i < 0\}$.

For each $n \in \mathbb{N}$ construct a last-passage growth model on $\mathcal{L}$ that represents the TASEP height function in the wedge. Let $\{\tau^n_{i,j} : (i,j) \in \mathcal{L}\}_{n\in\mathbb{N}}$ denote a sequence of independent collections of i.i.d. exponential rate 1 random variables. We need an extra index ℓ to denote the shifting. Define weights

$$\omega^{n,\ell}_{i,j} = c\left(\frac{i-\ell}{n}\right)^{-1}, \quad (i,j) \in \mathcal{L}. \tag{3.2}$$

For $\ell \in \mathbb{Z}$ and $n \in \mathbb{N}$ assign to site $(i,j) \in \mathcal{L}$ the random variable $\omega^{n,\ell}_{i,j}\tau^n_{i,j}$. Given lattice points $(a,b),(u,v) \in \mathcal{L}$, $\Pi((a,b),(u,v))$ is the set of lattice paths $\pi = \{(a,b) = (i_0,j_0),(i_1,j_1),...,(i_p,j_p) = (u,v)\}$ whose admissible steps satisfy

$$(i_l,j_l) - (i_{l-1},j_{l-1}) \in \{(1,0),(0,1),(-1,1)\}. \tag{3.3}$$

In the case that $(a,b) = (0,1)$ we simply denote this set by $\Pi(u,v)$. For $(u,v) \in \mathcal{L}$, $\ell \in \mathbb{R}$ and $n \in \mathbb{N}$ denote the *wedge last passage time*

$$T^{n,\ell}(u,v) = \max_{\pi\in\Pi(u,v)} \sum_{(i,j)\in\pi} \omega^{n,\ell}_{i,j}\tau^n_{i,j} \tag{3.4}$$

with boundary conditions

$$T^{n,\ell}(u,v) = 0 \quad \text{for} \quad (u,v) \in \partial\mathcal{L}. \tag{3.5}$$

Admissible steps (3.3) come from the properties of the TASEP height function. Notice that $(0,1)$ is in fact never used in a maximizing path.

To describe macroscopic last passage times define, for $(x, y) \in \mathcal{W}$ and $q \in \mathbb{R}$,

$$\Gamma^q(x,y) = \sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} \Big\{ \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s)-q)}\, ds \Big\}. \tag{3.6}$$

Theorem 3.1. *For all $q \in \mathbb{R}$ and (x, y) in the interior of $\mathcal{W}$*

$$\lim_{n\to\infty} n^{-1} T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor, \lfloor ny\rfloor) = \Gamma^q(x,y) \quad a.s. \tag{3.7}$$

Remark 3.2. In a constant rate c environment the wedge last passage limit is

$$\lim_{n\to\infty} \frac{1}{n} T^n(\lfloor nx\rfloor, \lfloor ny\rfloor) = c^{-1}\gamma(x,y) = c^{-1}\left(\sqrt{x+y}+\sqrt{y}\right)^2. \tag{3.8}$$

The limit $\gamma(x, y)$ is concave, but this is not true in general for $\Gamma^0(x, y)$. In some special cases concavity still holds, such as if the function $c(x)$ is nonincreasing if $x < 0$ and nondecreasing if $x > 0$.

To prove Theorem 3.1 we approximate $c(x)$ with step functions. Let $-\infty = a_1 < a_2 < ... < a_{L-1} < a_L = +\infty$, and consider the lower semicontinuous step function

$$c(x) = \sum_{m=1}^{L-1} r_m \mathbf{1}_{(a_m,a_{m+1})}(x) + \sum_{m=2}^{L-1} \min\{r_{m-1}, r_m\}\mathbf{1}_{\{a_m\}}(x). \tag{3.9}$$

Proposition 3.3. *Let $c(x)$ be given by (3.9). Then limit (3.7) holds.*

On the way to Proposition 3.3 we state preliminary lemmas that will be used for pieces of paths. We write c_i for the rate values instead of r_i to be consistent with the notation in Theorem 2.1.

Lemma 3.4. *Assume that there is a unique discontinuity $a_2 = 0$ for the speed function $c(x)$ in (3.9). Then for $y > 0$*

$$\lim_{n\to\infty} n^{-1} T^{n,0}(0, \lfloor ny\rfloor) = \frac{4y}{\min\{c_1, c_2\}} = \int_0^1 \frac{\gamma(0,y)}{c(0)}\, ds \quad a.s.$$

Proof: The upper bound in the limit is immediate from domination with constant rates $c(0)$.

For the lower bound we spell out the details for the case $c_1 \geq c_2$. Let $\varepsilon > 0$. To bound $T^{n,0}(0, \lfloor ny\rfloor)$ from below force the path to go through points $(0, 1)$, $\{(\lfloor ny\varepsilon\rfloor, (k-1)\lfloor ny\varepsilon\rfloor) : k = 1, \ldots, \lfloor\varepsilon^{-1}\rfloor\}$ and $(0, \lfloor ny\rfloor)$. For $1 \leq k < \lfloor\varepsilon^{-1}\rfloor$ let $T^n(R^n_k)$ be the last passage time from $(\lfloor ny\varepsilon\rfloor, (k-1)\lfloor ny\varepsilon\rfloor)$ to $(\lfloor ny\varepsilon\rfloor, k\lfloor ny\varepsilon\rfloor)$. R^n_k refers to the parallelogram that contains all the admissible paths from $(\lfloor ny\varepsilon\rfloor, (k-1)\lfloor ny\varepsilon\rfloor)$ to $(\lfloor ny\varepsilon\rfloor, k\lfloor ny\varepsilon\rfloor)$. Each R^n_k lies to the right of $x = 0$ and therefore in the c_2-rate area. (See Fig. 3.5.)

Let $0 < \delta < \varepsilon {c_2}^{-1}\gamma(0, y)$. A large deviation estimate (Theorem 4.1 in Seppäläinen, 1998) gives a constant $C = C(c_2, y, \varepsilon, \delta)$ such that

$$\mathbb{P}\left\{T^n_{c_2}(R^n_k) \leq n(\varepsilon {c_2}^{-1}\gamma(0,y)-\delta)\right\} \leq e^{-Cn^2}. \tag{3.10}$$

By a Borel-Cantelli argument, for large n,

$$T^{n,0}(0, \lfloor ny\rfloor) \geq \sum_{k=1}^{\lfloor\varepsilon^{-1}\rfloor-1} T^n(R^n_k) \geq n(\lfloor\varepsilon^{-1}\rfloor - 1)(\varepsilon {c_2}^{-1}\gamma(0,y)-\delta).$$

This suffices for the conclusion. □

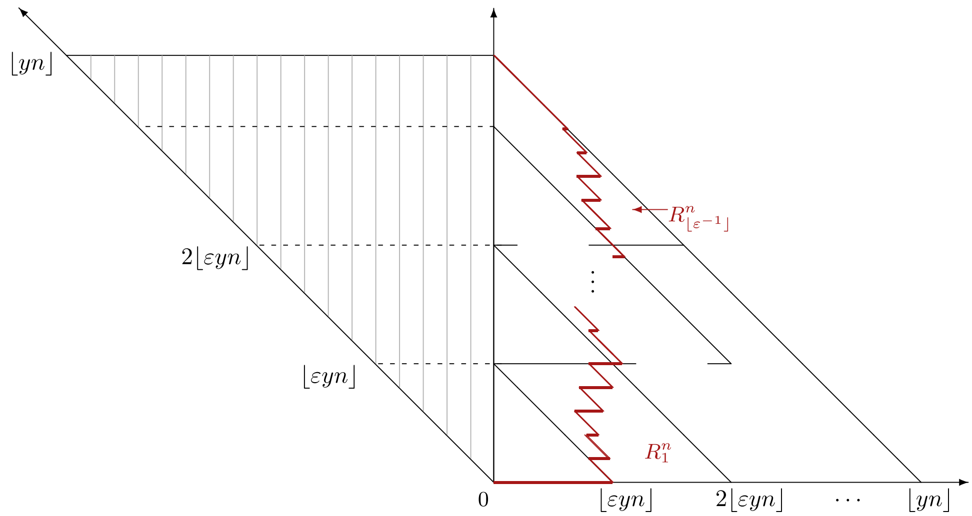

FIGURE 3.5. A possible microscopic path forced to go through opposite corners of the parallelograms R_k^n. The striped area left of $x = 0$ is the c_1-rate region.

Remark 3.5. This lemma shows why it is convenient to use a lower semi-continuous speed function. A path that starts and ends at the same discontinuity stays mostly in the low rate region to maximize its weight. This translates macroscopically to the formula for the limiting time constant obtained in the lemma, involving only the value of c at the discontinuity. If the speed function is not lower semi-continuous, we can state the same result using left and right limits.

Lemma 3.6. *Let $a = 0 < b < +\infty$ be discontinuities for the step speed function $c(x)$ and $c(x) = r$ for $a < x < b$. Take $z \in [0, b]$. Let $\widetilde{T}^n(\lfloor nz \rfloor, \lfloor ny \rfloor)$ be the wedge last passage time from $(0, 1)$ to $(\lfloor nz \rfloor, \lfloor ny \rfloor)$ subject to the constraint that the path has to stay in the r-rate region $(a, b) \times (0, +\infty)$, except possibly for the initial and final steps. Then*

$$\lim_{n\to\infty} n^{-1}\widetilde{T}^n(\lfloor nz \rfloor, \lfloor ny \rfloor) = r^{-1}\gamma(z, y) \quad a.s. \tag{3.11}$$

Same statement holds if $b \le z \le 0$.

Proof: The upper bound $\overline{\lim}\, n^{-1}\widetilde{T}^n(\lfloor nz \rfloor, \lfloor ny \rfloor) \le r^{-1}\gamma(z, y)$ is immediate by putting constant rates r everywhere and dropping the restrictions on the path. For the lower bound adapt the steps of the proof in Lemma 3.4. □

Lemma 3.6 is a place where we cannot allow accumulation of discontinuities for the speed function.

Before proceeding to the proof of Proposition 3.3 we make a simple but important observation about the macroscopic paths $\mathbf{x}(s) = (x^1(s), x^2(s))$, $s \in [0, 1]$, in $\mathcal{H}(x, y)$ for the case where $c(x)$ is a step function (3.9).

Lemma 3.7. *There exists a constant $C = C(x, y, c(\cdot - q))$ such that the supremum in (3.6) comes from paths in $\mathcal{H}(x, y)$ that consist of at most C line segments. Apart from the first and last segment, these segments can be of two types: segments that*

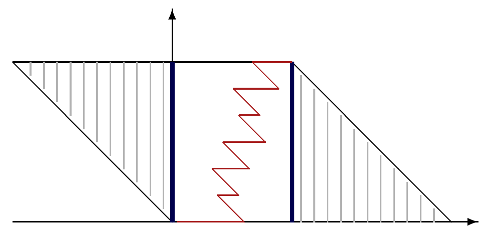

Figure 3.6. A possible microscopic path described in Lemma 3.6. The path has to stay in the unshaded region.

go from one discontinuity of $c(\cdot - q)$ to a neighboring discontinuity, and vertical segments along a discontinuity.

Proof: Path $\mathbf{x}$ is a union of subpaths $\{\mathbf{x}_j\}$ along which $c(x_j^1(s) - q)$ is constant, except possibly at the endpoints. Given such a subpath $(\mathbf{x}_j(s) : t_j \le s \le t_{j+1})$, concavity of γ and Jensen's inequality imply that the line segment ϕ_j that connects $\mathbf{x}_j(t_j)$ to $\mathbf{x}_j(t_{j+1})$ dominates:

$$\int_{t_j}^{t_{j+1}} \frac{\gamma(\mathbf{x}_j'(s))}{c(x_j^1(s) - q)}\, ds \le \int_{t_j}^{t_{j+1}} \frac{\gamma(\phi_j'(s))}{c(\phi_j^1(s) - q)}\, ds.$$

Consequently we can restrict to paths that are unions of line segments.

To bound the number of line segments, observe first that the number of segments that go from one discontinuity to a neighboring discontinuity is bounded. The reason is that the restriction $\mathbf{x}'(s) \in \mathcal{W}$ forces such a segment to increase at least one of the coordinates by the distance between the discontinuities.

Additionally there can be subpaths that touch the same discontinuity more than once without touching a different discontinuity. Lower semi-continuity of $c(\cdot)$ and Jensen's inequality show again that the vertical line segment that stays on the discontinuity dominates such a subpath. Consequently there can be at most one (vertical) line segment between two line segments that connect distinct discontinuities. □

Next a lemma about the continuity of Γ^q. We write $\Gamma^q((a,b),(x,y))$ for the value in (3.6) when the paths go from (a,b) to $(x,y) \in (a,b) + \mathcal{W}$.

Lemma 3.8. *Fix $z, w > 0$. Then there exists a constant $C = C(z, w, c(\cdot - q)) < \infty$ such that for all $0 < \delta \le 1$ and $0 \le a \le z$*

$$\Gamma^q((a,0),(z,\delta)) - \Gamma^q((a,0),(z,0)) \le C\sqrt{\delta}, \tag{3.12}$$

and for $0 \le b \le w$

$$\Gamma^q((-b,b),(-w,w+\delta)) - \Gamma^q((-b,b),(-w,w)) \le C\sqrt{\delta}. \tag{3.13}$$

Proof: Pick $\delta \in (0,1]$ and consider the point (z,δ) in $\mathcal{W}$. For any $\mathbf{x} = (x^1(s), x^2(s)) \in \mathcal{H}(z,\delta)$ set

$$I(\mathbf{x}, q) = \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x^1(s) - q)}\, ds. \tag{3.14}$$

Let $\varepsilon > 0$ and assume that $\phi = (\phi^1, \phi^2) \in \mathcal{H}(z,\delta)$ is a path such that $\Gamma^q(z,\delta) - I(\phi, q) < \varepsilon$. Lemma 3.7 implies that we can decompose ϕ into disjoint linear segments ϕ_j so that $\phi = \sum_{j=1}^M \phi_j$ and $\phi_j : [s_{j-1}, s_j] \to \mathcal{W}$. Here $\sum_j \phi_j$ means path concatenation.

We can find segments $\phi_{j(k)}$, $1 \le k \le N$, such that

$$\phi^1_{j(k)}(s_{j(k)-1}) < \phi^1_{j(k)}(s_{j(k)}), \quad \phi^1_{j(k)}(s_{j(k)}) = \phi^1_{j(k+1)}(s_{j(k+1)-1}),$$

$\phi^1_{j(1)}(s_{j(1)-1}) = 0$, and $\phi^1_{j(N)}(s_{j(N)}) = z$. In other words, the projections of the segments $\phi_{j(k)}$ cover the interval $[0,z]$ without overlap and without backtracking.

We bound the contribution of the remaining path segments to $I(\phi,q)$. Let $J = \bigcup_{k=1}^{N-1}[s_{j(k)}, s_{j(k+1)-1}]$ be the leftover portion of the time interval $[0,1]$. The subpath $\phi(s)$, $s \in [s_{j(k)}, s_{j(k+1)-1}]$, (possibly) eliminated from between $\phi_{j(k)}$ and $\phi_{j(k+1)}$ satisfies $\phi^1(s_{j(k)}) = \phi^1(s_{j(k+1)-1})$. Note that $\gamma(a,b) \le 2a + 4b$ for $(a,b) \in \mathcal{W}$ and $\int_0^1 (\phi^2)'(s)\,ds = \delta$. We can bound as follows:

$$\begin{aligned}\int_J \frac{\gamma((\phi^1)'(s), (\phi^2)'(s))}{c(\phi^1(s) - q)}\,ds &\le C \int_J \gamma((\phi^1)'(s), (\phi^2)'(s))\,ds \\ &\le C \int_J \big(2(\phi^1)'(s) + 4(\phi^2)'(s)\big)\,ds \\ &\le C \int_J 2(\phi^1)'(s)\,ds + C\int_0^1 4(\phi^2)'(s)\,ds \\ &= 0 + 4C\delta. \end{aligned} \tag{3.15}$$

Set $t_k = s_{j(k)-1} < u_k = s_{j(k)}$. Define a horizontal path w from $(0,0)$ to $(z,0)$ with segments

$$w_k(s) = \big(\phi^1_{j(k)}(s), 0\big), \quad \text{for } \; t_k \le s \le u_k, \tag{3.16}$$

and constant on the complementary time set J.

To get the lemma, we estimate

$$\begin{aligned}\Gamma^q(z,\delta) - \varepsilon \le I(\phi,q) &= \int_J \frac{\gamma(\phi'(s))}{c(\phi^1(s) - q)}\,ds + \int_{[0,1]\setminus J} \frac{\gamma(\phi'(s))}{c(\phi^1(s) - q)}\,ds \\ &\le C\delta + \sum_{k=1}^N \Big(I(\phi_{j(k)}, q) - I(w_k, q)\Big) + \Gamma^q(z,0) \\ &\le C\delta + C' \sum_{k=1}^N \int_{t_k}^{u_k} \big(\gamma(\phi'_{j(k)}(s)) - \gamma(w'_k(s))\big)\,ds + \Gamma^q(z,0) \\ &\le C\delta + C' \sum_{k=1}^N \bigg(\int_{t_k}^{u_k} (\phi^2)'_{j(k)}(s)\,ds + \\ &\qquad + 2\int_{t_k}^{u_k} \sqrt{(\phi^2)'_{j(k)}(s)}\sqrt{(\phi^1)'_{j(k)}(s) + (\phi^2)'_{j(k)}(s)}\,ds \bigg) + \Gamma^q(z,0) \\ &\le C\delta + C' \sum_{k=1}^N \bigg(\int_{t_k}^{u_k} (\phi^2)'_{j(k)}(s)\,ds \bigg)^{\frac12} \times \\ &\qquad \times \bigg(\int_{t_k}^{u_k} \big((\phi^1)'_{j(k)}(s) + (\phi^2)'_{j(k)}(s)\big)\,ds \bigg)^{\frac12} + \Gamma^q(z,0)\end{aligned}$$

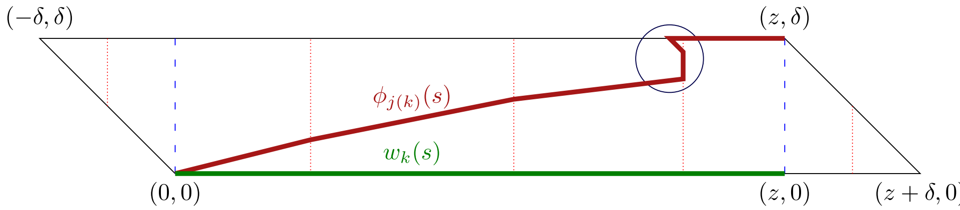

Figure 3.7. A possible macroscopic path from $(0,0)$ to (z,δ). The dotted vertical lines are discontinuity columns of $c(\cdot - q)$. The error from eliminating the segments outside the two vertical dashed lines and from eliminating pathologies (like the circled part) is of order δ and a comparison with the horizontal path leads to an error of order $\sqrt{\delta}$.

$$
\begin{aligned}
&\le C\delta + C'\Big(\sum_{k=1}^{N}\int_{t_k}^{u_k}(\phi^2)'_{j(k)}(s)\,ds\Big)^{\frac12}\times\\
&\qquad\times\Big(\sum_{k=1}^{N}\int_{t_k}^{u_k}\big((\phi^1)'_{j(k)}(s)+(\phi^2)'_{j(k)}(s)\big)\,ds\Big)^{\frac12}+\Gamma^q(z,0)\\
&\le C\delta + C'\sqrt{\delta}\sum_{k=1}^{N}\int_{t_k}^{u_k}\big((\phi^1)'_{j(k)}(s)+(\phi^2)'_{j(k)}(s)\big)\,ds+\Gamma^q(z,0)\\
&\le C\delta + C'\sqrt{\delta}\sqrt{z+\delta}+\Gamma^q(z,0)\\
&\le C\delta + C'\sqrt{\delta}\sqrt{z}+\Gamma^q(z,0).
\end{aligned}
$$

The first inequality (3.12) follows for $a = 0$ by letting ε go to 0. It also follows for all $a \in [0, z]$ by shifting the origin to a which replaces z with $z - a$.

For the second inequality (3.13) the arguments are analogous, so we omit them. □

Corollary 3.9. *Fix* $(x, y) \in \mathcal{W}$. *Then there exists* $C = C(x, y, c(\cdot - q)) < \infty$ *such that for all* $0 < \delta \le 1$

$$\Gamma^q(x, y+\delta) - \Gamma^q(x, y) < C\sqrt{\delta}. \tag{3.17}$$

Proof: Let $A((a, b), (x, y))$ be the parallelogram with sides parallel to the boundaries of the wedge, north-east corner the point (x, y) and south-west corner at (a, b). If $(a, b) = (0, 0)$ we simply write $A(x, y)$.

Let $\varepsilon > 0$. Let ϕ^ε a path such that $\Gamma^q(x, y+\delta) - I(\phi^\varepsilon, q) < \varepsilon$. Let u be the point where ϕ^ε first intersects the north or the east boundary of $A(x, y)$. Without loss of generality assume it is the north boundary and so $u = (a, y)$ for some $a \in [-y, x]$.

Then,

$$\begin{aligned}\Gamma^q(x,y+\delta)-\varepsilon &\le I(\phi^\varepsilon,q)\\ &\le \Gamma^q(a,y)+\Gamma^q((a,y),(x,y+\delta))\\ &= \Gamma^q(a,y)+\Gamma^q((a,y),(x,y))+\Gamma^q((a,y),(x,y+\delta))-\\ &\quad -\Gamma^q((a,y),(x,y))\\ &\le \Gamma^q(x,y)+\Gamma^q((a,y),(x,y+\delta))-\Gamma^q((a,y),(x,y)). \end{aligned} \tag{3.18}$$

The last inequality gives

$$\Gamma^q(x,y+\delta)-\Gamma^q(x,y)\le \Gamma^q((a,y),(x,y+\delta))-\Gamma^q((a,y),(x,y))+\varepsilon\le C\sqrt{\delta}+\varepsilon \tag{3.19}$$

by Lemma 3.8. Let ε decrease to 0 to prove the Corollary. □

Proof of Proposition 3.3: Fix (x,y) in the interior of $\mathcal{W}$. For $\mathbf{x}=(x^1(s),x^2(s))\in\mathcal{H}(x,y)$ set

$$I(\mathbf{x},q)=\int_0^1\frac{\gamma(\mathbf{x}'(s))}{c(x^1(s)-q)}\,ds. \tag{3.20}$$

We prove first

$$\varliminf_{n\to\infty} n^{-1}T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor)\ge\Gamma^q(x,y)\equiv\sup_{\mathbf{x}(\cdot)\in\mathcal{H}(x,y)} I(\mathbf{x},q). \tag{3.21}$$

It suffices to consider macroscopic paths of the type

$$\mathbf{x}(s)=\sum_{j=1}^{H}\mathbf{x}_j(s)\mathbf{1}_{[s_j,s_{j+1})}(s) \tag{3.22}$$

where $H\in\mathbb{N}$, $\mathbf{x}_j$ is the straight line segment from $\mathbf{x}(s_j)$ to $\mathbf{x}(s_{j+1})$, $c(x_1(s)-q)=r_{m_j}$ is constant for $s\in(s_j,s_{j+1})$, and by continuity $\mathbf{x}_j(s_{j+1})=\mathbf{x}_{j+1}(s_{j+1})$.

Let π^n be the microscopic path through points $(0,1)$, $\{\lfloor n\mathbf{x}_j(s_j)\rfloor : 1\le j\le H\}$ and $(\lfloor nx\rfloor,\lfloor ny\rfloor)$ constructed so that its segments π^n_j satisfy these requirements:

(i) π^n_j lies inside the region where $\omega^{n,\lfloor nq\rfloor}_{i,k}=r^{-1}_{m_j}$ is constant, except possibly for the initial and final step;

(ii) π^n_j maximizes passage time between its endpoints $\lfloor n\mathbf{x}_j(s_j)\rfloor$ and $\lfloor n\mathbf{x}_{j+1}(s_{j+1})\rfloor$ subject to the above requirement.

Let

$$T^{n,\lfloor nq\rfloor}_j=\max_{\pi^n_j}\sum_{(i,k)\in\pi^n_j}\omega^{n,\lfloor nq\rfloor}_{i,k}\tau_{i,k} \tag{3.23}$$

denote the last-passage time of a segment subject to these constraints. Observe that the proofs of Lemmas 3.4 and 3.6 do not depend on the shift parameter q, therefore

$$\lim_{n\to\infty} n^{-1}T^{n,\lfloor nq\rfloor}_j=\frac{\gamma(\mathbf{x}_j(s_j)-\mathbf{x}_{j+1}(s_j))}{r_{m_j}}=\int_{s_j}^{s_{j+1}}\frac{\gamma(\mathbf{x}_j'(s))}{c(x^1_j(s)-q)}\,ds.$$

Adding up the segments gives the lower bound:

$$\begin{aligned}\varliminf_{n\to\infty} n^{-1}T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor)&\ge\varliminf_{n\to\infty}\sum_j n^{-1}T^{n,\lfloor nq\rfloor}_j\\ &=\sum_j\int_{s_j}^{s_{j+1}}\frac{\gamma(\mathbf{x}_j'(s))}{c(x^1(s)-q)}\,ds=I(\mathbf{x},q).\end{aligned}$$

Now for the complementary upper bound

$$\varlimsup_{n\to\infty} n^{-1}T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor)\leq \Gamma^q(x,y). \tag{3.24}$$

Each microscopic path to $(\lfloor nx\rfloor,\lfloor ny\rfloor)$ is contained in nA for a fixed parallelogram $A\subseteq\mathcal{W}$ with sides parallel to the wedge boundaries. Pick $\varepsilon>0$. Let $r_*>0$ be a lower bound on all the rate values that appear in the set A. Find $\delta>0$ such that $|\gamma(v)-\gamma(w)|<\varepsilon r_*$ for all $v,w\in A$ with $|v-w|<\delta$ and $\delta\leq 1$ so that Corollary 3.9 is valid.

Consider an arbitrary microscopic path from $(0,1)$ to $(\lfloor nx\rfloor,\lfloor ny\rfloor)$. Given the speed function and q, there is a fixed upper bound $Q=Q(x,y)$ on the number of segments of the path that start at one discontinuity column $(\lfloor na_i\rfloor+\lfloor nq\rfloor)\times\mathbb{N}$ and end at a neighboring discontinuity column $(\lfloor na_{i\pm1}\rfloor+\lfloor nq\rfloor)\times\mathbb{N}$. The reason is that there is an order n lower bound on the number of lattice steps it takes to travel between distinct discontinuities in nA.

Fix $K\in\mathbb{N}$ and partition the interval $[0,y]$ evenly by $b_j=jy/K$, $0\leq j\leq K$, so that $y/K<\delta/Q$. Make the partition finer by adding the $y-$coordinates of the intersection points of discontinuity lines $\{a_i+q\}\times\mathbb{R}_+$ with the boundary of A.

Let π^n be the maximizing microscopic path. We decompose π^n into path segments $\{\pi^n_j : 0\leq j<M_n\}$ by looking at visits to discontinuity columns $(\lfloor na_i\rfloor+\lfloor nq\rfloor)\times\mathbb{N}$, both repeated visits to the same column and visits to a column different from the previous one. Let $\{0=b_{k_0}\leq b_{k_1}\leq b_{k_2}\leq ...\leq b_{k_{M_n-1}}\leq b_{k_{M_n}}=y\}$ be a sequence of partition points and $\{0=x_0,\, x_1=a_{m_1}+q,\, x_2=a_{m_2}+q,\dots,x_{M_n}=x\}$ a sequence where x_j for $0<j<M_n$ are discontinuity points of the shifted speed function $c(\cdot\,-q)$. We can create the path segments and these sequences with the property that segment π^n_j starts at $(\lfloor nx_j\rfloor,l)$ with l in the range $\lfloor nb_{k_j}\rfloor\leq l\leq\lfloor nb_{k_j+1}\rfloor$ and ends at $(\lfloor nx_{j+1}\rfloor,l')$ with $\lfloor nb_{k_{j+1}}\rfloor\leq l'\leq\lfloor nb_{k_{j+1}+1}\rfloor$. In an extreme case the entire path π^n can be a single segment that does not touch discontinuity columns.

In order to have a fixed upper bound on the total number M_n of segments, uniformly in n, we insist that for $0<j<M_n-1$ the labels satisfy:

(i) For odd j, π^n_j starts and ends at the same discontinuity column $(\lfloor nx_j\rfloor,\cdot)$. The rate relevant for segment π^n_j is $r_{\ell_j}=c(a_{m_j})$.

(ii) For even j, π^n_j starts and ends at different neighboring discontinuity columns, and except for the initial and final points, does not touch any discontinuity column and visits only points that are in a region of constant rate r_{ℓ_j}.

The above conditions may create empty segments. This is not harmful. Replace Q with $2Q+2$ to continue having the uniform upper bound $M_n\leq Q$.

Let $T(\pi^n_j)$ be the total weight of segment π^n_j. Let $\tilde\pi^n_j$ be the maximal path from $(\lfloor nx_j\rfloor,\lfloor nb_{k_j}\rfloor)$ to $(\lfloor nx_{j+1}\rfloor,\lfloor nb_{k_{j+1}+1}\rfloor)$ in an environment with constant weights $\omega_{i,j}=r^{-1}_{\ell_j}$ everywhere on the lattice, with total weight T^n_j. $T^n_j\geq T(\pi^n_j)$, up to an error from the endpoints of π^n_j.

Theorem 4.2 in Seppäläinen (1998) gives a large deviation bound for T^n_j. Consider a constant rate r environment and the maximal weight $T\big((\lfloor nu_1\rfloor,\lfloor nv_1\rfloor),(\lfloor nu_2\rfloor,\lfloor nv_2\rfloor)\big)$ between two points (u_1,v_1) and (u_2,v_2) such that their lattice versions can be connected by admissible paths for all n. Then there exists a positive constant C such that for n large enough,

$$\mathbb{P}\Big\{T\big((\lfloor nu_1\rfloor,\lfloor nv_1\rfloor),(\lfloor nu_2\rfloor,\lfloor nv_2\rfloor)\big) > nr^{-1}\gamma(u_2-u_1,v_2-v_1)+n\varepsilon\Big\} < e^{-Cn}. \tag{3.25}$$

There is a fixed finite collection out of which we pick the pairs $\{(x_j,b_{k_j}),(x_{j+1},b_{k_{j+1}+1})\}$ that determine the segments $\tilde{\pi}^n_j$. By (3.25) and the Borel-Cantelli lemma, a.s. for large enough n,

$$T^n_j \;\le\; nr^{-1}_{\ell_j}\gamma(x_{j+1}-x_j,b_{k_{j+1}+1}-b_{k_j})+n\varepsilon \quad \text{for } 0\le j<M_n. \tag{3.26}$$

Define $\delta_1>0$ by $y+\delta_1=\sum_{j=0}^{M_n-1}(b_{k_{j+1}+1}-b_{k_j})$. Since $y=\sum_{j=0}^{M_n-1}(b_{k_{j+1}}-b_{k_j})$ and by the choice of the mesh of the partition $\{b_k\}$, we have $\delta_1\le M_n\delta/Q\le\delta$. Think of $(x_{j+1}-x_j,b_{k_{j+1}+1}-b_{k_j})$, $0\le j<M_n$, as the successive segments of a macroscopic path from $(0,0)$ to $(x,y+\delta_1)$.

For sufficiently large n so that (3.26) is in effect,

$$\begin{aligned}T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor) &\le \sum_{j=1}^{M_n}T^n_j \le n\sum_{j=1}^{M_n}r^{-1}_{\ell_j}\gamma(x_{j+1}-x_j,b_{k_{j+1}+1}-b_{k_j})+nQ\varepsilon\\ &\le n\Gamma^q(x,y+\delta_1)+nQ\varepsilon\\ &\le n\Gamma^q(x,y)+nC\sqrt{\delta}+nQ\varepsilon.\end{aligned}$$

The last inequality came from Corollary 3.9. Let $\delta\to 0$. Since ε was arbitrary the upper bound (3.24) holds. □

Proof of Theorem 3.1: Fix (x,y). For each $\varepsilon>0$ we can find lower semicontinuous step functions c_1 and c_2 such that $\|c_1-c_2\|_\infty\le\varepsilon$ and on some compact interval, large enough to contain all the rates that can potentially influence $\Gamma^q(x,y)$, $c_1(x)\le c(x)\le c_2(x)$. When the weights in (3.2) come from speed function c_i let us write T_i for last passage times and Γ_i for their limits. An obvious coupling using common exponential variables $\{\tau_{i,j}\}$ gives

$$T_1^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor)\ge T^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor)\ge T_2^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor,\lfloor ny\rfloor).$$

Letting $\alpha>0$ denote a lower bound for $c(x)$ in the compact interval relevant for (x,y), we have this bound for $\mathbf{x}\in\mathcal{H}(x,y)$:

$$\begin{aligned}0\le\int_0^1\Big\{\frac{\gamma(\mathbf{x}'(s))}{c_1(x_1(s)-q)}-\frac{\gamma(\mathbf{x}'(s))}{c_2(x_1(s)-q)}\Big\}\,ds &\le \varepsilon\int_0^1\frac{\gamma(\mathbf{x}'(s))}{c_1^2(x_1(s)-q)}\,ds\\ &\le \varepsilon\alpha^{-2}\gamma(x,y).\end{aligned}$$

Therefore the limits also have the bound

$$0\le\Gamma_1^q(x,y)-\Gamma_2^q(x,y)\le C(x,y)\varepsilon.$$

From these approximations and the limits for T_i in Proposition 3.3 we can deduce Theorem 3.1. □

Proof of Theorem 2.1: We can construct the last passage times $G(x,y)$ of the corner growth model (2.2) with the same ingredients as the wedge last passage times $T^{n,0}(x,y)$ of (3.4), by taking $Y_{(i,j)}=\omega^{n,0}_{i-j,j}\tau^n_{i-j,j}$. Then $T^{n,0}(x,y)=G(x+y,y)$ and we can transfer the problem to the wedge. The correct speed function to use is now $c(x)=c_1\mathbf{1}\{x<0\}+c_2\mathbf{1}\{x\ge 0\}$. In this case the limit in Theorem 3.1 can be solved explicitly with calculus. We omit the details. □

4. Hydrodynamic limit

In this section we sketch the proof of the main result Theorem 2.2. This argument is from Seppäläinen (1999, 2001).

4.1. *Construction of the process and the variational coupling.* For each $n \in \mathbb{N}$ we construct a $\mathbb{Z}$-valued *height process* $z^n(t) = (z_i^n(t) : i \in \mathbb{Z})$. The height values obey the constraint

$$0 \le z_{i+1}^n(t) - z_i^n(t) \le 1. \tag{4.1}$$

Let $\{\mathcal{D}_i^n\}$ be a collection of mutually independent (in i and n) Poisson processes with rates c_i^n given by

$$c_i^n = c(n^{-1}i), \tag{4.2}$$

where $c(x)$ is the lower semicontinuous speed function. Dynamically, for each n and i, the height value z_i^n is decreased by 1 at event times of $\mathcal{D}_i^n$, provided the new configuration does not violate (4.1).

After we construct $z^n(t)$, we can define the exclusion process $\eta^n(t)$ by

$$\eta_i^n(t) = z_i^n(t) - z_{i-1}^n(t). \tag{4.3}$$

A decrease in z_i^n is associated with an exclusion particle jump from site i to $i+1$. Thus the z^n process keeps track of the current of the η^n-process, precisely speaking

$$J_i^n(t) = z_i^n(0) - z_i^n(t). \tag{4.4}$$

Assume that the processes z^n have been constructed on a probability space that supports the initial configurations $z^n(0) = (z_i^n(0))$ and the Poisson processes $\{\mathcal{D}_i^n\}$ that are independent of $(z_i^n(0))$. Next we state the envelope property that is the key tool for the proof of the hydrodynamic limit. Define a family of auxiliary height processes $\{\xi^{n,k} : n \in \mathbb{N},\, k \in \mathbb{Z}\}$ that grow upward from wedge-shaped initial conditions

$$\xi_i^{n,k}(0) = \begin{cases} 0, & \text{if } i \ge 0 \\ -i, & \text{if } i < 0. \end{cases} \tag{4.5}$$

The dynamical rule for the $\xi^{n,k}$ process is that $\xi_i^{n,k}$ jumps up by 1 at the event times of $\mathcal{D}_{i+k}^n$ provided the inequalities

$$\xi_i^{n,k} \le \xi_{i-1}^{n,k} \quad \text{and} \quad \xi_i^{n,k} \le \xi_{i+1}^{n,k} + 1 \tag{4.6}$$

are not violated. In particular $\xi_i^{n,k}$ attempts a jump at rate c_{i+k}^n.

Lemma 4.1 (Envelope Property)**.** *For each $n \in \mathbb{N}$, for all $i \in \mathbb{Z}$ and $t \ge 0$,*

$$z_i^n(t) = \sup_{k \in \mathbb{Z}} \{ z_k^n(0) - \xi_{i-k}^{n,k}(t) \} \quad a.s. \tag{4.7}$$

Equation (4.7) holds by construction at time $t = 0$, and it is proved by induction on jumps. For details see Lemma 4.2 in Seppäläinen (1999).

4.2. *The limit for ξ.* For $q, x \in \mathbb{R}$, $t > 0$ and for the speed function $c(x)$, define

$$g^q(x,t) = \inf\{y : (x,y) \in \mathcal{W}, \Gamma^q(x,y) \geq t\}. \tag{4.8}$$

$\Gamma^q(x,y)$ defined by (3.6) represents the macroscopic time it takes a ξ-type interface process to reach point (x,y). The level curve of Γ^q given by $g^q(\cdot,t)$ represents the limiting interface of a certain ξ-process, as stated in the next proposition.

Proposition 4.2. *For all $q, x \in \mathbb{R}$ and $t > 0$*

$$\lim_{n\to\infty} n^{-1}\xi^{n,\lfloor nq\rfloor}_{\lfloor nx\rfloor}(nt) = g^{-q}(x,t) \quad a.s. \tag{4.9}$$

Recall the lattice wedge $\mathcal{L}$ defined by (3.1). For $(i,j) \in \mathcal{L} \cup \partial\mathcal{L}$, let

$$L^{n,k}(i,j) = \inf\{t \geq 0 : \xi^{n,k}_i(t) \geq j\} \tag{4.10}$$

denote the time when $\xi^{n,k}_i$ reaches level j. The rules (4.5)–(4.6) give the boundary conditions

$$L^{n,k}(i,j) = 0 \quad \text{for} \quad (i,j) \in \partial\mathcal{L} \tag{4.11}$$

and for $(i,j) \in \mathcal{L}$ the recurrence

$$L^{n,k}(i,j) = \max\{L^{n,k}(i-1,j), L^{n,k}(i,j-1), L^{n,k}(i+1,j-1)\} + \beta^{n,k}_{i,j} \tag{4.12}$$

where $\beta^{n,k}_{i,j}$ is an exponential waiting time, independent of everything else. It represents the time $\xi^{n,k}_i$ waits to jump, *after* $\xi^{n,k}_i$ and its neighbors $\xi^{n,k}_{i-1}$, $\xi^{n,k}_{i+1}$ have reached positions that permit $\xi^{n,k}_i$ to jump from $j-1$ to j. The dynamical rule that governs the jumps of $\xi^{n,k}_i$ implies that $\beta^{n,k}_{i,j}$ has rate c^n_{i+k}.

Equations (3.4), (3.5), (4.11), and (4.12), together with the strong Markov property, imply that

$$\{L^{n,k}(i,j) : (i,j) \in \mathcal{L} \cup \partial\mathcal{L}\} \overset{\mathcal{D}}{=} \{T^{n,-k}(i,j) : (i,j) \in \mathcal{L} \cup \partial\mathcal{L}\}. \tag{4.13}$$

Consequently Theorem 3.1 gives the a.s. convergence $n^{-1}L^{n,\lfloor nq\rfloor}(\lfloor nx\rfloor, \lfloor ny\rfloor) \to \Gamma^{-q}(x,y)$, and this passage time limit gives limit (4.9).

Proof of Theorem 2.2: Given the initial configurations $\eta^n(0) = \{\eta^n_i(0) : i \in \mathbb{Z}\}$ that appear in hypothesis (2.10), define initial configurations $z^n(0) = \{z^n_i(0) : i \in \mathbb{Z}\}$ so that $z^n_0(0) = 0$ so that (4.3) holds at time $t = 0$. Hypothesis (2.10) implies that

$$\lim_{n\to\infty} n^{-1} z^n_{\lfloor nq\rfloor} = v_0(q) \quad \text{a.s.} \tag{4.14}$$

for all $q \in \mathbb{R}$, with v_0 defined by (2.6).

Construct the height processes z^n and define the exclusion processes η^n by (4.3). Define $v(x,t)$ by (2.9). From (4.3)–(4.4) we see that Theorem 2.2 follows from proving that for all $x \in \mathbb{R}, t \in \mathbb{R}^+$,

$$\lim_{n\to\infty} n^{-1} z^n_{\lfloor nx\rfloor}(nt) = v(x,t) \quad \text{a.s.} \tag{4.15}$$

Rewrite (4.7) with the correct scaling:

$$n^{-1} z^n_{\lfloor nx\rfloor}(nt) = \sup_{q\in\mathbb{R}}\left\{n^{-1} z^n_{\lfloor nq\rfloor}(0) - n^{-1}\xi^{\lfloor nq\rfloor}_{\lfloor nx\rfloor - \lfloor nq\rfloor}(nt)\right\}. \tag{4.16}$$

The proof of (4.15) is now to show that the right-hand side of (4.16) converges to the right-hand side of (2.9).

From (4.14), (4.16) and (4.9) we can prove that a.s.

$$\lim_{n\to\infty} n^{-1} z^n_{\lfloor nx \rfloor}(nt) = \sup_{q\in\mathbb{R}} \left\{ v_0(q) - g^{-q}(x-q,t) \right\} \equiv \tilde{v}(x,t). \tag{4.17}$$

The argument is the same as the one from equations (6.4)–(6.15) in Seppäläinen (1999) so we will not repeat it here.

Using (3.6) and (4.8) we can rewrite $\tilde{v}(x,t)$ as

$$\tilde{v}(x,t) = \sup_{q,y\in\mathbb{R}} \Big\{ v_0(q) - y : \exists \mathbf{x} \in \mathcal{H}(x-q,y) \text{ such that } \int_0^1 \frac{\gamma(\mathbf{x}'(s))}{c(x_1(s)+q)}\, ds \geq t \Big\}. \tag{4.18}$$

The final step is to prove $v(x,t) = \tilde{v}(x,t)$. The argument is identical to the one used to prove Proposition 4.3 in Seppäläinen (2001) so we omit it. With this we can consider Theorem 2.2 proved. □

5. Density profiles in two-phase TASEP

This section proves Corollary 2.4: assuming $c(x) = (1 - H(x))c_1 + H(x)c_2$, $c_1 \geq c_2$ and $\rho_0(x) \equiv \rho \in (0,1)$, we use variational formula (2.9) to obtain explicit hydrodynamic limits.

Remark 5.1. In light of Theorem 2.7, one can (instead of doing the following computations) guess the candidate solution for the scalar conservation law (2.20) and then check that it verifies the entropy conditions (2.22) - (2.24). The following computations do not require any knowledge of p.d.e. theory or familiarity with interface problems so we present them independently in this section.

Let

$$C^0(x,t,q) = \{ w \in C([0,t],\mathbb{R}) : w \text{ piecewise linear, } w(0) = q, w(t) = x \}. \tag{5.1}$$

To optimize in (2.9) we use a couple different approaches for different cases. We outline this and omit the details.

One approach is to separate the choice of the starting point q of the path. By setting

$$I(x,t,q) = \inf_{w\in C^0(x,t,q)} \left\{ \int_0^t c(w(s)) g\left(\frac{w'(s)}{c(w(s))} \right) ds \right\} \tag{5.2}$$

(2.9) becomes

$$v(x,t) = \sup_{q\in\mathbb{R}} \{ v_0(q) - I(x,t,q) \}. \tag{5.3}$$

We distinguish four cases according to the signs of x, q. Set

$$R_+(x,t) = \sup_{q>0} \{ v_0(q) - I(x,t,q) \}, \text{ if } x > 0, \tag{5.4}$$

$$L_-(x,t) = \sup_{q<0} \{ v_0(q) - I(x,t,q) \}, \text{ if } x < 0. \tag{5.5}$$

These functions are going to be used in Cases 1 and 2 below ($qx \geq 0$) where we can compute $I(x,t,q)$ directly.

However, there are values (x,t,q) for which the q-derivative of the expression in braces in (5.3) is a rational function with a quartic polynomial in the numerator. While an explicit formula for roots of a quartic exists, the solution is not attractive and it is not clear how to pick the right root. Instead we turn the problem into a two-dimensional maximization problem.

If $qx < 0$ the optimizing path w crosses the origin: $w(u) = 0$ for some u. It turns out convenient to find the optimal q for each crossing time u. For Case 3 $(q < 0, x > 0)$ set

$$\Phi(u,q) = q\rho - c_1 u g\left(\frac{-q}{uc_1}\right) - c_2(t-u)g\left(\frac{x}{(t-u)c_2}\right) \tag{5.6}$$

and

$$L_+(x,t) = \sup_{q<0, u\in[0,t]} \Phi(u,q). \tag{5.7}$$

For Case 4 $(q > 0, x < 0)$ the obvious modifications are

$$\Psi(u,q) = q\rho - c_2 u g\left(\frac{-q}{uc_2}\right) - c_1(t-u)g\left(\frac{x}{(t-u)c_1}\right) \tag{5.8}$$

and

$$R_-(x,t) = \sup_{q>0, u\in[0,t]} \Psi(u,q).$$

Rewrite (5.3) using functions $R_\pm$, $L_\pm$:

$$v(x,t) = \max\{R_+(x,t), L_+(x,t)\}\mathbf{1}\{x \ge 0\} + \max\{R_-(x,t), L_-(x,t)\}\mathbf{1}\{x < 0\}. \tag{5.9}$$

Proof of Corollary 2.4: We compute the functions $R_\pm$, $L_\pm$. The density profiles $\rho(x,t)$ are given then by the x-derivative of $v(x,t)$.
Case 1*:* $x \ge 0$, $q \ge 0$. Since $c_2 \le c_1$, the minimizing w of $I(x,t,q)$ is the straight line connecting $(0,q)$ to (t,x). In particular,

$$I(x,t,q) = c_2 t g\left(\frac{x-q}{tc_2}\right). \tag{5.10}$$

Then the resulting $R_+(x,t)$ is given by

$$R_+(x,t) = \begin{cases} -tc_2 g(\frac{x}{tc_2}) & \text{if } \rho \le \frac{1}{2}, \quad x < tc_2(1-2\rho) \\ \rho x - tc_2\rho(1-\rho), & \text{if } \rho \le \frac{1}{2}, \quad x \ge tc_2(1-2\rho) \\ \rho x - tc_2\rho(1-\rho), & \text{if } \rho > \frac{1}{2}, \end{cases} \tag{5.11}$$

Case 2*:* $x \le 0$, $q \le 0$. The minimizing path w can either be a straight line from $(0,q)$ to (t,x) or a piecewise linear path such that the set $\{t : w(t) = 0\}$ has positive Lebesgue measure. This last statement just says that the path might want to take advantage of the low rate at $x = 0$. We leave the calculus details to the reader and record the resulting minimum value of $I(x,t,q)$. Set $B = \sqrt{c_1(c_1 - c_2)}$.

$$I(x,t,q) = \begin{cases} \frac{-qc_1}{4B}\left(1 - \frac{B}{c_1}\right)^2 + \left(t - \frac{|x|-q}{B}\right)\frac{c_2}{4} - \frac{xc_1}{4B}\left(1 + \frac{B}{c_1}\right)^2, \\ \qquad\qquad \text{when } -(\sqrt{Bt} - \sqrt{|x|})^2 \le q, \; -Bt \le x < 0 \\ c_1 t g\left(\frac{x-q}{c_1 t}\right) \qquad \text{otherwise.} \end{cases} \tag{5.12}$$

The corresponding function $L_-(x,t)$ is given by

$$L_-(x,t) = \begin{cases} \rho x - tc_1\rho(1-\rho), & 0<\rho<\rho^*, x\in\mathbb{R} \\ \rho x - tc_1\rho(1-\rho), & \rho^*\le\rho\le\frac12, \quad x\le -tc_1(\rho-\rho^*) \\ -\left(t+\frac{x}{B}\right)\frac{c_2}{4}+\frac{xc_1}{4B}\left(1+\frac{B}{c_1}\right)^2, & \rho^*\le\rho\le\frac12, \quad x> -tc_1(\rho-\rho^*) \\ \rho x - tc_1\rho(1-\rho), & \frac12<\rho\le 1-\rho^*, \quad x< -tc_1(\rho-\rho^*) \\ -\left(t+\frac{x}{B}\right)\frac{c_2}{4}+\frac{xc_1}{4B}\left(1+\frac{B}{c_1}\right)^2, & \frac12<\rho\le 1-\rho^*, \quad -tc_1(\rho-\rho^*)\le x \\ -\left(t+\frac{x}{B}\right)\frac{c_2}{4}+\frac{xc_1}{4B}\left(1+\frac{B}{c_1}\right)^2, & 1-\rho^*<\rho<1, \quad -Bt\le x \\ -tc_1 g\left(\frac{x}{tc_1}\right), & 1-\rho^*<\rho<1, \quad -c_1t(2\rho-1)\le x< -Bt \\ \rho x - c_1t\rho(1-\rho), & 1-\rho^*<\rho<1, \quad x<-c_1t(2\rho-1). \end{cases} \tag{5.13}$$

***Case 3**:* $x>0$, $q\le 0$. Abbreviate $D=c_2^2-4c_1c_2\rho(1-\rho)$. First compute the q-derivative

$$\Phi_q(u,q) = \begin{cases} \rho-\frac12-\frac{q}{2uc_1}, & -uc_1\le q<0 \\ \rho & q<-uc_1. \end{cases} \tag{5.14}$$

If $\rho\ge 1/2$ then Φ_q is positive and the maximum value is when $q=0$ so we are reduced to Case 1. If $\rho<1/2$ the maximizing $q=uc_1(2\rho-1)$. Then

$$F(u)=\Phi\big(u,2uc_1(\rho-\tfrac12)\big) = -uc_1\rho(1-\rho)-c_2(t-u)g\left(\frac{x}{(t-u)c_2}\right),$$

with u-derivative

$$\frac{dF}{du} = -c_1\rho(1-\rho)+\frac{c_2}{4}\left(1-\frac{x^2}{(c_2(t-u))^2}\right).$$

Again we need to split two cases. If $\rho<\rho^*$ (equivalently $D>0$) and $x\le t\sqrt{D}$, the maximizing $u=t-x/\sqrt{D}$, otherwise $u=0$. If $\rho^*\le\rho<\frac12$ the derivative is negative so the maximizing u is still $u=0$. Together,

$$L_+(x,t) = \begin{cases} -tc_1\rho(1-\rho)+x\left(\frac12-\frac{\sqrt{D}}{2c_2}\right), & \rho<\rho^*, x\le t\sqrt{D} \\ -c_2tg\left(\frac{x}{tc_2}\right), & \rho<\rho^*, x\ge t\sqrt{D} \\ -c_2tg\left(\frac{x}{tc_2}\right), & \rho^*\le\rho\le 1. \end{cases} \tag{5.15}$$

***Case 4**:* $x\le 0, q\ge 0$. We treat this case in exactly the same way as Case 3, so we omit the details. Here we need the quantity $D_1=(c_1)^2-4c_1c_2\rho(1-\rho)$ and we compute

$$R_-(x,t) = \begin{cases} -tc_1g\left(\frac{x}{tc_1}\right), & \rho\le\frac12 \\ -tc_2\rho(1-\rho)+x\left(\frac12+\frac{\sqrt{D_1}}{2c_1}\right), & \frac12<\rho, -t\sqrt{D_1}\le x \\ -tc_1g\left(\frac{x}{tc_1}\right), & \frac12\le\rho\le 1, x<-t\sqrt{D_1} \end{cases} \tag{5.16}$$

Now compute $v(x,t)$ from (5.9). We leave the remaining details to the reader. □

6. Entropy solutions of the discontinuous conservation law

For this section, $c(x) = (1-H(x))c_1 + H(x)c_2$, $h(\rho) = \rho(1-\rho)$ and set $F(x,\rho) = c(x)h(\rho)$ for the flux function of the scalar conservation law (2.17) and $\widetilde{F}(x,\rho) = c(x)f(\rho)$ for the flux function of the particle system, where f is given by (2.7). (The difference between F and $\widetilde{F}$ is that the latter is $-\infty$ outside $0 \le \rho \le 1$.)

Adimurthi and Gowda (2003) prove that there exists a solution to the corresponding Hamilton-Jacobi equation

$$\begin{cases} V_t + c_1 h(V_x) = 0, & \text{if } x < 0, t > 0 \\ V_t + c_2 h(V_x) = 0, & \text{if } x > 0, t > 0 \\ V(x,0) = V_0(x) \end{cases} \tag{6.1}$$

such that V_x solves the scalar conservation law (2.17) with flux function $F(x,\rho)$ and V_x satisfies the entropy assumptions $(E_i), (E_b)$. $V(x,t)$ is given by

$$V(x,t) = \sup_{w(\cdot)} \left\{ V_0(w(0)) + \int_0^t (c(w(s))h)^*(w'(s))\,ds \right\}, \tag{6.2}$$

where the supremum is taken over piecewise linear paths $w \in C([0,t],\mathbb{R})$ that satisfy $w(t) = x$.

To apply the results of Adimurthi and Gowda (2003) to the profile $\rho(x,t)$ coming from our hydrodynamic limit, we only need to show that the variational descriptions match, in other words that we can replace F with $\widetilde{F}$ and the solution is still the same.

Proof of Theorem 2.7: Convex duality gives $(c(x)f)^*(y) = c(x)f^*(y/c(x))$ and so we can rewrite (2.9) as

$$v(x,t) = \sup_{w(\cdot)} \left\{ v_0(w(0)) + \int_0^t (c(w(s))f)^*(w'(s))\,ds \right\}. \tag{6.3}$$

Observe that for all $y \in \mathbb{R}$

$$(c(x)f)^*(y) \ge (c(x)h)^*(y), \tag{6.4}$$

with equality if and only if $y \in [-c_1, c_2]$. Since the supremum in (6.2) and (6.3) is taken over the same set of paths, (6.4) implies that

$$V(x,t) \le v(x,t). \tag{6.5}$$

The proof of the theorem is now reduced to proving that the supremum in (6.3) is achieved when $w'(s)c(w(s))^{-1} \in [-1,1]$, giving $V(x,t) = v(x,t)$.

To this end we rewrite $v(x,t)$ once more, this time as

$$v(x,t) = \max\{R_+(x,t), L_+(x,t)\}\mathbf{1}\{x \ge 0\} + \max\{R_-(x,t), L_-(x,t)\}\mathbf{1}\{x < 0\}$$

where the functions $R_\pm$, $L_\pm$ (as in the proof of Corollary 2.4) are defined by

$$R_+(x,t) = \sup_{q>0} \{v_0(q) - I(x,t,q)\}, \text{ if } x > 0, \tag{6.6}$$

$$L_-(x,t) = \sup_{q<0} \{v_0(q) - I(x,t,q)\}, \text{ if } x < 0, \tag{6.7}$$

where $I(x,t,q)$ is as in (5.2), and

$$L_+(x,t) = \sup_{q<0, u\in[0,t]} \left\{ v_0(q) - c_1 u g\left(\frac{-q}{uc_1}\right) - c_2(t-u) g\left(\frac{x}{(t-u)c_2}\right) \right\} \quad \text{if } x \ge 0, \tag{6.8}$$

and

$$R_-(x,t) = \sup_{q>0, u\in[0,t]} \left\{ v_0(q) - c_2 u g\left(\frac{-q}{uc_2}\right) - c_1(t-u) g\left(\frac{x}{(t-u)c_1}\right) \right\}, \quad x \le 0. \tag{6.9}$$

It suffices to show that the suprema that define $R_\pm$, $L_\pm$ are achieved when

$$w'(s)c(w(s))^{-1} \in [-1,1]. \tag{6.10}$$

We show this for L_+. The remaining cases are similar. In (6.8), as before, u is the time for which $w(u) = 0$. Let $\Phi(u,q)$ denote the expression in braces in (6.8) with q-derivative

$$\Phi_q(u,q) = \begin{cases} \rho_0(q) - \frac{1}{2} - \frac{q}{2uc_1}, & -uc_1 \le q < 0 \\ \rho_0(q), & q < -uc_1. \end{cases} \tag{6.11}$$

Observe that if $\Phi_q(u,q) = 0$ for some $q^* = q^*(u)$ then also q^* maximizes Φ. Otherwise the maximum is achieved at 0 and we are reduced to a different case. Assume that q^* exists. Then by (6.11)

$$\frac{-q^*}{u} = (1 - 2\rho_0(q^*))c_1 < c_1. \tag{6.12}$$

Therefore, the slope of the first segment of the maximizing path w satisfies (6.10).

The slope of the second segment is $x(t-u)^{-1}$. Assume that the piecewise linear path w defined by u and q^* is the one that achieves the supremum. Also assume $u > t - xc_2^{-1}$. Consider the path $\tilde{w}$ with $\tilde{w}(0) = q^*$ and $\tilde{w}(t - xc_2^{-1}) = 0$. Since g is decreasing, we only increase the value of Φ. Hence the supremum that gives L_+ cannot be achieved on w and this gives the desired contradiction. □

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