ALEA, Lat. Am. J. Probab. Math. Stat. 8, 1–17 (2011)

Description of limits of ranges of iterations of stochastic integral mappings of infinitely divisible distributions

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Abstract. For infinitely divisible distributions ρ on $\mathbb{R}^d$ the stochastic integral mapping $\Phi_f\,\rho$ is defined as the distribution of improper stochastic integral $\int_0^{\infty-} f(s)dX_s^{(\rho)}$, where $f(s)$ is a non-random function and $\{X_s^{(\rho)}\}$ is a Lévy process on $\mathbb{R}^d$ with distribution ρ at time 1. For three families of functions f with parameters, the limits of the nested sequences of the ranges of the iterations Φ_f^n are shown to be some subclasses, with explicit description, of the class L_∞ of completely selfdecomposable distributions. In the critical case of parameter 1, the notion of weak mean 0 plays an important role. Examples of f with different limits of the ranges of Φ_f^n are also given.

1. Introduction

Let $ID = ID(\mathbb{R}^d)$ be the class of infinitely divisible distributions on $\mathbb{R}^d$, where d is a fixed finite dimension. For a real-valued locally square-integrable function $f(s)$ on $\mathbb{R}_+ = [0,\infty)$, let

$$\Phi_f\,\rho = \mathcal{L}\left(\int_0^{\infty-} f(s)dX_s^{(\rho)}\right),$$

the law of the improper stochastic integral $\int_0^{\infty-} f(s)dX_s^{(\rho)}$ with respect to the Lévy process $\{X_s^{(\rho)}\colon s \geq 0\}$ on $\mathbb{R}^d$ with $\mathcal{L}(X_1^{(\rho)}) = \rho$. This integral is the limit in probability of $\int_0^t f(s)dX_s^{(\rho)}$ as $t \to \infty$. The domain of Φ_f, denoted by $\mathfrak{D}(\Phi_f)$, is the class of $\rho \in ID$ such that this limit exists. The range of Φ_f is denoted by $\mathfrak{R}(\Phi_f)$. If $f(s) = 0$ for $s \in (s_0,\infty)$, then $\Phi_f\,\rho = \mathcal{L}\big(\int_0^{s_0} f(s)dX_s^{(\rho)}\big)$ and $\mathfrak{D}(\Phi_f) = ID$. For many choices of f, the description of $\mathfrak{R}(\Phi_f)$ is known; they are quite diverse. A

Received by the editors August 31, 2010; accepted November 23, 2010.
2000 *Mathematics Subject Classification.* 60E07, 60G51, 60H05.
Key words and phrases. completely selfdecomposable distribution, infinitely divisible distribution, Lévy process, selfdecomposable distribution, stochastic integral mapping.

seminal example is $\mathfrak{R}(\Phi_f) = L = L(\mathbb{R}^d)$, the class of selfdecomposable distributions on $\mathbb{R}^d$, for $f(s) = e^{-s}$ (Wolfe, 1982, Sato, 1999, Rocha-Arteaga and Sato, 2003). The iteration Φ_f^n is defined by $\Phi_f^1 = \Phi_f$ and $\Phi_f^{n+1}\rho = \Phi_f(\Phi_f^n\rho)$ with $\mathfrak{D}(\Phi_f^{n+1}) = \{\rho \in \mathfrak{D}(\Phi_f^n) \colon \Phi_f^n\rho \in \mathfrak{D}(\Phi_f)\}$. Then

$$ID \supset \mathfrak{R}(\Phi_f) \supset \mathfrak{R}(\Phi_f^2) \supset \cdots .$$

We define the limit class

$$\mathfrak{R}_\infty(\Phi_f) = \bigcap_{n=1}^{\infty} \mathfrak{R}(\Phi_f^n).$$

If $f(s) = e^{-s}$, then $\mathfrak{R}(\Phi_f^n)$ is the class of n times selfdecomposable distributions and $\mathfrak{R}_\infty(\Phi_f)$ is the class L_∞ of completely selfdecomposable distributions, which is the smallest class that is closed under convolution and weak convergence and contains all stable distributions on $\mathbb{R}^d$. This sequence and the class L_∞ were introduced by Urbanik (1973) and studied by Sato (1980) and others. If $f(s) = (1-s)1_{[0,1]}(s)$, then $\mathfrak{R}_\infty(\Phi_f) = L_\infty$, which was established by Jurek (2004) and Maejima and Sato (2009); in this case $\mathfrak{R}(\Phi_f)$ is the class of s-selfdecomposable distributions in the terminology of Jurek (1985). The paper of Maejima and Sato (2009) showed $\mathfrak{R}_\infty(\Phi_f) = L_\infty$ in many cases including (1) $f(s) = (-\log s)1_{[0,1]}(s)$, (2) $s = \int_{f(s)}^\infty u^{-1}e^{-u}du$ $(0 < s < \infty)$, (3) $s = \int_{f(s)}^\infty e^{-u^2}du$ $(0 < s < s_0 = \sqrt{\pi}/2)$. The classes $\mathfrak{R}(\Phi_f)$ corresponding to (1)–(3) are the Goldie–Steutel–Bondesson class B, the Thorin class T (see Barndorff-Nielsen et al., 2006), and the class G of generalized type G distributions, respectively. These results pose a problem what classes other than L_∞ can appear as $\mathfrak{R}_\infty(\Phi_f)$ in general.

For $-\infty < \alpha < 2$, $p > 0$, and $q > 0$, we consider the three families of functions $\bar{f}_{p,\alpha}(s)$, $l_{q,\alpha}(s)$, and $f_\alpha(s)$ as in [S] (we refer to Sato 2010 as [S]). We define $\bar{\Phi}_{p,\alpha}$, $\Lambda_{q,\alpha}$, and Ψ_α to be the mappings Φ_f with $f(s)$ equal to these functions, respectively. In this paper we will prove the following theorem on the classes $\mathfrak{R}_\infty(\Phi_f)$ of those mappings. The case $\alpha = 1$ is delicate. There the notion of weak mean 0 plays an important role.

Theorem 1.1. (i) *If $\alpha \le 0$, $p \ge 1$, and $q > 0$, then*

$$\mathfrak{R}_\infty(\bar{\Phi}_{p,\alpha}) = \mathfrak{R}_\infty(\Lambda_{q,\alpha}) = \mathfrak{R}_\infty(\Psi_\alpha) = L_\infty.$$

(ii) *If $0 < \alpha < 1$, $p \ge 1$, and $q > 0$, then*

$$\mathfrak{R}_\infty(\bar{\Phi}_{p,\alpha}) = \mathfrak{R}_\infty(\Lambda_{q,\alpha}) = \mathfrak{R}_\infty(\Psi_\alpha) = L_\infty^{(\alpha,2)}.$$

(iii) *If $\alpha = 1$, $p \ge 1$, and $q = 1$, then*

$$\mathfrak{R}_\infty(\bar{\Phi}_{p,1}) = \mathfrak{R}_\infty(\Lambda_{1,1}) = \mathfrak{R}_\infty(\Psi_1) = L_\infty^{(1,2)} \cap \{\mu \in ID \colon \mu \text{ has weak mean } 0\}.$$

(iv) *If $1 < \alpha < 2$, $p \ge 1$, and $q > 0$, then*

$$\mathfrak{R}_\infty(\bar{\Phi}_{p,\alpha}) = \mathfrak{R}_\infty(\Lambda_{q,\alpha}) = \mathfrak{R}_\infty(\Psi_\alpha) = L_\infty^{(\alpha,2)} \cap \{\mu \in ID \colon \mu \text{ has mean } 0\}.$$

Let us explain the concepts used in the statement of Theorem 1.1. A distribution $\mu \in ID$ belongs to L_∞ if and only if its Lévy measure ν_μ is represented as

$$\nu_\mu(B) = \int_{(0,2)} \Gamma_\mu(d\beta) \int_S \lambda_\beta^\mu(d\xi) \int_0^\infty 1_B(r\xi) r^{-\beta-1} dr$$

for Borel sets B in $\mathbb{R}^d$, where Γ_μ is a measure on the open interval $(0,2)$ satisfying $\int_{(0,2)} (\beta^{-1} + (2-\beta)^{-1}) \Gamma_\mu(d\beta) < \infty$ and $\{\lambda_\beta^\mu \colon \beta \in (0,2)\}$ is a measurable family of probability measures on $S = \{\xi \in \mathbb{R}^d \colon |\xi| = 1\}$. This Γ_μ is uniquely determined by ν_μ and $\{\lambda_\beta^\mu\}$ is determined by ν_μ up to β of Γ_μ-measure 0 (see [S] and Sato, 1980). For a Borel subset E of the interval $(0,2)$, the class L_∞^E denotes, as in [S], the totality of $\mu \in L_\infty$ such that Γ_μ is concentrated on E. The classes $L_\infty^{(\alpha,2)}$ and $L_\infty^{(1,2)}$ appearing in Theorem 1.1 are for $E = (\alpha, 2)$ and $(1,2)$, respectively. Let $C_\mu(z)$ $(z \in \mathbb{R}^d)$, A_μ, and ν_μ be the cumulant function, the Gaussian covariance matrix, and the Lévy measure of $\mu \in ID$. A distribution $\mu \in ID$ is said to have weak mean m_μ if $\lim_{a\to\infty} \int_{1<|x|\le a} x\nu_\mu(dx)$ exists in $\mathbb{R}^d$ and if

$$C_\mu(z) = -\tfrac{1}{2}\langle z, A_\mu z\rangle + \lim_{a\to\infty} \int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle)\nu_\mu(dx) + i\langle m_\mu, z\rangle.$$

This concept was introduced by [S] recently. If $\mu \in ID$ has mean m_μ (that is, $\int_{\mathbb{R}^d} |x|\mu(dx) < \infty$ and $\int_{\mathbb{R}^d} x\mu(dx) = m_\mu$), then μ has weak mean m_μ (Remark 3.8 of [S]).

Section 2 begins with exact definitions of f_α, $\bar{f}_{p,\alpha}$, and $l_{q,\alpha}$ and expounds existing results concerning $\mathfrak{R}_\infty(\Phi_f)$. Then, in Section 3, we will prove Theorem 1.1. In Section 4 we will give examples of Φ_f for which $\mathfrak{R}_\infty(\Phi_f)$ is different from those appearing in Theorem 1.1. Section 5 gives some concluding remarks.

2. Known results

Let $-\infty < \alpha < 2$, $p > 0$, and $q > 0$ and let

$$\bar{g}_{p,\alpha}(t) = \frac{1}{\Gamma(p)} \int_t^1 (1-u)^{p-1} u^{-\alpha-1} du, \quad 0 < t \le 1,$$
$$j_{q,\alpha}(t) = \frac{1}{\Gamma(q)} \int_t^1 (-\log u)^{q-1} u^{-\alpha-1} du, \quad 0 < t \le 1,$$
$$g_\alpha(t) = \int_t^\infty u^{-\alpha-1} e^{-u} du, \quad 0 < t \le \infty.$$

Let $t = \bar{f}_{p,\alpha}(s)$ for $0 \le s < \bar{g}_{p,\alpha}(0+)$, $t = l_{q,\alpha}(s)$ for $0 \le s < j_{q,\alpha}(0+)$, and $t = f_\alpha(s)$ for $0 \le s < g_\alpha(0+)$ be the inverse functions of $s = \bar{g}_{p,\alpha}(t)$, $s = j_{q,\alpha}(t)$, and $s = g_\alpha(t)$, respectively. They are continuous, strictly decreasing functions. If $\alpha < 0$, then $\bar{g}_{p,\alpha}(0+)$, $j_{q,\alpha}(0+)$, and $g_\alpha(0+)$ are finite and we define $\bar{f}_{p,\alpha}(s)$, $l_{q,\alpha}(s)$, and $f_\alpha(s)$ to be zero for $s \ge \bar{g}_{p,\alpha}(0+)$, $j_{q,\alpha}(0+)$, and $g_\alpha(0+)$, respectively. Let $\bar{\Phi}_{p,\alpha}$, $\Lambda_{q,\alpha}$, and Ψ_α denote Φ_f with $f = \bar{f}_{p,\alpha}$, $l_{q,\alpha}$, and f_α, respectively. Let $K_{p,\alpha}$, $L_{q,\alpha}$, and $K_{\infty,\alpha}$ be the ranges of $\bar{\Phi}_{p,\alpha}$, $\Lambda_{q,\alpha}$, and Ψ_α, respectively. These mappings and classes were systematically studied in Sato (2006) and [S]. In the following cases we have explicit expressions:

$$\bar{f}_{1,\alpha}(s) = l_{1,\alpha}(s) = \begin{cases} (1-|\alpha|s)^{1/|\alpha|}\, 1_{[0,1/|\alpha|]}(s) & \text{for } \alpha < 0, \\ e^{-s} & \text{for } \alpha = 0, \\ (1+\alpha s)^{-1/\alpha} & \text{for } \alpha > 0, \end{cases}$$
$$\bar{f}_{p,-1}(s) = \{1 - (\Gamma(p+1)\, s)^{1/p}\}\, 1_{[0,1/\Gamma(p+1)]}(s), \quad p > 0,$$
$$l_{q,0}(s) = \exp(-(\Gamma(q+1)s)^{1/q}), \quad q > 0,$$
$$f_{-1}(s) = (-\log s)\, 1_{[0,1]}(s).$$

In the case $p = q = 1$ we have $\bar{\Phi}_{1,\alpha} = \Lambda_{1,\alpha}$ and $K_{1,\alpha} = L_{1.\alpha}$, which are in essence treated earlier by Jurek (1988, 1989); $\bar{\Phi}_{1,\alpha} = \Lambda_{1,\alpha}$ were studied by Maejima et al. (2010), and Maejima and Ueda (2010b) with the notation Φ_α. The mapping $\Lambda_{q,0}$ and the class $L_{q,0}$ with $q = 1, 2, \ldots$ coincide with those introduced by Jurek (1983) in a different form. A variant of Ψ_α is found in Grigelionis (2007).

A related family is

$$G_{\alpha,\beta}(t) = \int_t^\infty u^{-\alpha-1} e^{-u^\beta} du, \quad 0 < t \le \infty,$$

for $-\infty < \alpha < 2$ and $\beta > 0$. Let $t = G^*_{\alpha,\beta}(s)$ for $0 \le s < G_{\alpha,\beta}(0+)$ be the inverse function of $s = G_{\alpha,\beta}(t)$. If $\alpha < 0$, then $G_{\alpha,\beta}(0+)$ is finite and we define $G^*_{\alpha,\beta}(s) = 0$ for $s \ge G_{\alpha,\beta}(0+)$. Let $\Psi_{\alpha,\beta}$ denote Φ_f with $f = G^*_{\alpha,\beta}$. This was introduced by Maejima and Nakahara (2009) and studied by Maejima and Ueda (2010b) and, in the level of Lévy measures, by Maejima et al. (2011b). Clearly, $\Psi_{\alpha,1} = \Psi_\alpha$. We have

$$G^*_{-\beta,\beta}(s) = (-\log \beta s)^{1/\beta}\, 1_{[0,1/\beta]}(s), \quad \beta > 0.$$

Earlier the mappings $\Psi_{0,2}$ and $\Psi_{-\beta,\beta}$ were treated in Aoyama et al. (2008) and Aoyama et al. (2010), respectively; $\Psi_{-2,2}$ appeared also in Arizmendi et al. (2010).

Maejima and Sato (2009) proved the following two results.

Proposition 2.1. *Let $0 < t_0 \le \infty$. Let $h(u)$ be a positive decreasing function on $(0, t_0)$ such that $\int_0^{t_0} (1 + u^2) h(u) du < \infty$. Let $g(t) = \int_t^{t_0} h(u) du$ for $0 < t \le t_0$. Let $t = f(s)$, $0 \le s < g(0+)$, be the inverse function of $s = g(t)$ and let $f(s) = 0$ for $s \ge g(0+)$. Then $\mathfrak{R}_\infty(\Phi_f) = L_\infty$.*

Proposition 2.2. $\mathfrak{R}_\infty(\Psi_0) = L_\infty$.

It follows from Proposition 2.1 that $\mathfrak{R}_\infty(\Phi_f) = L_\infty$ for $f = \bar{f}_{p,\alpha}$ with $p \ge 1$ and $-1 \le \alpha < 0$, $f = l_{q,\alpha}$ with $q \ge 1$ and $-1 \le \alpha < 0$, $f = f_\alpha$ with $-1 \le \alpha < 0$, and $f = G^*_{\alpha,\beta}$ with $-1 \le \alpha < 0$ and $\beta > 0$. The function f_0 for $\Psi_0 = \Phi_{f_0}$ does not satisfy the condition in Proposition 2.1 but Proposition 2.2 is proved using the identity $\Psi_0 = \Lambda_{1,0} \Psi_{-1} = \Psi_{-1} \Lambda_{1,0}$.

In November 2007–January 2008, Sato wrote four memos, showing the part related to Ψ_α in (ii), (iii), and (iv) of Theorem 1.1. But assertion (iii) for Ψ_1 was shown with the set $\{\mu \in ID \colon \mu \text{ has weak mean } 0\}$ replaced by the set of $\mu \in L_\infty$ satisfying some condition related to (4.6) of Sato (2006). At that time the concept of weak mean was not yet introduced. Those memos showed that some proper subclasses of L_∞ appear as limit classes $\mathfrak{R}_\infty(\Phi_f)$.

Sato's memos were referred to by a series of papers Maejima and Ueda (2009a,b, 2010b,c) and Ichifuji et al. (2010). In Maejima and Ueda (2010a,c) they characterized $\mathfrak{R}(\Lambda^n_{1,\alpha})$, $-\infty < \alpha < 2$, for $n = 1, 2, \ldots$, in relation to a decomposability which they called α-selfdecomposability, and found $\mathfrak{R}_\infty(\Lambda_{1,\alpha})$ for $-\infty < \alpha < 2$. But the description of $\mathfrak{R}_\infty(\Lambda_{1,1})$ was similar to Sato's memos. In Maejima and Ueda (2010b) they showed that $\Psi_{\alpha,\beta}$ with $-\infty < \alpha < 2$ and $\beta > 0$ satisfies $\mathfrak{R}_\infty(\Psi_{\alpha,\beta}) = \mathfrak{R}_\infty(\Psi_\alpha)$, under the condition that $\alpha \ne 1 + n\beta$ for $n = 0, 1, 2, \ldots$. For $\Psi_{0,2}$ and $\Psi_{-\beta,\beta}$ with $\beta > 0$, this result was earlier obtained by Aoyama et al. (2010). Further it was shown in Maejima and Ueda (2009a) that $\mathfrak{R}_\infty(\Psi_\alpha) = \mathfrak{R}_\infty(\Lambda_{1,\alpha})$ for $-\infty < \alpha < 2$. An application of the result in Maejima and Ueda (2010c) was given in Ichifuji et al. (2010).

If $f(s) = b\,1_{[0,a]}(s)$ for some $a > 0$ and $b \neq 0$, then it is clear that $\mathfrak{R}_\infty(\Phi_f) = \mathfrak{R}(\Phi_f) = ID$. A first example of $\mathfrak{R}_\infty(\Phi_f)$ satisfying $L_\infty \subsetneqq \mathfrak{R}_\infty(\Phi_f) \subsetneqq ID$ was given by Maejima and Ueda (2009b); they showed that if $f(s) = b^{-[s]}$ for a given $b > 1$ with $[s]$ being the largest integer not exceeding s, then $\mathfrak{R}_\infty(\Phi_f) = L_\infty(b)$, the smallest class that is closed under convolution and weak convergence and contains all semi-stable distributions on $\mathbb{R}^d$ with b as a span; in this case $\mathfrak{R}(\Phi_f)$ is the class $L(b)$ of semi-selfdecomposable distributions on $\mathbb{R}^d$ with b as a span. See Sato (1999) for the definitions of semi-stability, semi-selfdecomposability, and span. See Maejima et al. (2000) for characterization of $L_\infty(b)$ as the limit of the class $L_n(b)$ of n times b-semi-selfdecomposable distributions and for description of the Lévy measures of distributions in $L_\infty(b)$. Recall that $L_\infty \subsetneqq L_\infty(b)$.

The following result is deduced easily from [S].

Proposition 2.3. *The assertions related to* $\Lambda_{q,\alpha}$ *in* (i), (ii), *and* (iv) *of Theorem 1.1 are true.*

Indeed, in [S], Theorem 7.3 says that $\Lambda_{q+q',\alpha} = \Lambda_{q',\alpha}\Lambda_{q,\alpha}$ for $\alpha \in (-\infty, 1)\cup(1, 2)$, $q > 0$, and $q' > 0$, and hence $\Lambda_{q,\alpha}^n = \Lambda_{nq,\alpha}$, and further, Theorem 7.11 combined with Proposition 6.8 describes, for $\alpha \in (-\infty, 1) \cup (1, 2)$, the class $\bigcap_{q>0} L_{q,\alpha}$, which equals $\bigcap_{q=1,2...} L_{q,\alpha}$.

3. Proof of Theorem 1.1

We prepare some lemmas. We use the terminology in [S] such as radial decomposition, monotonicity of order p, and complete monotonicity. In particular, our complete monotonicity implies vanishing at infinity. The location parameter γ_μ of $\mu \in ID$ is defined by

$$C_\mu(z) = -\tfrac{1}{2}\langle z, A_\mu z\rangle + \int_{\mathbb{R}^d} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle 1_{\{|x|\le 1\}}(x))\nu_\mu(dx) + i\langle \gamma_\mu, z\rangle.$$

Let $K^{\mathrm{e}}_{p,\alpha}$ [resp. $K^{\mathrm{e}}_{\infty,\alpha}$] denote the class of distributions $\mu \in ID$ for which there exist $\rho \in ID$ and a function q_t from $[0,\infty)$ into $\mathbb{R}^d$ such that $\int_0^t f_{p,\alpha}(s)dX_s^{(\rho)} - q_t$ [resp. $\int_0^t f_\alpha(s)dX_s^{(\rho)} - q_t$] converges in probability as $t \to \infty$ and the limit has distribution μ.

Lemma 3.1. *Let* $-\infty < \alpha < 2$ *and* $p > 0$*. The domains of* $\bar{\Phi}_{p,\alpha}$ *and* Ψ_α *are as follows:*

$$\mathfrak{D}(\bar{\Phi}_{p,\alpha}) = \mathfrak{D}(\Psi_\alpha) = \begin{cases} ID & \text{for } \alpha < 0,\\ \{\rho \in ID\colon \int_{|x|>1} \log|x|\,\nu_\rho(dx) < \infty\} & \text{for } \alpha = 0,\\ \{\rho \in ID\colon \int_{|x|>1} |x|^\alpha\,\nu_\rho(dx) < \infty\} & \text{for } 0 < \alpha < 1,\\ \{\rho \in ID\colon \int_{|x|>1} |x|\,\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\,\rho(dx) = 0, & \\ \qquad \lim_{a\to\infty}\int_1^a s^{-1}ds\int_{|x|>s} x\,\nu_\rho(dx) \text{ exists in } \mathbb{R}^d\} & \text{for } \alpha = 1,\\ \{\rho \in ID\colon \int_{|x|>1} |x|^\alpha\,\nu_\rho(dx) < \infty,\ \int_{\mathbb{R}^d} x\,\rho(dx) = 0\} & \text{for } 1 < \alpha < 2. \end{cases}$$

This is found in Sato (2006) or Theorems 4.2, 4.4 and Propositions 4.6, 5.1 of [S].

Lemma 3.2. *Let* $-\infty < \alpha < 2$ *and* $p > 0$*. The class* $K^{\mathrm{e}}_{p,\alpha}$ *[resp.* $K^{\mathrm{e}}_{\infty,\alpha}$*] is the totality of* $\mu \in ID$ *for which* ν_μ *has a radial decomposition* $(\lambda_\mu(d\xi), u^{-\alpha-1}\,k_\xi^\mu(u)du)$

such that $k_\xi^\mu(u)$ *is measurable in* (ξ,u) *and, for* λ_μ*-a. e.* ξ*, monotone of order* p [*resp. completely monotone*] *on* $\mathbb{R}_+^\circ=(0,\infty)$ *in* u*. The classes* $K_{p,\alpha}$ *and* $K_{\infty,\alpha}$*, that is, the ranges of* $\bar\Phi_{p,\alpha}$ *and* Ψ_α*, are as follows:*

$$K_{p,\alpha}=\begin{cases}K^{\mathrm{e}}_{p,\alpha} & \text{for } -\infty<\alpha<1,\\ \{\mu\in K^{\mathrm{e}}_{p,1}\colon \mu \text{ has weak mean } 0\} & \text{for } \alpha=1,\\ \{\mu\in K^{\mathrm{e}}_{p,\alpha}\colon \mu \text{ has mean } 0\} & \text{for } 1<\alpha<2,\end{cases}$$

$$K_{\infty,\alpha}=\begin{cases}K^{\mathrm{e}}_{\infty,\alpha} & \text{for } -\infty<\alpha<1,\\ \{\mu\in K^{\mathrm{e}}_{\infty,1}\colon \mu \text{ has weak mean } 0\} & \text{for } \alpha=1,\\ \{\mu\in K^{\mathrm{e}}_{\infty,\alpha}\colon \mu \text{ has mean } 0\} & \text{for } 1<\alpha<2.\end{cases}$$

See Theorems 4.18, 5.8, and 5.10 of [S]. Note that if μ is in $K^{\mathrm{e}}_{\infty,\alpha}$ or $K^{\mathrm{e}}_{p,\alpha}$ with $0<\alpha<2$, then $\int_{\mathbb{R}^d}|x|^\beta\mu(dx)<\infty$ for $\beta\in(0,\alpha)$ (Propositions 4.16 and 5.13 of [S]). It follows from the lemma above that $K^{\mathrm{e}}_{p,\alpha}\supset K^{\mathrm{e}}_{p',\alpha}$ and $K_{p,\alpha}\supset K_{p',\alpha}$ for $p<p'$ and that $K^{\mathrm{e}}_{\infty,\alpha}=\bigcap_{p>0}K^{\mathrm{e}}_{p,\alpha}$ and $K_{\infty,\alpha}=\bigcap_{p>0}K_{p,\alpha}$. The notation of $K^{\mathrm{e}}_{\infty,\alpha}$ and $K_{\infty,\alpha}$ comes from this property.

Lemma 3.3. *Let* $\rho\in L_\infty$*.*
(i) *Let* $0<\alpha<2$*. Then* $\int_{\mathbb{R}^d}|x|^\alpha\rho(dx)<\infty$ *if and only if* $\Gamma_\rho((0,\alpha])=0$ *and* $\int_{(\alpha,2)}(\beta-\alpha)^{-1}\,\Gamma_\rho(d\beta)<\infty$*.*
(ii) $\int_{|x|>1}\log|x|\,\rho(dx)<\infty$ *if and only if* $\int_{(0,2)}\beta^{-2}\,\Gamma_\rho(d\beta)<\infty$*.*

Proof: Assertion (i) is shown in Proposition 7.15 of [S]. Since

$$\begin{aligned}\int_{|x|>1}\log|x|\,\nu_\rho(dx)&=\int_{(0,2)}\Gamma_\rho(d\beta)\int_S\lambda_\beta^\rho(d\xi)\int_1^\infty(\log|r\xi|)r^{-\beta-1}dr\\&=\int_{(0,2)}\Gamma_\rho(d\beta)\int_1^\infty(\log r)r^{-\beta-1}dr=\int_{(0,2)}\beta^{-2}\Gamma_\rho(d\beta),\end{aligned}$$

assertion (ii) follows. □

Lemma 3.4. *Let* μ *and* ρ *be in* $L_\infty^{(1,2)}$*. Suppose that* $\Gamma_\rho(d\beta)=(\beta-1)b(\beta)\Gamma_\mu(d\beta)$ *and* $\lambda_\beta^\rho=\lambda_\beta^\mu$ *with a nonnegative measurable function* $b(\beta)$ *such that* $(\beta-1)^{-1}(b(\beta)-1)$ *is bounded on* $(1,2)$*. Then,* $\int_1^a s^{-1}ds\int_{|x|>s}x\nu_\rho(dx)$ *is convergent in* $\mathbb{R}^d$ *as* $a\to\infty$ *if and only if* μ *has weak mean* m_μ *for some* m_μ*.*

Proof: Notice that $b(\beta)$ is bounded on $(1,2)$ and that $\int_{|x|>1}|x|\nu_\rho(dx)<\infty$ by Lemma 3.3. We have

$$\begin{aligned}\int_1^a s^{-1}ds\int_{|x|>s}x\nu_\rho(dx)&=\int_1^a s^{-1}ds\int_{(1,2)}\Gamma_\rho(d\beta)\int_S\xi\lambda_\beta^\rho(d\xi)\int_s^\infty r^{-\beta}dr\\&=\int_{(1,2)}b(\beta)\Gamma_\mu(d\beta)\int_S\xi\lambda_\beta^\mu(d\xi)\int_1^a s^{-\beta}ds=I_1\quad\text{(say)}\end{aligned}$$

and

$$\int_{1<|x|\le a}x\nu_\mu(dx)=\int_{(1,2)}\Gamma_\mu(d\beta)\int_S\xi\lambda_\beta^\mu(d\xi)\int_1^a r^{-\beta}dr=I_2\quad\text{(say)}.$$

Hence

$$I_1-I_2=\int_{(1,2)}(b(\beta)-1)\Gamma_\mu(d\beta)\int_S\xi\lambda_\beta^\mu(d\xi)\int_1^a r^{-\beta}dr.$$

Since

$$\left|(b(\beta)-1)\int_1^a r^{-\beta}dr\right| \le (\beta-1)^{-1}|b(\beta)-1|$$

and $\int_1^a r^{-\beta}dr$ tends to $(\beta-1)^{-1}$, $I_1 - I_2$ is convergent in $\mathbb{R}^d$ as $a\to\infty$. Hence I_1 is convergent if and only if I_2 is convergent. □

Lemma 3.5. *Let f and h be locally square-integrable functions on $\mathbb{R}_+$. Assume that there is $s_0\in(0,\infty)$ such that $h(s)=0$ for $s\ge s_0$ and that Φ_h is one-to-one. Then $\Phi_f\Phi_h=\Phi_h\Phi_f$.*

Proof: Let $f_t(s)=f(s)\,1_{[0,t]}(s)$. Then $\Phi_{f_t}\Phi_h=\Phi_h\Phi_{f_t}$ by Lemma 3.6 of Maejima and Sato (2009). Let $\rho\in\mathfrak{D}(\Phi_f)$. Then $\Phi_{f_t}\rho\to\Phi_f\rho$ as $t\to\infty$ by the definition of Φ_f. Hence $\Phi_h\Phi_{f_t}\rho\to\Phi_h\Phi_f\rho$ by (3.1) of Maejima and Sato (2009). It follows that $\Phi_{f_t}\Phi_h\rho\to\Phi_h\Phi_f\rho$. Since the convergence of $\int_0^t f(s)dX_s^{(\Phi_h\rho)}$ in law implies its convergence in probability, $\Phi_h\rho$ is in $\mathfrak{D}(\Phi_f)$ and $\Phi_f\Phi_h\rho=\Phi_h\Phi_f\rho$. Conversely, suppose that $\rho\in ID$ satisfies $\Phi_h\rho\in\mathfrak{D}(\Phi_f)$. Then $\Phi_h\Phi_{f_t}\rho=\Phi_{f_t}\Phi_h\rho\to\Phi_f\Phi_h\rho$ as $t\to\infty$. Looking at (3.8) of Maejima and Sato (2009), we see that $\int_0^{s_0} h(s)\neq 0$ from the one-to-one property of Φ_h. Hence $\{\Phi_{f_t}\rho\colon t>0\}$ is precompact by the argument in pp. 138–139 of Maejima and Sato (2009). Hence, again from the one-to-one property of Φ_h, $\Phi_{f_t}\rho$ is convergent as $t\to\infty$, that is, $\rho\in\mathfrak{D}(\Phi_f)$. □

Lemma 3.6. *Let f be locally square-integrable on $\mathbb{R}_+$. Suppose that there is $\beta\ge 0$ such that any $\mu\in\mathfrak{R}(\Phi_f)$ has Lévy measure ν_μ with a radial decomposition $(\lambda_\mu(d\xi), u^\beta l_\xi^\mu(u)du)$ where $l_\xi^\mu(u)$ is measurable in (ξ,u) and decreasing on $\mathbb{R}_+^\circ$ in u. Then*

$$\mathfrak{R}_\infty(\Phi_f)\subset\mathfrak{R}_\infty(\Lambda_{1,-\beta-1})=L_\infty.$$

Proof: Clearly $l_\xi^\mu\ge 0$ for λ_μ-a. e. ξ. Since $\int_{|x|>1}\nu_\mu(dx)<\infty$, we have $\lim_{u\to\infty} l_\xi^\mu(u)=0$ for λ_μ-a. e. ξ. Hence we can modify $l_\xi^\mu(u)$ in such a way that $l_\xi^\mu(u)$ is monotone of order 1 in $u\in\mathbb{R}_+^\circ$. Recall that a function is monotone of order 1 on $\mathbb{R}_+^\circ$ if and only if it is decreasing, right-continuous, and vanishing at infinity (Proposition 2.11 of [S]). Then it follows from Theorem 4.18 or 6.12 of [S] that

$$\mathfrak{R}(\Phi_f)\subset\mathfrak{R}(\Lambda_{1,-\beta-1}). \tag{3.1}$$

Let us write $\Lambda=\Lambda_{1,-\beta-1}$ for simplicity. We have $\Phi_f\Lambda=\Lambda\Phi_f$ by virtue of Lemma 3.5, since Λ is one-to-one (Theorem 6.14 of [S]). If $\Phi_f\Lambda^n=\Lambda^n\Phi_f$ for some integer $n\ge 1$, then

$$\Phi_f\Lambda^{n+1}=\Phi_f\Lambda\Lambda^n=\Lambda\Phi_f\Lambda^n=\Lambda\Lambda^n\Phi_f=\Lambda^{n+1}\Phi_f.$$

Hence $\Phi_f\Lambda^n=\Lambda^n\Phi_f$ for $n=1,2,\ldots$. Now we claim that

$$\mathfrak{R}(\Phi_f^n)\subset\mathfrak{R}(\Lambda^n) \tag{3.2}$$

for $n=1,2,\ldots$. Indeed, this is true for $n=1$ by (3.1); if (3.2) is true for n, then any $\mu\in\mathfrak{R}(\Phi_f^{n+1})$ has expression

$$\mu=\Phi_f^{n+1}\rho=\Phi_f\Phi_f^n\rho=\Phi_f\Lambda^n\rho'=\Lambda^n\Phi_f\rho'=\Lambda^n\Lambda\rho''=\Lambda^{n+1}\rho''$$

for some $\rho\in\mathfrak{D}(\Phi_f^{n+1})$, $\rho'\in\mathfrak{D}(\Lambda^n)$ with $\Phi_f^n\rho=\Lambda^n\rho'$, and $\rho''\in\mathfrak{D}(\Lambda)$ with $\Phi_f\rho'=\Lambda\rho''$, which means (3.2) for $n+1$. It follows from (3.2) that $\mathfrak{R}_\infty(\Phi_f)\subset\mathfrak{R}_\infty(\Lambda)$. The equality $\mathfrak{R}_\infty(\Lambda)=L_\infty$ is from Proposition 2.3. □

Proof of the part related to $\mathfrak{R}_\infty(\Psi_\alpha)$ *in Theorem 1.1.* The result for $-1 \le \alpha \le 0$ is already known (see Propositions 2.1 and 2.2). But the proof below also includes this case. First, using Lemma 3.2, notice that Lemma 3.6 is applicable to $\Phi_f = \Psi_\alpha$ and $\beta = (-\alpha - 1) \vee 0$.

Case 1 ($-\infty < \alpha < 0$). We have $\mathfrak{D}(\Psi_\alpha) = ID$ in Lemma 3.1. Let us show that

$$\Psi_\alpha(L_\infty) = L_\infty. \tag{3.3}$$

Let $\rho \in L_\infty$ and $\mu = \Psi_\alpha \rho$. Then for $B \in \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the class of Borel sets in $\mathbb{R}^d$,

$$\begin{aligned}
\nu_\mu(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(f_\alpha(s)x)\nu_\rho(dx) = \int_0^\infty t^{-\alpha-1}e^{-t}dt \int_{\mathbb{R}^d} 1_B(tx)\nu_\rho(dx) \\
&= \int_0^\infty t^{-\alpha-1}e^{-t}dt \int_{(0,2)} \Gamma_\rho(d\beta) \int_S \lambda_\beta^\rho(d\xi) \int_0^\infty 1_B(tr\xi)r^{-\beta-1}dr \\
&= \int_{(0,2)} \Gamma(\beta-\alpha)\Gamma_\rho(d\beta) \int_S \lambda_\beta^\rho(d\xi) \int_0^\infty 1_B(u\xi)u^{-\beta-1}du.
\end{aligned}$$

Hence $\mu \in L_\infty$ with

$$\Gamma_\mu(d\beta) = \Gamma(\beta-\alpha)\Gamma_\rho(d\beta) \quad \text{and} \quad \lambda_\beta^\mu = \lambda_\beta^\rho. \tag{3.4}$$

Let us show the converse. Let $\mu \in L_\infty$. In order to find $\rho \in L_\infty$ satisfying $\Psi_\alpha\rho = \mu$, it suffices to choose Γ_ρ, λ_β^ρ, A_ρ, and γ_ρ such that (3.4) holds and

$$A_\mu = \int_0^\infty f_\alpha(s)^2 ds A_\rho, \tag{3.5}$$

$$\gamma_\mu = \int_0^{\infty-} f_\alpha(s)ds \left(\gamma_\rho + \int_{\mathbb{R}^d} x(1_{\{|f_\alpha(s)x|\le 1\}} - 1_{\{|x|\le 1\}})\nu_\rho(dx)\right) \tag{3.6}$$

(see Proposition 3.18 of [S]). This choice is possible, because $\inf_{\beta\in(0,2)} \Gamma(\beta-\alpha) > 0$, $\int_0^\infty f_\alpha(s)ds = \int_0^\infty t^{-\alpha}e^{-t}dt = \Gamma(1-\alpha)$, $\int_0^\infty f_\alpha(s)^2 ds = \int_0^\infty t^{1-\alpha}e^{-t}dt = \Gamma(2-\alpha)$, and

$$\begin{aligned}
&\int_0^\infty f_\alpha(s)ds \int_{\mathbb{R}^d} |x|\,|1_{\{|f_\alpha(s)x|\le 1\}} - 1_{\{|x|\le 1\}}|\,\nu_\rho(dx) \\
&\quad = \int_0^\infty t^{-\alpha}e^{-t}dt \int_{\mathbb{R}^d} |x|\,|1_{\{|tx|\le 1\}} - 1_{\{|x|\le 1\}}|\,\nu_\rho(dx) \\
&\quad = \int_0^1 t^{-\alpha}e^{-t}dt \int_{1<|x|\le 1/t} |x|\,\nu_\rho(dx) + \int_1^\infty t^{-\alpha}e^{-t}dt \int_{1/t<|x|\le 1} |x|\,\nu_\rho(dx) \\
&\quad = \int_{|x|>1} |x|\,\nu_\rho(dx) \int_0^{1/|x|} t^{-\alpha}e^{-t}dt + \int_{|x|\le 1} |x|\,\nu_\rho(dx) \int_{1/|x|}^\infty t^{-\alpha}e^{-t}dt < \infty,
\end{aligned}$$

since $\int_0^{1/|x|} t^{-\alpha}e^{-t}dt \sim (1-\alpha)^{-1}|x|^{\alpha-1}$ as $|x| \to \infty$ and $\int_{1/|x|}^\infty t^{-\alpha}e^{-t}dt \sim |x|^\alpha e^{-1/|x|}$ as $|x| \downarrow 0$. Therefore (3.3) is true. It follows that $\Psi_\alpha^n(L_\infty) = L_\infty$ for $n = 1, 2, \ldots$. Hence $\mathfrak{R}_\infty(\Psi_\alpha) \supset L_\infty$. On the other hand, $\mathfrak{R}_\infty(\Psi_\alpha) \subset L_\infty$ by virtue of Lemma 3.6.

Case 2 ($0 \le \alpha < 1$). Since $\mathfrak{D}(\Psi_\alpha)$ is as in Lemma 3.1, it follows from Lemma 3.3 that

$$L_\infty \cap \mathfrak{D}(\Psi_\alpha) = \begin{cases} \{\rho \in L_\infty \colon \int_{(0,2)} \beta^{-2}\,\Gamma_\rho(d\beta) < \infty\}, & \alpha = 0, \\ \{\rho \in L_\infty^{(\alpha,2)} \colon \int_{(\alpha,2)} (\beta-\alpha)^{-1}\,\Gamma_\rho(d\beta) < \infty\}, & 0 < \alpha < 1. \end{cases}$$

We have

$$\Psi_\alpha(L_\infty\cap\mathfrak{D}(\Psi_\alpha))=L_\infty^{(\alpha,2)},\tag{3.7}$$

where $L_\infty^{(0,2)}=L_\infty$. Indeed, if $\rho\in L_\infty\cap\mathfrak{D}(\Psi_\alpha)$ and $\mu=\Psi_\alpha\rho$, then we have $\mu\in L_\infty^{(\alpha,2)}$ and (3.4), using $\Gamma(\beta-\alpha)=(\beta-\alpha)^{-1}\Gamma(\beta-\alpha+1)$ for $0\le\alpha<1$. Conversely, if $\mu\in L_\infty^{(\alpha,2)}$, then we can find $\rho\in L_\infty\cap\mathfrak{D}(\Psi_\alpha)$ satisfying $\mu=\Psi_\alpha\rho$ in the same way as in Case 1; when $\alpha=0$, we have $\int_{(0,2)}\beta^{-2}\Gamma_\rho(d\beta)<\infty$ since $\Gamma_\rho(d\beta)=\beta(\Gamma(\beta+1))^{-1}\Gamma_\mu(d\beta)$ and $\int_{(0,2)}\beta^{-1}\Gamma_\mu(d\beta)<\infty$. Hence (3.7) holds. Now we have

$$\Psi_\alpha^n(L_\infty\cap\mathfrak{D}(\Psi_\alpha^n))=L_\infty^{(\alpha,2)}\tag{3.8}$$

for $n=1,2,\ldots$. Indeed, it is true for $n=1$ by (3.7) and, if (3.8) is true for n, then

$$\begin{aligned}L_\infty^{(\alpha,2)}&=\Psi_\alpha^n(L_\infty\cap\mathfrak{D}(\Psi_\alpha^n))=\Psi_\alpha^n(L_\infty^{(\alpha,2)}\cap\mathfrak{D}(\Psi_\alpha^n))\\&=\Psi_\alpha^n(\Psi_\alpha(L_\infty\cap\mathfrak{D}(\Psi_\alpha))\cap\mathfrak{D}(\Psi_\alpha^n))\\&=\Psi_\alpha^n(\Psi_\alpha(L_\infty\cap\mathfrak{D}(\Psi_\alpha^{n+1})))=\Psi_\alpha^{n+1}(L_\infty\cap\mathfrak{D}(\Psi_\alpha^{n+1})).\end{aligned}$$

It follows from (3.8) that $L_\infty^{(\alpha,2)}\subset\mathfrak{R}_\infty(\Psi_\alpha)$. Next we claim that

$$\mathfrak{R}(\Psi_\alpha)\cap L_\infty\subset L_\infty^{(\alpha,2)}.\tag{3.9}$$

Let $\mu\in\mathfrak{R}(\Psi_\alpha)\cap L_\infty$. Then μ has a radial decomposition $(\lambda_\mu(d\xi),r^{-\alpha-1}\,k_\xi^\mu(r)dr)$ with the property stated in Lemma 3.2. On the other hand,

$$\begin{aligned}\nu_\mu(B)&=\int_{(0,2)}\Gamma_\mu(d\beta)\int_S\lambda_\beta^\mu(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr\\&=\int_S\overline{\lambda}_\mu(d\xi)\int_{(0,2)}\Gamma_\xi^\mu(d\beta)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr\end{aligned}$$

for $B\in\mathcal{B}(\mathbb{R}^d)$, as there are a probability measure $\overline{\lambda}_\mu$ on S and a measurable family $\{\Gamma_\xi^\mu\}$ of measures on $(0,2)$ satisfying $\int_{(0,2)}(\beta^{-1}+(2-\beta)^{-1})\Gamma_\xi^\mu(d\beta)=\text{const}$ such that $\Gamma_\mu(d\beta)\lambda_\beta^\mu(d\xi)=\overline{\lambda}_\mu(d\xi)\Gamma_\xi^\mu(d\beta)$. Hence, by the uniqueness in Proposition 3.1 of [S], there is a positive, finite, measurable function $c(\xi)$ such that $\lambda_\mu(d\xi)=c(\xi)\overline{\lambda}_\mu(d\xi)$ and, for λ_μ-a. e. ξ, $r^{-\alpha-1}\,k_\xi^\mu(r)dr=c(\xi)^{-1}\left(\int_{(0,2)}r^{-\beta-1}\Gamma_\xi^\mu(d\beta)\right)dr$. Hence $k_\xi^\mu(r)=c(\xi)^{-1}\int_{(0,2)}r^{\alpha-\beta}\Gamma_\xi^\mu(d\beta)$, a. e. r. Since $k_\xi^\mu(r)$ is completely monotone, it vanishes as r goes to infinity. Hence $\Gamma_\xi^\mu((0,\alpha])=0$ for λ_μ-a. e. ξ. Hence $\Gamma_\mu((0,\alpha])=0$, that is, $\mu\in L_\infty^{(\alpha,2)}$, proving (3.9). Now, using Lemma 3.6, we obtain $\mathfrak{R}_\infty(\Psi_\alpha)\subset\mathfrak{R}(\Psi_\alpha)\cap L_\infty\subset L_\infty^{(\alpha,2)}$.

Case 3 ($\alpha=1$). Let us show that

$$\Psi_1(L_\infty\cap\mathfrak{D}(\Psi_1))=L_\infty^{(1,2)}\cap\{\mu\in ID\colon\text{weak mean }0\}.\tag{3.10}$$

Let $\rho\in L_\infty\cap\mathfrak{D}(\Psi_1)$, that is, $\rho\in L_\infty^{(1,2)}$, $\int_{(1,2)}(\beta-1)^{-1}\Gamma_\rho(d\beta)<\infty$, $\int_{\mathbb{R}^d}x\rho(dx)=0$, and $\lim_{a\to\infty}\int_1^a s^{-1}ds\int_{|x|>s}x\nu_\rho(dx)$ exists in $\mathbb{R}^d$. Let $\mu=\Psi_1\rho$. Then, as in Case 1, $\mu\in L_\infty^{(1,2)}$ and (3.4) holds with $\alpha=1$. By Lemma 3.2, μ has weak mean 0. Conversely, let $\mu\in L_\infty^{(1,2)}\cap\{\mu\in ID\colon\text{weak mean }0\}$. Choose $\rho\in L_\infty^{(1,2)}$ such that $\Gamma_\rho(d\beta)=(\Gamma(\beta-1))^{-1}\Gamma_\mu(d\beta)$, $\lambda_\beta^\rho=\lambda_\beta^\mu$, $A_\rho=A_\mu$, and $\gamma_\rho=-\int_{|x|>1}x\nu_\rho(dx)$ (note that $\int_{(1,2)}(\beta-1)^{-1}\Gamma_\rho(d\beta)<\infty$ and hence $\int_{|x|>1}|x|\nu_\rho(dx)<\infty$ by Lemma 3.3). Then $\int_{\mathbb{R}^d}x\rho(dx)=0$ (see Lemma 4.3 of [S]). Since μ has weak mean, $\int_1^a s^{-1}ds\int_{|x|>s} x\nu_\rho(dx)$ is convergent as $a\to\infty$ by application of Lemma 3.4 with $b(\beta)=1/\Gamma(\beta)$. Hence $\rho\in\mathfrak{D}(\Psi_1)$. We have $\nu_{\Psi_1\rho}=\nu_\mu$, $A_{\Psi_1\rho}=A_\mu$, and $\Psi_1\rho$ has weak mean 0. Among distributions $\mu'\in ID$ having $\nu_{\mu'}=\nu_\mu$ and $A_{\mu'}=A_\mu$, only one distribution has weak mean 0. Hence $\Psi_1\rho=\mu$. This proves (3.10). We have

$$\Psi_1^n(L_\infty\cap\mathfrak{D}(\Psi_1^n))=L_\infty^{(1,2)}\cap\{\mu\in ID\colon \text{weak mean } 0\},\qquad n=1,2,\ldots \tag{3.11}$$

from (3.10) by the same argument as in Case 2. Hence

$$L_\infty^{(1,2)}\cap\{\mu\in ID\colon \text{weak mean } 0\}\subset\mathfrak{R}_\infty(\Psi_1). \tag{3.12}$$

Next

$$\mathfrak{R}(\Psi_1)\cap L_\infty\subset L_\infty^{(1,2)}\cap\{\mu\in ID\colon \text{weak mean } 0\}. \tag{3.13}$$

Indeed, $\mathfrak{R}(\Psi_1)\cap L_\infty\subset L_\infty^{(1,2)}$ by the same argument as in Case 2. Any $\mu\in\mathfrak{R}(\Psi_1)$ has weak mean 0 by Lemma 3.2. Now it follows from Lemma 3.6 that

$$\mathfrak{R}_\infty(\Psi_1)\subset L_\infty^{(1,2)}\cap\{\mu\in ID\colon \text{weak mean } 0\}. \tag{3.14}$$

Case 4 ($1<\alpha<2$). We show that

$$\Psi_\alpha(L_\infty\cap\mathfrak{D}(\Psi_\alpha))=L_\infty^{(\alpha,2)}\cap\{\mu\in ID\colon \text{mean } 0\}. \tag{3.15}$$

Let $\rho\in L_\infty\cap\mathfrak{D}(\Psi_\alpha)$, that is, $\rho\in L_\infty^{(\alpha,2)}$, $\int_{(\alpha,2)}(\beta-\alpha)^{-1}\Gamma_\rho(d\beta)<\infty$, and $\int_{\mathbb{R}^d}x\rho(dx)=0$ (Lemmas 3.1 and 3.3). Let $\mu=\Psi_\alpha\rho$. Then $\mu\in L_\infty^{(\alpha,2)}$ and (3.4) holds. Hence $\int_{\mathbb{R}^d}|x|\mu(dx)<\infty$ by Lemma 3.3 and μ has mean 0 by Lemma 3.2. Conversely, if $\mu\in L_\infty^{(\alpha,2)}\cap\{\mu\in ID\colon \text{mean } 0\}$, then we can find $\rho\in L_\infty\cap\mathfrak{D}(\Psi_\alpha)$ satisfying $\Psi_\alpha\rho=\mu$, similarly to Case 3. Hence (3.15) is true. It follows that

$$\Psi_\alpha^n(L_\infty\cap\mathfrak{D}(\Psi_\alpha^n))=L_\infty^{(\alpha,2)}\cap\{\mu\in ID\colon \text{mean } 0\},\qquad n=1,2,\ldots$$

similarly to Cases 2 and 3. Hence

$$L_\infty^{(\alpha,2)}\cap\{\mu\in ID\colon \text{mean } 0\}\subset\mathfrak{R}_\infty(\Psi_\alpha). \tag{3.16}$$

We can also prove

$$\mathfrak{R}(\Psi_\alpha)\cap L_\infty\subset L_\infty^{(\alpha,2)}\cap\{\mu\in ID\colon \text{mean } 0\}$$

similarly to Cases 2 and 3. Hence the reverse inclusion of (3.16) follows from Lemma 3.6. □

Proof of the part related to $\mathfrak{R}_\infty(\bar\Phi_{p,\alpha})$ *in Theorem 1.1.* We assume $p\geq 1$. Since monotonicity of order $p\in[1,\infty)$ implies monotonicity of order 1 (Corollary 2.6 of [S]), it follows from Lemma 3.2 that Lemma 3.6 is applicable with $\beta=(-\alpha-1)\vee 0$. Hence $\mathfrak{R}_\infty(\bar\Phi_{p,\alpha})\subset L_\infty$. If $\rho\in L_\infty\cap\mathfrak{D}(\bar\Phi_{p,\alpha})$ and $\bar\Phi_{p,\alpha}\rho=\mu$, then $\rho\in L_\infty^{(\alpha,2)}$

(understand that $L_\infty^{(\alpha,2)} = L_\infty$ for $\alpha \le 0$) and, noting that

$$\begin{aligned}\nu_\mu(B) &= \int_0^\infty ds \int_{\mathbb{R}^d} 1_B(\bar f_{p,\alpha}(s)x)\nu_\rho(dx)\\ &= \frac{1}{\Gamma(p)}\int_0^1 t^{-\alpha-1}(1-t)^{p-1}dt \int_{\mathbb{R}^d} 1_B(tx)\nu_\rho(dx)\\ &= \frac{1}{\Gamma(p)}\int_0^1 t^{-\alpha-1}(1-t)^{p-1}dt \int_{(0,2)} \Gamma_\rho(d\beta)\int_S \lambda_\beta^\rho(d\xi)\int_0^\infty 1_B(tr\xi)r^{-\beta-1}dr\\ &= \int_{(0,2)} \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta-\alpha+p)}\Gamma_\rho(d\beta)\int_S \lambda_\beta^\rho(d\xi)\int_0^\infty 1_B(u\xi)u^{-\beta-1}du\end{aligned}$$

and recalling Lemmas 3.1 and 3.3, we obtain $\mu \in L_\infty^{(\alpha,2)}$ with

$$\Gamma_\mu(d\beta) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta-\alpha+p)}\Gamma_\rho(d\beta) \quad \text{and} \quad \lambda_\beta^\mu = \lambda_\beta^\rho. \tag{3.17}$$

Now the proof of assertions (i), (ii), and (iv) can be given in parallel to the corresponding assertions for Ψ_α. Note that, if $-\infty < \alpha < 1$, then

$$\int_0^\infty \bar f_{p,\alpha}(s)ds \int_{\mathbb{R}^d} |x|\,|1_{\{|\bar f_{p,\alpha}(s)x|\le 1\}} - 1_{\{|x|\le 1\}}|\,\nu_\rho(dx) < \infty$$

similarly. We also use the fact that $k_\xi^\mu(r)$ vanishes at infinity if it is monotone of order $p \in [1,\infty)$.

For assertion (iii) in the case $\alpha = 1$, we have to find another way, as Lemma 3.4 is not applicable if $\beta > 1$. Let us show

$$\bar\Phi_{p,1}(L_\infty \cap \mathfrak{D}(\bar\Phi_{p,1})) = L_\infty^{(1,2)} \cap \{\mu \in ID\colon \text{weak mean } 0\}. \tag{3.18}$$

Suppose that $\rho \in L_\infty \cap \mathfrak{D}(\bar\Phi_{p,1})$ and $\bar\Phi_{p,1}\rho = \mu$. Then $\rho \in L_\infty^{(1,2)}$, $\int_{(1,2)}(\beta-1)^{-1}\Gamma_\rho(d\beta) < \infty$, $\mu \in L_\infty^{(1,2)}$ with (3.17), and μ has weak mean 0 by Lemma 3.2. Conversely, suppose that $\mu \in L_\infty^{(1,2)}$ with weak mean 0. As in [S], let $\mathfrak{M}^L$ be the class of Lévy measures of infinitely divisible distributions on $\mathbb{R}^d$ and let $\bar\Phi_{p,1}^L$ be the transformation of Lévy measures associated with the mapping $\bar\Phi_{p,1}$. Define $\Gamma_0(d\beta) = \frac{\Gamma(\beta-1+p)}{\Gamma(\beta-1)}\Gamma_\mu(d\beta)$. Then $\int_{(1,2)}(2-\beta)^{-1}\Gamma_0(d\beta) < \infty$. Define

$$\nu_0(B) = \int_{(1,2)} \Gamma_0(d\beta)\int_S \lambda_\beta^\mu(d\xi)\int_0^\infty 1_B(r\xi)r^{-\beta-1}dr$$

for $B \in \mathcal{B}(\mathbb{R}^d)$. We have $\nu_0 \in \mathfrak{M}^L$. We see

$$\begin{aligned}\nu_\mu(B) &= \int_{(1,2)} \frac{\Gamma(\beta-1)}{\Gamma(\beta-1+p)}\Gamma_0(d\beta)\int_S \lambda_\beta^\mu(d\xi)\int_0^\infty 1_B(u\xi)u^{-\beta-1}du\\ &= \int_0^\infty ds\int_{\mathbb{R}^d} 1_B(\bar f_{p,1}(s)x)\nu_0(dx)\end{aligned}$$

from the calculation above. Since $\nu_\mu \in \mathfrak{M}^L$, we have $\nu_0 \in \mathfrak{D}(\bar\Phi_{p,1}^L)$ and $\bar\Phi_{p,1}^L\nu_0 = \nu_\mu$. Then it follows from Theorem 4.10 of [S] that ν_μ has a radial decomposition $(\lambda_\mu(d\xi), u^{-2}k_\xi^\mu(u)du)$ such that $k_\xi^\mu(u)$ is measurable in (ξ,u) and, for λ_μ-a.e. ξ, monotone of order p in $u \in \mathbb{R}_+^\circ$. Hence $\mu \in \mathfrak{R}(\bar\Phi_{p,1})$ from Lemma 3.2. Since $\bar\Phi_{p,1}^L\nu_0 = \nu_\mu$ and $\bar\Phi_{p,1}^L$ is one-to-one (Theorem 4.9 of [S]), we have $\mu = \bar\Phi_{p,1}\rho$ for some $\rho \in \mathfrak{D}(\bar\Phi_{p,1})$ with $\nu_\rho = \nu_0$. It follows that $\rho \in L_\infty$. This finishes the proof of

(3.18). Now we can show (3.11)–(3.14) with $\bar{\Phi}_{p,1}$ in place of Ψ_1 similarly to Case 3 in the preceding proof. □

Proof of the part related to $\mathfrak{R}_\infty(\Lambda_{q,\alpha})$ in Theorem 1.1. Since we have Proposition 2.3, it remains only to consider $\Lambda_{1,1}$. But the assertion for $\mathfrak{R}_\infty(\Lambda_{1,1})$ is obviously true, since $\Lambda_{1,1} = \bar{\Phi}_{1,1}$. □

4. Some examples of $\mathfrak{R}_\infty(\Phi_f)$

We present some examples of Φ_f for which the class $\mathfrak{R}_\infty(\Phi_f)$ is different from those appearing in Theorem 1.1.

Define T_a, the dilation by $a \in \mathbb{R}\setminus\{0\}$, as $(T_a\mu)(B) = \int 1_B(ax)\mu(dx) = \mu((1/a)B)$, $B \in \mathcal{B}(\mathbb{R}^d)$, for measures on $\mathbb{R}^d$. Define P_t, the raising to the convolution power $t > 0$, in such a way that, for $\mu \in ID$, $P_t\mu$ is an infinitely divisible distribution with characteristic function $\widehat{P_t\mu}(z) = \widehat{\mu}(z)^t$. The mappings T_a (restricted to ID), P_t, and Φ_f are commutative with each other. A measure μ on $\mathbb{R}^d$ is called symmetric if $T_{-1}\mu = \mu$. A distribution μ on $\mathbb{R}^d$ is called shifted symmetric if $\mu = \rho * \delta_\gamma$ with some symmetric distribution ρ and some δ-distribution δ_γ. Let $ID_{\mathrm{sym}} = ID_{\mathrm{sym}}(\mathbb{R}^d)$ [resp. $ID_{\mathrm{sym}}^{\mathrm{shift}} = ID_{\mathrm{sym}}^{\mathrm{shift}}(\mathbb{R}^d)$] denote the class of symmetric [resp. shifted symmetric] infinitely divisible distributions on $\mathbb{R}^d$.

Example 4.1. Let $f(s) = b1_{[0,a]}(s) - b1_{(a,2a]}(s)$ with $a > 0$ and $b \neq 0$. Then $\mathfrak{R}_\infty(\Phi_f) = ID_{\mathrm{sym}}$.

Indeed, for $\rho \in ID$,

$$C_{\Phi_f\rho}(z) = \int_0^a C_\rho(bz)ds + \int_a^{2a} C_\rho(-bz)ds = aC_\rho(bz) + aC_\rho(-bz) = C_{P_aT_b(\rho * T_{-1}\rho)}(z)$$

for $z \in \mathbb{R}^d$, and hence $\Phi_f\rho = P_aT_b(\rho * T_{-1}\rho)$. Define $U\rho = P_{1/2}\rho * T_{-1}P_{1/2}\rho$. Then $U\rho \in ID_{\mathrm{sym}}$ for any $\rho \in ID$. If $\rho \in ID_{\mathrm{sym}}$, then $U\rho = \rho$. Hence $U^n\rho = U\rho$ for $n = 1, 2, \ldots$. Since $\Phi_f = P_aT_bP_2U = P_{2a}T_bU$, we have $\Phi_f^n = P_{2a}^nT_b^nU = UP_{2a}^nT_b^n$ and $U = \Phi_f^nP_{1/(2a)}^nT_{1/b}^n$. Hence $\mathfrak{R}_\infty(\Phi_f) = \mathfrak{R}(U) = ID_{\mathrm{sym}}$.

Example 4.2. Let $f(s) = b1_{[0,a]}(s) - b1_{(a,a+c]}(s)$ with $a > 0$, $c > 0$, $a \neq c$, and $b \neq 0$. Then $\mathfrak{R}_\infty(\Phi_f) = ID_{\mathrm{sym}}^{\mathrm{shift}}$.

To see this, notice that

$$C_{\Phi_f\rho}(z) = aC_\rho(bz) + cC_\rho(-bz) = (a+c)(a_1C_{T_b\rho}(z) + (1-a_1)C_{T_b\rho}(-z))$$

for $\rho \in ID$, where $a_1 = a/(a+c)$. That is, $\Phi_f\rho = P_{a+c}T_b(P_{a_1}\rho * P_{1-a_1}T_{-1}\rho)$. Let us define $V\rho = P_{a_1}\rho * P_{1-a_1}T_{-1}\rho$. Note that V is the stochastic integral mapping Φ_f in the case $a + c = 1$ and $b = 1$. We have

$$V^n\rho = P_{a_n}\rho * P_{1-a_n}T_{-1}\rho \tag{4.1}$$

for $n = 1, 2, \ldots$, where a_n is given by $a_n = 1 - a_1 + a_{n-1}(2a_1 - 1)$. Indeed, if (4.1) is true for n, then it is true for $n + 1$ in place of n, since

$$\begin{aligned} V^{n+1}\rho &= P_{a_n}V\rho * P_{1-a_n}T_{-1}V\rho = P_{a_n}V\rho * P_{1-a_n}VT_{-1}\rho \\ &= P_{a_n}(P_{a_1}\rho * P_{1-a_1}T_{-1}\rho) * P_{1-a_n}(P_{a_1}T_{-1}\rho * P_{1-a_1}\rho) \\ &= P_{a_na_1+(1-a_n)(1-a_1)}\rho * P_{a_n(1-a_1)+(1-a_n)a_1}T_{-1}\rho \\ &= P_{a_{n+1}}\rho * P_{1-a_{n+1}}T_{-1}\rho. \end{aligned}$$

We see that $0 < a_n < 1$ for all n. We have $\Phi_f^n = P_{a+c}^n T_b^n V^n = V^n P_{a+c}^n T_b^n$ and $V^n = P_{1/(a+c)}^n T_{1/b}^n \Phi_f^n = \Phi_f^n P_{1/(a+c)}^n T_{1/b}^n$. Therefore $\mathfrak{R}(\Phi_f^n) = \mathfrak{R}(V^n)$ and hence $\mathfrak{R}_\infty(\Phi_f) = \mathfrak{R}_\infty(V)$. Next let us show that

$$\mathfrak{R}_\infty(V) = ID_{\text{sym}}^{\text{shift}}. \tag{4.2}$$

If $\rho \in ID_{\text{sym}}$, then $V\rho = \rho$. Hence $ID_{\text{sym}} \subset \mathfrak{R}_\infty(V)$. If $\rho = \delta_\gamma$, then $V\rho = \delta_{a_1\gamma} * \delta_{-(1-a_1)\gamma} = \delta_{(2a_1-1)\gamma}$. Now $\delta_\gamma = V\delta_{(1/(2a_1-1))\gamma}$, since $a_1 \neq 1/2$. Hence all δ-distributions are in $\mathfrak{R}(V^n)$ and hence in $\mathfrak{R}_\infty(V)$. Since $\mathfrak{R}_\infty(V)$ is closed under convolution, we obtain $ID_{\text{sym}}^{\text{shift}} \subset \mathfrak{R}_\infty(V)$. To show the converse, assume that $\mu \in \mathfrak{R}_\infty(V)$. Then $\mu = V^n \rho_n$ for some $\rho_n \in ID$. It follows from (4.1) that $\nu_\mu = a_n \nu_{\rho_n} + (1-a_n) T_{-1}\nu_{\rho_n}$. Let $\sigma_n \in ID$ be such that $(A_{\sigma_n}, \nu_{\sigma_n}, \gamma_{\sigma_n}) = (0, \nu_{\rho_n}, 0)$. It follows from $a_n = 1 - a_1 + a_{n-1}(2a_1 - 1)$ and from $0 < a_n < 1$ that $a_n \to 1/2$ as $n \to \infty$. Hence $a_n > 1/3$ for all large n. We see that the set $\{\sigma_n : n = 1, 2, \ldots\}$ is precompact, since $\nu_{\sigma_n} \leq a_n^{-1} \nu_\mu \leq 3\nu_\mu$ for all large n. Thus we can choose a subsequence $\{\sigma_{n_k}\}$ convergent to some $\mu' \in ID$. Since $\int \varphi(x) \nu_{\sigma_{n_k}}(dx) \to \int \varphi(x) \nu_{\mu'}(dx)$ for any bounded continuous function φ which vanishes on a neighborhood of the origin and since $a_n \to 1/2$, we obtain $\nu_\mu = (1/2)\nu_{\mu'} + (1/2) T_{-1}\nu_{\mu'}$. Hence ν_μ is symmetric. Hence $\mu * \delta_{-\gamma_\mu}$ is symmetric. It follows that $\mu \in ID_{\text{sym}}^{\text{shift}}$. This proves (4.2) and therefore $\mathfrak{R}_\infty(\Phi_f) = ID_{\text{sym}}^{\text{shift}}$.

Example 4.3. Let $\alpha < 0$. Let $h(s)$ be one of $f_\alpha(s)$, $\bar{f}_{p,\alpha}(s)$, and $l_{q,\alpha}(s)$ ($p \geq 1$, $q > 0$). Let $s_0 = \sup\{s : h(s) > 0\}$. Then $0 < s_0 < \infty$. Define

$$f(s) = \begin{cases} h(s), & 0 \leq s \leq s_0, \\ -h(2s_0 - s), & s_0 < s \leq 2s_0, \\ 0, & s > 2s_0. \end{cases}$$

Then $\mathfrak{R}_\infty(\Phi_f) = L_\infty \cap ID_{\text{sym}}$.

Proof is as follows. First, recall that $\mathfrak{D}(\Phi_f) = \mathfrak{D}(\Phi_h) = ID$. We have, for $\rho \in ID$,

$$\begin{aligned} C_{\Phi_f \rho}(z) &= \int_0^{s_0} C_\rho(h(s)z)ds + \int_{s_0}^{2s_0} C_\rho(-h(2s_0 - s)z)ds \\ &= \int_0^{s_0} C_\rho(h(s)z)ds + \int_0^{s_0} C_\rho(-h(s)z)ds \\ &= C_{\Phi_h \rho}(z) + C_{\Phi_h T_{-1}\rho}(z). \end{aligned}$$

It follows that $\Phi_f \rho = \Phi_h(\rho * T_{-1}\rho) = \Phi_h P_2 U \rho = U P_2 \Phi_h \rho$, where U is the mapping used in Example 4.1. It follows that $\Phi_f^n = \Phi_h^n P_2^n U = U P_2^n \Phi_h^n$ for $n = 1, 2, \ldots$. Hence $\mathfrak{R}(\Phi_f^n) \subset \mathfrak{R}(\Phi_h^n) \cap ID_{\text{sym}}$. Conversely, assume that $\rho \in \mathfrak{R}(\Phi_h^n) \cap ID_{\text{sym}}$. Then $\mu = \Phi_h^n \rho$ for some ρ and $T_{-1}\mu = \Phi_h^n T_{-1}\rho$. Since Φ_h is one-to-one (see [S]), we have $\rho = T_{-1}\rho$. Hence $\Phi_f^n \rho = \Phi_h^n P_2^n U \rho = \Phi_h^n P_2^n \rho = P_2^n \mu$ and thus $\mu = \Phi_f^n P_{1/2}^n \rho \in \mathfrak{R}(\Phi_f^n)$. In conclusion, $\mathfrak{R}(\Phi_f^n) = \mathfrak{R}(\Phi_h^n) \cap ID_{\text{sym}}$ and hence $\mathfrak{R}_\infty(\Phi_f) = \mathfrak{R}_\infty(\Phi_h) \cap ID_{\text{sym}} = L_\infty \cap ID_{\text{sym}}$.

Example 4.4. Let $h(s)$ and s_0 be as in Example 4.3. Define

$$f(s) = \begin{cases} h(s_0 - s), & 0 \le s \le s_0, \\ h(s - s_0), & s_0 < s \le 2s_0, \\ -h(3s_0 - s), & 2s_0 < s \le 3s_0, \\ 0, & s > 3s_0. \end{cases}$$

Then $\mathfrak{R}_\infty(\Phi_f) = L_\infty \cap ID_{\mathrm{sym}}^{\mathrm{shift}}$.

To see this, notice that

$$\begin{aligned} C_{\Phi_f \rho}(z) &= \int_0^{s_0} C_\rho(h(s_0 - s)z)ds + \int_{s_0}^{2s_0} C_\rho(h(s - s_0)z)ds \\ &\quad + \int_{2s_0}^{3s_0} C_\rho(-h(3s_0 - s)z)ds \\ &= \int_0^{s_0} C_\rho(h(s)z)ds + \int_0^{s_0} C_\rho(h(s)z)ds + \int_0^{s_0} C_\rho(-h(s)z)ds \\ &= 2C_{\Phi_h\rho}(z) + C_{\Phi_h\rho}(-z) \\ &= 3(\tfrac{2}{3}C_{\Phi_h\rho}(z) + \tfrac{1}{3}C_{\Phi_h\rho}(-z)). \end{aligned}$$

Hence $\Phi_f\rho = P_3 V \Phi_h \rho$, where $V\rho = P_{2/3}\rho * P_{1/3}T_{-1}\rho$. This mapping V is a special case of V in Example 4.2 with $a_1 = 2/3$. Hence (4.1) holds with $a_n = 2^{-1}(1 + 3^{-n})$ and $1 - a_n = 2^{-1}(1 - 3^{-n})$. Now $\Phi_f^n = P_3^n V^n \Phi_h^n = \Phi_h^n P_3^n V^n = V^n P_3^n \Phi_h^n$. Hence $\mathfrak{R}(\Phi_f^n) \subset \mathfrak{R}(\Phi_h^n) \cap \mathfrak{R}(V^n)$. It follows that $\mathfrak{R}_\infty(\Phi_f) \subset \mathfrak{R}_\infty(\Phi_h) \cap \mathfrak{R}_\infty(V) = L_\infty \cap ID_{\mathrm{sym}}^{\mathrm{shift}}$ from Theorem 1.1 and (4.2). Let us also show the converse inclusion $L_\infty \cap ID_{\mathrm{sym}}^{\mathrm{shift}} \subset \mathfrak{R}_\infty(\Phi_f)$. It is enough to show

$$\mathfrak{R}(\Phi_h^n) \cap ID_{\mathrm{sym}}^{\mathrm{shift}} \subset \mathfrak{R}(\Phi_f^n). \tag{4.3}$$

For any $\gamma \in \mathbb{R}^d$ we have

$$C_{\Phi_h\delta_\gamma}(z) = \int_0^{s_0} C_{\delta_\gamma}(h(s)z)ds = i\int_0^{s_0} \langle \gamma, h(s)z\rangle ds = ic\langle\gamma, z\rangle = C_{\delta_{c\gamma}}(z),$$

where $c = \int_0^{s_0} h(s)ds > 0$. That is, $\Phi_h\delta_\gamma = \delta_{c\gamma}$. Hence $\Phi_f\delta_\gamma = P_3\Phi_h V\delta_\gamma = P_3\Phi_h(\delta_{(2/3)\gamma} * \delta_{-(1/3)\gamma}) = \Phi_h\delta_\gamma = \delta_{c\gamma}$. Hence $\Phi_f^n\delta_\gamma = \delta_{c^n\gamma}$ and $\delta_\gamma = \Phi_f^n\delta_{c^{-n}\gamma}$. Hence all δ-distributions are in $\mathfrak{R}(\Phi_f^n)$. Similarly all δ-distributions are in $\mathfrak{R}(\Phi_h^n)$. Let $\mu \in \mathfrak{R}(\Phi_h^n) \cap ID_{\mathrm{sym}}^{\mathrm{shift}}$. Then $\mu * \delta_\gamma \in \mathfrak{R}(\Phi_h^n) \cap ID_{\mathrm{sym}}$ for some γ. Letting $\mu' = \mu * \delta_\gamma$, we have $\mu' = \Phi_h^n\rho'$ for some ρ'. Since $\mu' = T_{-1}\mu' = \Phi_h^n T_{-1}\rho'$, we have $\rho' = T_{-1}\rho'$ from the one-to-one property of Φ_h. Thus $V^n\rho' = \rho'$ and $\Phi_f^n\rho' = \Phi_h^n P_3^n\rho' = P_s^n\mu'$. Hence $\mu' = P_{1/3}^n\Phi_f^n\rho' = \Phi_f^n P_{1/3}^n\rho' \in \mathfrak{R}(\Phi_f^n)$. It follows that $\mu = \mu' * \delta_{-\gamma} \in \mathfrak{R}(\Phi_f^n)$. This proves (4.3). Hence $\mathfrak{R}_\infty(\Phi_f) = L_\infty \cap ID_{\mathrm{sym}}^{\mathrm{shift}}$.

Example 4.5. Let $b > 1$. Let $f(s) = b1_{[0,1]}(s) + 1_{(1,2]}(s)$. Let $L_\infty(b)$ be the class mentioned in Section 2. Then $L_\infty(b) \subset \mathfrak{R}_\infty(\Phi_f) \subsetneqq ID$. We do not know whether $\mathfrak{R}_\infty(\Phi_f)$ equals $L_\infty(b)$.

Let us show that $L_\infty(b) \subset \mathfrak{R}_\infty(\Phi_f)$. For $0 < \alpha \le 2$ define $\mathfrak{S}_\alpha(b) = \mathfrak{S}_\alpha(b, \mathbb{R}^d)$ as follows: $\rho \in \mathfrak{S}_\alpha(b)$ if and only if ρ is a δ-distribution or a non-trivial α-semi-stable distribution with b as a span, that is,

$$\mathfrak{S}_\alpha(b) = \{\rho \in ID\colon P_{b^\alpha}\rho = T_b\rho * \delta_\gamma \text{ for some } \gamma \in \mathbb{R}^d\}.$$

We have $C_{\Phi_f\rho}(z) = C_\rho(bz) + C_\rho(z)$ for $\rho \in ID$, that is, $\Phi_f\rho = T_b\rho * \rho$. If $\rho \in \mathfrak{S}_\alpha(b)$ with $P_{b^\alpha}\rho = T_b\rho * \delta_\gamma$, then $\mu = \Phi_f\rho$ satisfies $\mu = T_b\rho * \rho = P_{b^\alpha}\rho * \delta_{-\gamma} * \rho = P_{b^\alpha+1}\rho * \delta_{-\gamma}$ and $\mu \in \mathfrak{S}_\alpha(b)$. If $\mu \in \mathfrak{S}_\alpha(b)$ with $P_{b^\alpha}\mu = T_b\mu * \delta_{\gamma'}$, then $\mu = \Phi_f\rho$ for $\rho = P_{1/(b^\alpha+1)}(\mu * \delta_{(1/(b+1))\gamma'}) \in \mathfrak{S}_\alpha(b)$. Therefore $\Phi_f(\mathfrak{S}_\alpha(b)) = \mathfrak{S}_\alpha(b)$. Hence $\mathfrak{S}_\alpha(b) \subset \mathfrak{R}(\Phi_f^n)$ for $0 < \alpha \leq 2$ and $n = 1, 2, \ldots$. It follows from Proposition 3.2 of Maejima and Sato (2009) that $\mathfrak{R}(\Phi_f^n)$ is closed under convolution and weak convergence. Hence $L_\infty(b) \subset \mathfrak{R}(\Phi_f^n)$ and thus $L_\infty(b) \subset \mathfrak{R}_\infty(\Phi_f)$. In order to show $\mathfrak{R}_\infty(\Phi_f) \subsetneqq ID$, let μ be such that $\nu_\mu = \delta_a$ with $a \neq 0$. Suppose that $\mu = \Phi_f\rho$ for some $\rho \in ID$. Then $\nu_\mu = T_b\nu_\rho + \nu_\rho$. If $\nu_\rho \neq 0$, then the support of ν_ρ contains at least one point $a' \neq 0$ and hence the support of ν_μ contains at least two points $\{a', ba'\}$, which is absurd. If $\nu_\rho = 0$, then $\nu_\mu = 0$, which is also absurd. Therefore $\mu \notin \mathfrak{R}(\Phi_f)$ and hence $\mu \notin \mathfrak{R}_\infty(\Phi_f)$.

5. Concluding remarks

The limit class $\mathfrak{R}_\infty(\Phi_f)$ is not known in many cases. For instance it is not known for the following choices of $f(s)$: $l_{q,1}(s)$ with $q \in (0,1) \cup (1,\infty)$ in [S]; $\bar{f}_{p,\alpha}(s)$ with $p \in (0,1)$ and $\alpha \in (-\infty, 2)$ in [S]; $\cos(2^{-1}\pi s)$ in Maejima et al. (2011a); $e^{-s}\,1_{[0,c]}(s)$ with $c \in (0,\infty)$ in Pedersen and Sato (2005); $G^*_{\alpha,\beta}(s)$ with $\alpha \in [1,2)$ and $\beta > 0$ satisfying $\alpha = 1 + n\beta$ for some $n = 0, 1, \ldots$ in Maejima and Ueda (2010b). Another instance is $\Phi_f = \Upsilon^\alpha$ with $\alpha \in (0,1)$ related to the Mittag-Leffler function, introduced in Barndorff-Nielsen and Thorbjørnsen (2006).

Consider, as in Sato (2007), a stochastic integral mapping

$$\Phi_f\,\rho = \mathcal{L}\left(\int_{0+}^{a} f(s)dX_s^{(\rho)}\right)$$

with $0 < a < \infty$ for a function $f(s)$ locally square-integrable on the interval $(0, a]$ and study $\mathfrak{R}_\infty(\Phi_f) = \bigcap_{n=1}^\infty \mathfrak{R}(\Phi_f^n)$. Under appropriate choices of f we obtain $\mathfrak{R}_\infty(\Phi_f)$ equal to $L_\infty^{(0,\alpha)} \cap ID_0$ with $\alpha \in (1,2)$, $L_\infty^{(0,\alpha)} \cap ID_0 \cap \{\mu \in ID \colon \mu$ has drift $0\}$ with $\alpha \in (0,1)$, or a certain subclass of $L_\infty^{(0,1)} \cap ID_0$. This will be shown in a forthcoming paper.

It is an interesting problem what other classes can appear as $\mathfrak{R}_\infty(\Phi_f)$.

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