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Spiders in random environment

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Abstract. A spider consists of several, say N, particles. Particles can jump independently according to a random walk if the movement does not violate some given restriction rules. If the movement violates a rule it is not carried out. We consider random walk in random environment (RWRE) on $\mathbb{Z}$ as underlying random walk. We suppose the environment $\omega = (\omega_x)_{x\in\mathbb{Z}}$ to be elliptic, with positive drift and nestling, so that there exists a unique positive constant κ such that $\mathbf{E}[((1-\omega_0)/\omega_0)^\kappa] = 1$. The restriction rules are kept very general; we only assume transitivity and irreducibility of the spider. The main result is that the speed of a spider is positive if $\kappa/N > 1$ and null if $\kappa/N < 1$. In particular, if $\kappa/N < 1$ a spider has null speed but the speed of a (single) RWRE is positive.

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1. Introduction and results

To begin with, let us give a simple example of a spider. Imagine there are two particles performing nearest neighbor random walks on $\mathbb{Z}$ in continuous time. These particles are tied together with a rope of a certain length $s \in \mathbb{N}$. As long as the rope is not tight their movements are independent. If the rope is tight (the two particles are at a distance s from each other) the rope prevents the particles to jump *away* from each other.

In these notes, we consider a spider on $\mathbb{Z}$ in a random environment. First, suppose that $\omega := (\omega_x)_{x\in\mathbb{Z}}$ is a sequence of positive i.i.d. random variables taking values in $(0,1)$. We denote by $\mathbf{P}$ the distribution of ω and by $\mathbf{E}$ the corresponding expectation. In the example above, we first choose an environment ω at random according to the law $\mathbf{P}$ and we describe the position of our two particles by the vector $S(t) = (S_1(t), S_2(t))$ where $S_i(t)$, $i = 1, 2$, is the position of particle i at time t. As long as $|S_1(t)-S_2(t)| < s$, the two particles behave like two independent random walks in random environment. If $|S_1(t) - S_2(t)| = s$, their movements are dependent in order to prevent that $|S_1(t) - S_2(t)| > s$. For instance, let the first particle be in x_1 and the second in x_2. Then, if $|x_1 - x_2| < s$ the first particle jumps to $x_1 + 1$ with rate $\omega^+_{x_1} := \omega_{x_1+1}$ or to site $x_1 - 1$ with rate $\omega^-_{x_1} := 1 - \omega_{x_1}$. Analogously the second one moves to $x_2 + 1$ with rate $\omega^+_{x_2}$ or to site $x_2 - 1$ with $\omega^-_{x_2}$. If $|x_1 - x_2| = s$ and $x_1 < x_2$ the first leg may only jump to the right with rate $\omega^+_{x_1}$ and the second to the left with rate $\omega^-_{x_2}$. In the case $x_1 < x_2$ the roles of the two legs are interchanged.

More generally we can consider a spider with N legs, that is to say N interacting particles. The particles move independently as long as their movement does not violate some restriction rules concerning their positions. In this case we denote by $S(t) = (S_1(t), S_2(t), \dots, S_N(t))$ the positions of the N particles at time t where $S_i(t)$ represents the position of particle i at time t.

This model gained recently an interest in evolutionary dynamics and molecular cybernetics. At the moment, to our knowledge, there are just a few theoretical papers on this model. In Antal et al. (2007), Antal, Krapivsky and Mallick obtained the speed and diffusion constants for 1-dimensional spiders and in Antal and Krapivsky (2007), Antal and Krapivsky made the first study for non-Markovian spiders. In Gallesco et al. (2011), Gallesco, Müller and Popov study qualitative properties, as recurrence, transience, ergodicity and positive rate of escape of spiders in a quite general setting. We refer to the lecture notes of Zeitouni (2004) for a general overview on random walks in random environments (RWRE). The main result of this paper, Theorem 1.1, is that in random environment on $\mathbb{Z}$ the speed of a spider may be zero even if the speed of a (single) RWRE is positive. This is in contrast with the results in Antal et al. (2007) and in Section 4.1 of Gallesco et al. (2011) that the positive speed of a homogeneous random walk implies positive speed of the spider.

It is convenient to adopt the following notations of Gallesco et al. (2011). Recall that the spider is described through $S(t) = (S_1(t), \dots, S_N(t))$. The first leg defines the position of the spider: the position of the spider at time t is $S_1(t)$. The spider is defined through a set L of local configurations at 0, that is a finite subset of $\{(x_1, x_2, \dots, x_N) : x_1 = 0,\ x_2, \dots, x_N \in \mathbb{Z}\}$. Actually, the set L corresponds to all possible configurations for the spider at position 0. Since in this note we only consider transitive spiders the set of local configurations at position x (that is when

$x_1 = x$) can be obtained by translating the set L by x. Denoting by Θ_x the shift by x, we have

$$L_x = \Theta_x L = \{(x,\ x_2, \dots, x_N) \in \mathbb{Z}^N : (0, x_2 - x, \dots, x_N - x) \in L\}.$$

Let

$$\mathcal{V} = \bigcup_{x \in \mathbb{Z}} L_x.$$

For elements in $\mathcal{V}$ we write $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$. The transition rates $Q^S = Q^S(\omega) = (q^S(\mathbf{x}, \mathbf{y}))_{(\mathbf{x},\mathbf{y}) \in \mathcal{V}^2}$ of the spider are defined as follows: let $\mathbf{x}, \mathbf{y} \in \mathcal{V}$ then

- if $\|\mathbf{x} - \mathbf{y}\| = 1$ (where $\|\cdot\|$ is the usual ℓ_1-norm) and i is the coordinate such that $x_i \neq y_i$,

$$q^S(\mathbf{x}, \mathbf{y}) = \begin{cases} \omega^+_{x_i} & \text{if } y_i = x_i + 1 \\ \omega^-_{x_i} & \text{if } y_i = x_i - 1, \end{cases}$$

- otherwise,

$$q^S(\mathbf{x}, \mathbf{y}) = 0.$$

Now, following Gallesco et al. (2011), we define the spider graph. For a given realization ω of our environment, define the graph $\mathcal{G} = \mathcal{G}(\omega) = (\mathcal{V}, \mathcal{E}(\omega))$ such that an edge $e = (\mathbf{x}, \mathbf{y}) \in \mathcal{V} \times \mathcal{V}$ belongs to $\mathcal{E}(\omega)$ if and only if $q^S(\mathbf{x}, \mathbf{y}) > 0$. As the sequence ω takes values in $(0, 1)^{\mathbb{Z}}$, the spider graph is deterministic. In the rest of these notes we will assume the irreducibility of the spider walk which is implied by the two following conditions on $\mathcal{G}$ for almost all realizations of ω:

(i) L is a connected subgraph of the spider graph $\mathcal{G}$,
(ii) there exists at least one edge between L and L_1.

Condition (i) is not necessary for the irreducibility of the spider, nevertheless it is assumed in this stronger form to reduce the technical part of the proofs. We assume the following conditions on our random environment:

(iii) $\mathbf{E}[\ln \rho_0] < 0$, with $\rho_0 := \frac{\omega_0^-}{\omega_0^+}$,
(iv) there exists $0 < \delta < 1/2$ such that $\mathbf{P}[\delta \leq \omega_0^+ \leq 1 - \delta] = 1$,
(v) $\mathbf{P}[\omega_0^+ > 1/2] > 0$ and $\mathbf{P}[\omega_0^+ \leq 1/2] > 0$.

Condition (iii) implies that the RWRE is transient to the right, see Solomon (1975). Condition (iv) is the usual uniform ellipticity condition. Condition (v) corresponds to the fact that our environment is nestling. Observe that for non-nestling random environments it is possible to show that every spider, satisfying (i)+(ii), has positive speed. Furthermore, conditions (iii)-(v) imply that there exists a unique $\kappa > 0$, such that

$$\mathbf{E}[\rho_0^\kappa] = 1.$$

We denote by $\mathtt{P}^{\mathbf{x}}_\omega$ the quenched law of the spider starting at $\mathbf{x}$ in the environment ω and by $\mathtt{E}^{\mathbf{x}}_\omega$ the corresponding expectation. Finally, we denote by $\mathbb{P}^{\mathbf{x}} := \mathbf{P} \times \mathtt{P}^{\mathbf{x}}_\omega$ and $\mathbb{E}^{\mathbf{x}}$ the annealed probability and expectation for the spider starting at $\mathbf{x}$.

We define the speed of a spider as

$$v = \lim_{t \to \infty} \frac{S_1(t)}{t}$$

if the limit exists. Let us consider a spider starting at some initial position $\mathbf{x}_0 \in L$ and define the stopping time

$$\mathcal{T} := \inf\{s > 0 : S_1(s) > 0 \text{ and } S(s) = \Theta_{S_1(s)}\mathbf{x}_0\}.$$

The main result of these notes is the following theorem.

Theorem 1.1. *Consider a spider with N legs. Under conditions (i)-(v), the speed v of the spider is well-defined and we have $\mathbb{P}$-a.s.*

$$v = \frac{\mathbb{E}[S_1(\mathcal{T})]}{\mathbb{E}[\mathcal{T}]} > 0 \quad \text{if } \tfrac{\kappa}{N} > 1$$

and

$$v = 0 \quad \text{if } \tfrac{\kappa}{N} < 1.$$

In particular, this implies that the positivity of the speed of a spider only depends on the number of legs N and not on the set L. Our technique is not fine enough to deal with the critical case $\kappa = N$. Nevertheless, we are inclined to believe that in this case, independently of the set L, the speed of the spider should be zero.

2. Notations and auxiliary results

We will denote by K_1, K_2, ... the "important" constants (those that can be used far away from the place where they appear for the first time) and by C_1, C_2, ... the "local" ones (those that are used only in a small neighbourhood of the place where they appear for the first time), restarting the numeration at the beginning of each section in the latter case.

An important ingredient of our proofs is the analysis of the potential associated to the environment, which was introduced by Sinai (1982). The potential, denoted by $V = (V(x), x \in \mathbb{Z})$ is a function of the environment and is defined as follows:

$$V(x) = \begin{cases} \sum_{i=0}^{x-1} \ln \frac{\omega_i^-}{\omega_i^+}, & x > 0, \\ 0, & x = 0, \\ \sum_{i=x+1}^{0} \ln \frac{\omega_i^+}{\omega_i^-}, & x < 0. \end{cases}$$

2.1. *Reversible measure of a spider.* Let us first give an example to illustrate the construction of the spider graph $\mathcal{G}$. Consider a spider with 3 legs, that is $N = 3$, and the following set L of restrictions:

$$L = \{(0,1,2), (0,1,3), (0,2,3), (0,2,4)\}.$$

Figure 2.1 shows the set of local configurations and a part of the spider graph $\mathcal{G}$. In the spider graph, the horizontal axis corresponds to the positions of the spider and the vertical axis to the local configurations. While the spider graph is deterministic the transition rates associated to each edge of $\mathcal{G}$ depend on the realization ω of our random environment.

Now, given a couple (N, L), consider the continuous time Markov process $S = (S(t))_{t\geq 0}$ on the spider graph $\mathcal{G}$. Observe that we use the same notation $S(t)_{t\geq 0}$ for two different processes: the spider on $\mathbb{Z}^N$ and the Markov process on $\mathcal{G}$. It should always be clear from the context to which of these we are referring to.

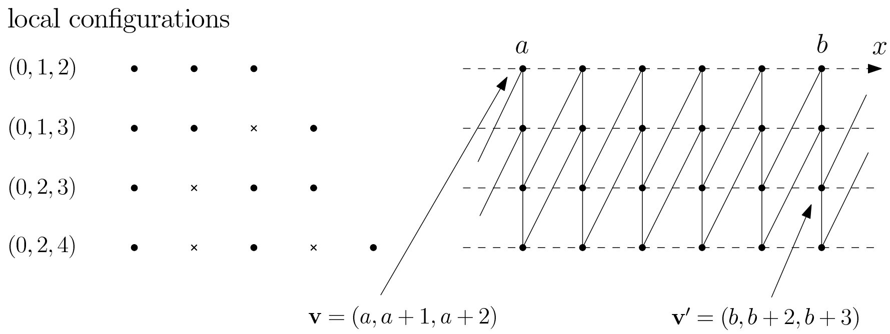

FIGURE 2.1. Structure of the spider graph $\mathcal{G}$. The elements of L are represented on the left.

Let $\theta_x = e^{-V(x)} + e^{-V(x-1)}$. Note that θ_x is the reversible measure at point x for a single random walk on $\mathbb{Z}$. Then, the process S is $\mathbf{P}$-a.s. reversible with reversible measure

$$\pi(\mathbf{x}) = \prod_{i=1}^{N} \theta_{x_i} \tag{2.1}$$

for all $\mathbf{x} \in \mathcal{V}$. Using condition (iv) we obtain that for all $x \in \mathbb{Z}$

$$K_1 e^{-V(x)} \leq \theta_x \leq K_2 e^{-V(x)}$$

for K_1 and K_2 two positive constants and

$$|V(x+1) - V(x)| \leq \ln \frac{1-\delta}{\delta}.$$

Using these inequalities and the fact that L is finite, we obtain that there exists two finite positive constants K_3 and K_4 such that

$$K_3 e^{-NV(x_1)} \leq \pi(\mathbf{x}) \leq K_4 e^{-NV(x_1)} \tag{2.2}$$

for all $\mathbf{x} \in \mathcal{V}$. Now, let $I = [a, b] \cap \mathbb{Z}$ be a finite interval. Consider the graph $\mathcal{G}_I = (\mathcal{V}_I, \mathcal{E}_I) \subset \mathcal{G}$ with

$$\mathcal{V}_I = \bigcup_{x \in I} L_x \tag{2.3}$$

$$\mathcal{E}_I = \Big\{ e = (\mathbf{v}, \mathbf{w}) \in \mathcal{E} \text{ such that } \mathbf{v}, \mathbf{w} \in \mathcal{V}_I \Big\}. \tag{2.4}$$

Then, consider the process $\hat{S}$ which is the restriction of the process S on the graph $\mathcal{G}_I$. As the graph $\mathcal{G}_I$ is a subgraph of $\mathcal{G}$, the reversible measure (2.1) is also reversible for the process $\hat{S}$. Moreover as the graph $\mathcal{G}_I$ is finite we can normalize the reversible measure (2.1) to obtain the invariant probability measure $\hat{\pi}$ of $\hat{S}$,

$$\hat{\pi}(\mathbf{x}) = \Big(\sum_{\mathbf{x} \in \mathcal{V}_I} \prod_{j=1}^{N} \theta_{x_j} \Big)^{-1} \pi(\mathbf{x})$$

for all $\mathbf{x} \in \mathcal{V}_I$.

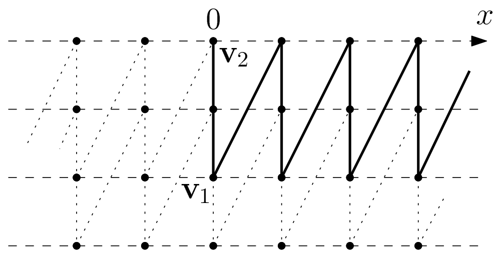

FIGURE 2.2. Example of graph $\mathcal{G}'$ (in straight lines) for the spider of Figure 2.1.

2.2. *Transience of the spider.* It was shown in Solomon (1975) that, under condition (iii), a single random walk is $\mathbb{P}$-a.s. transient to $+\infty$. The following proposition shows that this is still the case for a spider.

Proposition 2.1. *Under the hypothesis (i)-(iv) a spider is always transient, that is,*

$$\lim_{t\to\infty} S_1(t) = \infty,\ \mathbb{P}\text{-}a.s.$$

Proof: Consider the electrical network associated to $\mathcal{G}(\omega)$ by putting on each edge $e = (\mathbf{x}, \mathbf{y}) \in \mathcal{E}$ the resistance $R_e = R_{\mathbf{x},\mathbf{y}} = (q^S(\mathbf{x}, \mathbf{y})\pi(\mathbf{x}))^{-1}$. By condition (ii), there exists in L a vertex $\mathbf{v}_1$ which is linked to L_1 by some edge in $\mathcal{E}$. In the same way, there exists also a vertex $\mathbf{v}_2$ which is linked to L_{-1} by some edge in $\mathcal{E}$. By conditions (i) and (ii), we can choose a path γ_0 from $\mathbf{v}_1$ to $\mathbf{v}_2$. As $\mathcal{G}$ is homogeneous, we can iterate this construction to all the sets L_x, $x \geq 1$ and thus consider the linear sub-electrical network $\mathcal{G}'(\omega) = (\mathcal{V}', \mathcal{E}')$, see Figure 2.2. If I is an interval of $\mathbb{N}$, we define

$$\mathcal{V}'(I) = \mathcal{V}_I \cap \mathcal{V}'$$

and

$$\mathcal{E}'(I) = \Big\{ e = (\mathbf{v}, \mathbf{w}) \in \mathcal{E}' \text{ such that } \mathbf{v}, \mathbf{w} \in \mathcal{V}'(I) \Big\}.$$

Now, it is easy to compute the resistance $R_\infty = R_\infty(\omega)$ of $\mathcal{G}'(\omega)$ and to show that it is finite. By definition,

$$R_\infty := \lim_{n\to\infty} R_n$$

where

$$R_n = \sum_{e\in\mathcal{E}'([0,n])} R_e.$$

By condition (iv), we have

$$R_n \leq \frac{1}{\delta} \sum_{\mathbf{y}\in\mathcal{V}'([0,n])} \pi^{-1}(\mathbf{y}).$$

Using inequality (2.2), we obtain

$$R_n \leq \frac{K_3|L|}{\delta} \sum_{i=0}^{n} e^{NV(i)}. \tag{2.5}$$

Let us first show that $\lim_{n\to\infty} R_n < \infty$ **P**-a.s. As $V(x)$, for $x \geq 0$, is a sum of bounded i.i.d. random variables, by the Strong Law of Large Numbers, we have

$$\lim_{n\to\infty} \frac{V(n)}{n} = \mathbf{E}[\ln \rho_0] < 0, \qquad \mathbf{P}\text{-a.s.}$$

Now, take $\varepsilon > 0$ sufficiently small such that $(\mathbf{E}[\ln \rho_0] + \varepsilon) < 0$. This implies that $V(n) < (\mathbf{E}[\ln \rho_0] + \varepsilon)n$ **P**-a.s. Then, the general term $e^{NV(n)}$ of (2.5) is dominated by $e^{N(\mathbf{E}[\ln \rho_0]+\varepsilon)n}$ which is the general term of a convergent series. This shows that

$$R_\infty = \lim_{n\to\infty} R_n < \infty, \quad \mathbf{P}\text{-a.s.}$$

As $\mathcal{G}'(\omega)$ is a sub-network of $\mathcal{G}(\omega)$ with **P**-a.s. finite resistance, by the Rayleigh's Monotonicity Law (see for example Doyle and Snell, 1984) we deduce that the effective resistance of $\mathcal{G}(\omega)$ is **P**-a.s. finite, which implies that a spider on $\mathcal{G}(\omega)$ is transient for **P**-almost all ω. □

Remark 2.2. In fact, Proposition 2.1 does hold in a more general context. Assume condition (iv) and let (N, L) define a spider. Then, one can show that the RWRE is recurrent iff the spider is recurrent. This follows from the fact that one can show that the RWRE and the spider are roughly equivalent as electrical networks. We refer to Gallesco et al. (2011) where these questions are discussed for a general spider.

2.3. *Upper bound on the probability of confinement.* In this section we want to deduce an upper estimate for the probability of confinement of a spider on a finite interval. Fix a couple (N, L) and let $I = [a, b] \cap \mathbb{Z}$, $a, b \in \mathbb{Z}$, be a finite interval and

$$\tau_{\{a,b\}} = \inf\{s > 0 : S_1(s) = a \text{ or } S_1(s) = b\}.$$

We want to bound from above $\mathtt{P}^{\mathbf{x}}_\omega[\tau_{\{a,b\}} > t]$ uniformly over all initial positions $\mathbf{x} = (x_1, \ldots, x_N)$ such that $a < x_1 < b$. As L is finite, $d = \max_{\mathbf{u},\mathbf{v}\in L} \|\mathbf{u} - \mathbf{v}\|_\infty$ is finite (where $\|\cdot\|_\infty$ is the usual ∞-norm in $\mathbb{Z}^N$). Let $b_1 = b + d$ and define $I_1 = [a, b_1] \cap \mathbb{Z}$ and

$$H = \max_{x\in I_1} \Big(\max_{y\in[x,b_1]} V(y) - \min_{y\in[a,x)} V(y) \Big). \tag{2.6}$$

Also, let

$$m = \arg\min_{x\in I_1} V(x).$$

We will show the following

Proposition 2.3. *Let* $[a, b]$ *be a finite interval. We have*

$$\mathtt{P}^{\mathbf{x}}_\omega[\tau_{\{a,b\}} > t] \leq \exp\Big\{ - \frac{t}{K_5(b-a)^5 e^{NH}} \Big\} \tag{2.7}$$

with K_5 *a positive constant.*

Proof: First we use the following trick: consider the interval $I_2 := [a, b_2] \supset I_1$, where $b_2 = b + 2d$ and an interval $(b_2, b_3]$ such that $b_3 - b_2 = d$. On the subinterval $(b_1, b_3]$, we modify the environment such that $V(x) = V(m)$ for every $x \in (b_1, b_3]$, see Figure 2.3.

Consider now the process $\hat{S}$ on the graph $\mathcal{G}_{I_2}$ (see (2.3) and (2.4) for the definition of $\mathcal{G}_{I_2}$) and define

$$\tau' = \inf\{s > 0 : \hat{S}(s) \in L_{b_2}\}.$$

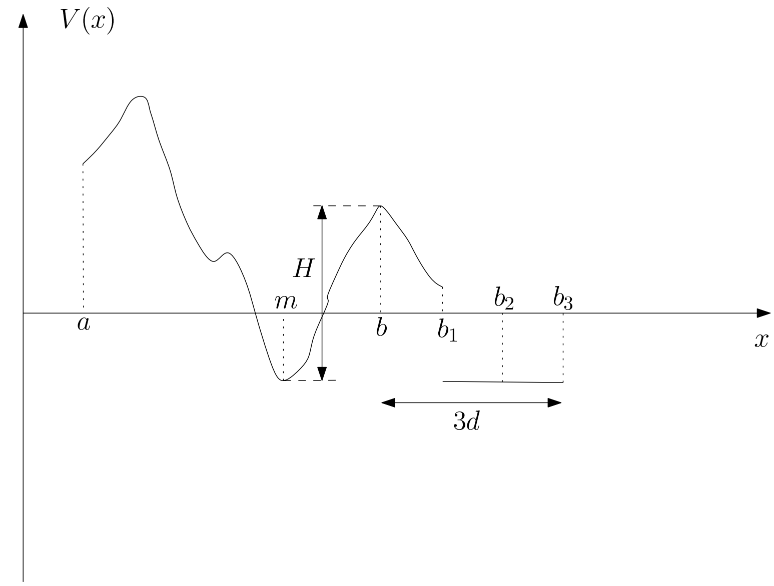

FIGURE 2.3. Potential extention technique.

Since, $\mathtt{P}_\omega^{\mathbf{x}}[\tau_{\{a,b\}} > t] \le \mathtt{P}_\omega^{\mathbf{x}}[\tau' > t]$, we focus from now on on finding an upper bound for $\mathtt{P}_\omega^{\mathbf{x}}[\tau' > t]$.

To this end, we construct a lower bound for the spectral gap λ of the process $\hat{S}$ using Theorem 3.2.1 of Saloff-Coste (1997). For all pairs of vertices $(\mathbf{x}, \mathbf{y})$ of $\mathcal{G}_{I_2}$, choose a path in $\mathcal{E}_{I_2}$ going from $\mathbf{x}$ to $\mathbf{y}$. We denote this path $\gamma(\mathbf{x}, \mathbf{y})$ and let $\Gamma = \{\gamma(\mathbf{x}, \mathbf{y}) : (\mathbf{x}, \mathbf{y}) \in \mathcal{V}_{I_2} \times \mathcal{V}_{I_2}\}$. Then, the latter theorem states that $\lambda \ge 1/A$ where

$$A = \max_{e \in \mathcal{E}_{I_2}} \Big\{ R_e \sum_{\mathbf{x}, \mathbf{y} \in \mathcal{V}_{I_2} : e \in \gamma(\mathbf{x}, \mathbf{y})} |\gamma(\mathbf{x}, \mathbf{y})| \hat{\pi}(\mathbf{x}) \hat{\pi}(\mathbf{y}) \Big\} \tag{2.8}$$

and R_e is the resistance of edge e as defined in subsection 2.2. Now, let us define a set of paths Γ that will give a good lower bound for the spectral gap λ. We start by enumerating the elements of the set L. The shift Θ induces the same enumeration on all the sets L_x for $x \in \mathbb{Z}$. If $\mathbf{y} \in L_x$ for some x, we will denote by $n(\mathbf{y})$ the number associated to $\mathbf{y}$. Furthermore, let us fix two local configurations $\mathbf{r}_1$ and $\mathbf{r}_2$ of L such that the edge $e = (\Theta_x \mathbf{r}_1, \Theta_{x+1} \mathbf{r}_2) \in \mathcal{E}$ for all $x \in \mathbb{Z}$. Let $\mathbf{x} = (x_1, \dots, x_N)$ and $\mathbf{y} = (y_1, \dots, y_N)$ be two vertices of $\mathcal{G}_{I_2}$, we will now choose a path $\gamma(\mathbf{x}, \mathbf{y})$ as follows:

- if $\mathbf{x}$ and $\mathbf{y}$ are such that $x_1 = y_1$ then consider the set of all paths that are contained in $\mathcal{E}_{x_1}$ (see (2.4) for the definition of $\mathcal{E}_{x_1}$) which go from $\mathbf{x}$ to $\mathbf{y}$. Assume $n(\mathbf{x}) < n(\mathbf{y})$. In this case, we choose the path $(\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_N, \mathbf{y})$ which minimizes the number $n(\mathbf{x}_1) \dots n(\mathbf{x}_N)$ in the following sense: $n(\mathbf{x}_1) \dots n(\mathbf{x}_{N_1})$ is smaller than $n(\mathbf{x}'_1) \dots n(\mathbf{x}'_{N_2})$ if $N_1 < N_2$. If $N_1 = N_2 = N$, we use the lexicographical order to decide which is the smallest one, that is, $n(\mathbf{x}_1) \dots n(\mathbf{x}_N)$ is smaller than $n(\mathbf{x}'_1) \dots n(\mathbf{x}'_N)$ if there exists $k \le N$ such that $n(\mathbf{x}_i) = n(\mathbf{x}'_i)$ for $i \le k$ and $n(\mathbf{x}_k) < n(\mathbf{x}'_k)$. If $n(\mathbf{x}) > n(\mathbf{y})$, define $\gamma(\mathbf{x}, \mathbf{y})$ as the inverse path of $\gamma(\mathbf{y}, \mathbf{x})$;

- if $\mathbf{x}$ and $\mathbf{y}$ are such that $x_1 \neq y_1$. Assume first that $x_1 < y_1$. Observe that there exists an element $\mathbf{z}$ of $\mathcal{V}_{x_1}$ such that $\mathbf{z} = \Theta_{x_1}\mathbf{r}_1$. Then, by the method above, we go from $\mathbf{x}$ to $\mathbf{z}$. From $\mathbf{z}$, we go to $\mathbf{x}' \in \mathcal{V}_{x_1+1}$ such that $\mathbf{x}' = \Theta_{x_1+1}\mathbf{r}_2$. From now on, we iterate the process to reach some $\mathbf{z}'$ such that $z_1' = y_1$. Finally, we again use the method above to go from $\mathbf{z}'$ to $\mathbf{y}$. If $x_1 > y_1$, define $\gamma(\mathbf{x},\mathbf{y})$ as the inverse path of $\gamma(\mathbf{y},\mathbf{x})$.

Thus, we have constructed the set Γ we will use in the rest of this proof.

Now, let us find an upper bound of A from (2.8). First let us define

$$A(e) = R_e \sum_{\mathbf{x},\mathbf{y}\in\mathcal{V}_{I_2}: e\in\gamma(\mathbf{x},\mathbf{y})} |\gamma(\mathbf{x},\mathbf{y})|\hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{y})$$

for all $e \in \mathcal{E}_{I_2}$. Let us find a uniform upper bound of $A(e)$ over all $e \in \mathcal{E}_{I_2}$. Let $e = (\mathbf{z},\mathbf{w})$. Using condition (iv), we obtain that

$$A(e) \le \frac{1}{\delta\hat{\pi}(\mathbf{z})} \sum_{\mathbf{x},\mathbf{y}\in\mathcal{V}_{I_2}: e\in\gamma(\mathbf{x},\mathbf{y})} |\gamma(\mathbf{x},\mathbf{y})|\hat{\pi}(\mathbf{x})\hat{\pi}(\mathbf{y}).$$

Then, as $|\gamma(\mathbf{x},\mathbf{y})|$ is uniformly bounded by $|L|(b_2-a)$, using inequalities (2.2) we obtain

$$A(e) \le C_3(b_2-a)D^{-1}e^{NV(z_1)} \sum_{\mathbf{x},\mathbf{y}\in\mathcal{V}_{I_2}: e\in\gamma(\mathbf{x},\mathbf{y})} e^{-N(V(x_1)+V(y_1))},$$

where $D = \sum_{\mathbf{x}\in\mathcal{V}_{I_2}} \prod_{i=1}^N \theta_{x_i}$ and C_3 is a positive constant.

Now, using the rough upper bound

$$|\{\mathbf{x},\mathbf{y} \in \mathcal{V}_{I_2} : e \in \gamma(\mathbf{x},\mathbf{y})\}| \le (b_2-a+1)^2|L|^2,$$

by the construction of Γ, we have

$$A(e) \le C_4(b_2-a)^3 D^{-1} \exp\{NV(z_1) - N(\min_{x_1\le z_1} V(x_1) + \min_{y_1\ge w_1} V(y_1))\}$$

with C_4 a positive constant. Now, observe that by (2.2)

$$D^{-1}\exp\{-N \min_{y_1\ge w_1} V(y_1)\} \le \frac{1}{K_3}$$

and by definition of H (see (2.6))

$$\max_{z_1\in I_2}[V(z_1) - \min_{x_1\le z_1} V(x_1)] \le H.$$

We obtain

$$A(e) \le C_5(b_2-a)^3 e^{NH}.$$

By condition (iv) note that there exists a positive constant C_6 such that $(b_2-a) \le C_6(b-a)$. Thus, we obtain

$$A = \max_{e\in\mathcal{E}_{I_2}} A(e) \le C_7(b-a)^3 e^{NH}.$$

and with Theorem 3.2.1 of Saloff-Coste (1997),

$$\lambda \ge \frac{1}{C_7(b-a)^3 e^{NH}}. \tag{2.9}$$

We are aiming now for a (uniform in $\mathbf{x} \in \mathcal{V}_{I_2}$) lower bound for $\mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(s) \in L_{b_2}]$. First, we recall the following fact: for $\mathbf{x}, \mathbf{y} \in \mathcal{V}_{I_2}$ and $s > 0$,

$$\Big|\mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(s) = \mathbf{y}] - \hat{\pi}(\mathbf{y})\Big| \leq \Big(\frac{\hat{\pi}(\mathbf{y})}{\hat{\pi}(\mathbf{x})}\Big)^{1/2} \exp\{-\lambda s\}, \tag{2.10}$$

see Corollary 2.1.5 in Saloff-Coste (1997). Furthermore, notice that

$$\mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(s) \in L_{b_2}] \geq \mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(s) = \mathbf{v}]$$

for any $\mathbf{v} \in L_{b_2}$.
Then, by inequalities (2.2) and condition (iv) for $\mathbf{x}$ such that $a < x_1 < b$ and $\mathbf{y}$ such that $y_1 = b_2$ we have

$$\Big(\frac{\hat{\pi}(\mathbf{y})}{\hat{\pi}(\mathbf{x})}\Big)^{1/2} \leq \Big(\frac{K_4}{K_3}\Big)^{\frac{1}{2}} e^{\frac{N}{2}(V(x_1)-V(b_2))} \leq e^{C_8(b-a)},$$

where C_8 is a positive constant to be chosen later. Note that we can take C_8 arbitrary large. Hence, using inequality (2.9) and taking

$$s = 2C_7C_8(b-a)^4 e^{NH}$$

we obtain

$$\Big(\frac{\hat{\pi}(\mathbf{y})}{\hat{\pi}(\mathbf{x})}\Big)^{1/2} \exp\{-\lambda s\} \leq e^{-C_8(b-a)}.$$

Since the potential is constant and equals to $V(m)$ on the interval $[b_1, b_3]$, we obtain

$$\hat{\pi}(\mathbf{v}) \geq \frac{1}{2|L|(b_2-a)} \geq \frac{1}{2C_6|L|(b-a)}.$$

Suppose that C_8 is large enough so that

$$e^{-C_8(b-a)} \leq \frac{1}{4|L|C_6(b-a)}.$$

Using (2.10), we obtain

$$\mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(s) = \mathbf{v}] \geq \frac{1}{4C_6|L|(b-a)}. \tag{2.11}$$

Now, divide the time interval $[0,t]$ into $M := \lfloor \frac{t}{s} \rfloor$ subintervals of length s. Using (2.11) and the Markov property we obtain

$$\begin{aligned}
\mathtt{P}_\omega^{\mathbf{x}}[\tau_{\{a,b\}} > t] &\leq \mathtt{P}_\omega^{\mathbf{x}}[\tau' > t] \\
&\leq \mathtt{P}_\omega^{\mathbf{x}}[\hat{S}(sj) \notin L_{b_2}, j = 1, \dots, M] \\
&\leq \Big(1 - \frac{1}{4C_6|L|(b-a)}\Big)^M \\
&\leq \exp\Big\{-\frac{M}{4C_6|L|(b-a)}\Big\} \\
&\leq \exp\Big\{-\frac{t}{C_9(b-a)^5 e^{NH}}\Big\}
\end{aligned}$$

with C_9 a positive constant.
This concludes the proof of Proposition 2.3. □

2.4. *Probability of escape in a given direction.* We also need the following result. For $y \in \mathbb{Z}$, let

$$\tau_y = \inf\{s > 0 : S_1(s) = y\}. \tag{2.12}$$

We can adapt Lemma 3.4 of Comets and Popov (2003) in an elementary way to obtain the following upper bound for the probability of escape in a given direction.

Proposition 2.4. *For some $K_6 \in (0,\infty)$, we have for all $s > 0$, $\mathbf{x} \in \mathcal{V}$, $y \in \mathbb{Z}$*

$$\mathtt{P}_\omega^{\mathbf{x}}[\tau_y < s] \le K_6 \int_0^{s+1} \mathtt{P}_\omega^{\mathbf{x}}[S_1(u) = y] du.$$

3. Case $\kappa/N > 1$

This section is devoted to the proof of the positiveness of the speed of a spider when $\kappa/N > 1$.

Fix a couple (N, L) and let $\mathbf{x}_0 \in L$ be an initial configuration of the spider. In order to simplify notations, we will systematically omit the superscript $\mathbf{x}_0$ for the quenched and the annealed laws and expectations. Remember that

$$\mathcal{T} := \inf\{s > 0 : S_1(s) > 0 \text{ and } S(s) = \Theta_{S_1(s)}\mathbf{x}_0\}.$$

We will show that if $\frac{\kappa}{N} > 1$ then $\mathbb{E}[\mathcal{T}] < \infty$ which will imply by the Birkhoff's Ergodic Theorem that $v > 0$. First, for each $t > 1$, we define the set of "t-good" environments.

Definition 3.1. Fix $t > 1$ and let $0 < \varepsilon < 1$. Then, fix a finite absolute constant $K_7 > 0$ (i.e. K_7 does not depend on ω and t). A realization of the potential V is said to be t-good if we have

- $V(\lfloor -K_7 \ln t \rfloor) \ge \frac{2+\varepsilon}{N} \ln t$,
- $V(\lceil K_7 \ln t \rceil) \le -\frac{2+\varepsilon}{N} \ln t$,
- $\max_{i\in[\lfloor -K_7 \ln t\rfloor, \lceil K_7 \ln t\rceil]} \max_{j\ge i}(V(j) - V(i)) \le \frac{1-\varepsilon}{N} \ln t$.

We will call Λ_t the set of t-good environments. See Figure 3.4.

The following decomposition is the key of our analysis.

$$\begin{aligned}\mathbb{P}[\mathcal{T} > t] &= \int_\Omega \mathtt{P}_\omega[\mathcal{T} > t] d\mathbf{P}(\omega)\\ &\le \sup_{\omega\in\Lambda_t} \mathtt{P}_\omega[\mathcal{T} > t] + \mathbf{P}[\Lambda_t^c].\end{aligned} \tag{3.1}$$

In the two following subsections we will show that both terms of the right-hand side of (3.1) are integrable in t and thus $\mathbb{E}[\mathcal{T}] < \infty$. We start with the term $\mathbf{P}[\Lambda_t^c]$.

3.1. *Upper bound on $\boldsymbol{P}[\Lambda_t^c]$.* By definition of Λ_t we obtain that

$$\begin{aligned}\mathbf{P}[\Lambda_t^c] &\le \mathbf{P}\Big[V(\lfloor -K_7 \ln t\rfloor) < \frac{2+\varepsilon}{N}\ln t\Big] + \mathbf{P}\Big[V(\lceil K_7 \ln t\rceil) > -\frac{2+\varepsilon}{N}\ln t\Big]\\ &\quad + \mathbf{P}\Big[\max_{i\in[\lfloor -K_7 \ln t\rfloor, \lceil K_7 \ln t\rceil]} \max_{j\ge i}(V(j) - V(i)) > \frac{1-\varepsilon}{N}\ln t\Big]\\ &\le 2\mathbf{P}\Big[V(\lceil K_7 \ln t\rceil) > -\frac{2+\varepsilon}{N}\ln t\Big]\\ &\quad + \mathbf{P}\Big[\max_{i\in[0, 2\lceil K_7 \ln t\rceil]} \max_{j\ge i}(V(j) - V(i)) > \frac{1-\varepsilon}{N}\ln t\Big].\end{aligned}$$

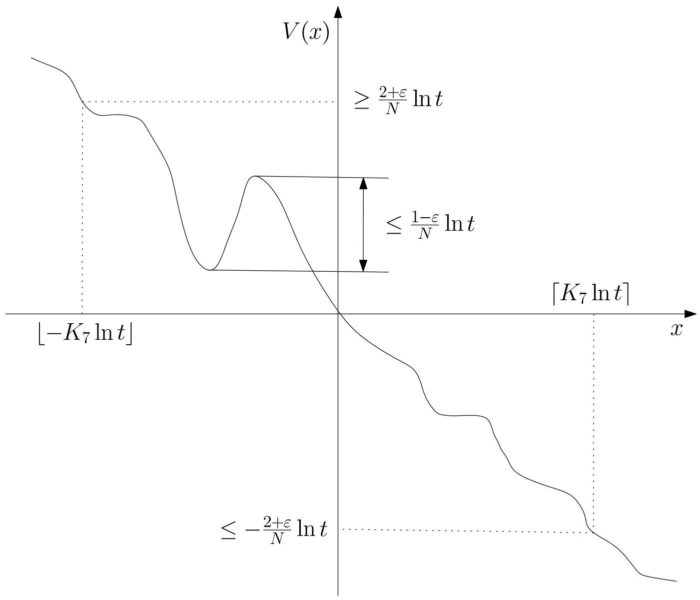

FIGURE 3.4. On the definition of Λ_t.

Let us define

$$A_t = \Big\{V(\lceil K_7 \ln t\rceil) > -\frac{2+\varepsilon}{N}\ln t\Big\}$$

and

$$B_t = \Big\{\max_{i\in[0,2\lceil K_7\ln t\rceil]}\max_{j\geq i}(V(j)-V(i)) > \frac{1-\varepsilon}{N}\ln t\Big\}.$$

Now, we will show that we can choose K_7 large enough such that $\int_0^\infty \mathbf{P}[A_t]dt$ is finite. Observe that as $\varepsilon < 1$ we have

$$\begin{aligned}\mathbf{P}[A_t] &= \mathbf{P}\Big[V(\lceil K_7 \ln t\rceil) > -\frac{2+\varepsilon}{N}\ln t\Big]\\ &\leq \mathbf{P}\Big[V(\lceil K_7 \ln t\rceil) > -\frac{3}{N}\ln t\Big]\\ &\leq \mathbf{P}\Big[\frac{|V(\lceil K_7 \ln t\rceil) - \mathbf{E}[V(1)]\lceil K_7 \ln t\rceil|}{\lceil K_7 \ln t\rceil} > a\Big]\end{aligned}$$

for $a = -\frac{\mathbf{E}[V(1)]}{2}$ if $K_7 > -\frac{6}{\mathbf{E}[V(1)]N}$. As $V(x)$, $x > 0$, is a sum of bounded i.i.d. random variables, we can apply Cramér's Theorem to obtain that

$$\mathbf{P}\Big[\frac{|V(\lceil K_7 \ln t\rceil) - \mathbf{E}[V(1)]\lceil K_7 \ln t\rceil|}{\lceil K_7 \ln t\rceil} > a\Big] \leq e^{-I(a)K_7\ln t}$$

with $I(\cdot)$ the large deviation function defined as

$$I(x) = \sup_{l>0}[lx - \ln \mathbf{E}[lV(1)]].$$

Taking $K_7 > \frac{1}{I(a)} \vee -\frac{6}{\mathbf{E}[V(1)]N}$, we obtain that

$$\mathbf{P}[A_t] \leq e^{-C_1 \ln t} = \frac{1}{t^{C_1}}$$

with $C_1 > 1$. This shows that $\int_0^\infty \mathbf{P}[A_t]dt$ is finite.

Now, let us show that, with the choice of K_7 above, $\int_0^\infty \mathbf{P}[B_t]dt$ is finite too. We have

$$\begin{aligned}\mathbf{P}[B_t] &= \mathbf{P}\Big[\max_{i\in[0,2\lceil K_7 \ln t\rceil]} \max_{j\geq i}(V(j)-V(i)) > \frac{1-\varepsilon}{N}\ln t\Big] \\ &\leq \sum_{i=0}^{2\lceil K_7 \ln t\rceil} \mathbf{P}\Big[\max_{j\geq i}(V(j)-V(i)) > \frac{1-\varepsilon}{N}\ln t\Big].\end{aligned}$$

The estimate (2.7) in Fribergh et al. (2010) yields

$$\mathbf{P}[B_t] \leq C_2 \frac{\lceil K_7 \ln t\rceil}{t^{\frac{\kappa}{N}(1-\varepsilon)}}$$

with C_2 a positive finite constant. As $\frac{\kappa}{N} > 1$, we can choose ε sufficiently small such that $\frac{\kappa}{N}(1-\varepsilon) > 1$, this shows the integrability of $\mathbf{P}[B_t]$.

3.2. *Upper bound on* $\sup_{\omega\in\Lambda_t} \mathtt{P}_\omega[\mathcal{T} > t]$. Let us denote $x_l = \lfloor -K_7 \ln t\rfloor$ and $x_r = \lceil K_7 \ln t\rceil$. Recall that the initial configuration of the spider is $\mathbf{x}_0 \in L$. We use the following decomposition

$$\begin{aligned}\mathtt{P}_\omega[\mathcal{T} > t] &= \mathtt{P}_\omega\Big[\mathcal{T} > t, \tau_{x_r} > \frac{t}{2}\Big] + \mathtt{P}_\omega\Big[\mathcal{T} > t, \tau_{x_r} \leq \frac{t}{2}\Big] \\ &\leq \mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}\Big] + \mathtt{P}_\omega\Big[\mathcal{T} > t, \tau_{x_r} \leq \frac{t}{2}\Big].\end{aligned}$$

3.2.1. *Upper bound on* $\mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}\Big]$. We write

$$\begin{aligned}\mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}\Big] &= \mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}, \tau_{x_r} > \tau_{x_l}\Big] + \mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}, \tau_{x_r} < \tau_{x_l}\Big] \\ &\leq \mathtt{P}_\omega[\tau_{x_r} > \tau_{x_l}] + \mathtt{P}_\omega\Big[\tau_{\{x_l,x_r\}} > \frac{t}{2}\Big]. \end{aligned} \tag{3.2}$$

Let us first treat the second term of the right-hand side of (3.2). Observe that on the interval $[x_l, x_r+d]$ (where d is from subsection 2.3), we have for $\omega \in \Lambda_t$, $e^{NH} \leq C_1 t^{1-\varepsilon}$ with C_1 a positive constant. Using Proposition 2.3 we obtain immediately that

$$\mathtt{P}_\omega\Big[\tau_{\{x_l,x_r\}} > \frac{t}{2}\Big] \leq \exp\Big\{-\frac{t^\varepsilon}{C_2(\ln t)^5}\Big\} \tag{3.3}$$

with C_2 a positive constant.

For the first term of the right-hand side of (3.2) let us write

$$\begin{aligned}\mathtt{P}_\omega[\tau_{x_r} > \tau_{x_l}] &= \mathtt{P}_\omega\Big[\tau_{x_r} > \tau_{x_l}, \tau_{\{x_l,x_r\}} > \frac{t}{2}\Big] + \mathtt{P}_\omega\Big[\tau_{x_r} > \tau_{x_l}, \tau_{\{x_l,x_r\}} \leq \frac{t}{2}\Big] \\ &\leq \mathtt{P}_\omega\Big[\tau_{\{x_l,x_r\}} > \frac{t}{2}\Big] + \mathtt{P}_\omega[\tau_{x_l} < t]. \end{aligned} \tag{3.4}$$

We can bound from above the first term of the right-hand side of (3.4) using (3.3).

In order to bound from above the second term (3.4), we use Proposition 2.4 and (2.2) to obtain

$$\begin{aligned}
\mathtt{P}_\omega[\tau_{x_l} < t] &\leq K_6 \int_0^{t+1} \mathtt{P}_\omega^{\mathbf{x}_0}[S_1(u) = x_l]du \\
&= K_6 \int_0^{t+1} \sum_{\mathbf{y}\in L_{x_l}} \mathtt{P}_\omega^{\mathbf{x}_0}[S(u) = \mathbf{y}]du \\
&= K_6 \int_0^{t+1} \sum_{\mathbf{y}\in L_{x_l}} \frac{\pi(\mathbf{y})}{\pi(\mathbf{x}_0)} \mathtt{P}_\omega^{\mathbf{y}}[S(u) = \mathbf{x}_0]du \\
&\leq K_6 |L|(t+1) \frac{\pi(\mathbf{y})}{\pi(\mathbf{x}_0)} \\
&\leq C_3 t e^{-NV(x_l)}
\end{aligned}$$

with C_3 a positive constant.
For $\omega \in \Lambda_t$, we have that $V(x_l) > \frac{2+\varepsilon}{N} \ln t$. Hence,

$$\mathtt{P}_\omega[\tau_{x_l} < t] \leq \frac{C_4}{t^{1+\varepsilon}} \tag{3.5}$$

with C_4 a positive constant. Eventually, by (3.2), (3.3), (3.4) and (3.5), we obtain

$$\mathtt{P}_\omega\Big[\tau_{x_r} > \frac{t}{2}\Big] \leq 2\exp\Big\{-\frac{t^\varepsilon}{C_2(\ln t)^5}\Big\} + \frac{C_4}{t^{1+\varepsilon}} \tag{3.6}$$

for $\omega \in \Lambda_t$.

3.2.2. *Upper bound on* $\mathtt{P}_\omega\Big[\mathcal{T} > t, \tau_{x_r} \leq \frac{t}{2}\Big]$. Let $\mathfrak{F}_{x_r}$ be the σ-field generated by the process S up to the stopping time τ_{x_r}. Using the Markov property, we obtain

$$\begin{aligned}
\mathtt{P}_\omega\Big[\mathcal{T} > t, \tau_{x_r} \leq \frac{t}{2}\Big] &= \mathtt{E}_\omega\Big[1_{\{\tau_{x_r}\leq t/2\}} \mathtt{P}_\omega[\mathcal{T} > t \mid \mathfrak{F}_{x_r}]\Big] \\
&\leq \mathtt{E}_\omega\Big[1_{\{\tau_{x_r}\leq t/2\}} \mathtt{P}_\omega^{S(\tau_{x_r})}\Big[\mathcal{T} > \frac{t}{2}\Big]\Big] \\
&\leq \mathtt{P}_\omega\Big[\tau_{x_r} \leq \frac{t}{2}\Big] \times \max_{\mathbf{y}\in L_{x_r}} \mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}\Big]. \\
&\leq \max_{\mathbf{y}\in L_{x_r}} \mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}\Big].
\end{aligned}$$

The next step is to bound uniformly in $\mathbf{y}$ the quantity $\mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}\Big]$ for $\mathbf{y} \in L_{x_r}$. We use the following decomposition

$$\begin{aligned}
\mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}\Big] &= \mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}, \tau_0 < t\Big] + \mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}, \tau_0 \geq t\Big] \\
&\leq \mathtt{P}_\omega^{\mathbf{y}}[\tau_0 < t] + \mathtt{P}_\omega^{\mathbf{y}}\Big[\mathcal{T} > \frac{t}{2}, \tau_0 \geq t\Big].
\end{aligned} \tag{3.7}$$

To bound from above the first term of the right-hand side of (3.7), we use Proposition 2.4 to obtain

$$\mathtt{P}_\omega^{\mathbf{y}}[\tau_0 < t] \leq \frac{C_4}{t^{1+\varepsilon}} \tag{3.8}$$

for $\omega \in \Lambda_t$.

For the second term of the right-hand side of (3.7) we start by defining

$$\mathcal{T}' = \inf\{s > 0 : S(s) = \Theta_{S_1(s)}\mathbf{x}_0\}.$$

Then, let Υ be the number of movements it takes the spider to be in local configuration $\mathbf{x}_0$ for the first time.
Formally, if $\Xi = (\Xi(n))_{n\geq 0} = (\Xi_1(n), \dots, \Xi_N(n))_{n\geq 0}$ is the jump chain (recall that the jump chain of a jump Markov process is the sequence of states visited by the Markov process) associated to the jump Markov process S we have

$$\Upsilon = \min\{n \geq 1 : \Xi(n) = \Theta_{\Xi_1(n)}\mathbf{x}_0\}.$$

Observe that by condition (iv) and the facts that the process is irreducible and L is finite, $\Upsilon < \infty$ $\mathbb{P}$-a.s. Now, let $(T_i)_{i\geq 1}$ be the sequence of jump times of the process S. Observe that

$$\begin{aligned} &\mathtt{P}^{\mathbf{y}}_\omega[\exists s \in [0, 2|L|] : S(s) = \Theta_{S_1(s)}\mathbf{x}_0] \\ &\quad = \mathtt{P}^{\mathbf{y}}_\omega[T_1 + T_2 + \dots + T_\Upsilon \leq 2|L|] \\ &\quad \geq \mathtt{P}^{\mathbf{y}}_\omega[T_1 + T_2 + \dots + T_\Upsilon \leq 2|L|, \Upsilon \leq |L|] \\ &\quad \geq \mathtt{P}^{\mathbf{y}}_\omega[T_1 + T_2 + \dots + T_{|L|} \leq 2|L|, \Upsilon \leq |L|] \\ &\quad = \mathtt{P}^{\mathbf{y}}_\omega[T_1 + T_2 + \dots + T_{|L|} \leq 2|L|]\mathtt{P}^{\mathbf{y}}_\omega[\Upsilon \leq |L|]. \end{aligned} \tag{3.9}$$

By condition (iv), there exists $\eta > 0$ such that

$$\mathtt{P}^{\mathbf{y}}_\omega[\Upsilon \leq |L|] \geq \eta \tag{3.10}$$

for all $\mathbf{y}$. Using Markov's inequality and condition (iv) we have

$$\begin{aligned} \mathtt{P}^{\mathbf{y}}_\omega[T_1 + T_2 + \dots + T_{|L|} \leq 2|L|] &\geq 1 - \frac{|L|}{2|L|} \\ &\geq 1 - \frac{1}{2} \\ &\geq \frac{1}{2}. \end{aligned} \tag{3.11}$$

Therefore, using (3.10), (3.11) and (3.9) we obtain

$$\max_{\mathbf{y}\in L_{x_r}} \mathtt{P}^{\mathbf{y}}_\omega[\exists s \in [0, 2|L|] : S(s) = \Theta_{S_1(s)}\mathbf{x}_0] \geq \frac{\eta}{2}.$$

The next step is to divide the interval $[0, \frac{t}{2}]$ into $\lfloor \frac{t}{4|L|} \rfloor$ intervals of size $2|L|$ and observe that by the Markov property,

$$\mathtt{P}^{\mathbf{y}}_\omega\Big[\mathcal{T}' > \frac{t}{2}\Big] \leq \Big(1 - \frac{\eta}{2}\Big)^{\lfloor \frac{t}{4|L|} \rfloor}$$

for all $\mathbf{y} \in L_{x_r}$. As $\{S(0) = \mathbf{y}, \mathcal{T} > t/2, \tau_0 > t\} \subset \{S(0) = \mathbf{y}, \mathcal{T}' > \frac{t}{2}\}$ for all $\mathbf{y} \in L_{x_r}$ and t sufficiently large, we obtain

$$\mathtt{P}^{\mathbf{y}}_\omega\Big[\mathcal{T} > \frac{t}{2}, \tau_0 > t\Big] \leq \Big(1 - \frac{\eta}{2}\Big)^{\lfloor \frac{t}{4|L|} \rfloor} \tag{3.12}$$

for all $\mathbf{y} \in L_{x_r}$.

To sum up, by (3.6), (3.8) and (3.12) we obtain

$$\sup_{\omega\in\Lambda_t} \mathtt{P}_\omega[\mathcal{T} > t] \leq 2\exp\Big\{ -\frac{t^\varepsilon}{C_2(\ln t)^5}\Big\} + 2\frac{C_4}{t^{1+\varepsilon}} + \Big(1 - \frac{\eta}{2}\Big)^{\lfloor \frac{t}{4|L|} \rfloor}. \tag{3.13}$$

This shows that $\int_0^\infty \sup_{\omega\in\Lambda_t} \mathtt{P}_\omega[\mathcal{T} > t]dt$ is finite.

3.3. *Positiveness of the speed.* In this subsection we show that the speed of the spider is positive $\mathbb{P}$-a.s. if $\kappa/N > 1$. Let $\zeta_0 = 0$ and

$$\zeta_n = \inf\{j > \zeta_{n-1}, \Xi_1(j) > \Xi_1(\zeta_{n-1}) \text{ and } \Xi(j) = \Theta_{\Xi_1(j)}\mathbf{x}_0\}$$

for $n \geq 1$.
Since the sequence $(\zeta_{n+1} - \zeta_n)_{n\geq 0}$ is ergodic under the annealed measure $\mathbb{P}$, we can apply the Birkhoff's Ergodic Theorem to obtain that

$$\lim_{n\to\infty} \frac{\zeta_n}{n} = \mathbb{E}[\zeta_1] = \mathbb{E}[\mathcal{T}] < \infty, \ \mathbb{P}\text{-a.s.}$$

where the last equality will be shown below.

Now, take $\zeta_n \leq m < \zeta_{n+1}$, we obtain

$$\Xi_1(\zeta_n) - (\zeta_{n+1} - \zeta_n) \leq \Xi_1(m) < \Xi_1(\zeta_n) + (\zeta_{n+1} - \zeta_n)$$

which implies

$$\frac{\Xi_1(\zeta_n) - (\zeta_{n+1} - \zeta_n)}{n}\frac{n}{\zeta_{n+1}} \leq \frac{\Xi_1(m)}{m} < \frac{\Xi_1(\zeta_n) + (\zeta_{n+1} - \zeta_n)}{n}\frac{n}{\zeta_n}. \tag{3.14}$$

Observe that the sequence $(\Xi_1(\zeta_{n+1}) - \Xi_1(\zeta_n))_{n\geq 0}$ is also ergodic under $\mathbb{P}$. Therefore we can apply the Birkhoff's Ergodic Theorem to obtain

$$\lim_{m\to\infty} \frac{\Xi_1(\zeta_m)}{m} = \mathbb{E}[\Xi_1(\zeta_1)] = \mathbb{E}[S_1(\mathcal{T})] > 0, \ \mathbb{P}\text{-a.s.}$$

where the last equality follows from the fact that $\Xi_1(\zeta_1) = S_1(\mathcal{T})$. Now, let $m \to \infty$ in (3.14). As the spider is transient to the right, we have also $n \to \infty$ $\mathbb{P}$-a.s. Thus, we can deduce that

$$\lim_{m\to\infty} \frac{\Xi_1(m)}{m} = \frac{\mathbb{E}[S_1(\mathcal{T})]}{\mathbb{E}[\mathcal{T}]} > 0, \quad \mathbb{P}\text{-a.s.} \tag{3.15}$$

The result (3.15), obtained for the embedded Markov chain, transfers to continuous time. Indeed, there exists a family $(e_i)_{i\geq 1}$ of exponential random variables of parameter 1, such that the nth jump of the continuous time random process S occurs at time $\sum_{i=1}^n e_i$. These random variables are independent of the environment and the discrete-time random walk. It follows that we can write $\mathcal{T} = \sum_{i=1}^{\zeta_1} e_i$. Hence, $\mathbb{E}[\mathcal{T}] = \mathbb{E}[\zeta_1]E[e_1] = \mathbb{E}[\zeta_1]$.

Let us denote by R_n the time of the nth jump of S. Then, take $R_n \leq t < R_{n+1}$, we obtain

$$\frac{\Xi_1(n)}{R_{n+1}} \leq \frac{S_1(t)}{t} < \frac{\Xi_1(n)}{R_n}$$

and consequently

$$\frac{\Xi_1(n)}{n}\frac{n}{R_{n+1}} \leq \frac{S_1(t)}{t} < \frac{\Xi_1(n)}{n}\frac{n}{R_n}. \tag{3.16}$$

Eventually, taking the limit $n \to \infty$ in inequality (3.16), using (3.15) and the law of large numbers for the sequence R_1, $R_2 - R_1$, $R_3 - R_2$, ..., we obtain

$$v = \lim_{t\to\infty} \frac{S_1(t)}{t} = \frac{\mathbb{E}[S_1(\mathcal{T})]}{\mathbb{E}[\mathcal{T}]} > 0 \ \mathbb{P}\text{-a.s.}$$

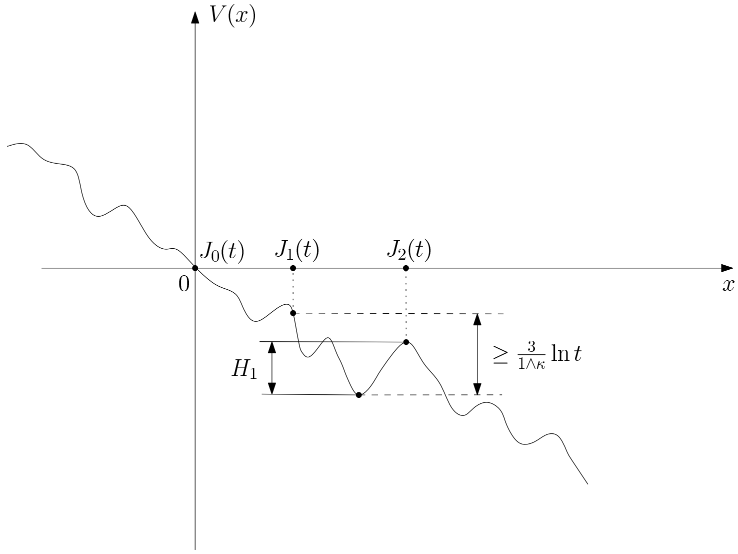

FIGURE 4.5. On the definition of the valleys.

4. Case $\kappa/N < 1$

In this last section, we show that if $\kappa/N < 1$, the speed of a spider is null. First, we need to introduce some notations. Following Fribergh et al. (2010) we will define the valleys of the potential V as follows. For $t > 1$, we define by induction the environment dependent sequence $(J_i(t))_{i\geq 1}$ by

$$J_0(t) = 0,$$

$$J_{i+1}(t) = \min\{j : \ j \geq J_i(t), V(J_i(t)) - \min_{l\in[J_i(t),j]} V(l) \geq \frac{3}{1\wedge\kappa}\ln t, \\ V(j) = \max_{l\geq j} V(l)\}.$$

In the following the dependence on t will be frequently omitted to ease the notations. The portion of the environment $[J_i, J_{i+1})$ is called the ith valley. In Fribergh et al. (2010), it is shown that for t large enough the valleys are descending in the sense that $V(J_{i+1}) < V(J_i)$ for all $i \geq 0$. Then, we define the depth of the ith valley as follows (see Figure 4.5)

$$H_i = \max_{J_i(t)\leq j<l<J_{i+1}(t))} (V(l) - V(j)).$$

Let us denote

$$\mathcal{L}_t(m, m') = \{i \geq 1 : [J_i, J_{i+1}) \cap [\lfloor m\rfloor, \lfloor m'\rfloor) \neq \emptyset\}.$$

We define $\nu_0 := \frac{\kappa}{N}$ and consider ν such that $\nu_0 < \nu < 1$. Then, take $\varepsilon = \varepsilon(t) = \frac{4\ln\ln t}{\ln t}$ and define

$$\mathcal{M} = \Big\{i \in \mathcal{L}_t(0, t^\nu) : H_i \geq \frac{1-\varepsilon}{\kappa}\ln t\Big\},$$

$$\Psi_t = \Big\{\omega : |\mathcal{M}| \geq \frac{1}{3}t^{\nu-\nu_0}\Big\}.$$

By Lemma 3.5 of Fribergh et al. (2010), on each subinterval of length t^{ν_0}, we find a valley of depth at least $\frac{1-\varepsilon}{\kappa}\ln t$ with probability at least 1/2 for sufficiently large t. Since the interval $[0, t^\nu]$ contains $t^{\nu-\nu_0}$ such intervals, we have

$$\mathbf{P}[\Psi_t] \geq 1 - \exp(-C_1 t^{\nu-\nu_0}).$$

For $i \in \mathcal{M}$, using the notation defined in (2.12), define $\sigma_i = \tau_{J_{i+1}+1} - \tau_{J_i+1}$ and let

$$s_0 = \frac{t}{4\gamma_2(\ln t)^4}.$$

Then, by Lemma 3.4 of Comets and Popov (2003) and the fact that $\kappa/N < 1$, for any $i \in \mathcal{M}$,

$$\begin{aligned} \mathtt{P}^{\mathbf{x}}_\omega[\sigma_i < s_0] &\leq 2\gamma_2 s_0 \exp\Big(-\frac{N}{\kappa}(\ln t - 4\ln\ln t)\Big) \\ &\leq 2\gamma_2 s_0 \exp\Big(-\ln t + 4\ln\ln t\Big) \\ &= 2\gamma_2 s_0 t^{-1}(\ln t)^4 \\ &= \frac{1}{2}. \end{aligned} \tag{4.1}$$

uniformly in $\mathbf{x}$, for sufficiently large t.

Define the family of random variables $\zeta_i = \mathbf{1}\{\sigma_i < s_0\}$, $i \in \mathcal{M}$. Observe that by the Markov property and (4.1), the sequence $(\zeta_i)_i$ is stochastically dominated by a sequence of independent Bernoulli$\{0,1\}$ random variables $(\eta_i)_i$ of parameter 1/2. Moreover, for t large enough, observe that we have

$$\frac{1}{3}s_0\frac{1}{3}t^{\nu-\nu_0} = \frac{1}{36\gamma_2(\ln t)^4}t^{1+\nu-\nu_0} > t.$$

With the same notation as in (2.12), since $|\mathcal{M}| \geq \frac{1}{3}t^{\nu-\nu_0}$ for $\omega \in \Psi_t$ and since the sequence $(\eta_i)_{i\geq 1}$ is i.i.d., we see using Cramér's theorem that for t large enough

$$\begin{aligned} \mathtt{P}_\omega[\tau_{\lfloor t^\nu \rfloor} < t] &\leq \mathtt{P}_\omega\Big[\sum_{i\in\mathcal{M}} \zeta_i > \frac{2}{3}|\mathcal{M}|\Big] \\ &\leq P\Big[\sum_{i\in\mathcal{M}} \eta_i > \frac{2}{3}|\mathcal{M}|\Big] \\ &\leq \exp\Big(-C_2 t^{\nu-\nu_0}\Big). \end{aligned}$$

From this last inequality, we immediately conclude that the speed of the spider is null $\mathbb{P}$-a.s.

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