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Functional macroscopic behavior of weighted random ball model

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Abstract. We consider a generalization of the weighted random ball model defined by

$$M(\mathbf{y}) = \int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} mh\left(\frac{\mathbf{y}-\mathbf{x}}{r}\right) N(d\mathbf{x}, dr, dm)$$

where N is a random Poisson measure on $\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$ with a product heavy tailed intensity measure and $h : \mathbb{R}^d \to \mathbb{R}$ is a fading function. This functional can serve as a basic model for transmission with fading effect. The convergence of the finite-dimensional distributions of related generalized random fields under various scalings is known in the particular case when h is the indicator function of the unit ball in $\mathbb{R}^d$, see Breton and Dombry (2009) and references therein. In the present paper, tightness and functional convergence are investigated. Using suitable moment estimates, we prove functional convergences for some parametric classes of configurations under the so-called large ball scaling and intermediate ball scaling. Convergence in the space of distributions is also discussed.

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1. Introduction

We consider weighted random balls in $\mathbb{R}^d$ generated by a Poisson random measure N_λ on $\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}$ with intensity

$$n_\lambda(d\mathbf{x}, dr, dm) = \lambda d\mathbf{x} F(dr) G(dm)$$

where $\lambda \in \mathbb{R}^+$ and F, G are probability measures on $\mathbb{R}^+$ and $\mathbb{R}$ respectively. For each 3-tuple $(\mathbf{x}, r, m)$, $\mathbf{x}$ represents the center of the Euclidean ball $B(\mathbf{x}, r)$ and r its radius, m stands for the weight of the ball. The parameter λ is interpreted as the intensity of the balls in $\mathbb{R}^d$. Such models are used for instance to represent a spatial communication network, see Kaj (2006), Yang and Petropulu (2003). In this case, $\mathbf{x}$ represents a station transmitting a signal, r the range of emission and m the intensity of the signal, see Breton and Dombry (2009) and references therein. Following this interpretation, the signal m transmitted by $\mathbf{x}$ is received in some $\mathbf{y} \in \mathbb{R}^d$ if and only if $\mathbf{y} \in B(\mathbf{x}, r)$, and the overall signal received from the stations at $\mathbf{y}$ is given by

$$\int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m \mathbf{1}_{B(\mathbf{x},r)}(\mathbf{y}) N_\lambda(d\mathbf{x}, dr, dm) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} m \mathbf{1}_{B(\mathbf{0},1)}((\mathbf{y}-\mathbf{x})/r) N_\lambda(d\mathbf{x}, dr, dm). \tag{1.1}$$

The mathematical study of such a quantity has some history and we refer in particular to Kaj and Taqqu (2008) (when $d = 1$), Kaj et al. (2007), Biermé et al. (2010) (when $G = \delta_1$, *i.e.* the weight are not considered) and Breton and Dombry (2009). From a modeling point of view, it is natural to consider that the signal transmitted by $\mathbf{x}$ and received in $\mathbf{y}$ fades when $\mathbf{y}$ gets away from the station $\mathbf{x}$. In order to take into account this phenomenon, we introduce a fading function h replacing $\mathbf{1}_{B(\mathbf{0},1)}$ in (1.1), more precisely the faded signal received at $\mathbf{y}$ from $\mathbf{x}$ is $mh((\mathbf{y}-\mathbf{x})/r)$ and, assuming moreover that no interference occurs between the stations, the quantity of signal received at $\mathbf{y}$ is now given by

$$M(\mathbf{y}) = \int_{\mathbb{R}^d \times \mathbb{R}^+ \times \mathbb{R}} mh((\mathbf{y}-\mathbf{x})/r) N_\lambda(d\mathbf{x}, dr, dm). \tag{1.2}$$

From a physical point of view, it is natural to assume that $h(\mathbf{y})$ is a radially non-increasing function with $h(\mathbf{0}) = 1$, $0 \le h(\mathbf{y}) \le 1$ and $\lim_{\|\mathbf{y}\| \to +\infty} h(\mathbf{y}) = 0$. The function h is said to be radially non-increasing if for all $\mathbf{y} \in \mathbb{R}^d$, the function $r \mapsto h(r\mathbf{y})$ is non-increasing on $[0, +\infty)$. However, from a mathematical point of view, more general assumptions will be enough, see assumption ($\mathbf{A}_3$) below.

In the sequel, we are more generally interested in the contribution

$$M(\mu) = \int_{\mathbb{R}^d} M(\mathbf{y}) \mu(d\mathbf{y})$$

of the model in a configuration of points $\mathbf{y}$ represented by a measure μ. For instance, the configuration reduced to the point $\mathbf{y}$ is represented by $\mu = \delta_{\mathbf{y}}$ and in this case $M(\delta_{\mathbf{y}}) = M(\mathbf{y})$. It is natural to consider finite positive measures μ on $\mathbb{R}^d$ but our study supports signed measures μ with finite total variation. In the sequel, we shall note $\mathcal{M}$ the set of such measures and we recall that, equipped with the total variation norm $\|\mu\|_{\mathcal{M}} = |\mu|(\mathbb{R}^d)$, $\mathcal{M}$ is a Banach space. Actually in order to make our study easier, and in contrast with Breton and Dombry (2009) and Biermé et al.

(2010), we shall consider signed measures μ with density, *i.e.* $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ for some $\phi \in L^1(\mathbb{R}^d)$. Setting $\tau_{\mathbf{x},r}h(\mathbf{y}) = h((\mathbf{y}-\mathbf{x})/r)$ and $\mu[f] = \int_{\mathbb{R}^d} f(\mathbf{y})\mu(d\mathbf{y})$ for $f \in L^1(\mathbb{R}^d, \mu)$, the Fubini theorem allows to rewrite

$$M(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} m\mu[\tau_{\mathbf{x},r}h] N_\lambda(d\mathbf{x}, dr, dm). \tag{1.3}$$

Note that the stochastic integral in (1.3) is well defined and the change in the order of integrals is justified when

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} |m\mu[\tau_{\mathbf{x},r}h]|\; n_\lambda(d\mathbf{x}, dr, dm)\\ \leq\;& \lambda|\mu|(\mathbb{R}^d)\left(\int_{\mathbb{R}^d}|h(\mathbf{x})|d\mathbf{x}\right)\left(\int_{\mathbb{R}^+} r^d F(dr)\right)\left(\int_{\mathbb{R}}|m|G(dm)\right) < +\infty. \end{aligned}\tag{1.4}$$

We will always suppose that all three integrals in (1.4) above are finite (precise assumptions on F, G and h are given in the set of conditions (**A**) below). Furthermore, in this case, the expected value of $M(\mu)$ is given by

$$\mathbb{E}[M(\mu)] = \lambda\mu(\mathbb{R}^d)\left(\int_{\mathbb{R}^d} h(\mathbf{y})d\mathbf{y}\right)\left(\int_{\mathbb{R}^+} r^d F(dr)\right)\left(\int_{\mathbb{R}} mG(dm)\right).$$

In order to investigate the macroscopic behavior of the generalized random field $(M(\mu))_{\mu\in\mathcal{M}}$, we apply a scaling $x \mapsto \rho x$, with $\rho < 1$. The scaling contracts the space $\mathbb{R}^d$ and is interpreted as zoom-out in the model. Note that when $\rho > 1$, the scaling becomes a dilation of $\mathbb{R}^d$ and is interpreted as zoom-in. In contrast to Biermé et al. (2010) and Breton and Dombry (2009) but like in Kaj et al. (2007), we focus in this article only on zoom-out (see below for further comments on the relation between this contribution and several related papers). In order to derive non-trivial asymptotics, the intensity λ of the Poisson measure is adapted to the scaling procedure by allowing $\lambda := \lambda(\rho)$ to depend on the zooming factor ρ. Note that the natural intensity $\lambda(\rho)$ corresponding to the scaling $x \mapsto \rho x$ is $\lambda(\rho) = \rho^{-d}\lambda$. After the scaling, the generalized field becomes

$$M_\rho(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} m\mu[\tau_{\mathbf{x},r}h] N_{\rho,\lambda(\rho)}(d\mathbf{x}, dr, dm) \tag{1.5}$$

where $N_{\rho,\lambda(\rho)}$ is the Poisson random measure with intensity

$$n_{\rho,\lambda(\rho)}(d\mathbf{x}, dr, dm) = \lambda(\rho)d\mathbf{x}F_\rho(dr)G(dm)$$

and F_ρ is the image measure of F under $r \mapsto \rho r$. We are finally led to investigate, for a proper normalization $n(\rho)$, the fluctuations of the rescaled and centered random field

$$\widetilde{M}_\rho(\mu) = n(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)]).$$

In order to derive non-trivial asymptotics for the model (1.5), the distributions F and G driving the behavior of the radius r and of the weights m, and the shape function h must satisfy some conditions, denoted by conditions (**A**), that we state now precisely:

- The probability G is assumed to belong to the normal domain of attraction of the α-stable distribution $S_\alpha(\sigma, b, \tau)$ with $\alpha \in (1, 2]$, *i.e.* if $X_1, \dots, X_n$ are *i.i.d.* with distribution G, $n^{-1/\alpha}(X_1 + \cdots + X_n) \Rightarrow S_\alpha(\sigma, b, \tau)$. According to Feller (1966, XVII.5), this is equivalent to the following estimate on the characteristic function φ_G of G:

$$\varphi_G(\theta) = 1 + i\theta\tau - \sigma^\alpha|\theta|^\alpha(1 + ib\varepsilon(\theta)\tan(\pi\alpha/2)) + o(|\theta|^\alpha) \quad \text{as } \theta \to 0. \qquad (\mathbf{A}_1)$$

In the case $\alpha \in (1, 2)$, a typical choice for G is a heavy-tailed distribution while for $\alpha = 2$, G may be any distribution with finite variance. Observe that since $\alpha > 1$, G has a finite moment of order 1.

- The probability F is assumed to have a regularly varying tail satisfying

$$\bar{F}(r) := \int_r^{+\infty} F(du) \sim_{r\to+\infty} C_\beta r^{-\beta} \quad \text{for some } d < \beta < \alpha d. \qquad (\mathbf{A}_2)$$

Here and in the sequel, $f(r) \sim_{r\to+\infty} g(r)$ indicates that

$$\lim_{r\to+\infty} f(r)/g(r) = 1.$$

Observe that, under ($\mathbf{A}_2$), the expectation of the volume of the random balls $\int_{\mathbb{R}^+} r^d F(dr)$ is finite (see Lemma A.3 below) and the bound (1.4) indeed holds true.

- The shape function h is assumed to be continuous almost everywhere and such that

$$h^*(\mathbf{x}) := \sup\{|h(r\mathbf{x})| \ : \ r \geq 1\} \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d). \qquad (\mathbf{A}_3)$$

Note that this implies that $h \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ and that if h is radially non-increasing, then $h^* = h$. Indeed, h^* is the smallest radially non-increasing function dominating h.

The convergences of the finite-dimensional distributions (*fdd*) of $\widetilde{M}_\rho$ were (essentially) already derived in Breton and Dombry (2009) under three different regimes depending on the behavior of $\lambda(\rho)\rho^\beta$ (the so-called large, small and intermediate ball regimes). In this note, we actually focus on the corresponding functional convergence for the generalized random fields $\left(\widetilde{M}_\rho(\mu)\right)_{\mu\in\mathcal{M}}$. Since there is no natural functional space in which the random function $\mu \in \mathcal{M} \mapsto \widetilde{M}_\rho(\mu)$ belongs (at least heuristically), we choose to consider a special parametric sub-family $(\mu_\mathbf{t})_{\mathbf{t}\in\mathbb{R}^p}$ of $\mathcal{M}$ and to investigate the tightness of the random fields $\left(\widetilde{M}_\rho(\mu_\mathbf{t})\right)_{\mathbf{t}\in\mathbb{R}^p}$. Our main contribution is to prove tightness in the space of continuous functions $\mathcal{C}(\mathbb{R}^p)$ of such random fields under suitable conditions on the family $(\mu_\mathbf{t})_{\mathbf{t}\in\mathbb{R}^p}$. As a by-product, we also obtain weak convergence of the random fields $\widetilde{M}_\rho$ in the space of distributions.

Let us conclude this introduction with some comments on the relations between this article and related papers. In Breton and Dombry (2009), *fdd* convergences are obtained for $(\widetilde{M}_\rho(\mu))_\mu$ when μ belongs to some special subspace $\mathcal{M}_{\alpha,\beta}$ on which

$$\int_{\mathbb{R}^d} |\mu(B(\mathbf{x}, r))|^\alpha d\mathbf{x} \leq C(r^p \wedge r^q) \quad \text{for some } p < \beta < q \qquad (1.6)$$

(roughly speaking the condition requires a control of the measures $\mu(B(\mathbf{x}, r))$ of both large and small balls, uniform in the centers of the balls). Moreover, it deals simultaneously with the macroscopic behavior (*i.e.* $\rho \to 0$ and F has a power law behavior in $+\infty$ of order $\beta > d$) and microscopic behavior (*i.e.* $\rho \to +\infty$ and F has a power law behavior in 0 of order $\beta < d$). In comparison, in this paper we deal only with the macroscopic behavior (*i.e.* $\rho \to 0$ and $\beta > d$) and for special measures

$\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y} \in \mathcal{M}$ with density $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. But, F is not any more assumed to have a density and we only assume the tail condition ($\mathbf{A}_2$). Moreover a shape function h is considered in the model (1.2) to take into account the fading of the signal in the communication network. But our main contribution in this setting is to derive tightness to obtain functional counterparts of the *fdd* convergences. In the particular case of the dimension $d = 1$, the model is related to the infinite Poisson model in Mikosch et al. (2002) and to the continuous flow reward in Kaj and Taqqu (2008). In both papers, the authors deal with the asymptotic behaviour of the random process corresponding in our setting to $(\widetilde{M}_\rho(1_{[0,t]}(y)dy))_{t\geq 0}$ and the issue of tightness is addressed in both papers.

The rest of the article is organized as follows: the main results are stated in Section 2 and proved in Section 3. Technical results are postponed in the Appendix.

2. Main results

First, we recall the finite-dimensional convergence for the generalized random field $\widetilde{M}_\rho = n(\rho)^{-1}(M_\rho(\mu) - \mathbb{E}[M_\rho(\mu)])$. Actually, we state a slight modification of the main results in Breton and Dombry (2009) replacing $\mathbf{1}_{B(\mathbf{0},1)}$ therein by a shape function h satisfying ($\mathbf{A}_3$). We shall abusively write $\mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, instead of $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. The symbol $\stackrel{fdd}{\Longrightarrow}$ stands for the *fdd* convergences and the symbol $\stackrel{\mathcal{X}}{\Longrightarrow}$ is also used throughout to indicate a functional convergence in the functional space $\mathcal{X}$ (for instance, in Theorem 2.4, $\mathcal{X} = \mathcal{C}(\mathbb{R}^d)$, the space of continuous function of $\mathbb{R}^d$).

Proposition 2.1. *Suppose conditions* ($\mathbf{A}$) *hold.*

(1) *(Large ball regime) If $\lambda(\rho)\rho^\beta \to +\infty$, then, setting $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha}$, we have as $\rho \to 0$:*

$$\widetilde{M}_\rho(\mu) \stackrel{fdd}{\Longrightarrow} Z_\alpha(\mu), \quad \mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$$

where Z_α is the stable field

$$Z_\alpha(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}^+} \mu[\tau_{\mathbf{x},r}h]M_\alpha(dr, d\mathbf{x})$$

with respect to the α-stable measure M_α with control measure $\sigma^\alpha C_\beta r^{-1-\beta}drd\mathbf{x}$ and constant skewness function b, where σ and b are related to G by ($\mathbf{A}_1$).

(2) *(Intermediate ball regime) If $\lambda(\rho)\rho^\beta \to a$ for some $a \in (0, +\infty)$, then, setting $n(\rho) = 1$, we have as $\rho \to 0$:*

$$\widetilde{M}_\rho(\mu) \stackrel{fdd}{\Longrightarrow} J_a(\mu), \quad \mu \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$$

where J_a is the compensated Poisson integral

$$J_a(\mu) = \int_{\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^+} m\mu[\tau_{\mathbf{x},r}h]\widetilde{N}_{\beta,a}(d\mathbf{x}, dr, dm)$$

with respect to the compensated Poisson random measure $\widetilde{N}_{\beta,a}$ with intensity $aC_\beta r^{-\beta-1}d\mathbf{x}drG(dm)$.

In the sequel, the finite-dimensional results are strengthened into functional convergence for a parametric sub-family of measures $\mu_{\mathbf{t}}(d\mathbf{y}) = \phi_{\mathbf{t}}(\mathbf{y})d\mathbf{y}, \mathbf{t} \in \mathbb{R}^p$, in $L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, so that the generalized random field $\widetilde{M}_\rho$ induces a p-dimensional random field $(\widetilde{M}_\rho(\mu_{\mathbf{t}}))_{\mathbf{t}\in\mathbb{R}^p}$. To that aim, we investigate the tightness in $\mathcal{C}(\mathbb{R}^p)$ of this induced p-dimensional random field. The proof relies on a Censov criterion and on moment estimates for increments presented in Section 3.2. Actually in our setting, the relevant increments are generalized increments on blocks defined as follows. Let $\mathbf{s}, \mathbf{t} \in \mathbb{R}^p$ be such that $s_i \le t_i, 1 \le i \le p$, and consider the corresponding block $[\mathbf{s}, \mathbf{t}] = \prod_{i=1}^p [s_i, t_i]$. The dimension of $[\mathbf{s}, \mathbf{t}]$ is given by the number of indices i such that $s_i < t_i$. Let say that a p-tuple $\mathbf{a} = (a_1, \dots, a_p)$ is adapted to $[\mathbf{s}, \mathbf{t}]$ whenever $a_i = 0$ when $s_i = t_i$, $1 \le i \le p$. The generalized increment of a random field $X = (X_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^p}$ on a block $[\mathbf{s}, \mathbf{t}]$ in $\mathbb{R}^p$ is defined (up to a factor ± 1) by

$$X([\mathbf{s}, \mathbf{t}]) := \sum_\epsilon (-1)^{p-\sum_{i=1}^p \epsilon_i} X(s_1 + \epsilon_1(t_1 - s_1), \cdots, s_p + \epsilon_p(t_p - s_p)) \tag{2.1}$$

where the sum above runs over $\epsilon \in \{0, 1\}^p$ adapted to $[\mathbf{s}, \mathbf{t}]$. Similarly for measure $\mu_{\mathbf{t}}$ with density $\phi_{\mathbf{t}}$, we define the increment on a block $[\mathbf{s}, \mathbf{t}]$ of $\phi_{\mathbf{t}}$ by

$$\phi_{[\mathbf{s},\mathbf{t}]} := \sum_\epsilon (-1)^{p-\sum_{i=1}^p \epsilon_i} \phi_{s_1+\epsilon_1(t_1-s_1),\dots,s_p+\epsilon_p(t_p-s_p)} \tag{2.2}$$

where again the sum runs over $\epsilon \in \{0, 1\}^p$ adapted to $[\mathbf{s}, \mathbf{t}]$. Such generalized increments are easy to handle in our context and Example 2.3 below confirms, for uniform-type measures, that such increments make sense in our setting. In order to control such increments, we introduce our main condition on the densities $\phi_{\mathbf{t}}$ of $\mu_{\mathbf{t}}$: we say that the family of densities $(\phi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^p}$ satisfies property (P_γ) for $\gamma \ge 1$ if for all $T > 0$, there exists some constant $C_T > 0$ such that for any $[\mathbf{s}, \mathbf{t}] \subset [-T, T]^p$,

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma \le C_T \prod_{i:s_i<t_i} |t_i - s_i|. \tag{P_γ}$$

Here, $\|\phi\|_\gamma$ stands for the $L^\gamma(\mathbb{R}^p)$-norm of ϕ. The following examples justify that condition (P_γ) is natural.

Example 2.2. Let $(\mu_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^p}$ be the family of (signed) uniform measures on the blocks $[\mathbf{0}, \mathbf{t}] = \prod_{i=1}^d [0, t_i]$, more precisely, $\phi_{\mathbf{t}} = \operatorname{sign}(t_1) \cdots \operatorname{sign}(t_p) \mathbf{1}_{[\mathbf{0},\mathbf{t}]}$. Then, we verify that for any non-degenerated block $[\mathbf{s}, \mathbf{t}] \subset \mathbb{R}^p$, $\phi_{[\mathbf{s},\mathbf{t}]} = \mathbf{1}_{[\mathbf{s},\mathbf{t}]}$ almost everywhere so that

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma = \prod_{1\le i\le p} |t_i - s_i|.$$

In the case of a degenerated block $[\mathbf{s}, \mathbf{t}] \subset [-T, T]^p$, we have

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma = \prod_{i:s_i=t_i} |s_i| \prod_{i:s_i<t_i} |t_i - s_i| \le C_T \prod_{i:s_i<t_i} |t_i - s_i|$$

with $C_T = \max(1, T)^p$. Hence (P_γ) holds true for all $\gamma \ge 1$. Such uniform-type measures are used to analyze cumulative workload in one-dimensional model, see Mikosch et al. (2002), Kaj and Taqqu (2008).

Example 2.3. Let $(\mu_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^p}$ be a family of measures with densities $\phi_{\mathbf{t}}$ such that for all $I \subset \{1, \dots, p\}$ and all $T > 0$,

$$\Big\| \sup_{t\in[-T,T]^p} \partial_I \phi_{\mathbf{t}}(\mathbf{y}) \Big\|_\gamma < +\infty \tag{2.3}$$

where ∂_I is the differential operator defined for $I = \{i_1, \dots, i_k\}$ by $\partial_I = \partial^k / \partial t_{i_1} \cdots \partial t_{i_k}$. The following condition (P'_γ) (that implies condition (P_γ)) is satisfied: for any $T > 0$, there exists some constant $C_T > 0$ such that for any $[\mathbf{s}, \mathbf{t}] \subset [-T, T]^p$,

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma \leq C_T \prod_{i: s_i < t_i} |t_i - s_i|^\gamma. \qquad (P'_\gamma)$$

This fact is justified in Section 3.3.

We now state our results for the functional convergences in the large ball and intermediate ball regime. Recall that the limits Z_α and J below are defined in Proposition 2.1 and the notation $\xrightarrow{\mathcal{C}(\mathbb{R}^p)}$ stands for the weak convergence in $\mathcal{C}(\mathbb{R}^p)$.

Theorem 2.4. *Suppose conditions* $(\mathbf{A})$ *hold. Let* $\mu_\mathbf{t}(d\mathbf{y}) = \phi_\mathbf{t}(\mathbf{y})d\mathbf{y}, \mathbf{t} \in \mathbb{R}^p$, *be a parametric family of measures in* $L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ *satisfying conditions* (P_1) *and* (P_α).

(1) *(Large ball regime) If* $\lambda(\rho)\rho^\beta \to +\infty$, *then, setting* $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha}$, *we have as* $\rho \to 0$:
$$\widetilde{M}_\rho(\mu_\mathbf{t}) \xrightarrow{\mathcal{C}(\mathbb{R}^p)} Z_\alpha(\mu_\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^p.$$
(2) *(Intermediate ball regime) If* $\lambda(\rho)\rho^\beta \to a > 0$, *then, setting* $n(\rho) = 1$, *we have as* $\rho \to 0$:
$$\widetilde{M}_\rho(\mu_\mathbf{t}) \xrightarrow{\mathcal{C}(\mathbb{R}^p)} J_a(\mu_\mathbf{t}), \quad \mathbf{t} \in \mathbb{R}^p.$$

As a by-product of the moment estimates used to prove tightness, we obtain Hölder-regularity properties in the case $\alpha = 2$. This is the content of the following result.

Proposition 2.5. *Suppose conditions* $(\mathbf{A})$ *hold with* $\alpha = 2$.

(1) *If* G *has a finite variance and the family of measures* $\mu_\mathbf{t}(d\mathbf{y}) = \phi_\mathbf{t}(\mathbf{y})d\mathbf{y}$ *satisfies* (P_1) *and* (P_2), *then the Gaussian limit process* $(Z_2(\mu_\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^p}$ *of Theorem 2.4 is* γ*-Hölder for all* $\gamma < \frac{3d-\beta}{2d}$.
(2) *If there is* $k \geq 2p$ *such that* $h \in L^k(\mathbb{R}^d)$, G *has finite moment of order* k, *and the family of measures* $\mu_\mathbf{t}(d\mathbf{y}) = \phi_\mathbf{t}(\mathbf{y})d\mathbf{y}$ *satisfies* (P_1) *and* (P_k), *then the limit process* $(J_a(\mu_\mathbf{t}))_{\mathbf{t}\in\mathbb{R}^p}$ *of Theorem 2.4 is* γ*-Hölder for all* $\gamma < \frac{3d-\beta}{2d} - \frac{p}{k}$.

Observe that in dimension $d = 1$, we recover at the limit the fractional Brownian motion obtained at the limit in Kaj and Taqqu (2008) with the Hölder-regularity $\gamma < (3-\beta)/2 \in (\frac{1}{2}, 1)$.

Finally, we consider functional convergence in the space of distributions. Let $\mathcal{D}(\mathbb{R}^d)$ be the space of smooth compactly supported functions, and $\mathcal{D}'(\mathbb{R}^d)$ be its dual, *i.e.* the space of distributions. We show that, for all $\rho > 0$, $\widetilde{M}(\rho)$ can be seen as a random distribution and state functional convergence in $\mathcal{D}'(\mathbb{R}^d)$.

Theorem 2.6. *Suppose conditions* $(\mathbf{A})$ *hold.*

(1) *For each* $\rho > 0$, M_ρ *induces a random distribution, i.e. the linear form*
$$M_\rho : \left\{ \begin{array}{ccc} \mathcal{D}(\mathbb{R}^d) & \to & \mathbb{R} \\ \phi & \mapsto & M_\rho(\phi(\mathbf{y})d\mathbf{y}) \end{array} \right.$$
is almost surely continuous.

(2) *(Large ball regime) If $\lambda(\rho)\rho^\beta \to +\infty$, then, setting $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha}$ in $\widetilde{M}_\rho(\mu)$, we have as $\rho \to 0$:*

$$\widetilde{M}_\rho(\mu) \overset{\mathcal{D}'(\mathbb{R}^p)}{\Longrightarrow} Z_\alpha(\mu), \quad \mu \in \mathcal{D}(\mathbb{R}^d).$$

(3) *(Intermediate ball regime) If $\lambda(\rho)\rho^\beta \to a > 0$, then, setting $n(\rho) = 1$ in $\widetilde{M}_\rho(\mu)$, we have as $\rho \to 0$:*

$$\widetilde{M}_\rho(\mu) \overset{\mathcal{D}'(\mathbb{R}^p)}{\Longrightarrow} J_a(\mu), \quad \mu \in \mathcal{D}(\mathbb{R}^d).$$

Remark 2.7. The result is still true if $\mathcal{D}(\mathbb{R}^d)$ is replaced by $\mathcal{C}_K^k(\mathbb{R}^d)$ the space of compactly supported functions of class $\mathcal{C}^k$ on $\mathbb{R}^d$ and $\mathcal{D}'(\mathbb{R}^d)$ is replaced by the dual of $\mathcal{C}_K^k(\mathbb{R}^d)$, $k \in \mathbb{N} \setminus \{0\}$.

3. Proofs

The proof of Theorem 2.4, as usual for functional convergences, consists of two arguments: *fdd* convergences and tightness. The first one is given in Proposition 2.1 which is a slight modification of Theorems 2.4 and 2.11 in Breton and Dombry (2009) whose changes are discussed in Section 3.1. The proof of tightness is given in Section 3.3 and it relies on moment estimates previously obtained in Section 3.2. Hölder regularity also relies on moments and cumulants estimates and Proposition 2.5 is proved in Section 3.4. Section 3.5 is devoted to the proof of functional convergence in the space of distributions $\mathcal{D}'(\mathbb{R}^d)$.

Recall that throughout the paper, we assume that conditions (**A**) hold, and in particular we consider $d < \beta < \alpha d$. All the asymptotics are considered as $\rho \to 0$.

3.1. *fdd convergences.* The results of Breton and Dombry (2009) do not apply directly since the model investigated therein is not exactly the same, see the discussion in page 180. However, *verbatim* changes in the proofs, with the following easy adaptations, shows that their results still apply in the present context and thus justify Proposition 2.1:

First a careful reading of the proofs in Breton and Dombry (2009) shows that the existence of the density f of F is used only in Lemma 3.2 therein. But this lemma can be replaced by Lemmas 2 and 3 in Kaj et al. (2007) deriving the same result but under the weaker assumption ($\mathbf{A}_2$). Observe in particular that $\bar{F}_\rho(1) = \bar{F}(1/\rho) \sim C_\beta \rho^\beta$ and that the continuity requirement in Lemmas 2 and 3 in Kaj et al. (2007) is ensured in our setting by condition ($\mathbf{A}_3$) and Lemma A.2 below.

Second, the bound (1.6) can be replaced by the condition below, justified in Lemma A.1 in the Appendix with $\gamma := \alpha$ therein: for $h \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ and $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, we have

$$\int_{\mathbb{R}^d} |\mu[\tau_{\mathbf{x},r}h]|^\alpha d\mathbf{x} \le C(r^d \wedge r^{\alpha d}).$$

3.2. *Moment estimates.* As we will see, our results on tightness and Hölder regularity strongly rely on moment estimates for the rescaled random field $\widetilde{M}_\rho$. Since the following properties of the moment are also interesting in their own right, they are stated in the following proposition.

Proposition 3.1. *Suppose conditions* (**A**) *hold.*

(1) *Let $0 < \gamma < \alpha$. There exists some constant $C := C(F, G, h, \alpha, \beta, \gamma, d)$, not depending on ρ and ϕ, such that for all $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y} \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ and all $\rho > 0$, we have*

$$\mathbb{E}\left[\left|\widetilde{M}_\rho(\mu)\right|^\gamma\right] \leq C\left[\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha}\right]^{\gamma/\alpha} \|\phi\|_\alpha^{\frac{\gamma(\beta-d)}{(\alpha-1)d}} \|\phi\|_1^{\frac{\gamma(\alpha d-\beta)}{(\alpha-1)d}}. \tag{3.1}$$

(2) *Suppose that $\alpha = 2$ and G has a finite moment of order $k \in \mathbb{N} \setminus \{0, 1\}$, $h \in L^1(\mathbb{R}^d) \cap L^k(\mathbb{R}^d)$. There exists some constant $C := C(F, G, h, \alpha, \beta, k, d)$, not depending on ρ and ϕ, such that for all $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y} \in L^1(\mathbb{R}^d) \cap L^k(\mathbb{R}^d)$ and all $\rho > 0$,*

$$\left|c_k\big(\widetilde{M}_\rho(\mu)\big)\right| \leq C\frac{\lambda(\rho)\rho^\beta}{n(\rho)^k} \|\phi\|_k^{\frac{k(\beta-d)}{(k-1)d}} \|\phi\|_1^{\frac{k(kd-\beta)}{(k-1)d}}, \tag{3.2}$$

where $c_k\big(\widetilde{M}_\rho(\mu)\big)$ is the cumulant of order k of $\widetilde{M}_\rho(\mu)$.

As a by-product of these moment estimates and the finite-dimensional convergence (Proposition 2.1), we obtain the following result stating the convergence of moments:

Corollary 3.2. *Suppose conditions (**A**) hold and $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. Let $0 < \gamma < \alpha$.*

(1) *If $\lambda(\rho)\rho^\beta \to +\infty$, then, setting $n(\rho) = (\lambda(\rho)\rho^\beta)^{1/\alpha}$ in $M_\rho(\mu)$, we have as $\rho \to 0$:*

$$\mathbb{E}\left[\left|\widetilde{M}_\rho(\mu)\right|^\gamma\right] \to \mathbb{E}\left[|Z_\alpha(\mu)|^\gamma\right].$$

(2) *If $\lambda(\rho)\rho^\beta \to a$, then, setting $n(\rho) = 1$ in $M_\rho(\mu)$, we have as $\rho \to 0$:*

$$\mathbb{E}\left[\left|\widetilde{M}_\rho(\mu)\right|^\gamma\right] \to \mathbb{E}\left[|J_a(\mu)|^\gamma\right].$$

In the case $\alpha = 2$, if furthermore G has a finite moment of order $n \geq 2$ and $h, \phi \in L^1(\mathbb{R}^d) \cap L^n(\mathbb{R}^d)$, then the above convergence of moments holds for all $0 \leq \gamma \leq n$.

Proof of Proposition 3.1.

First point: The estimate for $\mathbb{E}\left[\left|\widetilde{M}_\rho(\mu)\right|^\gamma\right]$ relies on the following expression of the fractional moment (see von Bahr and Esseen, 1965, Gaigalas, 2006 or Kaj and Taqqu, 2008, Eq. (60)): if X is a random variable with characteristic function $\varphi_X(t) = \mathbb{E}[\exp(itX)]$, then we have, for $1 < \gamma < 2$,

$$\mathbb{E}[|X|^\gamma] = A(\gamma)\int_0^{+\infty} (1 - |\varphi_X(\theta)|^2)\theta^{-1-\gamma}d\theta \tag{3.3}$$

where

$$A(\gamma) = \left(\int_0^{+\infty} (1 - \cos(x))x^{-1-\gamma}d\mathbf{x}\right)^{-1} < +\infty.$$

Since $\widetilde{M}_\rho(\mu)$ is a Poisson integral, its characteristic function is given by (see Lemma A.5)

$$\varphi_{\widetilde{M}_\rho(\mu)}(\theta) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+} \Psi_G\left(n(\rho)^{-1}\theta\mu[\tau_{\mathbf{x},r}h]\right)\lambda(\rho)d\mathbf{x}F_\rho(dr)\right)$$

with $\Psi_G(u) = \int_{\mathbb{R}} (e^{ium} - 1 - ium) G(dm)$. Hence,

$$\begin{aligned}
& 1 - |\varphi_{\widetilde{M}_\rho(\mu)}(\theta)|^2 \\
= \;& 1 - \exp\left(\int_{\mathbb{R}^d \times \mathbb{R}^+} 2\mathrm{Re}\Big(\Psi_G\left(n(\rho)^{-1}\theta\mu[\tau_{\mathbf{x},r}h]\right)\Big) \lambda(\rho) d\mathbf{x} F_\rho(dr) \right) \\
\leq \;& 1 - \exp\left(-2 \int_{\mathbb{R}^d \times \mathbb{R}^+} \left| \Psi_G\left(n(\rho)^{-1}\theta\mu[\tau_{\mathbf{x},r}h]\right) \right| \lambda(\rho) d\mathbf{x} F_\rho(dr) \right) \\
\leq \;& 1 - \exp\left(-2C(G) n(\rho)^{-\alpha} \lambda(\rho) |\theta|^\alpha \int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu[\tau_{\mathbf{x},r}h]|^\alpha \, d\mathbf{x} F_\rho(dr) \right). \quad (3.4)
\end{aligned}$$

Using Lemma A.4 in the Appendix with $\gamma := \alpha > \beta/d$, we have

$$\int_{\mathbb{R}^d \times \mathbb{R}^+} |\mu[\tau_{\mathbf{x},r}h]|^\alpha d\mathbf{x} F_\rho(dr) \leq \rho^\beta \widetilde{C}(\phi) \quad (3.5)$$

where $\widetilde{C}(\phi)$ is given by

$$\widetilde{C}(\phi) \;=\; C(F) \frac{(\alpha-1)\beta d}{(\alpha d - \beta)(\beta - d)} (\|\phi\|_\alpha \|h\|_1)^{\frac{\alpha(\beta-d)}{(\alpha-1)d}} (\|\phi\|_1 \|h\|_\alpha)^{\frac{(\alpha d - \beta)\alpha}{(\alpha-1)d}}$$

and $C(F)$ is a constant depending only on F. Plugging the bounds (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned}
& \mathbb{E}\left[\left| \widetilde{M}_\rho(\mu) \right|^\gamma \right] \\
& \leq A(\gamma) \int_0^{+\infty} \left(1 - \exp\left(-2C(G) n(\rho)^{-\alpha} \lambda(\rho) \rho^\beta |\theta|^\alpha \tilde{C}(\phi) \right) \right) \theta^{-1-\gamma} d\theta \\
& = A(\gamma) A(\alpha,\gamma) \left(\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha} \right)^{\gamma/\alpha} (2C(G)\widetilde{C}(\phi))^{\gamma/\alpha} \quad (3.6)
\end{aligned}$$

with a straightforward change of variables in (3.6) and

$$A(\alpha,\gamma) = \int_0^{+\infty} (1 - \exp(-\theta^\alpha)) \theta^{-1-\gamma} d\theta < +\infty.$$

This gives the result (3.1) with the constant

$$\begin{aligned}
& C(F,G,h,\alpha,\beta,\gamma,d) \\
= \;& A(\gamma)A(\alpha,\gamma) \left(2C(G)C(F) \frac{(\alpha-1)\beta d}{(\alpha d - \beta)(\beta - d)} \|h\|_1^{\frac{\alpha(\beta-d)}{(\alpha-1)d}} \|h\|_\alpha^{\frac{(\alpha d-\beta)\alpha}{(\alpha-1)d}} \right)^{\gamma/\alpha}.
\end{aligned}$$

The case $1 < \gamma < \alpha$ is proved. The case $0 < \gamma \leq 1$ comes from the previous case applied to any $\gamma' \in (1,\alpha)$ combined with the Jensen inequality which implies $\mathbb{E}[|X|^\gamma] \leq \mathbb{E}[|X|^{\gamma'}]^{\gamma/\gamma'}$.

Second point: We use the general form of the cumulant of a Poisson integral as recalled in Lemma A.5. Using Lemma A.4 with $\gamma := k \geq 2 > \beta/d$, we have:

$$\begin{aligned}
& c_k(\widetilde{M}_\rho(\mu)) \\
& = \frac{\lambda(\rho)}{n(\rho)^k} \left(\int_{\mathbb{R}} m^k G(dm) \right) \left(\int_{\mathbb{R}^d \times \mathbb{R}^+} [\mu[\tau_{\mathbf{x},r}h]]^k d\mathbf{x} F_\rho(dr) \right) \\
& \leq \frac{\lambda(\rho)\rho^\beta}{n(\rho)^k} C(F) \frac{(k-1)\beta d}{(kd-\beta)(\beta-d)} (\|\phi\|_k \|h\|_1)^{\frac{k(\beta-d)}{(k-1)d}} (\|\phi\|_1 \|h\|_k)^{\frac{(kd-\beta)k}{(k-1)d}} \left(\int_{\mathbb{R}} m^k G(dm) \right).
\end{aligned}$$

This gives the bound (3.2) with the constant

$$\begin{aligned} & C(F,G,h,\alpha,\beta,k,d) \\ = \; & C(F)\frac{(k-1)\beta d}{(kd-\beta)(\beta-d)}\|h\|_1^{\frac{k(\beta-d)}{(k-1)d}}\|h\|_k^{\frac{(kd-\beta)k}{(k-1)d}}\left(\int_{\mathbb{R}} m^k G(dm)\right). \end{aligned}$$

□

Proof of Corollary 3.2.

We use the following basic result: if X_n weakly converges to X as $n \to +\infty$ and $\mathbb{E}\left[|X_n|^{\gamma'}\right]$ is bounded, then the convergence of moments $\mathbb{E}\left[|X_n|^{\gamma}\right] \to \mathbb{E}\left[|X|^{\gamma}\right]$ holds for any $0<\gamma<\gamma'$ (in this case, the family $|X_n|^\gamma$ is indeed bounded in $L^{\gamma'/\gamma}$ and hence equi-integrable). As a consequence of this result, the finite-dimensional convergence stated in Proposition 2.1 and the moment estimates obtained in Proposition 3.1 yield the convergence of moments for $0<\gamma<\alpha$ and $0<\gamma<n$ respectively. The case $\gamma=n$ should be proved separately but we omit the details. □

3.3. *Tightness.* This section is devoted to the second step in the proof of Theorem 2.4, *i.e.* tightness. More precisely, we show that:

Proposition 3.3. *Under the assumptions of Theorem 2.4, the family of random fields $(\widetilde{M}_\rho(\mu_{\mathbf{t}}))_{\mathbf{t}\in\mathbb{R}^p}, \rho \le 1$, is tight in $\mathcal{C}(\mathbb{R}^p)$.*

The proof of Proposition 3.3 relies on a suitable control of the moment of the generalized increments of $\widetilde{M}_\rho(\mu_{\mathbf{t}})$ with the following Censov criterion for which we refer to Deshayes and Picard (1984, p. 16) or to Bickel and Wichura (1971).

Proposition 3.4. *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of random fields on $\mathbb{R}^p$ such that:*

(1) *The family of random variable $(X_n(\mathbf{0}))_{n\in\mathbb{N}}$ is tight;*
(2) *For all $T>0$, there are constants $\gamma>0$, $\delta>1$ and $C_T>0$ such that, for all $[\mathbf{s},\mathbf{t}]\subset[-T,T]^p$ and $n\in\mathbb{N}$,*

$$\mathbb{E}\left[|X_n([\mathbf{s},\mathbf{t}])|^\gamma\right] \le C_T \prod_{i:s_i\neq t_i}|t_i-s_i|^\delta.$$

Remark 3.5. Proposition 3.4 is a weaker version of the original criterion given in Deshayes and Picard (1984) that we have adapted to our setting. Note that $\prod_{i:s_i\neq t_i}|t_i-s_i|$ is the k-dimensional Lebesgue measure of $[\mathbf{s},\mathbf{t}]$, where k is the true dimension of the block $[\mathbf{s},\mathbf{t}]$. In Deshayes and Picard (1984), the Lebesgue measure is replaced by general Radon measure with diffuse marginals. Originally, the criterion is expressed in terms of bounds on the tails $\mathbb{P}(|X_n([\mathbf{s},\mathbf{t}])|>x)$ instead of the moments $\mathbb{E}\left[|X_n([\mathbf{s},\mathbf{t}])|^\gamma\right]$. Moreover, it is explained in Deshayes and Picard (1984) that when k-dimensional faces are involved, it is enough to check the condition on blocks $[\mathbf{s},\mathbf{t}]$ such that $s_i=t_i=0$ for the degenerated dimensions. For the sake of clearness, we do not insist on such general conditions.

Proof of Proposition 3.3.

We apply Proposition 3.4 to the family $X_\rho(\mathbf{t})=\widetilde{M}_\rho(\mu_{\mathbf{t}})$, $\rho\le 1$. To that aim, observe first that the tightness of $X_\rho(\mathbf{0})$ is a consequence of the one-dimensional convergence stated in Proposition 2.1. Furthermore, using the definitions (2.1) and (2.2) together with the linearity of the generalized random field $\widetilde{M}_\rho$, we see easily that

$$X_\rho([\mathbf{s},\mathbf{t}])=\widetilde{M}_\rho(\mu_{[\mathbf{s},\mathbf{t}]}) \quad \text{with} \quad \mu_{[\mathbf{s},\mathbf{t}]}(d\mathbf{y})=\phi_{[\mathbf{s},\mathbf{t}]}(\mathbf{y})d\mathbf{y}.$$

Proposition 3.1−1) gives the following moment estimate for $0<\gamma<\alpha$:

$$\begin{aligned}\mathbb{E}\left[|X_\rho([\mathbf{s},\mathbf{t}])|^\gamma\right] &= \mathbb{E}\left[|\widetilde{M}_\rho(\mu_{[\mathbf{s},\mathbf{t}]})|^\gamma\right]\\ &\le C\left[\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha}\right]^{\gamma/\alpha}\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\alpha^{\frac{\gamma(\beta-d)}{(\alpha-1)d}}\|\phi_{[\mathbf{s},\mathbf{t}]}\|_1^{\frac{\gamma(\alpha d-\beta)}{(\alpha-1)d}}. \qquad (3.7)\end{aligned}$$

Using (P_1) and (P_α) for $\phi_{[\mathbf{s},\mathbf{t}]}$, we obtain:

$$\begin{aligned}\mathbb{E}\left[|X_\rho([\mathbf{s},\mathbf{t}])|^\gamma\right] &\le C\left[\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha}\right]^{\gamma/\alpha}\left(\prod_{i:s_i<t_i}|t_i-s_i|\right)^{\frac{\gamma(\beta-d)}{\alpha(\alpha-1)d}+\frac{\gamma(\alpha d-\beta)}{(\alpha-1)d}}\\ &\le C\left[\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha}\right]^{\gamma/\alpha}\left(\prod_{i:s_i<t_i}|t_i-s_i|\right)^{\frac{\gamma}{\alpha}(1+\alpha-\beta/d)}. \qquad (3.8)\end{aligned}$$

Now, observe that the sequence $\lambda(\rho)\rho^\beta/n(\rho)^\alpha$ is bounded since, under the two asymptotics investigated, $\lambda(\rho)\rho^\beta/n(\rho)^\alpha$ converges to some finite constant as $\rho\to 0$. Furthermore since $d<\beta<\alpha d$ and $1+\alpha-\beta/d>1$, the exponent $\frac{\gamma}{\alpha}(1+\alpha-\beta/d)$ is (strictly) larger than 1 for γ close enough to α. This proves the second condition in Proposition 3.4. Proposition 3.3 and thus Theorem 2.4 easily follow. □

We finish this section with the proof of Example 2.3 where the differentiability condition (2.3) is stated to be sufficient for (P'_γ).

Proof for Example 2.3. Let $\mathbf{s}\le\mathbf{t}$. We first assume that, for all $1\le i\le p$, $-T\le s_i<t_i\le T$. We have

$$\phi_{[\mathbf{s},\mathbf{t}]}(\mathbf{y})=\int_{[\mathbf{s},\mathbf{t}]}\partial_{\{1,\dots,p\}}\phi_{\mathbf{u}}(\mathbf{y})d\mathbf{u}.$$

Using Hölder inequality, we have:

$$\begin{aligned}\left|\int_{[\mathbf{s},\mathbf{t}]}\partial_{\{1,\dots,p\}}\phi_{\mathbf{u}}(\mathbf{y})d\mathbf{u}\right|^\gamma &\le |[\mathbf{s},\mathbf{t}]|^{\gamma-1}\int_{[\mathbf{s},\mathbf{t}]}\left|\partial_{\{1,\dots,p\}}\phi_{\mathbf{u}}(\mathbf{y})\right|^\gamma d\mathbf{u}\\ &\le |[\mathbf{s},\mathbf{t}]|^\gamma\sup_{t\in[-T,T]^p}|\partial_{\{1,\dots,p\}}\phi_{\mathbf{u}}(\mathbf{y})|^\gamma\end{aligned}$$

so that

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma\le\left\|\sup_{t\in[-T,T]^p}\partial_{\{1,\dots,p\}}\phi_{\mathbf{t}}(\mathbf{y})\right\|_\gamma^\gamma\prod_{i=1}^p|t_i-s_i|^\gamma.$$

In general, if $\mathbf{s}\le\mathbf{t}$, let I be the set of indices such that $s_i<t_i$. We show similarly that

$$\|\phi_{[\mathbf{s},\mathbf{t}]}\|_\gamma^\gamma\le\left\|\sup_{t\in[-T,T]^p}\partial_I\phi_{\mathbf{t}}(\mathbf{y})\right\|_\gamma^\gamma\prod_{i\in I}|t_i-s_i|^\gamma.$$

□

3.4. *Hölder-regularity when* $\alpha=2$. In this section, we prove Proposition 2.5 where Hölder-regularity is stated for the limit $Z_2(\mu_{\mathbf{t}})$ (resp. $J_a(\mu_{\mathbf{t}})$) when G has finite variance (resp. finite moments of any order). In particular, we set $\alpha=2$ and we assume $\beta\in(d,2d)$. Hölder-regularity is proven using again moment estimates for increments. However since we have not found in the literature Hölder-regularity result relying on generalized increments in dimension $p\ge1$, we use standard increments $Z_2(\mu_{\mathbf{t}})-Z_2(\mu_{\mathbf{s}})$ (resp. $J_a(\mu_{\mathbf{t}})-J_a(\mu_{\mathbf{s}})$) that will be controlled thanks to the following elementary observation: suppose the family $(\phi_{\mathbf{t}})_{\mathbf{t}\in\mathbb{R}^p}$ satisfies condition (P_γ), then for all $T>0$, for all $[\mathbf{s},\mathbf{t}]\subset[-T,T]^p$,

$$\|\phi_{\mathbf{s}}-\phi_{\mathbf{t}}\|_\gamma^\gamma \le C_T p^\gamma \|\mathbf{s}-\mathbf{t}\|_\infty^\gamma. \tag{3.9}$$

To see this, rewrite a standard increment as a sum of d block increments along axis. More precisely, for $\mathbf{s},\mathbf{t}\in\mathbb{R}^p$, define $\mathbf{u}^i\in\mathbb{R}^p$ by $u_j^i=t_j$ for $j\le i$ and $u_j^i=s_j$ for $j>i$ so that, for each $0\le i\le p-1$, $\mathbf{u}^i$ and $\mathbf{u}^{i+1}$ only differ from at most one coordinate. For $\gamma>1$, we have

$$\|\phi_{\mathbf{s}}-\phi_{\mathbf{t}}\|_\gamma^\gamma \le p^{\gamma-1}\sum_{i=1}^p \left\|\phi_{[\mathbf{u}^{i-1},\mathbf{u}^i]}\right\|_\gamma^\gamma$$

so that condition (P_γ) implies

$$\|\phi_{\mathbf{s}}-\phi_{\mathbf{t}}\|_\gamma^\gamma \le C_T p^{\gamma-1}\ \sum_{i=1}^p |t_i-s_i| \le C_T p^\gamma\ \|\mathbf{s}-\mathbf{t}\|_\infty.$$

Remark 3.6. The method consisting in controlling the moment of standard increments by generalized increments in (3.9) cannot be used in Section 3.3. Indeed, similar bounds as in (3.7) but for standard increments $\widetilde{M}_\rho(\mu_{\mathbf{t}})-\widetilde{M}_\rho(\mu_{\mathbf{s}})$ combined with (3.9) would yield

$$\mathbb{E}[|\widetilde{M}_\rho(\mu_{\mathbf{t}})-\widetilde{M}_\rho(\mu_{\mathbf{s}})|^\gamma]\le CC_T p^\gamma\left[\frac{\lambda(\rho)\rho^\beta}{n(\rho)^\alpha}\right]^{\gamma/\alpha}\|\mathbf{s}-\mathbf{t}\|_\infty^{\frac{\gamma}{\alpha}(1+\alpha-\beta/d)}.$$

instead of (3.8) which is far from a bound in $\|\mathbf{s}-\mathbf{t}\|^\delta$ with $\delta>p$ required in the criterion for tightness relying on standard increments (see Kunita, 1990, Th. 1.4.1). The trick works when $\alpha=2$ because in this case, the limit is Gaussian and we can artificially increase the exponent in the bound thanks to the relation between moments and variance, see (3.10).

Proof of Proposition 2.5.
First point: We assume that G has a finite variance, $h\in L^1(\mathbb{R}^d)\cap L^2(\mathbb{R}^d)$ and (P_γ) holds for $\gamma=1,2$. Using Lemma A.1, we have, for $\mu(d\mathbf{y})=\phi(\mathbf{y})d\mathbf{y}$,

$$\begin{aligned}
\mathrm{Var}(Z_2(\mu)) &= \int_{\mathbb{R}^d}|\mu[\tau_{\mathbf{x},r}h]|^2\,\frac{\sigma^\alpha C_2}{r^{1+\beta}}drd\mathbf{x}\\
&\le \int_{\mathbb{R}^d}(r^d\|\phi\|_1^2\|h\|_2^2\wedge r^{2d}\|\phi\|_2^2\|h\|_1^2)\sigma^\alpha C_2 r^{-1-\beta}dr\\
&= \sigma^\alpha C_2\left(\frac{\|\phi\|_2^2\|h\|_1^2}{2d-\beta}\left(\frac{\|\phi\|_1^2\|h\|_2^2}{\|\phi\|_2^2\|h\|_1^2}\right)^{2-\beta/d}\right.\\
&\qquad\left.+\frac{\|\phi\|_1^2\|h\|_2^2}{\beta-d}\left(\frac{\|\phi\|_1^2\|h\|_2^2}{\|\phi\|_2^2\|h\|_1^2}\right)^{1-\beta/d}\right)\\
&= \frac{\sigma^\alpha C_2 d}{(2d-\beta)(\beta-d)}\left(\|\phi\|_2^{\beta/d-1}\|h\|_1^{\beta/d-1}\|\phi\|_1^{2-\beta/d}\|h\|_2^{2-\beta/d}\right)^2\\
&\le C\|\phi\|_2^{2(\beta/d-1)}\|\phi\|_1^{2(2-\beta/d)}
\end{aligned}$$

for some finite constant C. Next, the properties (P_1) and (P_2) together with (3.9) entail that for all $\mathbf{s},\mathbf{t}\in\mathbb{R}^p$:

$$\mathrm{Var}(Z_2(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))\quad\le\quad C\|\mathbf{s}-\mathbf{t}\|_\infty^{3-\beta/d}.$$

We derive easily the moment of order $2n$ of the Gaussian random variable $Z_2(\mu_{\mathbf{s}} - \mu_{\mathbf{t}})$,

$$\begin{aligned}\mathbb{E}[Z_2(\mu_{\mathbf{s}}-\mu_{\mathbf{t}})^{2n}] &= \frac{(2n-1)!}{(n-1)!2^{n-1}}\operatorname{Var}(Z_2(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^n \\ &\le C\frac{(2n-1)!}{(n-1)!2^{n-1}}\|\mathbf{s}-\mathbf{t}\|_\infty^{(3-\beta/d)n}.\end{aligned} \tag{3.10}$$

Using a standard regularity criterion for random fields (see e.g. Kunita, 1990, Th. 1.4.1), $Z_2(\mu_{\mathbf{t}})$ has, almost surely, Hölder-continuous path for all index strictly less than $\frac{3d-\beta-pd/n}{2d}$. Letting n go to $+\infty$, we obtain that $Z_2(\mu_{\mathbf{t}})$ is, almost surely, Hölder-continuous for all index strictly less than $\frac{3d-\beta}{2d}$.

Second point: With $\mu = \mu_{\mathbf{s}} - \mu_{\mathbf{t}}$, Proposition 3.1 entails

$$\left|c_k(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))\right| \le C\frac{\lambda(\rho)\rho^\beta}{n(\rho)^k}\|\phi_{\mathbf{s}}-\phi_{\mathbf{t}}\|_k^{\frac{k(\beta-d)}{(k-1)d}}\|\phi_{\mathbf{s}}-\phi_{\mathbf{t}}\|_1^{\frac{k(kd-\beta)}{(k-1)d}}, \tag{3.11}$$

Now observe that in the intermediate regime, $\lambda(\rho)\rho^\beta/n(\rho)^k$ remains bounded since it converges to a as $\rho \to 0$. Furthermore, using properties (P_1) and (P_k) together with (3.9) in (3.11), we deduce

$$\left|c_k(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))\right| \le C\|\mathbf{s}-\mathbf{t}\|_\infty^{1+k-\beta/d}.$$

Next, recall that the moments of a random variable X are expressed in terms of its cumulants by the so-called complete Bell polynomials, *i.e.* $\mathbb{E}[X^k] = B_k(c_1(X),\dots,c_k(X))$ with

$$B_k(c_1,\dots,c_k) = \sum_{i_1+2i_2\cdots+ki_k=k} K_n(i_1,\dots,i_n)c_1^{i_1}\dots c_k^{i_k}$$

where $i_1,\dots,i_k$ are non-negative integers and $K_k(i_1,\dots,i_k)$ are coefficients whose explicit (involved) form is not required in our argument. Since we have

$$c_1(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^{i_1}\dots c_k(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^{i_k} = 0$$

when $i_1 \neq 0$, we can assume, without loss of generality, that $i_1 = 0$. For all $\mathbf{s},\mathbf{t} \in \mathbb{R}^d$ such that $\|\mathbf{s}-\mathbf{t}\|_\infty \le 1$, we have

$$\begin{aligned}c_2(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^{i_2}\dots c_k(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^{i_k} &\le C\|\mathbf{s}-\mathbf{t}\|_\infty^{\sum_{l=2}^n(l+1-\beta/d)i_l} \\ &\le C\|\mathbf{s}-\mathbf{t}\|_\infty^{k+(1-\beta/d)(k/2)}\end{aligned}$$

where we use $\sum_{l=2}^k li_l = k$ and $\sum_{l=2}^k i_l \le k/2$ together with $\|\mathbf{s}-\mathbf{t}\|_\infty \le 1$. We deduce

$$\mathbb{E}\left[\left(\widetilde{M}_\rho(\mu_{\mathbf{s}}-\mu_{\mathbf{t}})\right)^k\right] \le C\|\mathbf{s}-\mathbf{t}\|_\infty^{(3-\beta/d)k/2}$$

and using Corollary 3.2,

$$\mathbb{E}\left[(J_a(\mu_{\mathbf{s}}-\mu_{\mathbf{t}}))^k\right] \le C\|\mathbf{s}-\mathbf{t}\|_\infty^{(3-\beta/d)k/2}.$$

From the standard regularity criterion for random fields (see e.g. Kunita, 1990, Th. 1.4.1), $J_a(\mu_{\mathbf{t}})$ has, almost surely, Hölder-continuous path for all index strictly less than $(3-\beta/d)/2 - p/k$. □

3.5. *Convergence in distributions space.* The results on convergence in distributions space are based on Lemma 3.7 below relating the generalized random field $\left(\widetilde{M}_\rho(\mu)\right)_{\mu\in\mathcal{D}}$ and the parametric random field $\left(\widehat{M}_\rho(\mathbf{t})\right)_{\mathbf{t}\in\mathbb{R}^d}$ defined by

$$\begin{aligned}\widehat{M}_\rho(t) &= \int_0^{t_1}\cdots\int_0^{t_d}\widetilde{M}_\rho(\delta_{\mathbf{y}})d\mathbf{y}\\ &= \widetilde{M}_\rho(\mu_{\mathbf{t}}),\quad \mathbf{t}\in\mathbb{R}^d\end{aligned}$$

where $\mu_{\mathbf{t}}(d\mathbf{y}) = \mathrm{sign}(t_1)\cdots\mathrm{sign}(t_d)\mathbf{1}_{[\mathbf{0},\mathbf{t}]}(\mathbf{y})d\mathbf{y}$ is the parametric family of signed uniform measures given in Example 2.2. Recall that this family satisfies the property (P_γ) for all $\gamma\geq 1$ so that Theorem 2.4 indeed holds. We also define the continuous random fields on $\mathbb{R}^d$ which are the possible limits in $\mathcal{C}(\mathbb{R}^d)$ under, respectively, the large ball regime and the intermediate ball regime:

$$\widehat{Z}_\alpha(\mathbf{t}) = Z_\alpha(\mu_{\mathbf{t}}),\quad \widehat{J}_a(\mathbf{t}) = J_a(\mu_{\mathbf{t}}),\quad \mathbf{t}\in\mathbb{R}^d.$$

Lemma 3.7. *For all $\phi\in\mathcal{D}(\mathbb{R}^d)$,*

$$\widetilde{M}_\rho(\phi(\mathbf{y})d\mathbf{y}) = (-1)^d\int_{\mathbb{R}^d}\widehat{M}_\rho(\mathbf{y})\frac{\partial^d\phi}{\partial y_1\cdots\partial y_d}(\mathbf{y})d\mathbf{y}.$$

Proof: This follows from successive integration by parts:

$$\begin{aligned}&\widetilde{M}_\rho(\phi(\mathbf{y})d\mathbf{y})\\ =\ &\int_{\mathbb{R}^d}\widetilde{M}_\rho(\delta_{\mathbf{y}})\phi(\mathbf{y})d\mathbf{y}\\ =\ &(-1)^d\int_{\mathbb{R}^d}\left(\int_0^{y_1}\cdots\int_0^{y_d}\widetilde{M}_\rho(u_1,\ldots,u_d)\,du_1\cdots du_d\right)\left(\frac{\partial^d\phi}{\partial y_1\cdots\partial y_d}(\mathbf{y})\right)d\mathbf{y}\\ =\ &(-1)^d\int_{\mathbb{R}^d}\widehat{M}_\rho(\mathbf{y})\frac{\partial^d\phi}{\partial y_1\cdots\partial y_d}(\mathbf{y})d\mathbf{y}.\end{aligned}$$

□

We prove now Theorem 2.6. The proof relies on Lemma 3.7 and on Theorem 2.4 stating that $\widehat{M}_\rho$ converge in $\mathcal{C}(\mathbb{R}^d)$ to $\widehat{Z}_\alpha$ (resp. $\widehat{J}_a$) in the large ball (resp. intermediate ball) regime.

Proof of Theorem 2.6.
First point: The random field $\widetilde{M}_\rho$ is a bounded linear operator on $\mathcal{D}(\mathbb{R}^d)$. Indeed, for $\phi\in\mathcal{D}(\mathbb{R}^d)$ whose support is included in $[-T,T]^d$, Lemma 3.7 entails:

$$\left|\widetilde{M}_\rho(\phi(\mathbf{y})d\mathbf{y})\right| \leq (2T)^d\left(\sup_{[-T,T]^d}|\widehat{M}_\rho|\right)\left(\sup_{[-T,T]^d}\frac{\partial^d\phi}{\partial y_1\cdots\partial y_d}\right).$$

This proves the continuity of the linear application on $\mathcal{D}(\mathbb{R}^d)$ and hence that $\widetilde{M}_\rho$ can be seen as a random distribution.
Second and third point: Let $I:\mathcal{C}(\mathbb{R}^d)\to\mathcal{D}'(\mathbb{R}^d)$ be the canonical injection given by

$$I(f):\phi\mapsto\int_{\mathbb{R}^d}f(\mathbf{y})\phi(\mathbf{y})d\mathbf{y},\quad f\in\mathcal{C}(\mathbb{R}^d),$$

and define the differential operator $D : \mathcal{D}'(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ by

$$D(s) : \phi \mapsto (-1)^d s\left(\frac{\partial^d \phi}{\partial y_1 \cdots \partial y_d}\right), \quad s \in \mathcal{D}'(\mathbb{R}^d).$$

With these notations, Lemma 3.7 states that $\widetilde{M}_\rho = (D \circ I)(\widehat{M}_\rho)$. Since, the operators D and I are continuous, and $\widehat{M}_\rho$ converges in $\mathcal{C}(\mathbb{R}^d)$ as $\rho \to 0$ to $\widehat{Z}_\alpha$ (resp. $\widehat{J}_a$) in the large ball regime (resp. in the intermediate ball regime), the continuous mapping theorem implies that $\widetilde{M}_\rho$ weakly converges in $\mathcal{D}'(\mathbb{R}^d)$ to $(D \circ I)(\widehat{Z}_\alpha)$ (resp. to $(D \circ I)(\widehat{J}_\alpha)$). Finally, since weak convergence in $\mathcal{D}'(\mathbb{R}^d)$ implies *fdd* convergence on $\mathcal{D}(\mathbb{R}^d)$, we have $(D \circ I)(\widehat{Z}_\alpha) \overset{fdd}{=} Z_\alpha$ and $(D \circ I)(\widehat{J}_a) \overset{fdd}{=} J_a$. This shows that Z_α and J_a have modifications that are continuous on $\mathcal{D}(\mathbb{R}^d)$. □

Appendix A. Technical results

Lemma A.1. *Let $\gamma \geq 1$ and $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi \in L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d) \cap L^\gamma(\mathbb{R}^d)$. Then*

$$\int_{\mathbb{R}^d} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x} \leq (r^d \|\phi\|_1^\gamma \|h\|_\gamma^\gamma) \wedge (r^{\gamma d} \|\phi\|_\gamma^\gamma \|h\|_1^\gamma). \tag{A.1}$$

Proof: Since $h \in L^\gamma(\mathbb{R}^d)$ and $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi \in L^1(\mathbb{R}^d)$, the Hölder inequality entails:

$$\begin{aligned}\int_{\mathbb{R}^d} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x} &= \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} h\left(\frac{\mathbf{y}-\mathbf{x}}{r}\right)\phi(\mathbf{y})d\mathbf{y}\right|^\gamma d\mathbf{x} \\ &\leq \|\phi\|_1^{\gamma-1} \int_{\mathbb{R}^d\times\mathbb{R}^d} \left|h\left(\frac{\mathbf{y}-\mathbf{x}}{r}\right)\right|^\gamma |\phi(\mathbf{y})| d\mathbf{y}d\mathbf{x} \\ &= r^d \|\phi\|_1^{\gamma-1} \int_{\mathbb{R}^d\times\mathbb{R}^d} |h(\mathbf{y})|^\gamma |\phi(ry+x)| d\mathbf{y}d\mathbf{x} \\ &= r^d \|\phi\|_1^\gamma \|h\|_\gamma^\gamma. \end{aligned} \tag{A.2}$$

On the other hand, still using Hölder inequality but with $\phi \in L^\gamma(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d)$, we have:

$$\begin{aligned}\int_{\mathbb{R}^d} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x} &= r^{\gamma d} \int_{\mathbb{R}^d} \left|\int_{\mathbb{R}^d} h(\mathbf{y})\phi(ry+x)d\mathbf{y}\right|^\gamma d\mathbf{x} \\ &\leq r^{\gamma d} \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} |h(\mathbf{y})| d\mathbf{y}\right)^{\gamma-1} \int_{\mathbb{R}^d} |\phi(ry+x)|^\gamma |h(\mathbf{y})| d\mathbf{y}d\mathbf{x} \\ &\leq r^{\gamma d} \|\phi\|_\gamma^\gamma \|h\|_1^\gamma. \end{aligned} \tag{A.3}$$

The bounds (A.2) and (A.3) together entail (A.1). □

The following result proves the continuity required to apply Lemmas 2 and 3 instead of Lemma 6 in Kaj et al. (2007) in the modification in Section 3.1.

Lemma A.2. *Suppose that the fading function h satisfies ($\mathbf{A}_3$). For $\mu(d\mathbf{y}) \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$, the application $r \mapsto \int_{\mathbb{R}^d} \Psi_G(\mu[\tau_{\mathbf{x},r}h])\,d\mathbf{x}$ is continuous on $(0, +\infty)$. The same holds true for $\int_{\mathbb{R}^d} \Psi_\alpha(\mu[\tau_{\mathbf{x},r}h])\,d\mathbf{x}$ with*

$$\Psi_\alpha(\theta) = -\sigma^\alpha |\theta|^\alpha (1 + ib\varepsilon(\theta)\tan(\pi\alpha/2))$$

and σ, b given in ($\mathbf{A}_1$).

Proof: Observe first that, for $r_n \to r_0 > 0$, $n \to \infty$, we have:

$$\lim_{n\to\infty} \mu[\tau_{\mathbf{x},r_n} h] = \mu[\tau_{\mathbf{x},r_0} h] \quad d\mathbf{x}\text{-a.e.} \tag{A.4}$$

This is a standard application of Lebesgue's convergence theorem. Indeed, since h is almost-everywhere continuous, we have $\tau_{\mathbf{x},r_n} h(\mathbf{y}) \to \tau_{\mathbf{x},r_0} h(\mathbf{y})$, $n \to +\infty$. From the definition of h^*, the convergence is bounded for all $n \geq 1$:

$$|\tau_{\mathbf{x},r_n} h(\mathbf{y})\phi(\mathbf{y})| \leq \tau_{\mathbf{x},R} h^*(y)|\phi(\mathbf{y})|, \quad x, y \in \mathbb{R}^d$$

with $R = \sup\{r_n; n \geq 1\}$ and the bound is $d\mathbf{x}$-integrable since Lemma A.1 applied to $h^* \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$ rewrites

$$\int_{\mathbb{R}^d} \left| \int_{\mathbb{R}^d} \tau_{\mathbf{x},R} h^*(\mathbf{y})|\phi(\mathbf{y})| d\mathbf{y} \right|^\alpha d\mathbf{x} < +\infty.$$

A second application of Lebesgue's convergence theorem yields

$$\lim_{n\to+\infty} \int_{\mathbb{R}^d} \Psi_G(\mu(\tau_{\mathbf{x},r_n} h)) d\mathbf{x} = \int_{\mathbb{R}^d} \Psi_G(\mu(\tau_{\mathbf{x},r_n} h)) d\mathbf{x}.$$

The convergence (A.4) together with the continuity of Ψ_G imply indeed the pointwise convergence

$$\lim_{n\to+\infty} \Psi_G(\mu(\tau_{\mathbf{x},r_n} h)) = \Psi_G(\mu(\tau_{\mathbf{x},r_0} h)) \quad d\mathbf{x}\text{-a.e.}$$

The convergence is bounded by $C|\mu(\tau_{\mathbf{x},R} h^*)|^\alpha$ since $\Psi_G(u) \leq C|u|^\alpha$, which is integrable by Lemma A.1 applied to $h^* \in L^1(\mathbb{R}^d) \cap L^\alpha(\mathbb{R}^d)$. A similar proof holds for

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} \Psi_\alpha(\mu(\tau_{\mathbf{x},r_n} h)) d\mathbf{x} = \int_{\mathbb{R}^d} \Psi_\alpha(\mu(\tau_{\mathbf{x},r_n} h)) d\mathbf{x}.$$

□

The following result estimates the truncated moments of F. In particular, the condition $\beta > d$ ensures that F has a finite moment of order d.

Lemma A.3. *For $\delta > 0$, when $u \to +\infty$, we have:*

$$\int_0^u r^\delta F(dr) \sim \begin{cases} Cst & \text{if } \delta < \beta \\ \beta C_\beta \ln u & \text{if } \delta = \beta \\ \frac{\beta}{\delta-\beta} C_\beta u^{\delta-\beta} & \text{if } \delta > \beta \end{cases} \tag{A.5}$$

and for $0 < \delta < \beta$, when $u \to +\infty$, we have:

$$\int_u^{+\infty} r^\delta F(dr) \sim \frac{\beta}{\beta-\delta} C_\beta u^{\delta-\beta}. \tag{A.6}$$

Moreover when $\delta > \beta$, we have the global bound

$$\int_0^u r^\delta F(dr) \leq C u^{\delta-\beta}. \tag{A.7}$$

Proof: Let R be a random variable with distribution F. We have

$$\begin{aligned}\int_0^u r^\delta F(dr) &= \mathbb{E}\left[R^\delta \mathbf{1}_{R\le u}\right] = \int_\Omega \int_{\mathbb{R}^+} \mathbf{1}_{\{R\le u\}}\mathbf{1}_{\{t\le R^\delta\}} dt dP \\ &= \int_{\mathbb{R}^+} \mathbb{P}(t^{1/\delta} \le R \le u)dt = \int_0^{u^\delta} \mathbb{P}(t^{1/\delta} \le R \le u)dt \\ &= \delta \int_0^u \mathbb{P}(s \le R \le u) s^{\delta-1} ds \\ &= \delta \int_0^u \mathbb{P}(R \ge s)s^{\delta-1}ds - \delta\int_0^u \mathbb{P}(R > u)s^{\delta-1}ds. \qquad (A.8)\end{aligned}$$

But using condition ($\mathbf{A}_2$), we have:

$$\int_0^u \mathbb{P}(R\ge s)s^{\delta-1}ds \sim \begin{cases} \text{Cst} & \text{if } \delta<\beta \\ C_\beta \ln u & \text{if } \delta = \beta \\ \frac{C_\beta}{\delta-\beta}u^{\delta-\beta} & \text{if } \delta>\beta \end{cases}$$

and $\delta\int_0^u \mathbb{P}(R>u)s^{\delta-1}ds \sim C_\beta u^{\delta-\beta}$ from which (A.5) easily derives. Next,

$$\begin{aligned}\int_u^{+\infty} r^\delta F(dr) &= \mathbb{E}[R^\delta \mathbf{1}_{R\ge u}] = \int_\Omega\int_{\mathbb{R}^+} \mathbf{1}_{\{R\ge u\}}\mathbf{1}_{\{t\le R^\delta\}}dtdP \\ &= \int_{\mathbb{R}^+} \mathbb{P}(R\ge \max(t^{1/\delta},u))dt = \delta\int_{\mathbb{R}^+}\mathbb{P}(R\ge\max(s,u))s^{\delta-1}ds \\ &= \delta\int_0^u \mathbb{P}(R\ge u)s^{\delta-1}ds + \delta\int_u^{+\infty}\mathbb{P}(R\ge s)s^{\delta-1}ds \\ &\sim C_\beta u^{\delta-\beta} + \frac{\delta}{\beta-\delta}C_\beta u^{\delta-\beta} = \frac{\beta}{\beta-\delta}C_\beta u^{\delta-\beta}\end{aligned}$$

which is (A.6). Finally, since $\mathbb{P}(R\ge s)\le 1$, (A.8) entails $\int_0^u r^\delta F(dr) = O(u^\delta)$ so that together with (A.5), it is easy to derive (A.7). □

Lemma A.4. *Let $\gamma > \beta/d$ and $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ with $\phi\in L^1(\mathbb{R}^d)\cap L^\gamma(\mathbb{R}^d)$ and $h\in L^1(\mathbb{R}^d)\cap L^\gamma(\mathbb{R}^d)$. Then, for any $\rho>0$,*

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x}F_\rho(dr) \\ \le\ & M\rho^\beta \frac{(\gamma-1)\beta d}{(\gamma d-\beta)(\beta-d)}\left(\|\phi\|_\gamma\|h\|_1\right)^{\frac{\gamma(\beta-d)}{(\gamma-1)d}}\left(\|\phi\|_1\|h\|_\gamma\right)^{\frac{(\gamma d-\beta)\gamma}{(\gamma-1)d}},\end{aligned}$$

where M is a constant depending only on F.

Proof: Using Lemma A.1, we have:

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x}F_\rho(dr) \\ \le\ & \int_{\mathbb{R}^+} (r^d\|\phi\|_1^\gamma\|h\|_\gamma^\gamma)\wedge(r^{\gamma d}\|\phi\|_\gamma^\gamma\|h\|_1^\gamma)\ F_\rho(dr) \\ =\ & \int_{\mathbb{R}^+} (\rho^d r^d\|\phi\|_1^\gamma\|h\|_\gamma^\gamma)\wedge(\rho^{\gamma d}r^{\gamma d}\|\phi\|_\gamma^\gamma\|h\|_1^\gamma)\ F(dr) \\ =\ & \|\phi\|_\gamma^\gamma\|h\|_1^\gamma\rho^{\gamma d}\int_0^{c/\rho} r^{\gamma d}F(dr) + \|\phi\|_1^\gamma\|h\|_\gamma^\gamma\rho^d\int_{c/\rho}^{+\infty} r^d\ F(dr)\end{aligned}$$

with $c = ((\|\phi\|_1\|h\|_\gamma)/(\|\phi\|_\gamma\|h\|_1))^{\frac{\gamma}{(\gamma-1)d}}$. Finally using the bound on the truncated moments of F in Lemma A.3 with $\delta := \gamma d > \beta$ in (A.5) and $\delta := d$ in (A.6), we derive

$$\begin{aligned}&\int_{\mathbb{R}^d\times\mathbb{R}^+} |\mu[\tau_{\mathbf{x},r}h]|^\gamma d\mathbf{x}F_\rho(dr)\\ \leq\quad & M\rho^\beta\left[\frac{\beta}{\gamma d-\beta}\|\phi\|_\gamma^\gamma\|h\|_1^\gamma c^{\gamma d-\beta} + \frac{\beta}{\beta-d}\|\phi\|_1^\gamma\|h\|_\gamma^\gamma c^{d-\beta}\right]\end{aligned}$$

where $M \in (0,+\infty)$ depends only on F. The result is obtained, after cancellation, by replacing c by its definition. □

The next result collects explicit formulas for the characteristic function and the cumulants of the rescaled and centered random variable $\widetilde{M}_\rho(\mu)$. It is based on standard results for Poisson integrals, see Kallenberg (2002).

Lemma A.5. (1) *The characteristic function of* $\widetilde{M}_\rho(\mu)$ *writes:*

$$\varphi_{\widetilde{M}_\rho(\mu)}(\theta) = \exp\left(\int_{\mathbb{R}^d\times\mathbb{R}^+\times\mathbb{R}} \Psi_G\left(n(\rho)^{-1}\theta\mu[\tau_{\mathbf{x},r}h]\right)\lambda(\rho)d\mathbf{x}F_\rho(dr)\right)$$

where $\Psi_G(u) = \int_{\mathbb{R}}(e^{ium}-1-ium)G(dm)$.

(2) *Suppose* G *has a finite moment of order* $k \geq 1$, $\mu(d\mathbf{y}) = \phi(\mathbf{y})d\mathbf{y}$ *with* $\phi \in L^1(\mathbb{R}^d)\cap L^k(\mathbb{R}^d)$ *and* $h \in L^1(\mathbb{R}^d)\cap L^k(\mathbb{R}^d)$. *Then* $\widetilde{M}_\rho(\mu)$ *has a finite moment of order* k *and its* k *first cumulants are given by:*

$$c_1(\widetilde{M}_\rho(\mu)) = 0,$$

$$c_l(\widetilde{M}_\rho(\mu)) = \frac{\lambda(\rho)}{n(\rho)^l}\left(\int_{\mathbb{R}} m^l\ G(dm)\right)\left(\int_{\mathbb{R}^d\times\mathbb{R}^+}(\mu[\tau_{\mathbf{x},r}h])^l d\mathbf{x}\ F_\rho(dr)\right),\ 2\leq l\leq k.$$

Note that the finiteness of $\int_{\mathbb{R}^d\times\mathbb{R}^+}|\mu[\tau_{\mathbf{x},r}h]|^l\ d\mathbf{x}F_\rho(dr)$ comes from Lemma A.1 when $\mu \in L^1(\mathbb{R}^d)\cap L^k(\mathbb{R}^d)$ and $h \in L^1(\mathbb{R}^d)\cap L^k(\mathbb{R}^d)$.

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