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I knew I should have taken that left turn at Albuquerque

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Abstract. We study the Laplacian-∞ path as an extreme case of the Laplacian-α random walk. Although, in the finite α case, there is reason to believe that the process converges to SLE_κ, with $\kappa = 6/(2\alpha+1)$, we show that this is not the case when $\alpha = \infty$. In fact, the scaling limit depends heavily on the lattice structure, and is not conformal (or even rotational) invariant.

1. Introduction

In recent years, much study has been devoted to the phenomena of conformally invariant scaling limits of processes in $\mathbb{Z}^2$, the two-dimensional Euclidean lattice. The invention of SLE, Schramm (2000), and subsequent development, have lead to many new results regarding such limits.

The first process considered by Schramm was loop-erased random walk, or LERW. This is a process in which one considers a random walk (on some graph) and then erases the loops in the path of that walk, obtaining a self-avoiding path. LERW was first defined in Lawler (1980). In Lawler et al. (2004), Lawler, Schramm and Werner proved that the scaling limit of LERW on $\mathbb{Z}^2$ is SLE_2.

LERW is related to another process, the so called *Laplacian-α random walk*, defined in Lyklema et al. (1986). In fact, LERW and Laplacian-1 random walk are the same process, Lawler (1987). For completeness, let us define the Laplacian-α random walk.

Let $\alpha \in \mathbb{R}$ be some real parameter. Let $G = (V, E)$ be a graph, and let w be a vertex (the target). Let $S \subset V$ be a set not containing w. Let $f_{w,S;G} : V \to [0, 1]$ be the function defined by setting $f_{w,S;G}(x)$ to be the probability that a random walk on G started at x hits w before S. $f_{w,S;G}$ is 1 at w, 0 on S and harmonic in $G \setminus (S \cup \{w\})$, and if the graph G is finite and connected, then it is the unique function satisfying these three conditions. Hence $f_{w,S;G}$ is usually called the *solution to the Dirichlet problem in G with boundary conditions* 1 *on w and* 0 *on S*.

Definition 1.1 (Laplacian-α random walk)**.** Let G be a graph. Let $s \neq w$ be vertices of G. The Laplacian-α random walk on G, starting at s with target w, is

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the process $(\gamma_t)_{t\geq 0}$ such that $\gamma_0 = s$, and such that for any $t > 0$ the distribution of γ_t given $\gamma_0, \gamma_1, \ldots, \gamma_{t-1}$ is

$$\mathbb{P}[\gamma_t = x \mid \gamma_0, \gamma_1, \ldots, \gamma_{t-1}] = \mathbf{1}_{\{x\sim\gamma_{t-1}\}} \cdot \frac{f^\alpha(x)}{\sum_{y\sim\gamma_{t-1}} f^\alpha(y)},$$

where $f = f_{w,\gamma[0,t-1];G}$ is the solution to the Dirichlet problem in G with boundary conditions 1 on w and 0 on $\gamma[0, t-1] = \{\gamma_0, \gamma_1, \ldots, \gamma_{t-1}\}$ and where $x \sim y$ means that x and y are neighbors in the graph G. The process terminates when first hitting w. Here and below we use the convention that $0^\alpha = 0$ even for $\alpha \leq 0$.

As already remarked, the case $\alpha = 1$ is equivalent to LERW in any graph, and therefore in two dimensional lattices has SLE_2 as its scaling limit. Another case which is understood is the case $\alpha = 0$ which is simply a random walk which chooses, at each step, equally among the possibilities which do not cause it to be trapped by its own past. Examine this process on the hexagonal (or honeycomb) lattice. This is a lattice with degree 3 so the walker has at most 2 possiblities at each step. A reader with some patience will be able to resolve some topological difficulties and convince herself that this process is exactly equivalent to an exploration of critical percolation on the *faces* of the hexagonal lattice (say with black-white boundary conditions and the edges between boundary vertices unavailable to the Laplacian random walk)[1]. This has SLE_6 as its scaling limit, see Smirnov (2001); Werner (2009),

Lawler (2006) gives an argument that leads one to expect that the scaling limit of the Laplacian-α random walk on $\mathbb{Z}^2$ should be SLE_κ, for $\kappa = \frac{6}{2\alpha+1}$, for the range of parameters $\alpha > -1/2$. In a public talk about this heuristic argument given in Oberwolfach in 2005 (which GK attended) Lawler stated (paraphrasing) that the argument can be trusted less and less as α increases. Therefore it seems natural to stress it as far as possible by setting $\alpha = \infty$. Let us define the process formally.

Definition 1.2 (Laplacian-∞ path)**.** Let G be a graph. Let $s \neq w$ be vertices of G. The Laplacian-∞ path on G, starting at s with target w, is the path $(\gamma_t)_{t\geq 0}$ such that $\gamma_0 = s$, and such that for any $t > 0$, given $\gamma_0, \gamma_1, \ldots, \gamma_{t-1}$, we set γ_t to be the vertex $x \sim \gamma_{t-1}$ that maximizes $f_{w,\gamma[0,t-1];G}(x)$ over all vertices adjacent to γ_{t-1}. If there is more than one maximum adjacent to γ_{t-1}, one is chosen uniformly among all maxima. The path terminates when first hitting w.

Note that except for the rule in the case of multiple maxima, the Laplacian-∞ path is not random.

The conjecture that the Laplacian-α random walk converges to SLE_κ for $\kappa = \frac{6}{2\alpha+1}$ naturally leads one to ask whether this also holds for $\alpha = \infty$; that is, does the Laplacian-∞ path on $\delta\mathbb{Z}^2$ converge to the (non-random) path SLE_0, as δ tends to 0? Specifically, if this is true, the conformal invariance of SLE_0 hints that this should hold regardless of the lattice one starts with, or at least for any rotated version of $\mathbb{Z}^2$.

We will show that, perhaps surprisingly, this is not the case. In fact, the process on $\mathbb{Z}^2$ can be described almost completely. Without further ado let us do so

[1]To the best of our knowledge this was first noted in Lawler (2006), in the last paragraph of §2.

Theorem 1.3. *There exists a universal constant C such that for any $(a,b) \in \mathbb{Z}^2$ with $a > |b| \geq C$ the following holds. The Laplacian-∞ path starting at $(0,0)$ with target (a,b) has $\gamma_t = (t,0)$ for all t with probability 1.*

If $a = b \geq C$ then $\gamma_t = (t,0)$ with probability $1/2$ and $\gamma_t = (0,t)$ with probability $1/2$.

In other words (and using the symmetries of the problem), the walker does the first step in the correct direction but then continues forward, missing the target (unless the target is extremely close to the axis which is the path of the walker) and goes on to infinity, never "turning left". If the target is on a diagonal the walker chooses among the two possible first steps equally. Comparing to SLE_0, which is a deterministic (conformal image of a) straight line from the start to the target, we see that the process is indeed deterministic, and is indeed a straight line, but is not (necessarily) aimed at the target, is not rotationally invariant and is not independent of the lattice — rotated versions of $\mathbb{Z}^2$ give rise to different scaling limits.

To explain the reason for this behavior in a single sentence, one may say that the pressure of the past of the process outweighs the pull of the target. For those interested in the proof, let us give a rough description of the ideas involved by applying them to prove the following lemma.

Lemma. *Let $t > 2$, and let $x \geq 1$. Let p_y be the probability that a random walk starting from some $y \in \mathbb{Z}^2$ avoids the interval $[(-x,0),(0,0)]$ up to time t. Then*

$$p_{(1,0)} > (1+c)p_{(0,1)}$$

for some absolute constant $c > 0$.

Proof sketch: Couple two walkers starting from these two points so that their paths are a reflection through the diagonal $\{(x,x) : x \in \mathbb{Z}\}$ until they first hit, and then they move together. This shows that $p_{(1,0)} \geq p_{(0,1)}$. Further, since it is possible for the walker starting from $(0,1)$, in 2 steps, to hit $(-1,0)$ without the other walker hitting the forbidden interval, we see that the difference $p_{(1,0)} - p_{(0,1)}$ is of the same order of magnitude as each of them. □

1.1. *Generalizations and speculations.* Although we use a planar argument, some simple adaptations of our methods should work also for higher dimensions; i.e. for Laplacian-∞ paths on $\mathbb{Z}^d$ (instead of reflecting through a diagonal, one needs to reflect through a hyperplane orthgonal to a vector of the form $e_1 \pm e_i$ where e_i is the i^{th} standard basis vector). Also, slight variations on the methods used can produce a similar result for the Laplacian-∞ path on the triangular lattice.

There is some awkwardness in our comparison to SLE since we prove our results on the whole plane, where SLE is not well defined. To rectify this one might examine our process in a large domain $\mathscr{D}$, directed at one point w on its boundary (i.e. solve the Dirichlet problem with boundary conditions 0 on the path γ and on $\partial\mathscr{D} \setminus w$ and 1 on w). This process may be readily compared to (time reversed) radial SLE_0. While we cannot analyze this process until the time it hits w, our methods do show that the process is a straight line along one of the axes until almost hitting the boundary of $\mathscr{D}$; this process is very far from radial SLE_0 which should be the *conformal image* of a straight line from 0 to w, that is, a smooth path but not necessarily a straight line, and definitely not necessarily aligned with

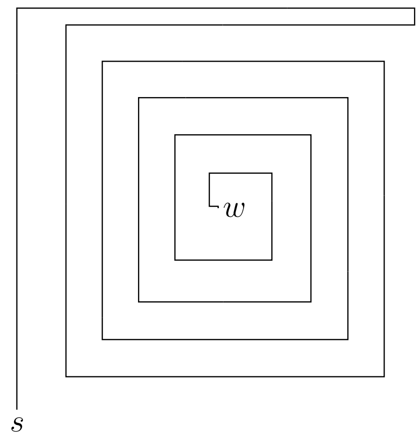

FIGURE 1.1. Laplacian-∞ on a 600×600 torus.

one of the axes. Another natural variation is starting from the boundary namely letting $s \in \partial\mathscr{D}$ and solving the Dirichlet problem with 0 on $\gamma \cup \partial\mathscr{D}$ and 1 on some $w \in \mathscr{D}$. Analyzing this process using our methods requires some more assumptions on $\mathscr{D}$ but it does work, for example, for $\mathscr{D}$ being a square and s not too close to one of the corners. One gets that the path is a straight line perpendicular to the boundary almost until hitting the facing boundary, at which point our analysis no longer works, but again, this is quite enough to see that the process is very far from SLE_0 (radial or chordal — depending on whether w is inside $\mathscr{D}$ or on its boundary). We will not prove either claim as they are similar to those that we do prove with only some minor additional technical difficulties.

We did some simulations on the behavior of the process on a 600×600 torus. Here the process must hit the target (this is easily seen on any finite graph). See Figure 1.1. As one can see the process does hit the target but takes its time to do so, turning only when it is about to hit its past. Some aspects of the picture could definitely do with some explanation: why does the process turn around quickly after the first round (the very top of the picture)? We have no proof and only a mildly convincing heuristic explanation for this behavior.

1.2. *Acknowledgements.* We wish to thank N. Aran for inspiring the name of the paper. The torus simulations would not have been possible without Timothy Davis' SuiteSparseQR, a library for fast solution of sparse self-adjoint linear equations.

2. Proof

Notation. $\mathbb{Z}^2$ denotes the discrete two dimensional Euclidean lattice; we denote the elements of $\mathbb{Z}^2$ by their complex counterparts, e.g. the vector $(1,2)$ is denoted by $1+2i$.

$\mathbb{P}_x$ and $\mathbb{E}_x$ respectively, denote the measure and expectation of a simple (nearest-neighbor, discrete time) random walk on $\mathbb{Z}^2$, $(X_t)_{t\geq 0}$, started at $X_0 = x$. For a set $S \subset \mathbb{Z}^2$, we denote by $T(S)$ the hitting time of S; that is

$$T(S) = \inf\{t \geq 0 \;:\; X_t \in S\},$$

Occasionally we will use $T(S)$ for a subset $S \subset \mathbb{C}$ that is not discrete, and in this case the hitting time of S is the first time the walk passes an edge that intersects S. We also use the notation $T(z,r) = T(\{w : |w-z| \geq r\})$, the exit time from the ball of radius r centered at z (we will always use it with the starting point inside the ball). For a vertex $z \in \mathbb{Z}^2$ we use the notation $T(z) = T(\{z\})$.

We will denote universal positive constants with c and C where c will refer to constants "sufficiently small" and C to constants "sufficiently large". We will number some of these constants for clarity.

We begin with an auxiliary lemma.

Lemma 2.1. *There exists a universal constant $C > 0$ such that the following holds. Let $w \in \mathbb{Z}^2$. Let $D = \{x + ix : x \in \mathbb{Z}\}$ be the discrete diagonal. Let $I = [-x, 0] \cap \mathbb{Z}$, for some $x > 0$. Then,*

$$\mathbb{P}_i[T(w) < T(I \cup D)] \leq C|w|^{-1/2}\, \mathbb{P}_i[T(w) < T(I)]. \tag{2.1}$$

Before starting the proof we need to apologize for some of the choices we made. It is well known that the probability that a random walk escapes from a corner of opening angle a to distance r is of the order $r^{-\pi/a}$. The case of $a = 2\pi$ was famously done by Kesten (1987)[2] and a simpler proof can be found in the book Lawler (1991, §2.4). The general case can be done using multiscale coupling to Brownian motion, but we could not find a suitable reference, and including a full proof would have weighed down on this paper. The reader is encouraged to verify that given the general $r^{-\pi/a}$ claim, both sides of (2.1) can be calculated explicitly. Thus, the exponent $1/2$ on the right side of (2.1) is not optimal, but is sufficient for our purpose and the proof is far simpler.

Proof of Lemma 2.1: Let us recall the aforementioned result regarding escape probabilities. Equations (2.37) and (2.38) of Lawler (1991, §2.4) tell us that for any $r > 0$,

$$\mathbb{P}_i[T(0,r) < T(I), \operatorname{Re}(X_{T(0,r)}) \geq 0] \geq c_1 r^{-1/2}, \tag{2.2}$$

for some universal constant $c_1 > 0$. We will also need the probability of escape from the diagonal D. This particular case is simple because for simple random walk the two projections $\operatorname{Re} X + \operatorname{Im} X$ and $\operatorname{Re} X - \operatorname{Im} X$ are *independent* one dimensional random walks. This makes it easy to calculate escape probabilities in a rhombus. Namely, if $S_r = \{x : |\operatorname{Re} x| + |\operatorname{Im} x| = r\}$ then the question whether, for random walk starting from i, $T(D) \leq T(S)$ or not, is equivalent to the question whether a one-dimensional random walk hits 1 before hitting r and before a second, independent one-dimensional random walk hits $\pm r$. Both are well known to be $\geq 1 - C/r$ so all-in all we get

$$\mathbb{P}_i[T(0,r) < T(D)] \leq \mathbb{P}_i[T(S_r) < T(D)] \leq C_1 r^{-1}. \tag{2.3}$$

Let $A(r,R) = \{z \in \mathbb{C} : r \leq |z| \leq R\}$ denote the closed annulus of inner radius r and outer radius R. Fix $r = \frac{|w|}{2}$. Without loss of generality, by adjusting the constant in the statement of the lemma, we can assume that r is large enough. Let $A = A(r/4, r)$. So $|w| > r$ and $w \notin A$. Let V be the set of all v with $\operatorname{Re}(v) \geq 0$ such that $\mathbb{P}_i[X_{T(0,r/2)} = v] > 0$. So $r/2 \leq |v| \leq r/2 + 1$ and $v \in A$. Let U be

[2] This result is not stated in Kesten (1987) explicitly but the upper bound can be inferred from results proved there (particularly lemma 6) easily, and the lower bound can be proved by the same methods.

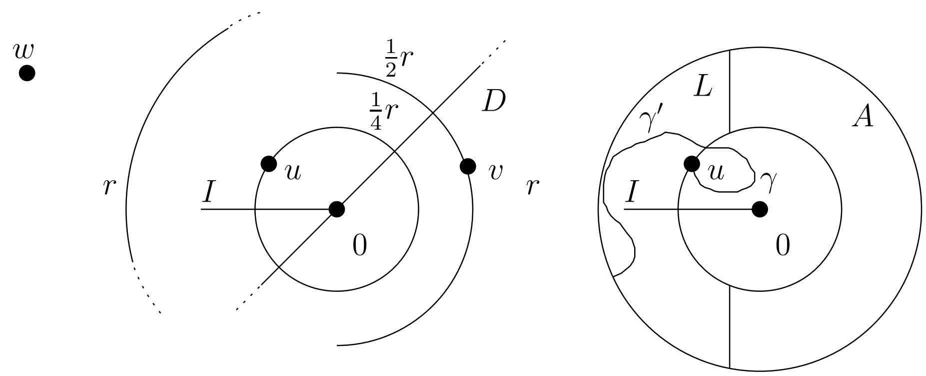

FIGURE 2.2. On the left, u, v and w. On the right, L, γ and γ'.

the set of all $u \in \mathbb{Z}^2$ such that $\mathbb{P}_i[X_{T(0,r/4-2)} = u] > 0$. Specifically, $|u| < r/4$ and $u \notin A$. See Figure 2.2, left.

Fix $u \in U$ and $v \in V$. Consider the function $f(z) = \mathbb{P}_z[T(w) < T(I)]$. This function is discrete-harmonic in the split ball $A(0,r) \setminus I$ and 0 on I. Thus, there exists a path $\gamma = (u = \gamma_0, \gamma_1, \ldots, \gamma_n)$ in $\mathbb{Z}^2$ from u to some $\gamma_n \notin A(0,r)$ such that $f(\cdot)$ is non-decreasing on γ; i.e. $f(\gamma_{j+1}) \geq f(\gamma_j)$ for all $0 \leq j \leq n-1$. See Figure 2.2, right.

We now examine the slightly-less-than-half of A, $L := \{x \in A : \mathrm{Re}\,(x) < -r/16\}$ and divide into two cases according to whether $\gamma \cap A$ is contained in L or not. In the second case, let $v' \in \gamma \cap (A \setminus L)$. By the discrete Harnack inequality, Lawler and Limic (2010, Theorem 6.3.9), we have $f(v) \geq cf(v')$ for some absolute constant. To aid the reader in using the reference efficiently, here are the sets we had in mind:

$$\begin{pmatrix}\mathbf{K}\ \&\ \mathbf{U} \\ \text{from Lawler \& Limic}\end{pmatrix} \qquad \begin{array}{l}\mathbf{K} = \{x \in \mathbb{R}^2 : \frac{1}{4} \leq |x| \leq 1 \text{ and } x_1 \geq -\frac{1}{16}\} \\ \mathbf{U} = \{x \in \mathbb{R}^2 : \frac{3}{16} < |x| < \frac{3}{2} \text{ and } x_1 > -\frac{1}{8}\}\end{array}$$

Note that $f(\cdot)$ is discrete harmonic in $\overline{r\mathbf{U}}$. Hence, since f is non-decreasing on γ, $f(v') \geq f(u)$, and in this case

$$f(v) \geq cf(u). \tag{2.4}$$

Showing (2.4) in the case that $\gamma \cap A \subset L$ is only slightly more complicated. Let γ' be the last portion of γ in L i.e. $\{\gamma_{m+1}, \ldots, \gamma_n\}$ where $m < n$ is maximal such that $\gamma_m \notin A$ (see Figure 2.2, right). In this case γ' divides L into two components, and $I \cap L$ lies completely in one of them. Assume for concreteness it is in the bottom one. Then every path crossing L counterclockwise will hit γ before hitting I. Examine therefore the event $\mathscr{E}$ that random walk starting from v will exit the slit annulus $A \setminus \{x : \mathrm{Re}\,(x) = -r/16, \mathrm{Im}\,(x) < 0\}$ by hitting the slit from its left side. By the invariance principle, Lawler and Limic (2010, §3.1), if r is sufficiently large then $\mathbb{P}(\mathscr{E}) > c_2$ for some constant $c_2 > 0$ independent of r, uniformly in $v \in V$. However, $\mathscr{E}$ implies that the random walk traversed L counterclockwise, hence it hits γ before hitting I. We get

$$\mathbb{P}_v[T(\gamma) < T(I \cup \{w\})] > c_2.$$

Since $f(X_t)$ is a martingale up to the first time $X.$ hits I or w, we may use the strong Markov property at the stopping time $T(\gamma)$ to get

$$f(v) \geq c_2\,\mathbb{E}[f(X_{T(\gamma)})\,|\,T(\gamma) < T(I \cup \{w\})] \geq c_2 f(u)$$

i.e. we have established (2.4) in both cases.

We now use the bounds on escape probabilities above to get

$$\begin{aligned}
&\mathbb{P}_i[T(w) < T(I \cup D)] \\
&\qquad \leq \mathbb{P}_i[T(0, r/4-2) < T(I \cup D)] \cdot \max_{u \in U} \mathbb{P}_u[T(w) < T(I \cup D)] \\
&\qquad \leq \mathbb{P}_i[T(0, r/4-2) < T(I \cup D)] \cdot \max_{u \in U} f(u) \\
\text{By (2.4)} &\qquad \leq \mathbb{P}_i[T(0, r/4-2) < T(D)] \cdot C_2 \min_{v \in V} f(v) \\
\text{By (2.3)} &\qquad \leq C_3 r^{-1} \cdot \min_{v \in V} f(v) \qquad (2.5)
\end{aligned}$$

where $C_2, C_3 > 0$ are universal constants. The lemma now follows from applying the strong Markov property at the stopping time $T(0, r/2)$,

$$\begin{aligned}
\mathbb{P}_i[T(w) < T(I)] &\geq \mathbb{P}_i[T(0, r/2) < T(I), \mathrm{Re}\,(X_{T(0,r/2)}) \geq 0] \cdot \min_{v \in V} f(v) \\
\text{By (2.2)} \qquad &\geq c r^{-1/2} \min_{v \in V} f(v) \\
\text{By (2.5)} \qquad &\geq c r^{1/2}\, \mathbb{P}_i[T(w) < T(I \cup D)].
\end{aligned}$$

□

We now turn to the main lemma, which uses the coupling argument sketched in the introduction.

Lemma 2.2. *There exist universal constants $C, \varepsilon > 0$ such that the following holds. Let $I = [-x, 0] \subset \mathbb{R}$, for some $x \geq 1$, and let $w \in \mathbb{Z}^2$ such that $|w| > C$. Then,*

$$\mathbb{P}_1[T(w) < T(I)] > \mathbb{P}_i[T(w) < T(I)](1 + \varepsilon).$$

(The proof will give $\varepsilon = 4^{-7}$.)

Proof: We couple two random walks on $\mathbb{Z}^2$ started at 1 and i, by constraining them to be the mirror image of each other around $D = \{x + ix \,:\, x \in \mathbb{Z}\}$ until they meet. When they do, they glue and continue walking together. In formulas, given the random walk (X_t), let (Y_t) be a random walk coupled to (X_t) as follows. Set $X_0 = 1$ and $Y_0 = i$. For $t > 0$, if $Y_{t-1} \neq X_{t-1}$, let $Y_t = i\overline{X_t}$. If $Y_{t-1} = X_{t-1}$, then let $Y_t = X_t$. It is immediate that (Y_t) is also a random walk.

Let $\tau = \min\{t \geq 0 \,:\, X_t = Y_t\}$, be the coupling time. For all $t \leq \tau$, $\mathrm{Re}\,(Y_t) = \mathrm{Im}\,(X_t)$ and $\mathrm{Im}\,(Y_t) = \mathrm{Re}\,(X_t)$. Hence, for any $t \leq \tau$, we have that $Y_t = X_t$ if and only if $Y_t, X_t \in D$. So we conclude that $\tau = T(D)$.

Now, let $T^1(I) = \min\{t \geq 0 \,:\, X_t \in I\}$ and $T^i(I) = \min\{t \geq 0 \,:\, Y_t \in I\}$ be the hitting times of I for $X.$ and $Y.$ respectively. Similarly, let $T^1(w), T^i(w)$ be the hitting times of w for $X.$ and $Y.$ respectively. Since $X_0 = 1$, we have that D separates X_0 from I, so $\tau = T(D) \leq T^1(I)$. Thus, $T^1(I) \geq T^i(I)$ always.

Let $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$ be the three events depicted in Figure 2.3. Formally, let $\mathscr{A}$ be the event $\{T(D) \leq T^1(w), T(D) \leq T^i(w)\}$. Let $\mathscr{B}$ be the event $\{T^1(w) < T(D) \leq T^i(w)\}$, and let $\mathscr{C}$ be the event $\{T^i(w) < T(D) \leq T^1(w)\}$. Note that $\mathscr{A}, \mathscr{B}$ and $\mathscr{C}$ are pairwise disjoint and their union is the whole space. Furthermore, either $\mathbb{P}[\mathscr{B}] = 0$ or $\mathbb{P}[\mathscr{C}] = 0$, depending on whether $\mathrm{Re}\,(w) < \mathrm{Im}\,(w)$ or $\mathrm{Re}\,(w) > \mathrm{Im}\,(w)$ respectively (if they are equal both events are empty).

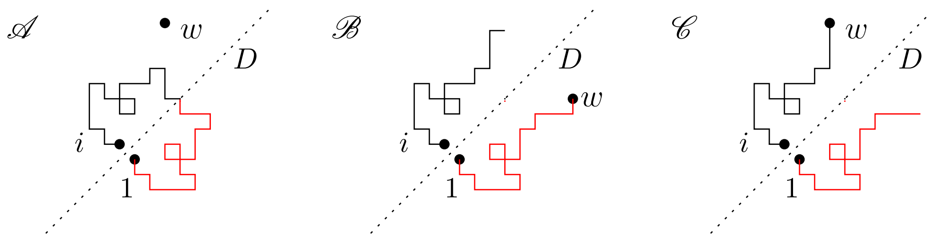

FIGURE 2.3. The events $\mathscr{A}$, $\mathscr{B}$ and $\mathscr{C}$.

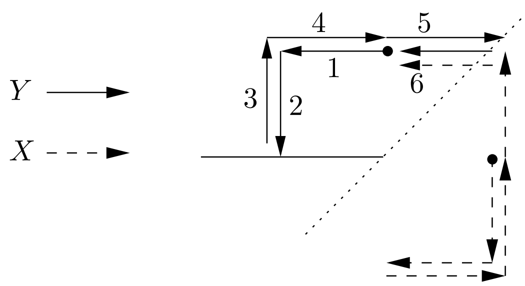

FIGURE 2.4. The event that gives 4^{-6} in the proof.

Now, on the event $\mathscr{A}$ we have that $T^i(w) = T^1(w)$. Thus, on the event $\mathscr{A}$, the event $\{T^i(w) < T^i(I)\}$ is contained in the event $\{T^1(w) < T^1(I)\}$. Hence,

$$\mathbb{P}[T^1(w) < T^1(I), \mathscr{A}] - \mathbb{P}[T^i(w) < T^i(I), \mathscr{A}] \\ = \mathbb{P}[T^i(I) < T(D) \le T^i(w) = T^1(w) < T^1(I)].$$

Next, consider the event $\{Y_1 = i-1, Y_2 = -1, Y_3 = i-1, Y_4 = i, Y_5 = 1+i, Y_6 = i\}$ (which is the same as the event $\{X_1 = 1-i, X_2 = -i, X_3 = 1-i, X_4 = 1, X_5 = 1+i, X_6 = i\}$, see Figure 2.4), which implies that $T^i(I) < T(D) \le T^i(w) = T^1(w)$. (Here we use that $x \ge 1$, so $-1 \in I$.) We have that

$$\mathbb{P}[T^1(w) < T^1(I), \mathscr{A}] - \mathbb{P}[T^i(w) < T^i(I), \mathscr{A}] \ge 4^{-6}\, \mathbb{P}_i[T(w) < T(I)]. \tag{2.6}$$

As for the event $\mathscr{B}$, we have that $\mathscr{B} \subset \{T^1(w) < T^1(I)\}$. So,

$$\mathbb{P}[T^1(w) < T^1(I), \mathscr{B}] - \mathbb{P}[T^i(w) < T^i(I), \mathscr{B}] = \mathbb{P}[\mathscr{B}] - \mathbb{P}[T^i(w) < T^i(I), \mathscr{B}] \ge 0 \tag{2.7}$$

Finally, the event $\mathscr{C}$ implies $T^i(w) < T(D)$ and therefore

$$\mathbb{P}[T^1(w) < T^1(I), \mathscr{C}] - \mathbb{P}[T^i(w) < T^i(I), \mathscr{C}] \ge \\ - \mathbb{P}[T^i(w) < T^i(I), \mathscr{C}] \ge - \mathbb{P}_i[T(w) < T(I \cup D)]. \tag{2.8}$$

Combining (2.6), (2.7) and (2.8), we get that

$$\mathbb{P}_1[T(w) < T(I)] - \mathbb{P}_i[T(w) < T(I)] \\ \ge 4^{-6}\, \mathbb{P}_i[T(w) < T(I)] - \mathbb{P}_i[T(w) < T(I \cup D)]. \tag{2.9}$$

We have not placed any restrictions on the constant C from the statement of the lemma so far. Let C_4 be the constant from Lemma 2.1. We now choose

$C \geq \left(C_4 4^7\right)^2$. Thus, if $|w| > C$ then by Lemma 2.1,

$$\mathbb{P}_i[T(w) < T(I \cup D)] < 4^{-7}\,\mathbb{P}_i[T(w) < T(I)].$$

Plugging this into (2.9) completes the proof of the lemma. □

The last piece of the puzzle is to determine the first step of the Laplacian-∞ path.

Lemma 2.3. *Let $w \in \mathbb{Z}^2$ with $\mathrm{Re}\,(w) > |\mathrm{Im}\, w|$. Let $(\gamma_t)_{t\geq 0}$ be the Laplacian-∞ path on $\mathbb{Z}^2$, started at $\gamma_0 = 0$ with target w. Then $\gamma_1 = 1$.*

If $\mathrm{Re}\,(w) = \mathrm{Im}\,(w) > 0$ then $\gamma_1 = 1$ with probability $\frac{1}{2}$ to be and $\gamma_1 = i$ with probability $\frac{1}{2}$.

Proof: We start with the case $\mathrm{Re}\,(w) > |\mathrm{Im}\,(w)|$. Recall that γ_1 is the neighbor e of 0 that maximizes the probability $\mathbb{P}_e[T(w) < T(0)]$.

As in the proof of Lemma 2.2, we couple two random walks $(X_t), (Y_t)$, starting at $X_0 = 1$ and $Y_0 = i$ respectively, by reflecting them around D. We use $T^1(0), T^1(w), T^i(0), T^i(w)$ to denote the hitting times of 0 and w by these walks, in the obvious way. Recall from the proof of Lemma 2.2, that the coupling time of these walks is $T(D)$, the hitting time of D.

Since D separates w from i, we have that $T^1(w) \leq T^i(w)$. Thus, the event $\{T^i(w) < T^i(0)\}$ implies the event $\left\{T^1(w) < T^1(0)\right\}$. Further, this inclusion is strict — the event that X hits w before D has positive probability. Hence

$$\mathbb{P}_1[T(w) < T(0)] > \mathbb{P}_i[T(w) < T(0)].$$

Showing that $\mathbb{P}_1[T(w) < T(0)] > \mathbb{P}_e[T(w) < T(0)]$ for $e = -1, -i$ is done likewise by reflecting through the imaginary line or the opposite diagonal $D^* = \{x - ix : x \in \mathbb{Z}\}$, respectively. This completes the proof of the case $\mathrm{Re}\,(w) > |\mathrm{Im}\,(w)|$. For the case $\mathrm{Re}\,(w) = \mathrm{Im}\,(w) > 0$, just note that the problem is now symmetric to reflection through the diagonal D, so $\mathbb{P}_1[T(w) < T(0)] = \mathbb{P}_i[T(w) < T(0)]$, so the walker chooses among them equally. Both are larger than the probabilities at -1 and $-i$, again by reflecting through the diagonal D^*. □

Proof of Theorem 1.3: We take $C > 0$ so that Lemma 2.2 holds with this constant C. We prove the theorem by induction on t. Let $w = a + ib$. The case of $t = 1$ is handled by Lemma 2.3. Assume therefore that $\gamma_s = s$ for all $0 \leq s \leq t-1$. Let $I = \{\gamma_s \ : \ 0 \leq s \leq t-1\}$. Let $f : \mathbb{Z}^2 \to [0,1]$ be the function $f(z) = \mathbb{P}_z[T(w) < T(I)]$. Translating by $-(t-1)$, since $|w - (t-1)| \geq |\mathrm{Im}\,(w)| > C$, we can use Lemma 2.2, to get that $f(t) > f(t-1+i)$. Reflecting through the real line and using Lemma 2.2 again we get $f(t) > f(t-1-i)$. Thus, $\gamma_t = t$ and the theorem is proved. □

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