



Ballistic Random Walk in a Random Environment with a Forbidden Direction

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Abstract. We consider a ballistic random walk in an i.i.d. random environment that does not allow retreating in a certain fixed direction. Homogenization and regeneration techniques combine to prove a law of large numbers and an averaged invariance principle. The assumptions are non-nestling and $1 + \varepsilon$ (resp. $2 + \varepsilon$) moments for the step of the walk uniformly in the environment, for the law of large numbers (resp. invariance principles). We also investigate invariance principles under fixed environments, and invariance principles for the environment-dependent mean of the walk.

1. Introduction

This paper studies random walk in a random environment (RWRE) on the d -dimensional integer lattice \mathbb{Z}^d . This is a basic model in the field of disordered or random media. The dimension d is in general any positive integer, but in certain results we are forced to distinguish between $d = 1$ and $d \geq 2$.

A general description of this model follows. An environment is a configuration of vectors of jump probabilities

$$\omega = (\omega_x)_{x \in \mathbb{Z}^d} \in \Omega = \mathcal{P}^{\mathbb{Z}^d},$$

where $\mathcal{P} = \{(p_z)_{z \in \mathbb{Z}^d} : p_z \geq 0, \sum_z p_z = 1\}$ is the simplex of all (infinite) probability vectors on \mathbb{Z}^d . We use the notation $\omega_x = (\pi_{x,x+y})_{y \in \mathbb{Z}^d}$ for the coordinates of

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ω_x . The space Ω is equipped with the canonical product σ -field \mathfrak{S} and with the natural shift $\pi_{xy}(T_z\omega) = \pi_{x+z, y+z}(\omega)$, for $z \in \mathbb{Z}^d$. On the space of environments (Ω, \mathfrak{S}) we are given a T -invariant probability measure \mathbb{P} such that the system $(\Omega, \mathfrak{S}, (T_z)_{z \in \mathbb{Z}^d}, \mathbb{P})$ is ergodic. All our results are for i.i.d. environments. This means that the random probability vectors $(\omega_x)_x$ are i.i.d. across the sites x under \mathbb{P} , or in other words that \mathbb{P} is an i.i.d. product measure on $\mathcal{P}^{\mathbb{Z}^d}$.

Here is how the random walk operates. First an environment ω is chosen from the distribution \mathbb{P} . The environment ω remains fixed for all time. Pick an initial state $z \in \mathbb{Z}^d$. The random walk in environment ω started at z is then the canonical Markov chain $\hat{X} = (X_n)_{n \geq 0}$ with state space \mathbb{Z}^d whose path measure P_z^ω satisfies

$$\begin{aligned} P_z^\omega(X_0 = z) &= 1 && \text{(initial state),} \\ P_z^\omega(X_{n+1} = y | X_n = x) &= \pi_{xy}(\omega) && \text{(transition probability).} \end{aligned}$$

The probability distribution P_z^ω on random walk paths is called the *quenched law*. The joint probability distribution

$$P_z(d\hat{X}, d\omega) = P_z^\omega(d\hat{X})\mathbb{P}(d\omega)$$

on walks and environments is called the *joint annealed law*, while its marginal on walks $P_z(d\hat{X}, \Omega)$ is called simply the *annealed law*. We will use \mathbb{E} , E_z^ω , and E_z to denote expectations under, respectively, \mathbb{P} , P_z^ω , and P_z .

Not much of interest has been proved for random walk in random environment in this generality, except in one dimension. In this paper and the subsequent paper Rassoul-Agha and Seppäläinen (2005b) we study the invariant distributions, laws of large numbers and central limit theorems for a class of ballistic walks in an i.i.d. environment that possess a “forbidden direction.” More precisely, we impose a non-nestling assumption that creates a drift in some spatial direction \hat{u} , and we also prohibit the walk from retreating in direction \hat{u} . However, there is some freedom in the choice of \hat{u} . The long term velocity v of the walk need not be in direction \hat{u} , although of course the assumptions will imply $\hat{u} \cdot v > 0$. These assumptions are introduced in Section 2. Given the hypotheses described above our results are complete, except for some moment assumptions that most certainly are not best possible.

Our main result is the central limit picture for this class of walks. As preparation for this we first need to establish the existence of suitable invariant distributions for the environment chain, and also prove a law of large numbers $n^{-1}X_n \rightarrow v$.

The first point of the fluctuation question is that the annealed invariance principle (functional central limit theorem) with nv centering is always valid. In other words, the distribution of the process $\{B_n(t) = n^{-1/2}(X_{[nt]} - [nt]v) : t \geq 0\}$ under P_0 converges to a Brownian motion with a certain diffusion matrix that we give a formula for. This will be the object of Section 4.

The second fluctuation issue is the quenched invariance principle under measures P_0^ω . Here we encounter the following dichotomy.

I. Suppose the walk is one-dimensional, or that the environment ω constrains the walk to be essentially one-dimensional in a sense made precise by Hypothesis (E) on page 124 below. Then process B_n does not satisfy a quenched invariance principle. However, under quenched mean centering a quenched invariance principle does hold. In other words, for \mathbb{P} -almost every ω , the process $\{\tilde{B}_n(t) = n^{-1/2}(X_{[nt]} - E_0^\omega(X_{[nt]})) : t \geq 0\}$ under P_0^ω converges weakly to a certain Brownian motion.

Furthermore, the centered quenched mean process $\{n^{-1/2}(E_0^\omega(X_{[nt]}) - ntv) : t \geq 0\}$ also satisfies an invariance principle.

II. In the complementary case the walk explores its environment thoroughly enough to suppress the fluctuations of the quenched mean. The variance

$$\mathbb{E}[|E_0^\omega(X_n) - E_0(X_n)|^2]$$

of the quenched mean is of order $n^{1-\delta}$ for $\delta > 0$, the centerings nv and $E_0^\omega(X_n)$ for the walk are asymptotically indistinguishable on the diffusive scale, and both processes B_n and \tilde{B}_n defined above satisfy a quenched invariance principle with the same limiting Brownian motion.

We can also view the dichotomy through the following decomposition of the fluctuations:

$$\begin{aligned} E_0[(X_n - nv)(X_n - nv)^t] \\ = \mathbb{E}[E_0^\omega(X_n X_n^t) - E_0^\omega(X_n)E_0^\omega(X_n)^t] + \mathbb{E}[(E_0^\omega(X_n) - nv)(E_0^\omega(X_n) - nv)^t]. \end{aligned} \tag{1.1}$$

Here, X^t denotes the transpose of the column vector X so that XX^t is a square matrix. In case I both terms contribute on the diffusive scale all the way to the limit. As the results in Section 5 show, the fluctuations described by the annealed central limit theorem decompose into quenched fluctuations of the walk around the quenched mean, plus the fluctuations of the quenched mean around nv . (The centering nv in (1.1) is not exactly the true mean $E_0(X_n)$, but the difference of the two is bounded as shown in Theorem 3.3.)

In case II the second term on the right hand side of (1.1) is negligible relative to the two other terms of order n . On the diffusive scale the fluctuations of the quenched mean are not felt.

This paper covers the type I fluctuation results in Section 5 which forms the main portion of the paper. The type II results are reported in the companion paper Rassoul-Agha and Seppäläinen (2005b). An appendix in the present paper contains a renewal lemma which is used in both papers. Despite our restriction to the forbidden direction case, we cautiously believe that the results and methods of proof are characteristic of more general ballistic random walks in random environment.

We turn to a brief overview of some past work. Two main approaches to laws of large numbers and central limit theorems for RWRE stand out in the literature.

Using regeneration techniques (see the renewal times σ_k in (3.7) below), Comets and Zeitouni (2004b) proves a law of large numbers, Comets and Zeitouni (2004a) proves an annealed central limit theorem, both in the case of a mixing environment, while Sznitman (2002) proves both results when the environment is product. Also, using these techniques, Zerner (2002) reduces the proof of the law of large numbers in product environments to that of a certain 0-1 law.

On the other hand, using homogenization techniques (see Section 3), Rassoul-Agha (2003) proves a law of large numbers in mixing environments and Rassoul-Agha and Seppäläinen (2005a) prove a quenched central limit theorem in space-time product environments. Furthermore, using this approach, Rassoul-Agha (2005) extends the result of Zerner (2002) to mixing environments.

The majority of results have been shown for the ballistic case where the velocity of escape does not vanish. Recent quenched central limit results in the diffusive case can be found in Bolthausen et al. (2003) and Sznitman and Zeitouni (2004).

For general overviews of the field and further literature we refer the reader to the lectures in Zeitouni (2004) and Sznitman (2004). The introduction of Rassoul-Agha and Seppäläinen (2005a) also presents a brief list of papers on central limit theorems for random walks in random environment.

To prove our central limit results we combine both homogenization and regeneration techniques. As groundwork we need suitable equilibrium measures and the law of large numbers. These have been proved in the past for more general ballistic walks, but typically with stringent assumptions of ellipticity and nearest-neighbor steps. Therefore, in Section 3, we reprove these results under our weaker assumptions. In particular, these preliminary results are then used in the paper Rassoul-Agha and Seppäläinen (2005b) that proves the quenched invariance principle for the genuinely multidimensional walks.

2. Assumptions

The type of random walk in random environment that we study in this article is defined by the following asymmetry condition. Given a nonzero vector \hat{u} , say the distribution \mathbb{P} on environments has *forbidden direction* $(-\hat{u})$ if

$$\mathbb{P} \left(\sum_{z: z \cdot \hat{u} \geq 0} \pi_{0z}(\omega) = 1 \right) = 1. \quad (2.1)$$

This condition says that $X_n \cdot \hat{u}$ never decreases along the walk.

Assumption (2.1) alone does not give the regeneration structure we need. Trivially, even $\mathbb{P}(\sum_{z: \hat{u} \cdot z > 0} \pi_{0z} = 0) = 1$ is possible, which permits an arbitrary $(d-1)$ -dimensional walk and the forbidden direction condition is vacuous. However, if we just assume $\mathbb{P}(\sum_{z: \hat{u} \cdot z > 0} \pi_{0z} > 0) > 0$, then there could still be a positive chance of having $x \neq y \in \mathbb{Z}^d$ such that $\pi_{0x} + \pi_{0y} = \pi_{x0} = \pi_{y0} = 1$. In this case, a walker starting at 0 will be stuck in the set $\{0, x, y\}$ forever and the hypothesis on the \hat{u} -direction is again useless. If we add on the assumption $\mathbb{P}(\sum_{z: \hat{u} \cdot z > 0} \pi_{0z} > 0) = 1$, then the walker will eventually move in the \hat{u} -direction, but the expected time of doing so could be infinite, leading to zero velocity in direction \hat{u} . This scenario is illustrated by the next example.

Example 2.1. Consider a two-dimensional product environment \mathbb{P} with marginal at 0 given by

$$\mathbb{P}(\omega_0 = p_i) = \mathbb{P}(\omega_0 = q_i) = 2^{-i},$$

where for $i \geq 1$

$$p_i = (1 - \alpha_i)\delta_{(1,0)} + \alpha_i\delta_{(0,1)} \text{ and } q_i = (1 - \alpha_i)\delta_{(1,0)} + \alpha_i\delta_{(0,-1)}$$

with $\alpha_i = 1 - 2^{-2i}$. The formulas above mean that if $\omega_x = p_i$ then from x the walk jumps to $x + (1, 0)$ with probability $1 - \alpha_i$, and to $x + (0, 1)$ with probability α_i . Note that the jump size is finite, which is even stronger than the moment hypothesis (M) below. Also, assumption (2.1) holds with $\hat{u} = (1, 0)$.

Define $\sigma_1 = \inf\{n : X_n \cdot \hat{u} \geq X_0 \cdot \hat{u} + 1\}$. Then

$$P_0(\sigma_1 > n) \geq \sum_{i \geq 1} \mathbb{P}(\omega_{(0,0)} = p_i, \omega_{(0,1)} = q_i) \alpha_i^n = \sum_{i \geq 1} 2^{-2i} (1 - 2^{-2i})^n$$

and

$$E_0(\sigma_1) = \sum_{n \geq 0} P_0(\sigma_1 > n) \geq \sum_{i \geq 1} 2^{-2i} \sum_{n \geq 0} (1 - 2^{-2i})^n = \sum_{i \geq 1} 2^{-2i} 2^{2i} = \infty.$$

These degenerate situations are prevented by a *non-nestling* condition in direction \hat{u} . That is our second assumption.

HYPOTHESIS (N). There exists a positive deterministic constant δ such that

$$\mathbb{P} \left(\sum_z z \cdot \hat{u} \pi_{0z} \geq \delta \right) = 1.$$

It is noteworthy at this point that with a forbidden direction $-\hat{u}$, any sensible uniform ellipticity assumption implies hypothesis (N). Indeed, if there exists a $\kappa > 0$ and an $x_0 \in \mathbb{Z}^d$ with $x_0 \cdot \hat{u} > 0$ and $\mathbb{P}(\pi_{0x_0} \geq \kappa) = 1$, then (N) is satisfied with $\delta = \kappa x_0 \cdot \hat{u} > 0$. In this sense, hypothesis (N) is a fairly weak assumption that insures a non-zero velocity in direction \hat{u} .

Most of the literature on random walks in random environment concentrates on walks with nearest-neighbor or bounded jumps. This restriction we do not need, but moment assumptions are necessary for laws of large numbers and central limit theorems. This is our next assumption.

HYPOTHESIS (M). There exist two deterministic positive constants M and p such that

$$\mathbb{P} \left(\sum_z |z|^p \pi_{0z} \leq M^p \right) = 1.$$

Here, and in the rest of this paper $|\cdot|$ denotes the l^1 -norm on \mathbb{Z}^d . Each time hypothesis (M) is invoked, a range of values will be given for p , such as $p > 1$ or $p \geq 2$.

3. Invariant measures and the law of large numbers

To prove the law of large numbers we adopt the point of view of the particle. More precisely, we consider the Markov process on Ω with transition kernel

$$\hat{\pi}(\omega, A) = P_0^\omega(T_{X_1} \omega \in A).$$

To apply the ergodic theorem one needs an invariant measure for the above process. Since the state space Ω is large, there will be many such measures. The one that is useful for us will be suitably “comparable” to the original measure \mathbb{P} . For integers n define σ -algebras $\mathfrak{S}_n = \sigma(\omega_x : x \cdot \hat{u} \geq n)$. Our first result is the existence and uniqueness theorem for the invariant distribution.

Theorem 3.1. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 1$. Then there exists a probability measure \mathbb{P}_∞ on (Ω, \mathfrak{S}) that is invariant for the Markov process with transition kernel $\hat{\pi}$ and has these properties:*

- (a) \mathbb{P}_∞ is absolutely continuous relative to \mathbb{P} when restricted to \mathfrak{S}_k with $k \leq 0$. Moreover, $\mathbb{P} = \mathbb{P}_\infty$ on \mathfrak{S}_1 and the two measures are mutually absolutely continuous on \mathfrak{S}_0 .
- (b) The Markov process with kernel $\hat{\pi}$ and initial distribution \mathbb{P}_∞ is ergodic.

- (c) *There can be at most one \mathbb{P}_∞ that is invariant under $\hat{\pi}$, absolutely continuous relative to \mathbb{P} on \mathfrak{S}_k for all $k \leq 0$, and satisfies (b). Moreover, $\mathbb{P}_n(A) = P_0(T_{X_n}\omega \in A)$ converges to $\mathbb{P}_\infty(A)$ for any $A \in \mathfrak{S}_k$ for any $k \leq 0$.*

An issue of interest in the RWRE literature is the equivalence (mutual absolute continuity) of \mathbb{P} and \mathbb{P}_∞ . To have equivalence of \mathbb{P} and \mathbb{P}_∞ on all negative half-spaces requires some sort of ellipticity. An example of strong enough ellipticity hypotheses that imply such absolute continuity can be found in Theorem 2 of Rassoul-Agha (2003). On page 124 below we introduce the ellipticity hypothesis (E) under which the CLT scenario II of the introduction is valid, as proved in the companion paper Rassoul-Agha and Seppäläinen (2005b). Here we wish to point out that this condition (E) is still too weak to imply mutual absolute continuity on all \mathfrak{S}_k , as illustrated by this example.

Example 3.2. Consider a two-dimensional product environment \mathbb{P} whose marginal at 0 is, with equal probability, one of $\{p_1, p_2, p_3\}$. Here,

$$p_1 = \frac{1}{2}(\delta_{(1,1)} + \delta_{(1,-1)}), \quad p_2 = \frac{1}{2}(\delta_{(1,1)} + \delta_{(1,0)}), \quad \text{and} \quad p_3 = \frac{1}{2}(\delta_{(1,-1)} + \delta_{(1,0)}).$$

Here again we have finite-size jumps, so that the moment hypothesis (M) is in force. So is the non-nestling hypothesis (N). As for ellipticity, only the weak version (hypothesis (E) below) is satisfied.

A computation shows that

$$f_n(\omega) := \sum_{x \cdot e_1 = -n} P_x^\omega(X_n = 0) = \frac{d\mathbb{P}_n}{d\mathbb{P}}(\omega).$$

Thus, this is an $L^1(\mathbb{P})$ -martingale relative to the filtration $\{\mathfrak{S}_{-n}\}_{n \geq 0}$ and by the martingale convergence theorem it converges in $L^1(\mathbb{P})$ to a limit f . Then, $\frac{d\mathbb{P}_\infty}{d\mathbb{P}} = f$ and $\mathbb{P}_\infty = \mathbb{P}_n$ on \mathfrak{S}_{-n} for all $n \geq 0$.

Consider the event $A = \{\omega_{(-1,0)} = p_1, \omega_{(-1,1)} = p_2, \omega_{(-1,-1)} = p_3\}$. A is \mathfrak{S}_{-1} -measurable and $\mathbb{P}(A) = 1/27 > 0$. But, if $\omega \in A$, $n \geq 1$, and x is such that $x \cdot e_1 < 0$, then $P_x^\omega(X_n = 0) = 0$. This implies that $\mathbb{P}_\infty(A) = \mathbb{P}_n(A) = 0$ for all $n \geq 1$.

Once one has a suitable invariant measure one can conclude a law of large numbers. Define the *drift* as

$$D(\omega) = E_0^\omega(X_1) = \sum_z z \pi_{0z}(\omega).$$

By part (a) of Theorem 3.1 the moment assumption (M) is also valid under \mathbb{P}_∞ , so there is no problem in defining

$$v = \mathbb{E}_\infty(D).$$

Here, naturally, \mathbb{E}_∞ is expectation under \mathbb{P}_∞ of Theorem 3.1. Define also $\sigma_1 = \inf\{n : X_n \cdot \hat{u} \geq X_0 \cdot \hat{u} + 1\}$. One has the following:

Theorem 3.3. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 1$. Then the following law of large numbers is satisfied:*

$$P_0 \left(\lim_{n \rightarrow \infty} n^{-1} X_n = v \right) = 1.$$

Moreover, $E_0(\sigma_1) < \infty$, $v = E_0(X_{\sigma_1})/E_0(\sigma_1)$, and

$$\sup_n |E_0(X_n) - nv| < \infty. \quad (3.1)$$

Theorems 3.1 and 3.3 correspond essentially to Theorems 2 and 3 of Rassoul-Agha (2003). However, we will reprove them since some of our assumptions are weaker than those of Rassoul-Agha (2003). We start with a lemma.

Lemma 3.4. *Let $d \geq 1$ and consider a T -invariant probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 1$. Then there exist strictly positive, finite constants $\tilde{C}_m(M, \delta, p)$, $\hat{C}_{\bar{p}}(M, \delta, p)$, and $\lambda_0(M, \delta, p)$ such that for all $x \in \mathbb{Z}^d$, $\lambda \in [0, \lambda_0]$, $n, m \geq 0$, and \mathbb{P} -a.e. ω ,*

$$E_x^\omega(|X_m - x|^{\bar{p}}) \leq M^{\bar{p}} m^{\bar{p}} \text{ for } 1 \leq \bar{p} \leq p, \quad (3.2)$$

$$E_x^\omega(e^{-\lambda X_n \cdot \hat{u}}) \leq e^{-\lambda x \cdot \hat{u}} (1 - \lambda \delta / 2)^n, \quad (3.3)$$

$$P_x^\omega(\sigma_1 > n) \leq e^\lambda (1 - \lambda \delta / 2)^n, \quad (3.4)$$

$$E_x^\omega(\sigma_1^m) \leq \tilde{C}_m, \quad (3.5)$$

$$E_x^\omega(|X_{\sigma_1} - x|^{\bar{p}}) \leq \hat{C}_{\bar{p}} \text{ for } 1 \leq \bar{p} < p. \quad (3.6)$$

Proof. Note first that $E_x^\omega(|X_m - x|^{\bar{p}}) = E_0^{T_x \omega}(|X_m|^{\bar{p}})$, with a similar equation for each of the above assertions. Therefore, without loss of generality, we will assume that $x = 0$. To prove (3.2) we write

$$E_0^\omega(|X_m|^{\bar{p}}) \leq m^{\bar{p}-1} \sum_{j=0}^{m-1} E_0^\omega \left(E_0^{T_{X_j} \omega}(|X_1|^{\bar{p}}) \right) \leq M^{\bar{p}} m^{\bar{p}}.$$

Concerning (3.3), observe that since $P_0^\omega(X_1 \cdot \hat{u} \geq 0) = 1$ \mathbb{P} -a.s., one has

$$\begin{aligned} E_x^\omega(e^{-\lambda X_n \cdot \hat{u}} | X_{n-1}) &= e^{-\lambda X_{n-1} \cdot \hat{u}} E_0^{T_{X_{n-1}} \omega}(e^{-\lambda X_1 \cdot \hat{u}}) \\ &\leq e^{-\lambda X_{n-1} \cdot \hat{u}} \left(1 - \lambda D(T_{X_{n-1}} \omega) \cdot \hat{u} + \lambda^p E_0^{T_{X_{n-1}} \omega}(|X_1|^p) \right) \\ &\leq e^{-\lambda X_{n-1} \cdot \hat{u}} (1 - \lambda \delta + M^p \lambda^p). \end{aligned}$$

We have used the fact that $D \cdot \hat{u} \geq \delta$ and $E_0^\omega(|X_1|^p) \leq M^p$, \mathbb{P} -a.s. Taking now the quenched expectation on both sides and iterating the procedure proves (3.3). Then (3.4) follows immediately. Indeed,

$$P_0^\omega(\sigma_1 > n) = P_0^\omega(X_n \cdot \hat{u} < 1) \leq e^\lambda E_0^\omega(e^{-\lambda X_n \cdot \hat{u}}) \leq e^\lambda (1 - \delta \lambda / 2)^n,$$

implying also (3.5). As for (3.6), we have

$$\begin{aligned} E_0^\omega(|X_{\sigma_1}|^{\bar{p}}) &= \sum_{m \geq 1} E_0^\omega(|X_m|^{\bar{p}}, \sigma_1 = m) \\ &\leq \sum_{m \geq 1} E_0^\omega(|X_m|^{\bar{p}})^{\frac{\bar{p}}{p}} P_0^\omega(X_{m-1} \cdot \hat{u} < 1)^{\frac{p-\bar{p}}{p}} \\ &\leq M^{\bar{p}} e^{\lambda \frac{p-\bar{p}}{p}} \sum_{m \geq 1} m^{\bar{p}} \left((1 - \lambda \delta / 2)^{\frac{p-\bar{p}}{p}} \right)^{m-1} < \infty. \quad \square \end{aligned}$$

Once $P_0(\sigma_1 < \infty) = 1$ one can set $\sigma_0 = 0$ and define, by induction, for $k \geq 1$:

$$\sigma_{k+1} = \inf\{n > \sigma_k : X_n \cdot \hat{u} \geq X_{\sigma_k} \cdot \hat{u} + 1\} < \infty, \quad P_0\text{-a.s.} \quad (3.7)$$

Using ideas from Sznitman and Zerner (1999), we next show an annealed renewal property.

Proposition 3.5. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 1$. Then*

(a) $E_0(\sigma_k) < \infty$ for all $k \geq 0$. Moreover,

$$P_0 \left(\lim_{k \rightarrow \infty} k^{-1} \sigma_k = E_0(\sigma_1) \right) = 1.$$

(b) Fix some integer $K \geq 0$ and define

$$\Theta_k = (\sigma_{k+1} - \sigma_k; X_{\sigma_{k+1}} - X_{\sigma_k}, X_{\sigma_{k+2}} - X_{\sigma_k}, \dots, X_{\sigma_{k+1}} - X_{\sigma_k}; \{\omega_x : |x \cdot \hat{u} - X_{\sigma_k} \cdot \hat{u}| \leq K\}).$$

Then $(\Theta_{m(2K+1)+m_0})_{m \geq 1}$ is i.i.d. under P_0 for any $m_0 \geq 0$. In particular, if $K = 0$, then $(\Theta_m)_{m \geq 0}$ is i.i.d. under P_0 .

Proof. First, by (3.5) we know that $E_0(\sigma_1) < \infty$. If one takes $m_0 = 0$ and $K = 0$ in (b), then $(\sigma_{k+1} - \sigma_k)_{k \geq 0}$ is a sequence of non-negative i.i.d. random variables under P_0 and the rest of (a) follows. So we will now prove (b).

Fix $k \geq 0$ and $L \geq k + 1 + 2K$ and define

$$\mathcal{G}_k = \sigma(\sigma_1, \dots, \sigma_{k+1}, X_1, \dots, X_{\sigma_{k+1}}, \{\omega_x : x \cdot \hat{u} \leq X_{\sigma_k} \cdot \hat{u} + K\}).$$

Let F be a bounded function on \mathbb{Z}^d paths \hat{X} and environments ω which depends on ω only through \mathfrak{S}_{-K} . Let H be a bounded \mathcal{G}_k -measurable function.

To understand where the independence here and in similar places elsewhere in the paper comes from, observe that a quenched probability such as $P_0^\omega(X_{\sigma_k} = x)$ is a function of $(\omega_z : z \cdot \hat{u} < x \cdot \hat{u})$, while any quenched expectation $E_x^\omega[f(X_0, X_1, X_2, \dots)]$ of the process restarted at x is a function of $(\omega_z : z \cdot \hat{u} \geq x \cdot \hat{u})$.

Since $\sigma_k < \infty$ P_0 -a.s., we can write

$$\begin{aligned} E_0[F(X_{\sigma_{L+}} - X_{\sigma_L}, T_{X_{\sigma_L}} \omega) H] &= \sum_x \mathbb{E}[E_0^\omega(H, X_{\sigma_{L-K}} = x)] \\ &\quad \times \mathbb{E}[E_x^\omega\{F(X_{\sigma_{K+}} - X_{\sigma_K}, T_{X_{\sigma_K}} \omega)\}] \\ &= \sum_x E_0[H, X_{\sigma_{L-K}} = x] \\ &\quad \times E_0[F(X_{\sigma_{K+}} - X_{\sigma_K}, T_{X_{\sigma_K}} \omega)] \\ &= E_0[H] E_0[F(X_{\sigma_{K+}} - X_{\sigma_K}, T_{X_{\sigma_K}} \omega)]. \end{aligned}$$

In the first equality we used

$$X_{\sigma_k} \cdot \hat{u} + K < X_{\sigma_{L-K-1}} \cdot \hat{u} + 1 \leq X_{\sigma_{L-K}} \cdot \hat{u} \leq X_{\sigma_L} \cdot \hat{u} - K.$$

With $L = m(2K+1) + m_0$ and $k = (m-1)(2K+1) + m_0$ the above shows that $\Theta_{m(2K+1)+m_0}$ is independent of $\mathcal{G}_{(m-1)(2K+1)+m_0}$ and distributed like Θ_K . Since Θ_k is \mathcal{G}_k -measurable, the i.i.d. property follows. If $K = 0$, then one can start m from 0, since $\Theta_K = \Theta_0$. \square

The next proposition defines the invariant measure through a limit and a formula.

Proposition 3.6. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 1$. Then there exists a probability measure \mathbb{P}_∞ such that the sequence $\mathbb{P}_n(A) = P_0(T_{X_n}\omega \in A)$ converges to $\mathbb{P}_\infty(A)$ for all $A \in \mathfrak{S}_{-k}$ and all $k \geq 0$. Moreover, if A is \mathfrak{S}_{-k} -measurable for some $k \geq 0$, then*

$$\mathbb{P}_\infty(A) = \frac{E_0\left(\sum_{m=\sigma_k}^{\sigma_{k+1}-1} \mathbb{I}\{T_{X_m}\omega \in A\}\right)}{E_0(\sigma_1)}. \quad (3.8)$$

Proof. Write

$$P_0(T_{X_n}\omega \in A) = P_0(\sigma_{k+1} > n, T_{X_n}\omega \in A) + P_0(\sigma_{k+1} \leq n, T_{X_n}\omega \in A).$$

The first term on the right-hand-side goes to 0 as n goes to infinity. As for the second term, we have

$$\begin{aligned} P_0(\sigma_{k+1} \leq n, T_{X_n}\omega \in A) &= \sum_{j \geq 1} P_0(\sigma_{k+j} \leq n < \sigma_{k+j+1}, T_{X_n}\omega \in A) \\ &= \sum_{\substack{j \geq 1, x, y \\ 0 \leq m \leq n}} P_0(X_m = x, \sigma_j = m, \sigma_{k+j} \leq n < \sigma_{k+j+1}, X_n = y, T_y\omega \in A) \\ &= \sum_{\substack{j \geq 1, x, y \\ 0 \leq m \leq n}} P_0(X_m = x, \sigma_j = m) P_0(\sigma_k \leq n - m < \sigma_{k+1}, X_{n-m} = y - x, T_{y-x}\omega \in A) \\ &= \sum_{\substack{j \geq 1 \\ 0 \leq m \leq n}} P_0(\sigma_j = m) P_0(\sigma_k \leq n - m < \sigma_{k+1}, T_{X_{n-m}}\omega \in A) \\ &= \sum_{m=0}^n P_0(\exists j \geq 1 : \sigma_j = n - m) P_0(\sigma_k \leq m < \sigma_{k+1}, T_{X_m}\omega \in A). \end{aligned}$$

In the third equality we have used the fact that $T_y A$ is $\mathfrak{S}_{y \cdot \hat{u} - k}$ -measurable, while

$$y \cdot \hat{u} = X_n \cdot \hat{u} \geq X_{\sigma_{k+j}} \cdot \hat{u} \geq X_{\sigma_j} \cdot \hat{u} + k = x \cdot \hat{u} + k.$$

Finally, by Proposition 3.5, $(\sigma_j - \sigma_{j-1})_{j \geq 1}$ are independent under P_0 and thus, by the renewal theorem Feller (1968, p. 313), we have that $P_0(\exists j \geq 1 : \sigma_j = n - m)$ converges to $E_0(\sigma_1)^{-1}$. This finishes the proof. \square

We are now ready to prove Theorems 3.1 and 3.3.

Proof of Theorem 3.1. Let \mathbb{P}_∞ be the measure in Proposition 3.6. For $k \leq 0$ and $A \in \mathfrak{S}_k$, $\hat{\pi}(\omega, A)$ is also \mathfrak{S}_k -measurable. Then the invariance of \mathbb{P}_∞ follows from passing to the limit in the equation $\mathbb{P}_{n+1} = \hat{\pi}\mathbb{P}_n$, utilizing the convergence proved in Proposition 3.6. Moreover, (3.8) shows the absolute continuity of \mathbb{P}_∞ relative to \mathbb{P} on each \mathfrak{S}_k and the absolute continuity of \mathbb{P} relative to \mathbb{P}_∞ on \mathfrak{S}_0 . Now, if A is \mathfrak{S}_1 -measurable, then

$$\mathbb{P}_n(A) = P_0(T_{X_n}\omega \in A) = \sum_x \int P_0^\omega(X_n = x) \mathbb{I}\{T_x\omega \in A\} \mathbb{P}(d\omega) = \mathbb{P}(A),$$

since the two integrands above are independent and \mathbb{P} is shift-invariant. This proves that $\mathbb{P} = \mathbb{P}_\infty$ on \mathfrak{S}_1 , finishing the proof of part (a).

Consider a bounded local function Ψ that is $\sigma(\omega_x : |x \cdot \hat{u}| \leq K)$ -measurable for some $K \geq 0$. Let $\Phi_k = \sum_{j=\sigma_k}^{\sigma_{k+1}-1} \Psi(T_{X_j} \omega)$. By Proposition 3.5, $(\Phi_{m(2K+1)+m_0})_{m \geq 1}$ is a sequence of i.i.d. random variables with a finite first moment under probability P_0 . This implies that

$$P_0 \left(\lim_{m \rightarrow \infty} m^{-1} \sum_{j=0}^{\sigma_{(m+1)(2K+1)}-1} \Psi(T_{X_j} \omega) = (2K+1)E_0(\Phi_K) \right) = 1. \quad (3.9)$$

Note that in (3.9) the sum started at 0 instead of σ_{2K+1} , since $P_0(\sigma_{2K+1} < \infty) = 1$ and Ψ is bounded. Define now $K_n = (m_n + 1)(2K + 1)$ such that $\sigma_{K_n} \leq n < \sigma_{K_{n+1}}$. Then

$$\left| n^{-1} \sum_{m=0}^{n-1} \Psi(T_{X_m} \omega) - n^{-1} \sum_{m=0}^{\sigma_{K_n}-1} \Psi(T_{X_m} \omega) \right| \leq n^{-1}(\sigma_{K_{n+1}} - \sigma_{K_n}) \|\Psi\|_\infty$$

goes to 0 P_0 -a.s., since $n^{-1}\sigma_{K_n}$ goes to 1. On the other hand, by Proposition 3.5, $k^{-1}\sigma_k$ converges to $E_0(\sigma_1) < \infty$. Hence, $n^{-1}K_n$ converges to $E_0(\sigma_1)^{-1}$ and $n^{-1}m_n$ converges to $[(2K+1)E_0(\sigma_1)]^{-1}$. Thus, one has

$$P_0 \left(\lim_{n \rightarrow \infty} n^{-1} \sum_{m=0}^{\sigma_{K_n}-1} \Psi(T_{X_m} \omega) = E_0(\Phi_K)/E_0(\sigma_1) \right) = 1.$$

Therefore, we have \mathbb{P} -a.s.

$$P_0^\omega \left(\lim_{n \rightarrow \infty} n^{-1} \sum_{m=0}^{n-1} \Psi(T_{X_m} \omega) = E_0(\Phi_K)/E_0(\sigma_1) \right) = 1. \quad (3.10)$$

Since the above quantity is \mathfrak{S}_{-K} -measurable, the same holds \mathbb{P}_∞ -a.s. Moreover, the constant limit cannot be anything other than $\mathbb{E}_\infty(\Psi)$. By bounded convergence, one then has

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{m=0}^{n-1} E_0^\omega[\Psi(T_{X_m} \omega)] = \mathbb{E}_\infty(\Psi), \quad \mathbb{P}_\infty\text{-a.s.}$$

Now approximate a general $\Psi \in L^1(\mathbb{P}_\infty)$ by a bounded local Ψ in the $L^1(\mathbb{P}_\infty)$ -sense. The above convergence is then true in the $L^1(\mathbb{P}_\infty)$ -sense for any $\Psi \in L^1(\mathbb{P}_\infty)$. The fact that all these limits are constants suffices for ergodicity. This follows from Section IV.2 in Rosenblatt's monograph Rosenblatt (1971), see especially Corollary 2, the definition at the top of p. 94, and Corollary 5. Part (b) is proved.

Once one has ergodicity and absolute continuity on half-spaces uniqueness follows. Indeed, recalling that (3.10) holds \mathbb{P} -a.s. and using bounded convergence once again implies that for any measurable bounded Ψ

$$\lim_{n \rightarrow \infty} n^{-1} \sum_{m=0}^{n-1} E_0(\Psi(T_{X_m} \omega)) = \mathbb{E}_\infty(\Psi),$$

determining \mathbb{P}_∞ uniquely. The rest of part (c) is true by construction of \mathbb{P}_∞ . \square

Proof of Theorem 3.3. We first prove a law of large numbers with limiting velocity

$$\bar{v} = \frac{E_0(X_{\sigma_1})}{E_0(\sigma_1)}$$

and then the approximation (3.1) with \bar{v} instead of v . Lastly we identify \bar{v} with $v = \mathbb{E}_\infty(D)$.

Recall that $E_0(|X_{\sigma_1}|) < \infty$. Using $K = 0$ and $m_0 = 0$ in part (b) of Proposition 3.5 one can see that $(X_{\sigma_k} - X_{\sigma_{k-1}}, \sigma_k - \sigma_{k-1})_{k \geq 1}$ is a sequence of i.i.d. random variables, under P_0 . Therefore, we have

$$P_0 \left(\lim_{k \rightarrow \infty} k^{-1}(X_{\sigma_k}, \sigma_k) = (E_0(X_{\sigma_1}), E_0(\sigma_1)) \right) = 1.$$

Now let $K_n = K(n) = \max\{j : \sigma_j \leq n\}$. We will sometimes use the notation $K(n)$ to avoid subscripts on subscripts. Clearly, $n^{-1}K_n$ converges P_0 -a.s. to $E_0(\sigma_1)^{-1}$.

Write

$$\frac{X_n}{n} = \frac{X_n - X_{\sigma_{K(n)+1}}}{n} + \frac{X_{\sigma_{K(n)+1}}}{K_n + 1} \frac{K_n + 1}{n}. \quad (3.11)$$

Using (3.6) with some $\bar{p} \in (1, p)$, we have that for any $\varepsilon > 0$

$$P_0(|X_n - X_{\sigma_{K(n)+1}}| > \varepsilon n) \leq \varepsilon^{-\bar{p}} n^{-\bar{p}} E_0(|X_n - X_{\sigma_{K(n)+1}}|^{\bar{p}}) \leq \varepsilon^{-\bar{p}} \hat{C}_{\bar{p}} n^{-\bar{p}}$$

is the general term of a convergent series. By Borel-Cantelli's Lemma, the first term on the right-hand-side of (3.11) goes to 0 P_0 -a.s. Therefore,

$$P_0 \left(\lim_{n \rightarrow \infty} n^{-1} X_n = \bar{v} \right) = 1.$$

This proves the law of large numbers with \bar{v} instead of v . Since

$$\{K_n = k\} = \left\{ \sum_{j=1}^k (\sigma_j - \sigma_{j-1}) \leq n < \sum_{j=1}^{k+1} (\sigma_j - \sigma_{j-1}) \right\},$$

$K_n + 1$ is a stopping time with respect to the filtration of the process $\{(X_{\sigma_k} - X_{\sigma_{k-1}}, \sigma_k - \sigma_{k-1}) : k \geq 1\}$. Moreover, $K_n + 1$ is bounded, and both σ_1 and X_{σ_1} are integrable by Lemma 3.4. By Wald's identity,

$$E_0 \left(\sum_{j=1}^{K_n+1} (X_{\sigma_j} - X_{\sigma_{j-1}}) \right) = E_0(1 + K_n) E_0(X_{\sigma_1})$$

and

$$E_0(\sigma_{K_n+1}) = E_0 \left(\sum_{j=1}^{K_n+1} (\sigma_j - \sigma_{j-1}) \right) = E_0(1 + K_n) E_0(\sigma_1).$$

From $X_n = \sum_{j=1}^{K_n+1} (X_{\sigma_j} - X_{\sigma_{j-1}}) - (X_{\sigma_{K_n+1}} - X_n)$ we get

$$\begin{aligned} E_0(X_n) &= E_0(1 + K_n) E_0(X_{\sigma_1}) - E_0(X_{\sigma_{K_n+1}} - X_n) \\ &= n\bar{v} + \bar{v} [E_0(1 + K_n) E_0(\sigma_1) - n] - E_0(X_{\sigma_{K_n+1}} - X_n) \\ &= n\bar{v} + \bar{v} E_0(\sigma_{K_n+1} - n) - E_0(X_{\sigma_{K_n+1}} - X_n). \end{aligned}$$

The last two terms above are bounded by a constant. To see it for the last one, note that

$$E_0^\omega(X_{\sigma_{K_n+1}} - X_n) = E_0^\omega [E_{X_n}^\omega(X_{\sigma_1} - X_0)] = E_0^\omega [E_0^{T_{X_n} \omega}(X_{\sigma_1})]$$

which is bounded by a constant \mathbb{P} -almost surely by (3.6). The same argument works for $E_0(\sigma_{K_n+1} - n)$ via (3.5). This proves (3.1) with \bar{v} instead of v . One consequence of this is that $n^{-1}E_0(X_n)$ converges to \bar{v} , as n goes to infinity.

On the other hand, by the ergodic theorem one has for \mathbb{P}_∞ -a.e. ω

$$P_0^\omega \left(\lim_{n \rightarrow \infty} n^{-1} \sum_{m=0}^{n-1} D(T_{X_m} \omega) = v = \mathbb{E}_\infty(D) \right) = 1.$$

Due to the absolute continuity of \mathbb{P} relative to \mathbb{P}_∞ on \mathfrak{S}_0 the above is also true \mathbb{P} -a.s. Due to the moment hypothesis (M), the drift D is bounded in norm by M , for \mathbb{P} -a.e. ω . Therefore, by bounded convergence, we conclude that $n^{-1}E_0(X_n) = n^{-1}E_0(\sum_{m=0}^{n-1} D(T_{X_m} \omega))$ converges to v , P_0 -a.s. Thus $\bar{v} = v$ and the proof of Theorem 3.3 is complete. \square

4. The annealed invariance principle

Let us start with more notation and definitions. We write Γ^t for the transpose of a vector or matrix Γ . An element of \mathbb{R}^d is regarded as a $d \times 1$ matrix, or column vector.

For a symmetric, non-negative definite $d \times d$ matrix Γ , a Brownian motion with diffusion matrix Γ is the \mathbb{R}^d -valued process $\{W(t) : t \geq 0\}$ such that $W(0) = 0$, W has continuous paths, independent increments, and for $s < t$ the d -vector $W(t) - W(s)$ has Gaussian distribution with mean zero and covariance matrix $(t - s)\Gamma$. The matrix Γ is *degenerate* in direction $\xi \in \mathbb{R}^d$ if $\xi^t \Gamma \xi = 0$. Equivalently, $\xi \cdot W(t) = 0$ almost surely.

For $t \geq 0$ let

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}}.$$

Here $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ for $x \in \mathbb{R}$. $D_{\mathbb{R}^d}([0, \infty))$ denotes the space of \mathbb{R}^d -valued cadlag paths endowed with the usual Skorohod topology, as developed for instance in Ethier and Kurtz (1986). The next theorem is the annealed invariance principle.

Theorem 4.1. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 2$. Then the distribution of the process $B_n(t)$ under P_0 converges weakly on the space $D_{\mathbb{R}^d}([0, \infty))$ to the distribution of a Brownian motion with diffusion matrix*

$$\mathfrak{D} = \frac{E_0[(X_{\sigma_1} - v\sigma_1)(X_{\sigma_1} - v\sigma_1)^t]}{E_0[\sigma_1]}.$$

The matrix \mathfrak{D} is degenerate in direction u if, and only if, u is orthogonal to the vector space spanned by $\{x - y : \mathbb{E}(\pi_{0x})\mathbb{E}(\pi_{0y}) > 0\}$.

Note that degeneracy in directions that are orthogonal to all $x - y$, where x and y range over admissible jumps, cannot be avoided. This can be seen from the simple example of a homogeneous random walk that chooses with equal probability between two jumps a and b . The diffusion matrix is then $\frac{1}{4}(a - b)(a - b)^t$.

Proof of Theorem 4.1. Using Lemma 3.4 one can check that non-nestling (N) in direction \hat{u} and the moment hypothesis (M) with $p > 2$ imply that $E_0(\sigma_1^2) < \infty$ and $E_0(\sup_{j \leq \sigma_1} |X_j|^2) < \infty$. Then, since $X_{\sigma_k} - v\sigma_k$ is a sum of square-integrable i.i.d. random variables under P_0 , the process $n^{-1/2}(X_{\sigma_{[nt]}} - v\sigma_{[nt]})$ converges to a Brownian motion with diffusion matrix $E_0[(X_{\sigma_1} - v\sigma_1)(X_{\sigma_1} - v\sigma_1)^t]$.

Next the invariance principle is transferred to the process $n^{-1/2}(X_{\sigma_{K(\lfloor nt \rfloor)}} - v\sigma_{K(\lfloor nt \rfloor)})$, where as before $K_n = K(n) = \max\{j : \sigma_j \leq n\}$. Now the limiting diffusion matrix is \mathfrak{D} . We omit this step as it can be found in the proof of Theorem 4.1 of Sznitman (2000). Note that the regeneration times τ_k in Sznitman (2000) are exactly our σ_k , and that the event $D = \infty$ in Sznitman (2000) is of full P_0 -measure in our case.

The last step passes the invariance principle to the process $B_n(t)$, via the following estimation.

$$\begin{aligned} & P_0 \left[\sup_{t \leq 1} |B_n(t) - n^{-1/2}(X_{\sigma_{K(\lfloor nt \rfloor)}} - v\sigma_{K(\lfloor nt \rfloor)})| \geq \varepsilon \right] \\ & \leq P_0 \left[\sup_{k \leq K_n} \sup_{\sigma_k \leq j < \sigma_{k+1}} |X_j - X_{\sigma_k}| \geq \varepsilon\sqrt{n}/2 \right] + P_0 \left[\sup_{k \leq K_n} (\sigma_{k+1} - \sigma_k) \geq \varepsilon\sqrt{n}/2|v| \right] \\ & \leq (n+1)P_0 \left[\sup_{0 \leq j < \sigma_1} |X_j| \geq \varepsilon\sqrt{n}/2 \right] + (n+1)P_0[\sigma_1 \geq \varepsilon\sqrt{n}/2|v|] \\ & \leq \frac{4(n+1)}{n\varepsilon^2} \left\{ E_0 \left[\sup_{0 \leq j < \sigma_1} |X_j|^2, \sup_{0 \leq j < \sigma_1} |X_j| \geq \varepsilon\sqrt{n}/2 \right] + |v|^2 E_0[\sigma_1^2, \sigma_1 \geq \varepsilon\sqrt{n}/2] \right\} \\ & \xrightarrow{n \rightarrow \infty} 0. \end{aligned}$$

We used the stationarity of $(\sigma_{k+1} - \sigma_k, X_{\sigma_k}, \dots, X_{\sigma_{k+1}-1})_{k \geq 0}$ and $K_n \leq n$ which follows from $\sigma_n \geq n$. In the last step we used $E_0(\sigma_1^2) < \infty$ and $E_0(\sup_{0 \leq j < \sigma_1} |X_j|^2) < \infty$. This proves the weak convergence of the process $B_n(t)$ under P_0 .

Next we address the degeneracy of the diffusion matrix. We will first show that this matrix is degenerate in direction u if, and only if, u is orthogonal to the vector space spanned by $\{x - v : \mathbb{E}(\pi_{0x}) > 0\}$. To this end, let $u \neq 0$ be such that $P_0(X_{\sigma_1} \cdot u = \sigma_1 v \cdot u) = 1$.

If x is such that $x \cdot \hat{u} > 0$ and $\mathbb{E}_0(\pi_{0x}) > 0$, then

$$P_0(X_{\sigma_1} = x, \sigma_1 = 1) \geq \mathbb{E}(\pi_{0x}) > 0$$

implies that $x \cdot u = v \cdot u$ and u is perpendicular to $x - v$.

If $z \neq 0$ is such that $z \cdot \hat{u} = 0$ and $\mathbb{E}(\pi_{0z}) > 0$, then choose x such that $x \cdot \hat{u} > 0$ and $\mathbb{E}(\pi_{0x}) > 0$. This is always possible, due to non-nestling (N). Then

$$P_0(X_{\sigma_1} = nz + x, \sigma_1 = n + 1) \geq \mathbb{E}(\pi_{0z})^n \mathbb{E}(\pi_{0x}) > 0$$

implies that $nz \cdot u + x \cdot u = (n+1)v \cdot u$, for all $n \geq 0$. This gives $z \cdot u = x \cdot u = v \cdot u$ and u is perpendicular to $z - v$.

If $\mathbb{E}(\pi_{00}) > 0$, then by non-nestling (N) there exists a vector $z \neq 0$ such that $\mathbb{E}(\pi_{00} \pi_{0z}) > 0$. If $z \cdot \hat{u} > 0$, then

$$P_0(X_{\sigma_1} = z, \sigma_1 = n + 1) \geq \mathbb{E}(\pi_{00}^n \pi_{0z}) > 0$$

and $z \cdot u = (n+1)v \cdot u$ implies that $v \cdot u = 0$. On the other hand, if $z \cdot \hat{u} = 0$, then choose x such that $x \cdot \hat{u} > 0$ and $\mathbb{E}(\pi_{0x}) > 0$. In this case, we have

$$P_0(X_{\sigma_1} = z + x, \sigma_1 = n + 2) \geq \mathbb{E}(\pi_{00}^n \pi_{0z}) \mathbb{E}(\pi_{0x}) > 0,$$

which again implies $(z+x) \cdot u = (n+2)v \cdot u$ and $v \cdot u = 0$.

The above discussion shows that if the diffusion matrix is degenerate in direction u , then u is orthogonal to the vector space spanned by $\{x - v : \mathbb{E}(\pi_{0x}) > 0\}$.

Inversely, if u is orthogonal to $x - v$ for each x such that $\mathbb{E}(\pi_{0x}) > 0$, then

$$E_0[u \cdot (X_{\sigma_1} - \sigma_1 v)] = \sum_{n \geq 1} E_0 \left[\mathbb{I}\{\sigma_1 = n\} \left| \sum_{k=0}^{n-1} u \cdot ((X_{k+1} - X_k) - v) \right| \right] = 0$$

and the diffusion matrix is degenerate in direction u .

To finish, we observe that the span of $\{x - v : \mathbb{E}(\pi_{0x}) > 0\}$ is the same as that of $\{x - y : \mathbb{E}(\pi_{0x})\mathbb{E}(\pi_{0y}) > 0\}$, which is the space that appears in the theorem. One part comes simply from $x - y = (x - v) - (y - v)$. Conversely,

$$x - v = \lim_{n \rightarrow \infty} n^{-1} \sum_{k=0}^{n-1} (x - (X_{k+1} - X_k))$$

shows that $x - v$ lies in the span of $x - y$ for admissible steps y . \square

5. The quenched invariance principle

Two types of centering of the quenched process will be considered: the (random) quenched mean $E_0^\omega(X_n)$ and its asymptotic limit nv . For $\omega \in \Omega$, define two scaled processes

$$B_n(t) = \frac{X_{[nt]} - [nt]v}{\sqrt{n}} \quad \text{and} \quad \tilde{B}_n(t) = \frac{X_{[nt]} - E_0^\omega(X_{[nt]})}{\sqrt{n}}.$$

Let Q_n^ω , respectively \tilde{Q}_n^ω , denote the distribution of B_n , respectively \tilde{B}_n , induced by P_0^ω on the Borel sets of $D_{\mathbb{R}^d}([0, \infty))$. The key to distinguishing between the two quenched scenarios I and II described in the Introduction is the following ellipticity hypothesis.

HYPOTHESIS (E). One has

$$\mathbb{P}(\forall z \neq 0 : \pi_{00} + \pi_{0z} < 1) > 0. \quad (5.1)$$

Moreover, the walk is not supported by any one-dimensional subspace. More precisely, if $\mathcal{J} = \{y : \mathbb{E}(\pi_{0y}) > 0\}$ is the set of all points that are accessible from 0 with one jump, then \mathcal{J} is not contained in any subspace of the kind $\mathbb{R}u = \{su : s \in \mathbb{R}\}$ for any $u \in \mathbb{R}^d$. In particular, this rules out the case $d = 1$.

It turns out that if we add assumption (E) to the earlier hypotheses (2.1), (N), and (M) with $p > 2$, then both laws Q_n^ω and \tilde{Q}_n^ω converge to a Brownian motion with the diffusion matrix \mathfrak{D} of Theorem 4.1, for \mathbb{P} -almost every ω . This is proved in the companion paper Rassoul-Agha and Seppäläinen (2005b).

What we show in the remainder of the present paper is that (E) is also *necessary* in order for the processes B_n to satisfy the quenched invariance principle. So we have an almost complete characterization of the central limit behavior of the non-nestling walk with a forbidden direction. The only shortcoming is that some of our moment hypotheses are almost certainly not best possible.

To be more precise, we show that if (E) fails while the other hypotheses are in force, then scenario I from the Introduction is realized. And additionally, if a degenerate situation is ruled out, the distributions $\{Q_n^\omega\}_{n \geq 1}$ are not tight for a.e. ω , so the processes B_n cannot satisfy a quenched central limit theorem. The proof is done separately for the two possible ways in which hypothesis (E) can fail:

(i) The walk can be supported on a one-dimensional subspace. Such a walk can be converted to the case $d = 1$, which is treated in Theorem 5.4.

(ii) We can have $d \geq 2$ but condition (5.1) fails. This case is treated in Theorem 5.11.

The qualitative insight we take from this is that the quenched CLT centered at nv demands enough random noise in the selection of the path of the walker under a fixed environment. Only then can the walker experience a sufficiently rich sample of environments so that the fluctuations of the quenched mean do not interfere at the diffusive scale.

5.1. *The one-dimensional case.* When $d = 1$ we take $\hat{u} = 1$, and so our random walk is totally asymmetric in the sense that $X_n \leq X_{n+1}$ holds almost surely. Define the events $V_i = \{\exists n \geq 0 : X_n = i\}$ and the coefficients

$$a_i(\omega) = (1 - \pi_{ii}(\omega))^{-1}(P_0^\omega(V_i) - P_1^\omega(V_i)).$$

Let $g(\omega) = D(\omega) - v$ and define

$$\Delta(\omega) = \sum_{i \geq 0} a_i(\omega)g(T_i\omega). \tag{5.2}$$

Two observations that will help seeing how various formulas emerge: in the one-dimensional case

$$\frac{d\mathbb{P}_\infty}{d\mathbb{P}} = \frac{(1 - \pi_{00})^{-1}}{\mathbb{E}[(1 - \pi_{00})^{-1}]},$$

and also

$$\mathbb{E}[|E_0^\omega(X_{\sigma_1}) - vE_0^\omega(\sigma_1)|^2] = \mathbb{E}[g^2/(1 - \pi_{00})^2].$$

The following constants will be the diffusion coefficients in our next theorem.

$$\kappa_m^2 = v^{-1} \mathbb{E}[|E_0^\omega(X_{\sigma_1}) - vE_0^\omega(\sigma_1)|^2] \sum_{i \geq 0} \mathbb{E} \left[\left| P_0^\omega(V_i) - \sum_{k=0}^i P_k^\omega(V_i) \mathbb{E}_\infty \pi_{0k} \right|^2 \right], \tag{5.3}$$

$$\kappa_q^2 = \mathbb{E}_\infty E_0^\omega \left(\left| X_1 - v - \sum_{z=0}^{X_1-1} \Delta(T_z\omega) \right|^2 \right). \tag{5.4}$$

We first state some properties of the diffusion constants.

Proposition 5.1. *Let $d = 1$ and consider a product probability measure \mathbb{P} on environments with forbidden direction $-\hat{u} = -1$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , and the moment hypothesis (M) with $p > 2$. Then*

(1) $\kappa_m > 0$ if, and only if,

$$\mathbb{P}(D = v) < 1. \tag{5.5}$$

Moreover, when $\kappa_m = 0$, one has $E_0^\omega(X_n) = nv$ and $Q_n^\omega = \tilde{Q}_n^\omega$, \mathbb{P} -a.s.

(2) $\kappa_q > 0$ if, and only if,

$$\mathbb{P}(\sup_z \pi_{0z} < 1) > 0. \tag{5.6}$$

(3) One has

$$\kappa_m^2 + \kappa_q^2 = E_0[|X_{\sigma_1} - v\sigma_1|^2]/E_0(\sigma_1),$$

which is precisely \mathfrak{D} from Theorem 4.1.

(4) We have this alternative representation:

$$\kappa_m^2 = \frac{v}{E_{0,0}(L)} \mathbb{E}[|E_0^\omega(X_{\sigma_1} - v\sigma_1)|^2]$$

where $E_{0,0}(L)$ is the expected first common point

$$L = \inf\{\ell > 0 : \exists n, m : X_n = \bar{X}_m = \ell\}$$

of two walks that start at the origin and run independently in a common environment.

We do not have an argument for the decomposition in (c) directly from the defining formulas. Instead, we justify it by a limit. In order not to interrupt the flow of ideas we will prove this proposition after the statement and proof of the next theorem. But before we proceed with the theorem, let us give two examples where one can compute the above diffusion coefficients.

Example 5.2. Consider a one-dimensional product environment \mathbb{P} with marginal satisfying:

$$\mathbb{P}(\pi_{01} + \pi_{02} = 1) = 1.$$

For convenience, we will use $p_x = \pi_{x,x+1}$ and $q_x = \pi_{x,x+2} = 1 - p_x$. Then,

$$P_0^\omega(V_i) - P_1^\omega(V_i) = -q_0(P_1^\omega(V_i) - P_2^\omega(V_i)) = \dots = (-1)^i q_0 \cdots q_{i-1}$$

and $v = \mathbb{E}D = 1 + \mathbb{E}q_0$. Hence

$$\begin{aligned} \kappa_m^2 &= (1 + \mathbb{E}q_0)^{-1} \text{Var}(E_0^\omega X_1) \sum_{i \geq 0} \mathbb{E}[|(P_0^\omega(V_i) - P_1^\omega(V_i)) + (P_1^\omega(V_i) - P_2^\omega(V_i))\mathbb{E}q_0|^2] \\ &= (1 + \mathbb{E}q_0)^{-1} \text{Var}(q_0) \left(1 + \sum_{i \geq 1} \mathbb{E}[(q_0 - \mathbb{E}q_0)q_1 \cdots q_{i-1}]^2\right) \\ &= \text{Var}(q_0) \frac{1 - \mathbb{E}q_0}{1 - \mathbb{E}(q_0^2)} \end{aligned}$$

while

$$\kappa_q^2 = E_0[|X_1 - v|^2] - \kappa_m^2.$$

The next example is used in the proof of Theorem 5.11.

Example 5.3. Consider a one-dimensional product environment \mathbb{P} with marginal satisfying

$$\mathbb{P}(\pi_{00} + \pi_{01} = 1) = 1.$$

Then $v = 1/E_0(\sigma_1) = (E_0[\pi_{01}^{-1}])^{-1}$ and $a_0(\omega) = \pi_{01}^{-1}(\omega)$ while $a_i(\omega) = 0$ for $i \geq 1$. This gives

$$\kappa_m^2 = (\mathbb{E}[\pi_{01}^{-1}])^{-3} \text{Var}(\pi_{01}^{-1}) = \frac{\text{Var}(E_0^\omega[v\sigma_1])}{E_0(\sigma_1)}, \quad (5.7)$$

$$\kappa_q^2 = (\mathbb{E}[\pi_{01}^{-1}])^{-3} \mathbb{E}\left[\frac{\pi_{00}}{\pi_{01}^2}\right] = \frac{\mathbb{E}[\text{Var}_0^\omega(v\sigma_1)]}{E_0(\sigma_1)}. \quad (5.8)$$

The following theorem treats the one-dimensional situation. The moment needed is defined by

$$p_0 = \frac{19}{6} + \frac{\sqrt{139}}{3} \cos\left(\frac{1}{3} \arccos\left(\frac{1504}{139^{3/2}}\right)\right). \quad (5.9)$$

This is approximately 7.06025.

Theorem 5.4. *Let $d = 1$ and consider a product probability measure \mathbb{P} on environments with forbidden direction $-\hat{u} = -1$ as in (2.1). Assume non-nestling (N) in direction \hat{u} and the moment hypothesis (M) with $p > p_0$. Then*

- (1) *For \mathbb{P} -almost every ω the distributions \tilde{Q}_n^ω converge weakly to the distribution of the process $\{\kappa_q B(t) : t \geq 0\}$ where $B(t)$ denotes standard one-dimensional Brownian motion.*
- (2) *Under \mathbb{P} , the scaled quenched mean process $n^{-1/2}\{E_0^\omega(X_{[nt]}) - nt\}$ converges in distribution to $\{\kappa_m B(t) : t \geq 0\}$.*
- (3) *Assume the non-degeneracy condition (5.5) holds. Then for \mathbb{P} -almost every ω , the sequence of distributions $\{Q_n^\omega : n \geq 1\}$ is not tight.*

Remark 5.5. *The crude moment bounds we use in the proof require the strong assumption $p > p_0$. We believe this can be improved, but do not venture into this.*

Proof. Given the moment p of the hypothesis of the theorem, henceforth we write

$$m = 2\bar{p} - 2$$

where \bar{p} is some value that satisfies $2 \leq \bar{p} < p$. The lower bound on \bar{p} ensures $m \geq 2$, which is convenient so that powers $m/2$ used below do not go below 1. Also, $m > 1$ is needed for applying Theorem 3.2 from Burkholder (1973) in the next proof.

Lemma 5.6. Δ is \mathfrak{S}_0 -measurable, $\mathbb{E}(|\Delta|^m) < \infty$, and $\mathbb{E}(\Delta) = 0$.

Proof of Lemma 5.6. The measurability statement is immediate from the definition of Δ . Concerning the moments, first note that the moment hypothesis (M) implies that g is uniformly bounded by $2M$. Second, note that combining the non-nestling hypothesis (N) with the moment hypothesis (M) and Hölder's inequality, one has

$$\delta \leq \sum_{x \geq 1} x \pi_{0x} \leq (1 - \pi_{00})^{(p-1)/p} M,$$

which says that π_{00} is uniformly bounded away from 1. Third, note that if we denote by $P_{i,j}^\omega$ the process of two independent walkers in the same environment ω , one starting at i and the other at j , then

$$a_i(\omega) = (1 - \pi_{ii}(\omega))^{-1} E_{0,1}^\omega(\mathbb{I}_{V_i} - \mathbb{I}_{\bar{V}_i}),$$

where \bar{V}_i is the event corresponding to the walker starting at 1. If we denote by L the first common point of the paths $\{X_n : n \geq 0\}$ and $\{\bar{X}_n : n \geq 0\}$, then we have

$$\begin{aligned} |a_i(\omega)| &\leq (M/\delta)^{p/(p-1)} \left| \sum_{j=1}^i P_{0,1}^\omega(L = j) E_{j,j}^\omega(\mathbb{I}_{V_i} - \mathbb{I}_{\bar{V}_i}) \right| \\ &\quad + (M/\delta)^{p/(p-1)} \left| E_{0,1}^\omega(\mathbb{I}_{V_i} - \mathbb{I}_{\bar{V}_i}, L > i) \right| \\ &\leq (M/\delta)^{p/(p-1)} P_{0,1}^\omega(L > i). \end{aligned}$$

The coefficients $P_0^\omega(V_i)$ and $P_1^\omega(V_i)$ are functions of $\omega_0, \dots, \omega_{i-1}$, hence independent of $(1 - \pi_{ii}(\omega))^{-1} g(T_i \omega)$. By (3.8) and the definition of v we have

$$\mathbb{E}[g/(1 - \pi_{00})] = \mathbb{E}_\infty(g) E_0(\sigma_1) = 0.$$

Therefore, under the measure \mathbb{P} , $\Delta_n = \sum_{i=0}^n a_i(\omega) g(T_i \omega)$ is a martingale relative to the filtration $\sigma(\omega_0, \dots, \omega_n)$. Then the Burkholder-Davis-Gundy inequality

Burkholder (1973, Theorem 3.2) gives us

$$\begin{aligned} \sup_n \mathbb{E}[|\Delta_n|^m] &\leq C \sup_n \mathbb{E} \left[\left| \sum_{i=0}^n |a_i|^2 \right|^{m/2} \right] \leq C' \mathbb{E} \left[\left| \sum_{i \geq 0} P_{0,1}^\omega(L > i) \right|^{m/2} \right] \\ &\leq C' \mathbb{E}[|E_{0,1}^\omega(L)|^{m/2}] \leq C' E_{0,1}[L^{m/2}]. \end{aligned}$$

In the last stage above we can take $P_{0,1} = P_0 \otimes P_1$ to be the process of two walkers with independent annealed distributions. This works because the two walkers do not meet until they visit site L for the first time. We can apply Lemma A.1 from the appendix to L , by taking $Y_i = X_{\sigma_i} - X_{\sigma_{i-1}}$ with distribution

$$P[Y_i = k] = \mathbb{E}[(1 - \pi_{00})^{-1} \pi_{0k}], \quad k \geq 1.$$

Bound (3.6) from Lemma 3.4 gives $E(Y_i^{\bar{p}}) < \infty$ for any $\bar{p} < p$. Then Lemma A.1 tells us that $E_{0,1}(L^{\bar{p}-1}) < \infty$ and we can take $m = 2\bar{p} - 2$ in the bound above.

The integrability of L and the earlier bound on $|a_i(\omega)|$ imply that the series (5.2) defining Δ converges absolutely, \mathbb{P} -almost surely. By the moment bound above and the martingale convergence theorem, Δ is also the $L^m(\mathbb{P})$ -limit of Δ_n . This gives $\mathbb{E}|\Delta|^m < \infty$ and $\mathbb{E}[\Delta] = 0$. \square

Remark 5.7. *A place where it should be possible to improve the moment hypotheses needed for the argument is the moment bound on Δ . For example, one has*

$$\sum_{i \geq 0} |a_i|^2 \leq \sum_{i \geq 0} P_{0,1}^\omega(L > i)^2 = E^\omega(L \wedge \bar{L}),$$

where \bar{L} is an independent copy of L . Then for the m -th moment of Δ_n we get the bound

$$\mathbb{E}[E^\omega(L \wedge \bar{L})^{m/2}] \leq \mathbb{E}[E^\omega(L^{m/4} \wedge \bar{L}^{m/4})^2] \leq \mathbb{E}[E^\omega(L^{m/4})^2].$$

Now m is divided by 4 instead of 2, but the drawback is that one needs to bound the quenched moment of L .

Define $\chi(0, \omega) = 0$, and for $x \geq 1$

$$\chi(x, \omega) = \sum_{y=0}^{x-1} \Delta(T_y \omega).$$

One can check that $a_0 = 1/(1 - \pi_{00})$ and, for $i \geq 1$,

$$\sum_{j=0}^i a_j(T_{i-j} \omega) \sum_{y>i-j} \pi_{0y}(\omega) = 0.$$

With the help of the above formula one can check that

$$E_0^\omega[\chi(X_1, \omega)] = g(\omega). \quad (5.10)$$

This implies that, for \mathbb{P} -a.e. ω ,

$$M_n = X_n - nv - \chi(X_n, \omega) \quad (5.11)$$

is a P_0^ω -martingale. Using the invariance and ergodicity of \mathbb{P}_∞ under the process $(T_{X_n} \omega)$, and the ergodic theorem, one can check that M_n satisfies the conditions of the martingale invariance principle; see, for example, Theorem (7.4) of Chapter 7 of Durrett (2004). So the distribution of $(n^{-1/2} M_{[nt]})_{t \geq 0}$ under P_0^ω converges weakly to a Brownian motion with diffusion coefficient $\mathbb{E}_\infty E_0^\omega(M_1^2)$, for \mathbb{P}_∞ -a.e. ω .

Since everything in this statement is \mathfrak{S}_0 -measurable, the same invariance principle holds for \mathbb{P} -a.e. ω .

To handle the random summation limit in $\chi(X_n, \omega)$ we decompose

$$\chi(X_n, \omega) = Z_n(\omega) + R_n \quad (5.12)$$

where $Z_n(\omega) = \sum_{y=0}^{\lfloor nv \rfloor} \Delta(T_y \omega)$ and R_n is the remaining part.

We can now outline the strategy for proving the invariance principles. From (5.11) and (5.12) we write

$$E_0^\omega(X_n) - nv = Z_n(\omega) + E_0^\omega(R_n) \quad (5.13)$$

and then

$$X_n - E_0^\omega(X_n) = M_n + R_n - E_0^\omega(R_n). \quad (5.14)$$

With (5.14) the quenched invariance principle for the processes

$$\tilde{B}_n(t) = n^{-1/2} \{X_{\lfloor nt \rfloor} - E_0^\omega(X_{\lfloor nt \rfloor})\}$$

follows from the invariance principle of M_n observed above, after we show that the errors R_n and $E_0^\omega(R_n)$ are negligible. For the quenched mean process

$$n^{-1/2} \{E_0^\omega(X_{\lfloor nt \rfloor}) - nt\}$$

we show that the process Z_n is close to a martingale under \mathbb{P} , and apply the martingale invariance principle again.

As a preparatory step we state an annealed bound on deviations of the walk.

Lemma 5.8. *For every $\bar{p} \in [2, p)$ there exists a constant $C = C(\bar{p}) < \infty$ such that, for $h > 0$ and $n \geq 1$,*

$$P_0(|X_n - nv| > h) \leq Ch^{-\bar{p}} n^{\bar{p}/2}. \quad (5.15)$$

Proof of Lemma 5.8. Recall the definition $K(n) = \max\{j : \sigma_j \leq n\}$. In one dimension $\sigma_{K(n)+1} = \inf\{k > n : X_k > X_n\}$ because X_n does not jump during any time interval $\sigma_i \leq n < \sigma_{i+1}$.

$$\begin{aligned} P_0(|X_n - nv| > h) &\leq P_0(|X_n - X_{\sigma_{K(n)+1}}| > h/3) \\ &\quad + P_0(|X_{\sigma_{K(n)+1}} - \sigma_{K(n)+1}v| > h/3) + P_0(\sigma_{K(n)+1} - n > h|v|^{-1}/3) \\ &\leq E_0 P_{X_n}^\omega(X_{\sigma_1} - X_0 > h/3) + P_0\left(\max_{k \leq n+1} |X_{\sigma_k} - \sigma_k v| > h/3\right) \\ &\quad + E_0 P_{X_n}^\omega(\sigma_1 > h|v|^{-1}/3). \end{aligned}$$

For the last inequality we restarted the walk at time n for the first and last probability, and used $K_n \leq n$ for the middle probability. From (3.6) we get a bound on the first probability:

$$E_0 P_{X_n}^\omega(X_{\sigma_1} - X_0 > h/3) \leq Ch^{-\bar{p}} \leq Ch^{-\bar{p}} n^{\bar{p}/2}.$$

The third probability is handled similarly via (3.5). For the middle probability note that $X_{\sigma_k} - \sigma_k v$ is a sum of mean zero i.i.d. random variables with finite \bar{p} -th

moment. Use first Doob's inequality, then the Burkholder-Davis-Gundy inequality Burkholder (1973, Theorem 3.2), and finally the Schwarz inequality:

$$\begin{aligned} P_0\left(\max_{k \leq n+1} |X_{\sigma_k} - \sigma_k v| > h/3\right) &\leq Ch^{-\bar{p}} E_0[|X_{\sigma_{n+1}} - \sigma_{n+1} v|^{\bar{p}}] \\ &\leq Ch^{-\bar{p}} E_0\left[\left|\sum_{k=1}^{n+1} (X_{\sigma_k} - X_{\sigma_{k-1}} - \sigma_k v + \sigma_{k-1} v)^2\right|^{\bar{p}/2}\right] \\ &\leq Ch^{-\bar{p}} (n+1)^{\bar{p}/2} E_0\left[\frac{1}{n+1} \sum_{k=1}^{n+1} |X_{\sigma_k} - X_{\sigma_{k-1}} - \sigma_k v + \sigma_{k-1} v|^{\bar{p}}\right] \leq C' h^{-\bar{p}} n^{\bar{p}/2}. \end{aligned}$$

Inequality (5.15) has been verified. \square

We turn to the first main task, bounding the errors R_n and $E_0^\omega(R_n)$; see (5.12). Fix $\varepsilon > 0$ and $\eta \in (0, 1/2 - 1/p)$, and let $\delta_n = n^{-\eta}$. The strategy is to develop summable deviation estimates for the errors and apply Borel-Cantelli. Summability of the estimates will require that η satisfies several conditions simultaneously. Then we show that for $p > p_0$ a single choice of η can satisfy all the requirements. Write

$$P_0(|R_n| > \varepsilon\sqrt{n}) \leq P_0(|X_n - nv| > n\delta_n) + \mathbb{P}(\mathcal{B}_n^\varepsilon) + \mathbb{P}(\mathcal{C}_n^\varepsilon) \quad (5.16)$$

where

$$\mathcal{B}_n^\varepsilon = \left\{ \max_{0 \leq i \leq n\delta_n} \left| \sum_{j=0}^i \frac{\Delta(T_{-j}\omega)}{\sqrt{n}} \right| > \frac{\varepsilon}{2} \right\} \quad \text{and} \quad \mathcal{C}_n^\varepsilon = \left\{ \max_{0 \leq i \leq n\delta_n} \left| \sum_{j=i}^{[n\delta_n]} \frac{\Delta(T_{-j}\omega)}{\sqrt{n}} \right| > \frac{\varepsilon}{2} \right\}.$$

If we pick $\bar{p} < p$ close enough to p so that

$$p^{-1} < \bar{p}^{-1} < \frac{1}{2} - \eta \quad (5.17)$$

to ensure $\bar{p}(\frac{1}{2} - \eta) > 1$, Lemma 5.8 shows that the first term on the right-hand side of (5.16) is summable:

$$\sum_n P_0(|X_n - nv| > n\delta_n) \leq C \sum_n n^{-\bar{p}(\frac{1}{2} - \eta)} < \infty. \quad (5.18)$$

To control the other two terms in (5.16), define

$$H_n = \sum_{k \geq 0} \Delta(T_{-n+k}\omega) \sum_{y > k} \mathbb{E}_\infty \pi_{0y}.$$

For the next calculations, recall that

$$\frac{d\mathbb{P}_\infty}{d\mathbb{P}} = \frac{(1 - \pi_{00})^{-1}}{\mathbb{E}[(1 - \pi_{00})^{-1}]} \quad \text{and} \quad v = \mathbb{E}_\infty D = \mathbb{E}_\infty \sum_{k \geq 0} \sum_{y > k} \pi_{0y}.$$

Then, for $m = 2\bar{p} - 2$,

$$\begin{aligned} \mathbb{E}[|H_n|^m] &= v^m \mathbb{E}\left[\left|v^{-1} \sum_{k \geq 0} \Delta(T_k\omega) \sum_{y > k} \mathbb{E}_\infty \pi_{0y}\right|^m\right] \\ &\leq v^{m-1} \sum_{k \geq 0} \mathbb{E}[|\Delta|^m] \sum_{y > k} \mathbb{E}_\infty \pi_{0y} \\ &= v^m \mathbb{E}[|\Delta|^m] < \infty. \end{aligned}$$

Also, under \mathbb{P} , H_n is a martingale difference relative to the filtration \mathfrak{G}_{-n} . To see that, it is enough to show that $\mathbb{E}(H_0 | \mathfrak{G}_1) = 0$, since $H_n = T_{-n}H_0$. And indeed:

$$\begin{aligned} \mathbb{E}[(1 - \pi_{00})^{-1}] \mathbb{E}[H_0 | \mathfrak{G}_1] &= \mathbb{E}[\Delta | \mathfrak{G}_1] + \sum_{k \geq 1} \Delta(T_k \omega) \sum_{y > k} \mathbb{E}[(1 - \pi_{00})^{-1} \pi_{0y}] \\ &= \mathbb{E} \left[(1 - \pi_{00})^{-1} \sum_{k \geq 0} \Delta(T_k \omega) \sum_{y > k} \pi_{0y} \middle| \mathfrak{G}_1 \right] \\ &= \mathbb{E}[(1 - \pi_{00})^{-1} g] = 0, \end{aligned}$$

where we have used (5.10) and $\mathbb{E}_\infty g = 0$. Moreover,

$$\begin{aligned} \sum_{k=1}^n (v \Delta(T_{-k} \omega) - H_k) &= \sum_{j \geq 0} \sum_{y > j} \mathbb{E}_\infty \pi_{0y} \sum_{k=1}^n (\Delta(T_{-k} \omega) - \Delta(T_{-k+j} \omega)) \\ &= \sum_{y > 1} \mathbb{E}_\infty \pi_{0y} \sum_{j=1}^{y-1} \sum_{k=1}^n (\Delta(T_{-k} \omega) - \Delta(T_{-k+j} \omega)) \\ &= \sum_{y > 1} \mathbb{E}_\infty \pi_{0y} \sum_{j=1}^{y-1} \sum_{k=0}^{j-1} (\Delta(T_{k-n} \omega) - \Delta(T_k \omega)), \end{aligned}$$

where the last line was obtained by adding (resp. subtracting) the necessary terms in the third summation when $j \geq n$ (resp. $j < n$). Thus

$$v \sum_{k=0}^n \Delta(T_{-k} \omega) = \sum_{k=1}^n H_k + U_n, \quad (5.19)$$

with

$$\begin{aligned} U_n(\omega) &= v \Delta(\omega) + A(T_{-n} \omega) - A(\omega), \\ A(\omega) &= \sum_{y > 1} \mathbb{E}_\infty (\pi_{0y}) \sum_{j=0}^{y-2} (y - j - 1) \Delta(T_j \omega). \end{aligned}$$

Now, similarly to the moment bound above for H_n ,

$$\mathbb{E}[|A|^m] \leq \mathbb{E} \left[\left| \sum_{j \geq 0} |\Delta(T_j \omega)| \sum_{y > j} y \mathbb{E}_\infty \pi_{0y} \right|^m \right] \leq [\mathbb{E}_\infty E_0^\omega X_1^2]^m \mathbb{E}[|\Delta|^m].$$

Then, using Doob's inequality, followed by the Burkholder-Davis-Gundy inequality Burkholder (1973, Theorem 3.2), and then by Jensen's inequality on the first term

below, and Chebyshev's inequality on the second term, one has

$$\begin{aligned}
\mathbb{P}(\mathcal{B}_n^\varepsilon) &\leq \mathbb{P}\left(\max_{i \leq n\delta_n} \left| \sum_{k=1}^i H_k \right| > \frac{v\varepsilon\sqrt{n}}{4}\right) + n\delta_n \max_{i \leq n\delta_n} \mathbb{P}\left(|U_i| > \frac{v\varepsilon\sqrt{n}}{4}\right) \\
&\leq Cn^{-m/2} \mathbb{E} \left| \sum_{k=1}^{\lfloor n\delta_n \rfloor} H_k \right|^m + C\delta_n n^{1-m/2} \\
&\leq Cn^{-m/2} \mathbb{E} \left| \sum_{k=1}^{\lfloor n\delta_n \rfloor} H_k^2 \right|^{m/2} + C\delta_n n^{1-m/2} \\
&\leq Cn^{-m/2} (n\delta_n)^{m/2-1} \mathbb{E} \sum_{k=1}^{\lfloor n\delta_n \rfloor} |H_k|^m + C\delta_n n^{1-m/2} \\
&\leq C(\delta_n^{m/2} + \delta_n n^{1-m/2}). \tag{5.20}
\end{aligned}$$

The same argument works also for $\mathbb{P}(\mathcal{C}_n^\varepsilon)$, by first writing

$$\left| \sum_{j=i}^{\lfloor n\delta_n \rfloor} \Delta(T_{-j}\omega) \right| \leq \left| \sum_{j=0}^{\lfloor n\delta_n \rfloor} \Delta(T_{-j}\omega) \right| + \left| \sum_{j=0}^{i-1} \Delta(T_{-j}\omega) \right|.$$

In order to have the Borel-Cantelli Lemma imply that R_n/\sqrt{n} goes to 0 \mathbb{P} -a.s. and, therefore, also

$$\max_{k \leq n} |R_k|/\sqrt{n} \text{ goes to } 0, \mathbb{P}\text{-a.s.} \tag{5.21}$$

we need

$$\sum_n (\delta_n^{m/2} + \delta_n n^{1-m/2}) = \sum_n (n^{-\eta m/2} + n^{-\eta+1-m/2}) < \infty. \tag{5.22}$$

Next, write

$$\begin{aligned}
\mathbb{P}(|E_0^\omega[R_n]| > \varepsilon\sqrt{n}) &\leq \mathbb{P}(|E_0^\omega[R_n, X_n - nv > n\delta_n]| > \varepsilon\sqrt{n}/4) \\
&\quad + \mathbb{P}(|E_0^\omega[R_n, nv - X_n > n\delta_n]| > \varepsilon\sqrt{n}/4) \\
&\quad + \mathbb{P}(\mathcal{B}_n^{\varepsilon/2}) + \mathbb{P}(\mathcal{C}_n^{\varepsilon/2})
\end{aligned}$$

The last two terms can be controlled as in (5.20). The first two are similar to each other, so we work only with the first one. To this end, we have

$$\begin{aligned}
& \mathbb{P}(|E_0^\omega[R_n, X_n - nv > n\delta_n]| > \varepsilon\sqrt{n}/4) \\
& \leq \frac{4}{\varepsilon\sqrt{n}} \sum_{x > nv + n\delta_n} \mathbb{E} \left[P_0^\omega(X_n = x) \sum_{y=[nv]}^{x-1} |\Delta(T_y\omega)| \right] \\
& \leq \frac{4}{\varepsilon\sqrt{n}} \sum_{x > nv + n\delta_n} \mathbb{E}[P_0^\omega(X_n = x)^{m/(m-1)}]^{1-1/m} \mathbb{E} \left[\left| \sum_{y=[nv]}^{x-1} |\Delta(T_y\omega)| \right|^m \right]^{1/m} \\
& \leq \frac{4}{\varepsilon\sqrt{n}} \sum_{x > nv + n\delta_n} P_0(X_n = x)^{1-1/m} \mathbb{E} \left[(x - [nv])^{m-1} \sum_{y=[nv]}^{x-1} |\Delta(T_y\omega)|^m \right]^{1/m} \\
& \leq \frac{4\mathbb{E}[|\Delta|^m]^{1/m}}{\varepsilon\sqrt{n}} \sum_{x > n\delta_n} x P_0(X_n - [nv] = x)^{1-1/m} \\
& \leq Cn^{-1/2 + \bar{p}(1-1/m)/2} \sum_{x > n\delta_n} x^{1-\bar{p}(1-1/m)} \leq Cn^{3/2 - 2\eta - \bar{p}(1/2 - \eta)(1-1/m)},
\end{aligned}$$

where the next to last inequality came from Lemma 5.8. Again, the Borel-Cantelli Lemma implies that

$$\max_{k \leq n} |E_0^\omega(R_k)|/\sqrt{n} \text{ goes to } 0, \mathbb{P}\text{-a.s.} \quad (5.23)$$

if

$$3/2 - 2\eta - \bar{p}(1/2 - \eta)(1 - 1/m) < -1. \quad (5.24)$$

Recall the definition (5.9) of p_0 . Now we observe that $p > p_0$ enables us to make all the Borel-Cantelli arguments simultaneously valid. As throughout this section, $m = 2\bar{p} - 2$.

Lemma 5.9. *Suppose $p > p_0$. Then we can choose $\eta \in (0, 1/2 - 1/p)$ so that there exists $\varepsilon_1 > 0$ such that, for all $\bar{p} \in (p - \varepsilon_1, p)$, requirements (5.17), (5.22), and (5.24) all hold, and thereby the conclusions (5.21) and (5.23) hold.*

Proof of Lemma 5.9. The first requirement is easy: for any $\eta \in (0, 1/2 - 1/p)$ there is a whole range $(p - \varepsilon_2, p)$ of \bar{p} -values that satisfy (5.17), and forcing \bar{p} closer to p cannot violate this condition.

The second condition (5.22) has two parts. The first one requires $1 < \eta m/2 = \eta(\bar{p} - 1)$, or $\eta > (\bar{p} - 1)^{-1}$. Observe that $\frac{1}{2}(5 + \sqrt{17}) \approx 4.6$ is the larger root of $(p - 1)^{-1} = 1/2 - 1/p$. Hence for $p > \frac{1}{2}(5 + \sqrt{17})$ and \bar{p} close enough to p , there is room to choose η so that $(\bar{p} - 1)^{-1} < \eta < 1/2 - 1/p$.

The second part of (5.22) requires $-\eta + 1 - m/2 < -1$ which is satisfied for any $\eta > 0$ if $\bar{p} \geq 3$. With $p > \frac{1}{2}(5 + \sqrt{17})$ and \bar{p} close enough to p this is guaranteed.

Finally we consider requirement (5.24). Take first $\eta = (\bar{p} - 1)^{-1}$ which was identified as a lower bound for η two paragraphs above. By getting rid of denominators, observe that (5.24) is satisfied iff

$$(2\bar{p} - 2)(5\bar{p} - 9) - \bar{p}(\bar{p} - 3)(2\bar{p} - 3) < 0.$$

By calculus we see that over the interval $(4, \infty)$, this happens iff p is larger than $p_0 = \frac{19}{6} + \frac{\sqrt{139}}{3} \cos(\frac{1}{3} \arccos(\frac{1504}{139^{3/2}}))$. Since the inequality above is strict, it remains in force if $\eta > (\bar{p} - 1)^{-1}$ is close enough to $(\bar{p} - 1)^{-1}$.

To summarize, as long as $p > p_0$, a pair (\bar{p}, η) can be found to satisfy all the requirements. \square

We are ready to prove Theorem 5.4. Part (a) follows from (5.14), and the martingale invariance principle applied to M_n , because (5.21) and (5.23) show that $R_n - E_0^\omega(R_n)$ is a negligible error. Formula (5.4) says simply that the diffusion coefficient is $\mathbb{E}_\infty E_0^\omega |M_1|^2$.

For part (b), combine (5.19) with (5.13) to write

$$\begin{aligned} E_0^\omega(X_n) - nv &= Z_n + E_0^\omega(R_n) \\ &= v^{-1} \sum_{k=1}^{[nv]} H_k(T_{[nv]}\omega) + v^{-1} U_{[nv]}(T_{[nv]}\omega) + E_0^\omega(R_n). \end{aligned} \quad (5.25)$$

By the calculation in (5.20), one can neglect U_n . By (5.23), one can also neglect $E_0^\omega(R_n)$. Then, using the ergodicity of \mathbb{P} , one can check that the martingale $\sum_{k=1}^n H_k$ satisfies the conditions of the martingale invariance principle Durrett (2004, Section 7.7). This implies (b), short of identifying the diffusion coefficient.

If we define κ_m^2 as the limiting diffusion coefficient for the process

$$n^{-1/2} \{E_0^\omega(X_{[nt]}) - nt v\},$$

then (5.25) implies $\mathbb{E}(H_0^2) = \kappa_m^2 v$. Then, to calculate $\mathbb{E}(H_0^2)$, use the fact that the partial sums of the infinite sum below form a martingale relative to the filtration $\sigma(\omega_0, \dots, \omega_n)$:

$$\begin{aligned} \mathbb{E}[H_0^2] &= \mathbb{E} \left[\left| \sum_{i \geq 0} g(T_i \omega) \sum_{k=0}^i a_{i-k}(T_k \omega) \sum_{y > k} \mathbb{E}_\infty \pi_{0y} \right|^2 \right] \\ &= \mathbb{E}[g^2 / (1 - \pi_{00})^2] \sum_{i \geq 0} \mathbb{E} \left[\left| \sum_{k=0}^i (P_k^\omega(V_i) - P_{k+1}^\omega(V_i)) \sum_{y > k} \mathbb{E}_\infty \pi_{0y} \right|^2 \right] \\ &= \mathbb{E}[|E_0^\omega(X_{\sigma_1}) - v E_0^\omega(\sigma_1)|^2] \sum_{i \geq 0} \mathbb{E} \left[\left| P_0^\omega(V_i) - \sum_{k=0}^i P_k^\omega(V_i) \mathbb{E}_\infty \pi_{0k} \right|^2 \right]. \end{aligned}$$

A comparison with (5.3) shows that this completes the proof of part (b).

As for part (c), use (5.25) to write

$$\frac{X_n - nv}{\sqrt{n}} = \frac{X_n - E_0^\omega(X_n)}{\sqrt{n}} + \frac{v^{-1} U_{[nv]}(T_{[nv]}\omega) + E_0^\omega(R_n)}{\sqrt{n}} + \frac{\sum_{k=1}^{[nv]} H_k(T_{[nv]}\omega)}{v \sqrt{n}}.$$

The first term above is tight under P_0^ω , for \mathbb{P} -a.e. ω , by part (a). We have shown that the second term goes to 0 in \mathbb{P} -probability. For the last term, we have also shown that it converges to a Gaussian under \mathbb{P} . This implies that

$$\mathbb{P} \left(\overline{\lim}_{n \rightarrow \infty} \frac{\sum_{k=1}^n H_k(T_n \omega)}{\sqrt{n}} \geq y \right) > 0, \quad \forall y.$$

Since the above is a tail event and \mathbb{P} is product, the third term cannot be bounded, for a fixed ω , unless $0 = \mathbb{E}(H_0^2) = v\kappa_m^2$. But, according to part (a) of Proposition 5.1, this is not allowed by the non-degeneracy condition. \square

We conclude this section by proving Proposition 5.1 about the diffusion coefficients.

Proof of Proposition 5.1. To prove part (a) one notices that in the sum in (5.3) the $i = 0$ term is positive unless the walk never leaves 0. But as observed in the proof of Lemma 5.6, π_{00} is bounded away from 1. Therefore, $\kappa_m = 0$ if, and only if, $(1 - \pi_{00})^{-1}g = 0$, \mathbb{P} -a.s. Part (a) follows.

To prove part (b) we will distinguish two cases. First, if $\mathbb{P}(\pi_{00} > 0) > 0$, then

$$\kappa_q^2 \geq v^2 (E_0 \sigma_1)^{-1} \mathbb{E} \left[\frac{\pi_{00}}{1 - \pi_{00}} \right] > 0.$$

On the other hand, if $\pi_{00} = 0$, \mathbb{P} -a.s., and if there exist two distinct points x and y such that $\mathbb{E}\pi_{0x}\pi_{0y} > 0$, then

$$\mathbb{E} \left[\prod_{i=0}^{y-1} \pi_{ix, (i+1)x} \prod_{i=0}^{x-1} \pi_{iy, (i+1)y} \right] > 0$$

and if $X_1 - v - \sum_{z=0}^{X_1-1} \Delta(T_z \omega) = 0$, P_0 -a.s., then

$$\sum_{z=0}^{xy-1} (1 - \Delta(T_z \omega))$$

is simultaneously equal to

$$\sum_{i=0}^{y-1} \sum_{z=0}^{x-1} (1 - \Delta(T_{z+ix} \omega)) = yv$$

and

$$\sum_{i=0}^{x-1} \sum_{z=0}^{y-1} (1 - \Delta(T_{z+iy} \omega)) = xv$$

which implies that $v = 0$ and is a contradiction.

The proof of part (c) uses uniform integrability that follows from having $p > 2$. Start with

$$E_0 \left[\left(\frac{X_n - nv}{\sqrt{n}} \right)^2 \right] = E_0 \left[\left(\frac{X_n - E_0^\omega(X_n)}{\sqrt{n}} \right)^2 \right] + \mathbb{E} \left[\left(\frac{E_0^\omega(X_n) - nv}{\sqrt{n}} \right)^2 \right]. \quad (5.26)$$

Let $2 < \tilde{p} < \bar{p} < p$. Estimate (5.15) implies that

$$\sup_n E_0 \left[\left| \frac{X_n - nv}{\sqrt{n}} \right|^{\tilde{p}} \right] < \infty,$$

which together with the annealed CLT (Theorem 4.1) implies

$$\lim_{n \rightarrow \infty} E_0 \left[\left(\frac{X_n - nv}{\sqrt{n}} \right)^2 \right] = \frac{E_0(|X_{\sigma_1} - \sigma_1 v|^2)}{E_0(\sigma_1)}.$$

Similarly on the right-hand side of (5.26) we get convergence to $\kappa_q^2 + \kappa_m^2$ by observing that uniform integrability follows from that already proved:

$$\mathbb{E} \left[\left| \frac{E_0^\omega(X_n) - nv}{\sqrt{n}} \right|^{\tilde{p}} \right] = \mathbb{E} \left[\left| E_0^\omega \left(\frac{X_n - nv}{\sqrt{n}} \right) \right|^{\tilde{p}} \right] \leq E_0 \left[\left| \frac{X_n - nv}{\sqrt{n}} \right|^{\tilde{p}} \right]$$

and then

$$E_0 \left[\left| \frac{X_n - E_0^\omega(X_n)}{\sqrt{n}} \right|^{\tilde{p}} \right] \leq 2^{\tilde{p}} E_0 \left[\left| \frac{X_n - nv}{\sqrt{n}} \right|^{\tilde{p}} \right] + 2^{\tilde{p}} \mathbb{E} \left[\left| \frac{E_0^\omega(X_n) - nv}{\sqrt{n}} \right|^{\tilde{p}} \right].$$

To prove part (d), recall that $K(n) = \max\{j : \sigma_j \leq n\}$ and observe that

$$\begin{aligned} \mathbb{E}[|E_0^\omega(X_n) - nv|^2] &= \mathbb{E} \left[\left| \sum_{k=0}^{n-1} E_0^\omega[g(T_{X_k}\omega)] \right|^2 \right] \\ &= \mathbb{E} \left[\left| E_0^\omega \left[\sum_{k=0}^{\sigma_{K(n)+1}-1} g(T_{X_k}\omega) \right] \right|^2 \right] + o(n). \end{aligned} \quad (5.27)$$

The bound $o(n)$ on the error above comes because of ergodicity of \mathbb{P}_∞ , boundedness of g and $\mathbb{E}_\infty g = 0$, and because $E_0^\omega(\sigma_{K(n)+1} - n) = E_0^\omega E_{X_n}^\omega(\sigma_1)$ is bounded.

Neglecting the error, (5.27) equals

$$\begin{aligned} &\mathbb{E} \left[E_0^\omega \left(\sum_x \sum_{i=0}^{\sigma_{K(n)+1}-1} \mathbb{I}\{X_i = x\} g(T_x\omega) \right) \right. \\ &\quad \left. \times E_0^\omega \left(\sum_y \sum_{j=0}^{\sigma_{K(n)+1}-1} \mathbb{I}\{X_j = y\} g(T_y\omega) \right) \right]. \end{aligned} \quad (5.28)$$

Manipulate each of the two expressions in the brackets as follows, noting that the time spent at site X_{σ_k} is $\sigma_{k+1} - \sigma_k$:

$$\begin{aligned} &E_0^\omega \left(\sum_z \sum_{k=0}^{\sigma_{K(n)+1}-1} \mathbb{I}\{X_k = z\} g(T_z\omega) \right) \\ &= E_0^\omega \left(\sum_z \sum_{k=0}^{\infty} \mathbb{I}\{X_{\sigma_k} = z, \sigma_k \leq n\} (\sigma_{k+1} - \sigma_k) g(T_z\omega) \right) \\ &= \sum_z \sum_{k=0}^{\infty} P_0^\omega(X_{\sigma_k} = z, \sigma_k \leq n) E_z^\omega(\sigma_1) g(T_z\omega) \\ &= \sum_z \sum_{k=0}^{\infty} P_0^\omega(X_{\sigma_k} = z, \sigma_k \leq n) \frac{g(T_z\omega)}{1 - \pi_{zz}(\omega)}. \end{aligned}$$

Combine the factors again, so that (5.27) turns into

$$\begin{aligned} &\mathbb{E}[|E_0^\omega(X_n) - nv|^2] = o(n) \\ &+ \mathbb{E} \left[\sum_{x,y} \sum_{i,j \geq 0} P_0^\omega(X_{\sigma_i} = x, \sigma_i \leq n) P_0^\omega(X_{\sigma_j} = y, \sigma_j \leq n) \frac{g(T_x\omega)}{1 - \pi_{xx}(\omega)} \frac{g(T_y\omega)}{1 - \pi_{yy}(\omega)} \right]. \end{aligned}$$

If, for example, $x < y$ then the factor $g(T_y\omega)(1 - \pi_{yy}(\omega))^{-1}$ is independent of the rest and is mean zero. Hence we simplify to

$$\begin{aligned} & \mathbb{E}[|E_0^\omega(X_n) - nv|^2] + o(n) \\ &= \mathbb{E}\left[\left|\frac{g(\omega)}{1 - \pi_{00}(\omega)}\right|^2\right] \mathbb{E}\left[\sum_x \sum_{i,j \geq 0} P_0^\omega(X_{\sigma_i} = x, \sigma_i \leq n) P_0^\omega(X_{\sigma_j} = x, \sigma_j \leq n)\right] \\ &= \mathbb{E}\left[\left|\frac{g(\omega)}{1 - \pi_{00}(\omega)}\right|^2\right] E_{0,0}(|X_{[0,n]} \cap \bar{X}_{[0,n]}|), \end{aligned} \tag{5.29}$$

where $X_I = \{X_i, i \in I\}$ and $P_{0,0} = \mathbb{E}P_{0,0}^\omega$ is the annealed process of two independent walks X and \bar{X} in a common environment ω , both starting at 0.

Define now $L_0 = 0$ and for $j \geq 1$,

$$L_j = \inf\{\ell > L_{j-1} : \exists n, m : X_n = \bar{X}_m = \ell\}.$$

Using similar arguments to those of Proposition 3.5 one shows that $(L_j - L_{j-1})_{j \geq 1}$ is an i.i.d. sequence under $P_{0,0}$. Since these are non-negative random variables, we have that L_j/j converges to $E_{0,0}(L_1)$.

Define $J_n = \max\{j : L_j \leq X_n \wedge \bar{X}_n\}$. Then

$$L_{J_n} \leq X_n \wedge \bar{X}_n \leq L_{J_n+1},$$

and Theorem 3.3 implies that J_n/n converges to $v/E_{0,0}(L_1)$. Since $J_n/n \in [0, 1]$, we have the same limit for $E_{0,0}(J_n)/n$. Dividing by n in (5.29) and letting $n \rightarrow \infty$ shows that the diffusion coefficient is given by

$$\kappa_m^2 = \frac{v}{E_{0,0}(L_1)} \mathbb{E}[|E_0^\omega(X_{\sigma_1} - v\sigma_1)|^2],$$

which proves part (d). □

5.2. *The restricted-path case.* Now we consider the case where $d \geq 1$ is arbitrary but

$$\mathbb{P}(\exists z \neq 0 : \pi_{00} + \pi_{0z} = 1) = 1. \tag{5.30}$$

Before the theorem we go through some preliminaries. Once an environment ω has been fixed, each $x \in \mathbb{Z}^d$ has a unique point $w(x) \in \mathbb{Z}^d \setminus \{x\}$ such that $\pi_{x,x}(\omega) + \pi_{x,w(x)}(\omega) = 1$, hence the name “restricted path.” Once we assume forbidden direction and non-nestling (N) with \hat{u} in addition to (5.30), any $p > 1$ in (M) implies uniform boundedness of the steps $w(x) - x$ and the existence of $\delta_1 > 0$ such that both $\pi_{x,w(x)}(\omega) \geq \delta_1$ and $(w(x) - x) \cdot \hat{u} \geq \delta_1$ uniformly over x and \mathbb{P} -almost every ω . It is convenient to replace the stopping times $\{\sigma_j\}$ with the stopping times $\lambda_0 = 0$,

$$\lambda_i = \inf\{n > \lambda_{i-1} : X_n \cdot \hat{u} \geq X_{\lambda_{i-1}} \cdot \hat{u} + \delta_1\}, \quad i \geq 1.$$

Replacing the increment 1 by δ_1 in (3.7) does not affect the role of the stopping times in the proofs of the LLN or the annealed CLT. Hence we can express the asymptotic velocity as $v = E_0(X_{\lambda_1})/E_0(\lambda_1)$ and the diffusion matrix of the annealed invariance principle as

$$\mathfrak{D} = \frac{E_0[(X_{\lambda_1} - \lambda_1 v)(X_{\lambda_1} - \lambda_1 v)^t]}{E_0[\lambda_1]} \tag{5.31}$$

As in Theorem 5.4, we will show next that the covariance matrix (5.31) of the annealed walk can be decomposed as the limiting covariance of the quenched walk plus the limiting covariance of the quenched mean process. Noticing that in the

case at hand X_{σ_1} is not random under P_0^ω , define for the quenched mean process the matrix \mathfrak{D}_m , analogously to (5.7), by

$$\mathfrak{D}_m = \frac{\mathbb{E}[(X_{\lambda_1} - E_0^\omega(\lambda_1)v)(X_{\lambda_1} - E_0^\omega(\lambda_1)v)^t]}{E_0[\lambda_1]}. \quad (5.32)$$

For the quenched walk, define the constant κ_0 , analogously to (5.8), by

$$\kappa_0^2 = \frac{\mathbb{E}[\text{Var}_0^\omega(\lambda_1)]}{E_0[\lambda_1]} = \left(\mathbb{E}\left[\frac{1}{1 - \pi_{00}}\right] \right)^{-1} \mathbb{E}\left[\frac{\pi_{00}}{(1 - \pi_{00})^2}\right].$$

The limit process for the quenched invariance principle is $\kappa_0 B(\cdot)v$ where $B(\cdot)$ is a standard one-dimensional Brownian motion. It has diffusion matrix $\mathfrak{D}_q = \kappa_0^2 v v^t$, which is exactly what needs to be added to (5.32) to yield (5.31).

Remark 5.10. *As it was in the one-dimensional case, $\kappa_m > 0$ if, and only if,*

$$\mathbb{P}(D = v) = \mathbb{P}\left(\omega : w(0) = \frac{v}{1 - \pi_{00}(\omega)}\right) < 1. \quad (5.33)$$

If this condition fails, Q_n^ω and \tilde{Q}_n^ω coincide. Secondly, $\kappa_q = 0$ if, and only if, $P_0(\pi_{00} = 0) = 1$, i.e. (5.6) fails to hold.

Theorem 5.11. *Let $d \geq 1$ and consider a product probability measure \mathbb{P} on environments with a forbidden direction $-\hat{u} \in \mathbb{R}^d \setminus \{0\}$ as in (2.1). Assume non-nestling (N) in direction \hat{u} , the moment hypothesis (M) with $p > 1$, and (5.30).*

- (1) *For \mathbb{P} -almost every ω the distributions \tilde{Q}_n^ω converge weakly to the distribution of the process $\{\kappa_0 B(t)v : t \geq 0\}$ where $B(t)$ denotes standard one-dimensional Brownian motion.*
- (2) *Under \mathbb{P} , the scaled quenched mean process $n^{-1/2}\{E_0^\omega(X_{[nt]}) - nt v\}$ converges weakly to a Brownian motion with diffusion matrix \mathfrak{D}_m .*
- (3) *Assume non-degeneracy condition (5.33) holds. Then for \mathbb{P} -almost every ω , the sequence of distributions $\{Q_n^\omega : n \geq 1\}$ is not tight.*

Proof. The environment ω uniquely determines a path $\{0 = z_0, z_1, z_2, \dots\}$ such that $z_{i+1} = w(z_i)$. Each new point z_i takes the walk to fresh environments. Consequently under P_0 the sequence $\{\omega_{z_i}\}$ of environments is also i.i.d. In particular, $\{\xi_i = z_i - z_{i-1}\}$ are i.i.d. random vectors with common distribution $P_0\{\xi_i = z\} = \mathbb{E}[\pi_{0z}(1 - \pi_{00})^{-1}]$ for $z \neq 0$ and mean $\tilde{v} = E_0 \xi_i$.

Under P_0^ω the walk X_n is confined to the path $\{z_i\}$. Let Y_n mark the location of the walk along this path:

$$X_n = z_{Y_n}.$$

Given ω , $\{Y_n\}$ is a nearest-neighbor random walk on nonnegative integers with transitions

$$P_0^\omega(Y_{n+1} = i | Y_n = i) = \pi_{00}(T_{z_i}\omega) = 1 - P_0^\omega(Y_{n+1} = i + 1 | Y_n = i).$$

The previously defined λ_i also give the hitting times $\lambda_i = \inf\{n : Y_n = i\}$. (This is why we use them here.) Furthermore, under P_0 X_{λ_1} has the same distribution as $\xi_1 = z_1$, and so $\tilde{v} = E_0 \xi_i = E_0(\lambda_1)v$.

The limiting velocity for Y_n is (P_0 -a.s.) $n^{-1}Y_n \rightarrow \gamma = (E_0 \lambda_1)^{-1}$. By Theorem 5.4 and Example 5.3, Y_n satisfies a quenched invariance principle

$$\left\{ \frac{Y_{[nt]} - E_0^\omega(Y_{[nt]})}{\sqrt{n}} : t \geq 0 \right\} \xrightarrow{\text{dist.}} \kappa_q B(\cdot) \quad (5.34)$$

with limiting variance

$$\kappa_q^2 = \gamma^3 \mathbb{E}[\text{Var}_0^\omega(\lambda_1)] = (E_0[\lambda_1])^{-2} \kappa_0^2.$$

Since the process $\kappa_q B(t)\tilde{v}$ is the same as the limit process $\kappa_0 B(t)v$ claimed in part (a) of the theorem, part (a) will follow from showing that

$$\lim_{n \rightarrow \infty} \max_{m \leq n} \frac{1}{\sqrt{n}} \left| \{X_m - E_0^\omega(X_m)\} - \{Y_m - E_0^\omega(Y_m)\} \tilde{v} \right| = 0 \quad P_0\text{-almost surely.} \quad (5.35)$$

Introduce the centered random vectors $\bar{z}_i = z_i - i\tilde{v}$. Decompose (5.35) into two tasks:

$$\lim_{n \rightarrow \infty} \max_{m \leq n} \frac{1}{\sqrt{n}} |\bar{z}_{Y_m} - \bar{z}_{[m\gamma]}| = 0 \quad P_0\text{-a.s.} \quad (5.36)$$

and

$$\lim_{n \rightarrow \infty} \max_{m \leq n} \frac{1}{\sqrt{n}} |\bar{z}_{[m\gamma]} - E_0^\omega(X_m) + E_0^\omega(Y_m)\tilde{v}| = 0 \quad P_0\text{-a.s.} \quad (5.37)$$

For (5.36),

$$\begin{aligned} & P_0 \left(\max_{m \leq n} |\bar{z}_{Y_m} - \bar{z}_{[m\gamma]}| \geq \varepsilon \sqrt{n} \right) \\ & \leq \sum_{m=1}^n \left\{ P_0(|Y_m - [m\gamma]| \geq n^{3/4}) \right. \end{aligned} \quad (5.38)$$

$$\left. + P_0 \left(\max_{k: |k - [m\gamma]| \leq n^{3/4}} |\bar{z}_k - \bar{z}_{[m\gamma]}| \geq \varepsilon \sqrt{n} \right) \right\}. \quad (5.39)$$

Both probabilities are controlled by standard large deviation estimates. For this we use the following general inequality.

Lemma 5.12. *Let $S_m = Z_1 + \dots + Z_m$ be a sum of i.i.d. mean zero random variables $\{Z_i\}$ with an exponential moment: $E(e^{\theta|Z_1|}) < \infty$ for some $\theta > 0$. Then there is a constant C determined by the distribution of $\{Z_i\}$ such that, for all $m \geq 1$ and $h > 0$,*

$$P(\max_{k \leq m} S_k \geq h) \leq e^{-C(\frac{h^2}{m} \wedge h)}. \quad (5.40)$$

Same bound is of course valid for $P(\min_{k \leq m} S_k \leq -h) = P(\max_{k \leq m} (-S_k) \geq h)$.

Proof of Lemma 5.12. Let $\phi(\theta) = \log Ee^{\theta Z_1}$. Note that $\phi(\theta) \geq 0$, and there exist constants $0 < A, \theta_0 < \infty$ such that $\phi(\theta) \leq A\theta^2$ for $|\theta| \leq \theta_0$. For $0 \leq \theta \leq \theta_0$, applying Doob's inequality to the martingale $e^{\theta S_k - k\phi(\theta)}$ gives

$$P(\max_{k \leq m} S_k \geq h) \leq e^{-\theta h + m\phi(\theta)} \leq e^{-\theta h + Am\theta^2}.$$

If $h/(2Am) < \theta_0$, pick $\theta = h/(2Am)$ to get the bound $e^{-h^2/(4Am)}$. If $h/(2Am) \geq \theta_0$, then

$$-\theta h + Am\theta^2 \leq -\theta h + \theta^2 h/(2\theta_0) = -\theta_0 h/2$$

where we chose $\theta = \theta_0$. □

The walk Y_n is not a sum of i.i.d. steps, but we can consider the i.i.d. sequence $\{\lambda_j - \lambda_{j-1}\}$ of successive sojourn times. With $m_0 = [m\gamma] + [n^{3/4}]$, $m_1 = ([m\gamma] -$

$[n^{3/4}]^+$, and $b \in (0, 1)$ a small positive number, the probability on line (5.38) is bounded as follows:

$$\begin{aligned}
& P_0(|Y_m - [m\gamma]| \geq n^{3/4}) \\
& \leq P_0(\lambda_{m_0} \leq m) + P_0(\lambda_{m_1} \geq m) \mathbb{I}_{\{m_1 > 0\}} \\
& \leq P_0(\lambda_{m_0} - E_0(\lambda_{m_0}) \leq -bn^{3/4}) + P_0(\lambda_{m_1} - E_0(\lambda_{m_1}) \geq bn^{3/4}) \mathbb{I}_{\{m_1 > 0\}} \\
& \leq e^{-C(\frac{bn^{3/2}}{m_0} \wedge n^{3/4})} + e^{-C(\frac{bn^{3/2}}{m_1} \wedge n^{3/4})} \mathbb{I}_{\{m_1 > 0\}} \\
& \leq 2e^{-Cn^{1/2}}.
\end{aligned} \tag{5.41}$$

A direct application of (5.40) to line (5.39) gives a bound $Ce^{-Cn^{1/4}}$. To summarize, we have the bound

$$P_0\left(\max_{m \leq n} |\bar{z}_{Y_m} - \bar{z}_{[m\gamma]}| \geq \varepsilon\sqrt{n}\right) \leq Cn(e^{-Cn^{1/2}} + e^{-Cn^{1/4}})$$

which is summable in n . By Borel-Cantelli, (5.36) has been verified.

For (5.37), note that by the boundedness of $\xi_i - \bar{v}$, $|\bar{z}_{[m\gamma]} - \bar{z}_i| \leq Cn$. Write

$$\begin{aligned}
|\bar{z}_{[m\gamma]} - E_0^\omega(X_m) + E_0^\omega(Y_m)\bar{v}| &= \left| \sum_{i=0}^m P_0^\omega(Y_m = i)(\bar{z}_{[m\gamma]} - \bar{z}_i) \right| \\
&\leq \sum_{i: |i - [m\gamma]| \geq n^{3/4}} P_0^\omega(Y_m = i) |\bar{z}_{[m\gamma]} - \bar{z}_i| \\
&\quad + \sum_{i: |i - [m\gamma]| < n^{3/4}} P_0^\omega(Y_m = i) |\bar{z}_{[m\gamma]} - \bar{z}_i| \\
&\leq Cn \sum_{i: |i - [m\gamma]| \geq n^{3/4}} P_0^\omega(Y_m = i) + \max_{i: |i - [m\gamma]| < n^{3/4}} |\bar{z}_{[m\gamma]} - \bar{z}_i| \\
&\leq CnP_0^\omega(|Y_m - [m\gamma]| \geq n^{3/4}) + \max_{i: |i - [m\gamma]| < n^{3/4}} |\bar{z}_{[m\gamma]} - \bar{z}_i|.
\end{aligned}$$

The last term is treated as was done above for (5.39). For the second last term a summable deviation bound develops as follows.

$$\begin{aligned}
& \mathbb{P}\left(\max_{m \leq n} nP_0^\omega(|Y_m - [m\gamma]| \geq n^{3/4}) \geq \varepsilon\sqrt{n}\right) \\
& \leq \sum_{m=1}^n \frac{n}{\varepsilon\sqrt{n}} P_0(|Y_m - [m\gamma]| \geq n^{3/4}) \\
& \leq Cn^{3/2}e^{-Cn^{1/2}}.
\end{aligned}$$

Above we repeated the estimate from line (5.41). To summarize, we have the bound

$$P_0\left(\max_{m \leq n} |\bar{z}_{[m\gamma]} - E_0^\omega(X_m) + E_0^\omega(Y_m)\bar{v}| \geq \varepsilon\sqrt{n}\right) \leq Cn^{3/2}(e^{-Cn^{1/2}} + e^{-Cn^{1/4}}),$$

and (5.37) also has been verified. This completes the proof of the convergence of \tilde{Q}_n^ω .

We turn to part (b) of the theorem. To see the convergence of the quenched mean process, note that $nv = [n\gamma]\tilde{v} + O(1)$ and write

$$\begin{aligned} \frac{E_0^\omega(X_n) - [n\gamma]\tilde{v}}{\sqrt{n}} &= \frac{E_0^\omega(X_n) - E_0^\omega(Y_n)\tilde{v} - \bar{z}_{[n\gamma]}}{\sqrt{n}} \\ &\quad + \frac{(E_0^\omega(Y_n) - [n\gamma])\tilde{v} + \bar{z}_{[n\gamma]}}{\sqrt{n}}. \end{aligned} \quad (5.42)$$

The first ratio after the equality sign tends to zero by (5.37).

Notice now that in (5.2), applied to the Y_n walk satisfying $P_0(Y_1 \in \{0, 1\}) = 1$, we have $a_i = 0$ for $i > 0$, and

$$\Delta(\omega) = \frac{E_0^\omega(Y_1) - \gamma}{1 - \pi_{00}(\omega)} = 1 - \gamma E_0^\omega(\lambda_1).$$

Therefore, (5.23) and (5.13) imply that on the diffusive scale the centered quenched mean process $E_0^\omega(Y_n) - [n\gamma]$ behaves like the sum of i.i.d. mean zero random variables $1 - \gamma E_0^{T_{z_j} \omega}(\lambda_1)$, $0 \leq j \leq [n\gamma] - 1$. Expanding $\bar{z}_{[n\gamma]}$, we see that the last term in (5.42) is asymptotically the same as

$$\frac{1}{\sqrt{n}} \sum_{j=0}^{[n\gamma]-1} (\xi_{j+1} - \gamma E_0^{T_{z_j} \omega}(\lambda_1)\tilde{v}). \quad (5.43)$$

Under \mathbb{P} this is a sum of mean zero i.i.d. random vectors, and this gives the invariance principle for $n^{-1/2}(E_0^\omega(X_{[nt]}) - nt\tilde{v})$.

Lastly, part (c) is proved by comparison with (b). Write

$$\begin{aligned} \frac{X_n - [n\gamma]\tilde{v}}{\sqrt{n}} &= \frac{X_n - E_0^\omega(X_n)}{\sqrt{n}} + \frac{E_0^\omega(X_n) - E_0^\omega(Y_n)\tilde{v} - \bar{z}_{[n\gamma]}}{\sqrt{n}} \\ &\quad + \frac{(E_0^\omega(Y_n) - [n\gamma])\tilde{v} + \bar{z}_{[n\gamma]}}{\sqrt{n}}. \end{aligned}$$

The first ratio after the equality sign is tight under P_0^ω by part (a). The second ratio tends to zero by (5.37). As observed above, the last ratio is asymptotically the same as the sum (5.43) of mean zero i.i.d. random vectors under \mathbb{P} . Hence under a fixed ω this sequence of normalized sums cannot be bounded unless the summands are degenerate. This is exactly the condition in the theorem. \square

Appendix A. A renewal process bound

We write $\mathbb{N}^* = \{1, 2, 3, \dots\}$ and $\mathbb{N} = \{0, 1, 2, \dots\}$. The setting for the next technical lemma is the following. $\{Y_i : i \in \mathbb{N}^*\}$ is a sequence of i.i.d. positive integer-valued random variables, and $\{\tilde{Y}_j : j \in \mathbb{N}^*\}$ is an independent copy of this sequence. Y will also denote a random variable with the same distribution as Y_1 . The corresponding renewal processes are defined by

$$S_0 = \tilde{S}_0 = 0, S_n = Y_1 + \dots + Y_n \text{ and } \tilde{S}_n = \tilde{Y}_1 + \dots + \tilde{Y}_n \text{ for } n \geq 1.$$

Let h be the largest positive integer such that the distribution of Y is supported on $h\mathbb{N}^*$. For $i, j \in h\mathbb{N}$ define

$$L_{i,j} = \inf\{\ell \geq 1 : \text{there exist } m, n \geq 0 \text{ such that } i + S_m = \ell = j + \tilde{S}_n\}.$$

The restriction $\ell \geq 1$ in the definition has the consequence that $L_{i,i} = i$ for $i > 0$ but $L_{0,0}$ is nontrivial.

Lemma A.1. *Let $1 \leq p < \infty$ be a real number, and assume $E(Y^{p+1}) < \infty$. Then there exists a finite constant C such that for all $i, j \in \mathbb{h}\mathbb{N}$,*

$$E(L_{i,j}^p) \leq C(1 + i^p + j^p). \quad (\text{A.1})$$

Proof. First some reductions. If $h > 1$, then we can consider the random variables $\{h^{-1}Y_i, h^{-1}\tilde{Y}_j\}$. Thus we may assume that $h = 1$.

For convenience, we assume $i \leq j$ in $L_{i,j}$. The cases $L_{i,i}$ for $i > 0$ were already observed to be trivial. If $i < j$ then $L_{i,j}$ has the same distribution as $i + L_{0,j-i}$. If we can prove the statement for all $L_{0,j}$, then

$$E(L_{i,j}^p) \leq 2^p i^p + 2^p E(L_{0,j-i}^p) \leq 2^p i^p + 2^p C(1 + (j-i)^p) \leq 2^p(C+1)(1 + i^p + j^p).$$

So we only need to consider the case $L_{0,j}$ for $0 \leq j < \infty$.

Next we remove the case $L_{0,0}$. Suppose we have the conclusion for $L_{0,j}$ for all $0 < j < \infty$. On the event $\{Y_1 = j\}$, $L_{0,0} = \tilde{L}_{j,0}$ computed with the partially shifted sequences $(Y_2, Y_3, Y_4, \dots; \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \dots)$, and furthermore $\tilde{L}_{j,0}$ is independent of Y_1 and distributed like $L_{0,j}$. Thus

$$\begin{aligned} E(L_{0,0}^p) &= \sum_{j=1}^{\infty} E(Y_1 = j, L_{0,0}^p) = \sum_{j=1}^{\infty} E(Y_1 = j, \tilde{L}_{j,0}^p) = \sum_{j=1}^{\infty} P(Y_1 = j) E(L_{0,j}^p) \\ &\leq C \sum_{j=1}^{\infty} P(Y_1 = j)(1 + j^p) = C(1 + E[Y^p]) < \infty. \end{aligned} \quad (\text{A.2})$$

Now we turn to proving the statement (A.1) for $L_{0,j}$, $0 < j < \infty$. The proof will make use of several Markov chains.

Denote the forward recurrence time of the pure renewal process $\{S_n : n \geq 0\}$ by

$$B_k^0 = \min\{m \geq 0 : k + m \in \{S_n : n \geq 0\}\}.$$

More generally, if we start the renewal process with a deterministic delay $Y_0 = r \in \mathbb{N}$ and then define $S_n = Y_0 + Y_1 + \dots + Y_n$ for $n \geq 0$, we denote the forward recurrence time by B_k^r , defined exactly as above. $\{B_k^r : k \geq 0\}$ is a Markov chain with initial state $B_0^r = r$ and transition probability

$$p(0, j) = P(Y = j + 1) \text{ for } j \geq 0, \text{ and } p(j, j - 1) = 1 \text{ for } j \geq 1.$$

The usefulness of this Markov chain is that $B_k^r = 0$ if, and only if, $k \in \{S_n : n \geq 0\}$.

The state space for the p -chain is $\Gamma = [0, \rho) \cap \mathbb{N}$ where

$$\rho = \sup\{y : P[Y = y] > 0\} \in \mathbb{N}^* \cup \{\infty\}$$

is the supremum of the support of Y . This chain is irreducible, and the assumption $h = 1$ makes it aperiodic. Its unique invariant distribution is given by

$$\pi(j) = \frac{P(Y > j)}{E(Y)}.$$

We also consider the joint chain (B_k^r, \tilde{B}_k^s) of two independent copies of the Markov chain B_k^r , started at $(r, s) \in \Gamma^2$, with transition

$$\bar{p}((x, y), (u, v)) = p(x, u)p(y, v).$$

The familiar argument from Markov chain theory shows that the irreducibility and aperiodicity of transition p implies the irreducibility of transition \bar{p} ; see Resnick (1992, p. 129-130) or Durrett (2004, p. 311). The unique invariant distribution of \bar{p} is $\bar{\pi}(i, j) = \pi(i)\pi(j)$. We are ready to prove (A.1) for bounded Y .

Case 1. $\rho < \infty$.

For $0 \leq i, j < \rho$, $L_{i,j}$ has the same distribution as the first return time

$$T_{(0,0)} = \inf\{k \geq 1 : (B_k^i, \tilde{B}_k^j) = (0, 0)\}$$

to state $(0, 0)$, given that the joint chain starts from (i, j) . The reason is that considering $i + S_m$ and $j + \tilde{S}_n$ is the same as starting the renewal processes with delays i and j . Since the state space of the \bar{p} -chain is a single positive recurrent class, we can pick for each state $(r, s) \in \Gamma^2$ a finite path to $(0, 0)$ of positive probability. Since there are only finitely many states, there exist $n_0 < \infty$ and $\delta > 0$ such that

$$\inf_{(r,s) \in \Gamma^2} P_{(r,s)}[T_{(0,0)} \leq n_0] \geq \delta.$$

From this and the Markov property follow uniform exponential tail bounds

$$\sup_{(r,s) \in \Gamma^2} P_{(r,s)}[T_{(0,0)} \geq n] \leq C\eta^n$$

for constants $0 < C < \infty$ and $0 < \eta < 1$, and thereby

$$A_0 \equiv \sup_{0 \leq j < \rho} E(L_{0,j}^p) < \infty.$$

To cover $j \geq \rho$ we can repeat the argument given above in (A.2). Let U be the stopping time for the sequence (Y_i) defined by $S_{U-1} < j \leq S_U$. On the event $S_U - j = B_j^0 = x > 0$, let $\tilde{L}_{x,0}$ be defined in terms of the partially restarted sequences $(Y_{U+1}, Y_{U+2}, Y_{U+3}, \dots; \tilde{Y}_1, \tilde{Y}_2, \tilde{Y}_3, \dots)$. $\tilde{L}_{x,0}$ is independent of B_j^0 and distributed like $L_{0,x}$. We have $L_{0,j} = j + \tilde{L}_{x,0}$ because $\tilde{L}_{x,0}$ ignores the distance from 0 to j .

$$\begin{aligned} E(L_{0,j}^p) &= j^p P(B_j^0 = 0) + \sum_{0 < x < \rho} E[B_j^0 = x, (j + \tilde{L}_{x,0})^p] \\ &\leq 2^p j^p + 2^p \sum_{0 < x < \rho} P(B_j^0 = x) E(L_{0,x}^p) \leq 2^p j^p + A_0. \end{aligned}$$

Case 2. $\rho = \infty$.

To handle this case we define another Markov chain on non-negative integers with transition q . The chain absorbs at zero so $q(0, 0) = 1$. The other transitions are

$$q(x, y) = P(B_x^0 = y), \text{ for } x \geq 1 \text{ and } y \geq 0.$$

Let $\{\zeta_i : i \geq 0\}$ denote the process with transition q . This chain represents a construction of $L_{0,j}$ where we take turns sampling variables Y_i and \tilde{Y}_k until the sums S_m and \tilde{S}_n meet. Assume now that $0 < j < \infty$. For $j = 0$ the description below would stop right away and not work correctly.

- (i) Put $\zeta_0 = j$.

(ii) First, construct $0 = S_0 < S_1 < S_2 < \dots$ until the first $S_m \geq j$. If this $S_m = j$ we are done and we set $\zeta_1 = 0$ and $L_{0,j} = S_m = j$. If S_m overshoots so that $S_m > j$, then let $\zeta_1 = S_m - j = B_j^0$ be the forward recurrence time at $j = \zeta_0$.

(iii) Now turn to construct $0 = \tilde{S}_0 < \tilde{S}_1 < \tilde{S}_2 < \dots$ until the first $\tilde{S}_n \geq \zeta_1$. If $\tilde{S}_n = \zeta_1$ we are done, and then $\zeta_2 = 0$ and $L_{0,j} = \zeta_0 + \zeta_1$. If $\tilde{S}_n > \zeta_1$ overshoots, then let $\zeta_2 = \tilde{S}_n - \zeta_1 = \tilde{B}_{\zeta_1}^0$ be the new forward recurrence time.

(iv) Next go back to add more terms to the first sum S_m . The earlier terms do not play a role. Given ζ_2 , we add new independent terms Y'_1, Y'_2, Y'_3, \dots until $Y'_1 + \dots + Y'_m \geq \zeta_2$. And the overshoot is then $\zeta_3 = Y'_1 + \dots + Y'_m - \zeta_2 = B_{\zeta_2}^0$, a forward recurrence time relative to a new renewal process independent of the earlier ones.

These steps are repeated until the sums meet. This happens at time

$$\nu_0 = \inf\{i \geq 1 : \zeta_i = 0\}$$

and then

$$\text{for } 0 < j < \infty, \quad L_{0,j} = \sum_{i=0}^{\nu_0} \zeta_i \quad \text{with } \zeta_0 = j. \quad (\text{A.3})$$

The term ζ_{ν_0} is zero but for conditioning it is convenient to extend the sum to ν_0 .

To make use of this we need two auxiliary bounds: a uniform bound on $E_x(\zeta_1^p)$, and a tail bound on ν_0 .

Lemma A.2. *Let $1 \leq p < \infty$, a real number. Then for some constant C independent of p , $E_x(\zeta_1^p) \leq CE(Y^{p+1})$ for all $x \geq 0$.*

Lemma A.3. *Assume $\rho = \infty$. Then there exist $0 < \theta < 1$ and $A < \infty$ such that $P_x(\nu_0 > n) \leq A\theta^n$ for all $x \geq 1$ and $n \geq 0$.*

Let us complete the proof of the main statement and then return to prove the auxiliary lemmas. Recall that $0 < j < \infty$. As all the summands are non-negative, we can manipulate the sums without regard to finiteness.

$$\begin{aligned} E(L_{0,j}^p) &= E_j \left[\left(\sum_{i=0}^{\nu_0} \zeta_i \right)^p \right] = E_j \left[\left(j + \sum_{i=1}^{\nu_0} \zeta_i \right)^p \right] \\ &\leq 2^p j^p + 2^p E_j \left[\left(\sum_{i=1}^{\nu_0} \zeta_i \right)^p \right]. \end{aligned}$$

It remains to show that the last expectation is bounded by a constant independently of j . Apply Minkowski's inequality (with a limit to infinitely many terms), note that the event $\{\nu_0 \geq i\}$ is measurable with respect to $\sigma\{\zeta_0, \dots, \zeta_{i-1}\}$, and apply Lemmas A.2 and A.3.

$$\begin{aligned} &\left\{ E_j \left[\left(\sum_{i=1}^{\infty} \mathbb{I}_{\{\nu_0 \geq i\}} \zeta_i \right)^p \right] \right\}^{1/p} \leq \sum_{i=1}^{\infty} \left\{ E_j(\mathbb{I}_{\{\nu_0 \geq i\}} \zeta_i^p) \right\}^{1/p} \\ &= \sum_{i=1}^{\infty} \left\{ E_j(\mathbb{I}_{\{\nu_0 \geq i\}} E_{\zeta_{i-1}}[\zeta_1^p]) \right\}^{1/p} \\ &\leq \sum_{i=1}^{\infty} C \{E(Y^{p+1})\}^{1/p} \{P_j(\nu_0 \geq i)\}^{1/p} \leq C_1 < \infty, \end{aligned}$$

bounded independently of j as required. This completes the proof of Lemma A.1. \square

It remains to prove the auxiliary lemmas used above.

Proof of Lemma A.2. This lemma is trivial for $x = 0$ because the ζ -chain absorbs at 0. For $x \geq 1$ a straightforward calculation, utilizing $S_n \geq n$, gives

$$\begin{aligned}
E_x(\zeta_1^p) &\leq C \sum_{k=1}^{\infty} P(B_x^0 \geq k) k^{p-1} = C \sum_{k=1}^{\infty} \sum_{n=1}^x \sum_{i=1}^x P(S_{n-1} = x - i, S_n \geq x + k) k^{p-1} \\
&= C \sum_{k=1}^{\infty} \sum_{n=1}^x \sum_{i=1}^x P(S_{n-1} = x - i) P(Y \geq i + k) k^{p-1} \\
&= C \sum_{k=1}^{\infty} \sum_{i=1}^x P(S_n = x - i \text{ for some } n) \sum_{m=i+k}^{\infty} P(Y = m) k^{p-1} \\
&\leq C \sum_{k=1}^{\infty} \sum_{i=1}^x \sum_{m=i+k}^{\infty} P(Y = m) k^{p-1} \\
&\leq C \sum_{m=2}^{\infty} P(Y = m) \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \mathbb{I}\{i + k \leq m\} k^{p-1} \\
&\leq C E[Y^{p+1}].
\end{aligned}$$

The third last inequality came from bounding probabilities by 1, the second last inequality from removing the upper bound x from the index i . \square

Proof of Lemma A.3. Since the transition p of B_k^r is positive recurrent, irreducible and aperiodic,

$$\lim_{x \rightarrow \infty} P(B_x^0 = 0) = \lim_{k \rightarrow \infty} p^k(0, 0) = \pi(0) > 0.$$

Fix $0 < \varepsilon_0 < 1$ and $m_0 < \infty$ so that $P(B_x^0 = 0) \geq \varepsilon_0$ for $x \geq m_0$.

Now we use the assumption $\rho = \infty$. Since the support of the distribution of Y is unbounded,

$$\varepsilon_1 = P(Y \geq 2m_0) > 0.$$

Then for $0 < x \leq m_0$

$$P(B_x^0 \geq m_0) \geq P(Y_1 \geq x + m_0) \geq \varepsilon_1.$$

In terms of the transition q , for $x < m_0$,

$$q^2(x, 0) \geq \sum_{y \geq m_0} q(x, y) q(y, 0) = \sum_{y \geq m_0} P(B_x^0 = y) P(B_y^0 = 0) \geq \varepsilon_0 \varepsilon_1,$$

while for $x \geq m_0$,

$$q(x, 0) = P(B_x^0 = 0) \geq \varepsilon_0 \geq \varepsilon_0 \varepsilon_1.$$

From this follows, for any $n \geq 2$, $x \geq 1$, and with $\gamma = 1 - \varepsilon_0 \varepsilon_1$,

$$\begin{aligned} P_x(\nu_0 > n) &= \sum_{z \geq 1} P_x(\nu_0 > n - 2, \zeta_{n-2} = z) P_z(\zeta_1 \neq 0, \zeta_2 \neq 0) \\ &\leq \sum_{1 \leq z < m_0} P_x(\nu_0 > n - 2, \zeta_{n-2} = z) P_z(\zeta_2 \neq 0) \\ &\quad + \sum_{z \geq m_0} P_x(\nu_0 > n - 2, \zeta_{n-2} = z) P_z(\zeta_1 \neq 0) \\ &\leq \gamma P_x(\nu_0 > n - 2) \leq \dots \leq \gamma^{\lfloor n/2 \rfloor}. \end{aligned}$$

□

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References

- E. Bolthausen, A. S. Sznitman and O. Zeitouni. Cut points and diffusive random walks in random environment. *Ann. Inst. H. Poincaré Probab. Statist.* **39** (3), 527–555 (2003). ISSN 0246-0203.
- D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability* **1**, 19–42 (1973).
- F. Comets and O. Zeitouni. Gaussian fluctuations for random walks in random mixing environments. *Israel J. Math.* (2004a). To appear.
- F. Comets and O. Zeitouni. A law of large numbers for random walks in random mixing environments. *Ann. Probab.* **32** (1B), 880–914 (2004b). ISSN 0091-1798.
- R. Durrett. *Probability: theory and examples*. Duxbury Advanced Series. Brooks/Cole–Thomson, Belmont, CA, third edition (2004). ISBN 0-534-42441-4.
- S. N. Ethier and T. G. Kurtz. *Markov processes*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York (1986). ISBN 0-471-08186-8. Characterization and convergence.
- W. Feller. *An introduction to probability theory and its applications. Vol. I*. Third edition. John Wiley & Sons Inc., New York (1968).
- F. Rassoul-Agha. The point of view of the particle on the law of large numbers for random walks in a mixing random environment. *Ann. Probab.* **31** (3), 1441–1463 (2003). ISSN 0091-1798.
- F. Rassoul-Agha. On the zero-one law and the law of large numbers for random walk in mixing random environment. *Electron. Comm. Probab.* **10**, 36–44 (electronic) (2005). ISSN 1083-589X.
- F. Rassoul-Agha and T. Seppäläinen. An almost sure invariance principle for random walks in a space-time i.i.d. random environment. *Probab. Theory Related Fields* **133** (3), 299–314 (2005a).
- F. Rassoul-Agha and T. Seppäläinen. Quenched invariance principle for multi-dimensional ballistic random walk in a random environment with a forbidden direction. *Ann. Probab.* (2005b). Submitted.
- S. Resnick. *Adventures in stochastic process*. Birkhäuser Boston Inc., Boston, MA (1992). ISBN 0-8176-3591-2.

- M. Rosenblatt. *Markov processes. Structure and asymptotic behavior*. Springer-Verlag, New York (1971). Die Grundlehren der mathematischen Wissenschaften, Band 184.
- A. S. Sznitman. Slowdown estimates and central limit theorem for random walks in random environment. *J. Eur. Math. Soc. (JEMS)* **2** (2), 93–143 (2000). ISSN 1435-9855.
- A. S. Sznitman. An effective criterion for ballistic behavior of random walks in random environment. *Probab. Theory Related Fields* **122** (4), 509–544 (2002). ISSN 0178-8051.
- A. S. Sznitman. Topics in random walk in random environment. In *Notes of the School and Conference on Probability Theory (Trieste, 2002)*, ICTP Lecture Series, pages 203–266 (2004).
- A. S. Sznitman and O. Zeitouni. An invariance principle for isotropic diffusions in random environment (2004). Preprint.
- A. S. Sznitman and M. Zerner. A law of large numbers for random walks in random environment. *Ann. Probab.* **27** (4), 1851–1869 (1999). ISSN 0091-1798.
- O. Zeitouni. *Random walks in random environments*, volume 1837 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2004). ISBN 3-540-20832-1. Lectures from the 31st Summer School on Probability Theory held in Saint-Flour, July 8–25, 2001, Edited by Jean Picard.
- M. P. W. Zerner. A non-ballistic law of large numbers for random walks in i.i.d. random environment. *Electron. Comm. Probab.* **7**, 191–197 (electronic) (2002). ISSN 1083-589X.