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Wiener integrals for centered Bessel and related processes, II

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Abstract. This is the second part of a series of papers on the construction of stochastic integrals of Wiener's type for the centered δ-dimensional Bessel processes (BES(δ)-processes in short) and their variants. The approach adopted in the present paper is via the Brascamp-Lieb inequality. This method works well for the BES(δ)-processes, BES(δ)-bridges with $\delta \geq 3$, the Brownian meander and their extensions described by a class of stochastic differential equations, but not for their powers. As we have seen in the first part, another approach via Hardy's L^2 inequality is effective for BES(δ)-processes with $\delta \geq 1$ and their powers. The method used in this paper is powerful to establish a family of accurate bounds on the distributions of these Wiener integrals.

1. Introduction

This paper, which is the second part of a series of papers, is concerned with the definition of stochastic integrals for non-random integrands $f = \{f(t); t \in [0,1]\}$ relative to a certain class of non-negative processes $x = \{x(t); t \in [0,1]\}$ starting at 0. The class of stochastic processes x we shall treat includes δ-dimensional Bessel processes (BES(δ)-processes in short), BES(δ)-bridges, Brownian meander and others which are, in general, described by stochastic differential equations with coefficients subject to certain conditions. Since these processes x satisfy $x(t) \geq 0$ for all $t \in [0,1]$ while $x(0) = 0$, they exhibit a singularity at least for t near 0, especially due to the 1/2-scaling property of the Bessel processes. One of the main purposes

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of the present paper is to demonstrate that such a singularity can be compensated by considering stochastic integrals relative to the centered processes $\hat{x}$ instead of x. In other words, the singularity is taken care of by the (deterministic) Stieltjes integrals relative to the mean values $\bar{x}$ of x.

In the study of stochastic processes, there is a deep parallel between certain stochastic integrals and the semi-convergent integrals (or principal values) in analysis; these semi-convergent (stochastic) integrals arise in the following:

(1) Compensation of jumps for Lévy processes and, indeed, existence of the Lévy processes, when the Lévy measure is not bounded. This goes back to Lévy.
(2) Jeulin's lemma and its variants (Jeulin (1980, p. 44); also Jeulin (1982), Pitman and Yor (1986, section 6)).
(3) The Varadhan type renormalization results Varadhan (1969); Rosen (1986); Dynkin (1988); Le Gall (1992).

Let us now formulate our problems more precisely. We shall consider the BES(δ)-processes $R \equiv R^\delta = \{R^\delta(t); t \in [0,1]\}$, the BES($\delta$)-bridges $r_b \equiv r_b^\delta = \{r_b^\delta(t); t \in [0,1]\}$ reaching $b \geq 0$ (i.e. $r_b(1) = b$) and the Brownian meander $m = \{m(t); t \in [0,1]\}$, see Revuz and Yor (1999), Yor (1992). In fact, our study concerning R^δ may be included in that of the more general family of solutions $X(t)$ of the stochastic differential equations (2.1) (with $a = 0$) below. All these processes start at 0: $R(0) = X(0) = r_b(0) = m(0) = 0$. In general, for a stochastic process $x = \{x(t); t \in [0,1]\}$, we denote its mean by $\bar{x}(t) = E[x(t)]$ and the centered process by $\hat{x}(t) = x(t) - \bar{x}(t)$, respectively. One of the goals of the series of papers is to define the stochastic integrals of Wiener's type

$$I(f;\hat{x}) = \int_0^1 f(t) d\hat{x}(t), \tag{1.1}$$

for a suitable class of (non-random) functions $f = \{f(t); t \in [0,1]\}$ relative to the centered processes $\hat{x}$ of x introduced above.

This paper presents an approach based on the Brascamp-Lieb inequality, which was originally exploited for applications to statistical mechanics or quantum physics Brascamp and Lieb (1976), for $x = R^\delta, r_b^\delta$ with $\delta \geq 3$, m and their extensions. One can actually define the Wiener integrals $I(f;\hat{x})$ for the centered processes of such x and for all $f \in L^2([0,1])$. Another approach which relies upon Hardy's L^2 inequality for $x = R^\delta$ with $\delta > 1$ was discussed in our first part Funaki, Hariya and Yor (2006) where it was shown that the Wiener integrals can be defined even for the centered processes of $x = (R^\delta)^\alpha$, the αth power of the BES(δ)-process R^δ, with $\delta > 0$ and $\alpha \in (0,2)$ satisfying that $\alpha \geq (2-\delta)_+$, for all f such that $f_\alpha \in L^2([0,1])$, where $f_\alpha(t) = t^{(\alpha-1)/2} f(t)$.

The Brascamp-Lieb inequality and Hardy's L^2 inequality have different virtues. The former inequality holds due to the effect of "squeezing random variables around their means", so that the variance always becomes smaller under certain kind of comparison. In this sense, this inequality is rather accurate and powerful; however, it only works directly for a limited class of processes. In fact, our main results in the present paper, formulated in Proposition 4.1 and Theorem 4.2 below, give the inequalities

$$E[\psi(I(f;\hat{x}))] \leq E[\psi(I(f;\hat{y}))] \tag{1.2}$$

for the pairs $(x, y) = (R^\delta, B), (r_b^\delta, \beta_b)$ with $\delta \geq 3$ or (m, B) and for every convex function ψ on $\mathbb{R}$ bounded below. Here, $B = \{B(t); t \in [0,1]\}$ is the (one-dimensional standard) Brownian motion and $\beta_b = \{\beta_b(t); t \in [0,1]\}$ is the Brownian bridge reaching b (i.e. $\beta_b(1) = b$), both starting at 0: $B(0) = \beta_b(0) = 0$. The class of the processes x can be generalized to the solutions X of the stochastic differential equations (2.1) satisfying the conditions (A.1) and (A.2) stated in Section 2, and the processes X_b obtained by conditioning X such that $X_b(1) = b$. On the other hand, as we have mentioned, Hardy's L^2 inequality works for the BES(δ)-processes with $\delta \geq 1$ (taking $\alpha = 1$). Thus, each approach has advantages and drawbacks, but complements each other.

As we shall see in Section 4, to apply the Brascamp-Lieb inequality, the log-concavity of the density functions $D = D(x)$ of the distributions of the processes $x = \{x(t); t \in [0,1]\}$ (with respect to the Wiener measure) on path space plays an essential role, see also Section 3. Section 5 points out that, as far as the L^2-estimates (i.e., the estimates (1.2) with $\psi(a) = a^2$) are concerned, the log-concavity of the partition functions $Z_h(\eta)$, which are defined from the density functions D, only at $\eta = 0$ is sufficient.

In order to define the Wiener integrals $I(f; x)$ for the processes x themselves, decomposing them into a sum of integrals relative to $\hat{x}$ and $\bar{x}$, we are led to analyze the Stieltjes integrals relative to $\bar{x}$, which are denoted by $I(f; \bar{x}) = \int_0^1 f(t)\, d\bar{x}(t)$, as well. Indeed, because of the singularity of $\dot{\bar{x}}(t)$ at $t = 0$ (and at $t = 1$ for $\bar{x} = \bar{r}_0$), we shall see that the integrals $I(f; \bar{x})$ converge under Jeulin's condition, which guarantees some integrability property related to the reciprocals $1/R^\delta(t)$ of the BES(δ)-processes, see Section 6.

Some applications of the results of this paper are made in Funaki and Ishitani (2006), with the study of the integration by parts formulae for the pinned or the standard Wiener measures restricted on a space of paths staying between two curves, in which the stochastic integrals $I(f; x) = \int_0^1 f(t)\, dx(t)$ relative to $x = r_b^3$ or m are needed. The functions f are determined from the derivatives of the curves in this setting, so that they are non-random.

2. A class of SDEs and examples

In this section, we introduce a class of stochastic processes taking values in $\mathbb{R}_+ = [0, \infty)$, described by certain stochastic differential equations (SDEs), including the BES(δ)-processes with $\delta \geq 3$, and then we define the generalized meanders.

For a given C^1-function $b : \mathbb{R}_+^\circ \to \mathbb{R}$, we consider the SDE on $\mathbb{R}_+^\circ = (0, \infty)$ of the form

$$\begin{cases} dX(t) = dB(t) + b(X(t))dt, & t \in [0,1], \\ X(0) = a > 0, \end{cases} \tag{2.1}$$

where $B(t)$ is the one-dimensional Brownian motion. We assume the following two conditions:

(A.1) $X(t)$ has no explosion and admits 0 as an entrance boundary Revuz and Yor (1999, p. 306).

(A.2) $b'(u) \leq 0$ for every $u \in \mathbb{R}_+^\circ$ and $b^2(u) + b'(u)$ is convex on $\mathbb{R}_+^\circ$.

Let P_a be the distribution of the unique solution $X = \{X(t); t \in [0,1]\}$ of the SDE (2.1) on the space $\mathcal{C}_+^\circ = C([0,1], \mathbb{R}_+^\circ)$. Since P_a is monotone in a (due to the comparison theorem, see, e.g. Theorem (3.7) on p. 394 of Revuz and Yor (1999)), it has a weak limit P_0 as $a \downarrow 0$, which is a distribution on the space $\mathcal{C}_+ = C([0,1], \mathbb{R}_+)$. The condition (A.1) implies that, P_0-a.s., $x \in \mathcal{C}_+$ starts at 0, but immediately leaves there and afterwards obeys the SDE (2.1) without coming back to 0. Namely, the point 0 is an entrance-not-exit boundary. It is well-known (see p. 108 of Itô and McKean (1965)) that this is equivalent to:

$$-\int_{0+}^{1} s(u)\, m(du) < \infty \quad \text{and} \quad \lim_{v \downarrow 0} m((v,1)) s(v) = -\infty,$$

where

$$s(u) = \int_1^u \exp\left(-\int_1^y 2b(z)dz\right) dy \quad \text{and} \quad m(du) = 2\exp\left(\int_1^u 2b(y)dy\right) du,$$

for $u \in \mathbb{R}_+^\circ$. The condition (A.2) implies that both functions

$$\varphi_1(u) = -\int_1^u b(y)dy \quad \text{and} \quad \varphi_2(u) = \frac{1}{2}\big(b^2(u) + b'(u)\big) \tag{2.2}$$

are convex on $\mathbb{R}_+^\circ$.

Example 2.1. (1) (BES(δ)-processes) *The drift is given by* $b(u) = (\delta-1)/2u$ *and* (A.1) *holds if* $\delta \geq 2$, *while* (A.2) *holds if* $\delta \geq 3$. *In fact, in this case, we have that* $\varphi_1(u) = -\frac{\delta-1}{2}\log u$ *and* $\varphi_2(u) = \frac{1}{8u^2}(\delta-1)(\delta-3)$ *for* $u \in \mathbb{R}_+^\circ$.
(2) (Norms of Ornstein-Uhlenbeck processes; *Pitman and Yor (1982), Exercise* (1.13) *on* p. 448 *of Revuz and Yor (1999)*) *The drift is given by* $b(u) = (\delta-1)/2u + bu/2$, *for some* $b \in \mathbb{R}$. *The condition* (A.2) *holds if* $\delta \geq 3$ *and* $b \leq 0$. *When* $b = 0$, *the* (*Euclidean*) *norm of the* δ*-dimensional OU-process is the BES*(δ)*-process.*
(3) (BES(δ)-processes with naive drift; p. 104 *of Yor (1984)*) $b(u) = (\delta-1)/2u + C$ *and* (A.2) *holds if* $\delta \geq 3$ *and* $C \geq 0$.

We next define the distribution $M_a^{\delta',\delta}$ on $\mathcal{C}_+$ of the generalized meander $m^{\delta',\delta} = \{m^{\delta',\delta}(t); t \in [0,1]\}, 0 < \delta' < \delta$, starting at $a \geq 0$ by

$$dM_a^{\delta',\delta} = z_{a,\delta',\delta}^{-1} x(1)^{-\delta'}\, dP_a^\delta, \tag{2.3}$$

where P_a^δ stands for the distribution of the BES(δ)-process starting at a and $z_{a,\delta',\delta}$ is the normalizing constant. Such processes are introduced in (3.7) on p. 43 of Yor (1992) and, in fact, $M_0^{\delta',\delta}$ is the distribution of the process $x(t) = \{r(t)^2 + R(t)^2\}^{1/2}$ with the BES(δ')-bridge $r(t) = r_0^{\delta'}(t)$ and the BES($\delta - \delta'$)-process $R(t) = R^{\delta-\delta'}(t)$. In particular, $M_0^{1,3}$ is the distribution of the Brownian meander.

3. Girsanov's formula

For $x \in \mathcal{C} = C([0,1], \mathbb{R})$ and $\alpha \in \mathbb{R}$, we denote $\sigma_\alpha = \inf\{t \in [0,1]; x(t) = \alpha\}$. We write $\sigma_\alpha > 1$ if $x(t) \neq \alpha$ for all $t \in [0,1]$. Let W_a be the Wiener measure on $\mathcal{C}$ starting at $a \geq 0$ and recall that P_a is the distribution on $\mathcal{C}_+$ of the solution X of the SDE (2.1).

Lemma 3.1. *Assume* (A.1) *and* $a > 0$. *Then we have*

$$dP_a = \exp\left\{-\varphi_1(x(1)) + \varphi_1(a) - \int_0^1 \varphi_2(x(t))\, dt\right\} 1_{\{\sigma_0 > 1\}}\, dW_a, \tag{3.1}$$

where φ_1 *and* φ_2 *are the functions defined by* (2.2).

Remark 3.2. (1) *Note that* $\sigma_0 > 1$ *implies* $\inf_{t\in[0,1]} x(t) > 0$ *so that the functional* $F(x) = \varphi_1(x(1)) + \int_0^1 \varphi_2(x(t))\,dt$ *appearing inside the exponential in the formula* (3.1) *is well-defined on this event. Note also that the condition "*$\sigma_0 > 1$*" may be replaced with "*$\sigma_0 \geq 1$*" because the probability* $W_a(\sigma_0 = 1)$ *is equal to* 0.
(2) (3.1) *implies that* $E^{W_a}[e^{-F}, \sigma_0 > 1] = e^{-\varphi_1(a)}$.
(3) *For the distribution* P_a^3 *of the BES*(3)*-process,* (3.1) *is equivalent to Imhof's relation, since* $dM_a^{1,3} = z^{-1}1_{\{\sigma_0>1\}}\,dW_a$ *with* $z = W_a(\sigma_0 > 1)$ *and* $\varphi_1(u) = -\log u, \varphi_2(u) \equiv 0$ *for* P_a^3.
(4) *Lemma* 3.1 *can be found as Exercise* (1.22) *on* p.451 *of Revuz and Yor (1999) for the BES*(δ)*-processes, as* (6.3) *of Pitman and Yor (1982) for the (squares of) norms of OU-processes.*

Proof. For every $0 < \eta < \min\{a, a^{-1}\}$, choose $\varphi_{1,\eta} \in C_b^2(\mathbb{R})$, which coincides with φ_1 on the interval $[\eta, \eta^{-1}]$, and define two functions $\varphi_{2,\eta}$ and b_η on $\mathbb{R}$, respectively, by $\varphi_{2,\eta} = \big\{(\varphi'_{1,\eta})^2 - \varphi''_{1,\eta}\big\}/2$ and $b_\eta = -\varphi'_{1,\eta}$ from $\varphi_{1,\eta}$. Note that $\varphi_{2,\eta}$ and b_η coincide with φ_2 and b on the interval $[\eta, \eta^{-1}]$, respectively. Let $P_{a;\eta}$ be the distribution on $\mathcal{C}$ of the solution $X = X_\eta$ of the SDE:

$$dX(t) = dB(t) + b_\eta(X(t))\,dt, \quad t \in [0,1],$$

starting at a: $X(0) = a$. Then, Girsanov's formula for $P_{a;\eta}$ implies that

$$dP_{a;\eta} = \exp\{-F_\eta(x) + \varphi_{1,\eta}(a)\}\,dW_a,$$

where F_η is the functional F inside the exponential in (3.1) (see Remark 3.2-(1)) defined with φ_1 and φ_2 replaced by $\varphi_{1,\eta}$ and $\varphi_{2,\eta}$, respectively.

On the event $K_\eta = \big\{x \in \mathcal{C}; \min\{\sigma_\eta, \sigma_{\eta^{-1}}\} \geq 1\big\}$, by the strong Markov property of x under P_a, we see that

$$P_a(\,\cdot\, \cap K_\eta) = P_{a;\eta}(\,\cdot\, \cap K_\eta).$$

Accordingly, for every $\Phi \in C_b(\mathcal{C})$, we have

$$E^{P_a}[\Phi, K_\eta] = E^{W_a}[\Phi e^{-F+\varphi_1(a)}, K_\eta],$$

because $F_\eta = F$ on the event K_η. Letting $\eta \downarrow 0$, since $K_\eta \nearrow \cup_{\lambda>0} K_\lambda = \{\sigma_0 > 1\}$ and $P_a(\sigma_0 > 1) = 1$ from the condition (A.1), we obtain (3.1). □

Let $P_{a,b}$ be the conditional probability of P_a under the condition that $x(1) = b > 0$, and let $W_{a,b}$ be the distribution of the Brownian bridge on $\mathcal{C}$ starting at a and reaching b. The following corollary for $P_{a,b}$ immediately follows by conditioning P_a in (3.1) as $x(1) = b$.

Corollary 3.3. *Assume* (A.1) *and* $a, b > 0$. *Then we have*

$$dP_{a,b} = Z_{a,b}^{-1} \exp\left\{-\int_0^1 \varphi_2(x(t))\,dt\right\} 1_{\{\sigma_0>1\}}\,dW_{a,b}, \tag{3.2}$$

where $Z_{a,b} = E^{W_{a,b}}[e^{-\int_0^1 \varphi_2(x(t))\,dt}, \sigma_0 > 1]\big(= e^{\varphi_1(b)-\varphi_1(a)}\big)$ *is the normalizing constant.*

One can let $b \downarrow 0$ and define $P_{a,b}$ for all $a, b \geq 0$ by noting the monotonicity due to the comparison result. In particular, $P_{a,b}^\delta$ is the distribution of the BES(δ)-bridge starting at a and reaching b, obtained by conditioning the distribution P_a^δ of the BES(δ)-process with $x(1) = b$.

4. Wiener integrals via Brascamp-Lieb inequality

Let $\mathcal{S}$ be the class of all step functions f on $[0,1]$, i.e. $f(t) = \sum_{k=1}^n f_k 1_{[t_{k-1},t_k)}(t)$ with $f_k \in \mathbb{R}, 1 \le k \le n$, $0 = t_0 < t_1 < \cdots < t_n = 1$ and $n = 1, 2, \ldots$. We define the stochastic integrals of such f's relative to the centered process $\hat{x}$ of x by

$$I(f;\hat{x}) = \sum_{k=1}^n f_k\big(\hat{x}(t_k) - \hat{x}(t_{k-1})\big).$$

The following proposition, which is valid for the solutions X of the SDE (2.1) satisfying the conditions (A.1) and (A.2), their conditioned processes X_b reaching $b \ge 0$ and the generalized meanders $m^{\delta',\delta}(t)$ with $\delta \ge 3$ and $0 < \delta' \le (\delta-1)/2$, is a consequence of the Brascamp-Lieb inequality. Recall the definitions of the processes B and β_b stated after (1.2).

Proposition 4.1. *Let (x, y) be the pairs of processes $(X, B), (X_b, \beta_b), b \ge 0$ or $(m^{\delta',\delta}, B)$ with $\delta \ge 3$ and $0 < \delta' \le (\delta-1)/2$. Then, the inequalities*

$$E[\psi(I(f;\hat{x}))] \le E[\psi(I(f;\hat{y}))], \tag{4.1}$$

hold for every $f \in \mathcal{S}$ and every convex function ψ on $\mathbb{R}$ bounded below.

For each $f \in L^2([0,1])$, one can find a sequence $f_n \in \mathcal{S}, n = 1, 2, \ldots$ such that $\|f_n - f\|_2 \to 0$ as $n \to \infty$, where $\|f\|_2 = \big\{\int_0^1 f^2(t)\,dt\big\}^{1/2}$ stands for the usual norm of the space $L^2([0,1])$. We shall denote the corresponding inner product by $\langle\ ,\ \rangle_2$. Since $E[I(f;\hat{B})^2] = \|f\|_2^2$ and $E[I(f;\hat{\beta}_b)^2] = \|f\|_2^2 - \langle f, 1\rangle_2^2 \le \|f\|_2^2$ for every $f \in \mathcal{S}$, (4.1) especially with $\psi(a) = a^2$ shows that $\{I(f_n;\hat{x})\}_n$ is a Cauchy sequence in $L^2(P)$ for $x = X, X_b$ or $m^{\delta',\delta}$. Thus one can define the stochastic integrals $I(f;\hat{x})$ for every $f \in L^2([0,1])$ relative to the centered processes $\hat{x}$ of such x's as the limits. Then, noting that (4.1) implies the uniform integrability of $\{\psi(I(f_n;\hat{x}))\}_n$ (see Lemma 4.6 below), Proposition 4.1 can be extended to all $f \in L^2([0,1])$.

Theorem 4.2. *The statement of Proposition 4.1 holds for all $f \in L^2([0,1])$.*

In the following, we shall give the proof of Proposition 4.1. The idea lies in the fact that the distributions of the processes x admit log-concave Girsanov densities with respect to the Wiener measure or the pinned Wiener measure as long as they start at $a > 0$ (and reach $b > 0$), cf. Lemma 3.1 and Corollary 3.3.

The first task is to establish the polygonal approximations for P_a. Those for $P_{a,b}$ are parallel and will be discussed later. Let $\wp = \{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be a finite partition of the interval $[0,1]$. Let $\pi_\wp : \mathcal{C} \to \mathcal{C}$ be the polygonalization of $x \in \mathcal{C}$ associated with $\wp$; namely, $\pi_\wp = \pi_{\wp,2} \circ \pi_{\wp,1}$ and $\pi_{\wp,1} : \mathcal{C} \to \mathbb{R}^n$ is defined by $\pi_{\wp,1}x \equiv ((\pi_{\wp,1}x)_k)_{k=1}^n = (x(t_k))_{k=1}^n$ for $x \in \mathcal{C}$, while $\pi_{\wp,2} : \mathbb{R}^n \to \mathcal{C}$ and $\pi_{\wp,2}\underline{x}$ describes the polygon determined from $\underline{x} = (x_k)_{k=1}^n \in \mathbb{R}^n$ supplemented by $x_0 = a$ as

$$(\pi_{\wp,2}\underline{x})(t) = \{(t - t_{k-1})x_k + (t_k - t)x_{k-1}\}/(t_k - t_{k-1}), \quad t \in [t_{k-1}, t_k], \tag{4.2}$$

for $1 \le k \le n$. We define $\mu_{a,\wp} \in \mathcal{P}(\mathbb{R}^n)$ for $a > 0$ by

$$d\mu_{a,\wp}(\underline{x}) = Z_\wp^{-1} \exp\{-F_\wp(\underline{x})\}\, 1_{\{\min_k x_k > 0\}}\, dW_a \circ \pi_{\wp,1}^{-1}(\underline{x}), \tag{4.3}$$

with the normalizing constant $Z_\wp \equiv Z_{a,\wp}$, where

$$F_\wp(\underline{x}) = \varphi_1(x_n) + \sum_{k=1}^n \varphi_2(x_k)(t_k - t_{k-1}), \quad \underline{x} = (x_k)_{k=1}^n \in (\mathbb{R}_+^\circ)^n.$$

We denote the family of all Borel probability measures on a topological space $\mathcal{E}$ by $\mathcal{P}(\mathcal{E})$.

Lemma 4.3. *If $a > 0$, $\mu_{a,\wp} \circ \pi_{\wp,2}^{-1}$ weakly converges to P_a on $\mathcal{C}_+$ as*

$$|\wp| = \max_{1 \le k \le n} (t_k - t_{k-1}) \to 0.$$

Proof. Let us consider $\tilde{P}_{a,\wp} \in \mathcal{P}(\mathcal{C})$ defined by

$$d\tilde{P}_{a,\wp}(x) = \tilde{Z}_\wp^{-1} \exp\{-F_\wp(\pi_{\wp,1}x)\} 1_{\{\sigma_0 > 1\}} \, dW_a(x),$$

with the normalizing constant $\tilde{Z}_\wp$. Then, $\tilde{P}_{a,\wp}$ weakly converges to P_a as $|\wp| \to 0$. To this end, we first prove the convergence of the normalizing factor by decomposing it as follows for sufficiently small $\eta > 0$:

$$\begin{aligned}\tilde{Z}_\wp &= E^{W_a}[e^{-F_\wp(\pi_{\wp,1}x)}, \sigma_0 > 1] \\ &= E^{W_a}[e^{-F_\wp(\pi_{\wp,1}x)}, K_\eta] + E^{W_a}[e^{-F_\wp(\pi_{\wp,1}x)}, \{\sigma_0 > 1\} \cap K_\eta^c] \\ &=: \tilde{Z}_{\wp,1,\eta} + \tilde{Z}_{\wp,2,\eta}.\end{aligned}$$

Since φ_1 and φ_2 are convex functions on $\mathbb{R}_+^\circ$, $\varphi_1(u), \varphi_2(u) \ge -C_1 u - C_2$ for some $C_1, C_2 > 0$. In particular, $-F_\wp(\underline{x})$ is at most linearly growing in $\underline{x}$ and we have that $\sup_\wp E^{W_a}[e^{-pF_\wp(\pi_{\wp,1}x)}] < \infty$ for all $p > 1$. This implies that

$$\tilde{Z}_{\wp,2,\eta} \le E^{W_a}[e^{-pF_\wp(\pi_{\wp,1}x)}]^{1/p} W_a(\{\sigma_0 > 1\} \cap K_\eta^c)^{1/q} \ \longrightarrow 0$$

as $\eta \downarrow 0$ uniformly in $\wp$, where $1/p + 1/q = 1$. On the other hand, for arbitrarily fixed $\eta > 0$, since $F_\wp(\pi_{\wp,1}x) \to F(x)$ as $|\wp| \to 0$ for every $x \in K_\eta$, we have that

$$\lim_{|\wp| \to 0} \tilde{Z}_{\wp,1,\eta} = E^{W_a}[e^{-F}, K_\eta].$$

We thus obtain that

$$\lim_{|\wp| \to 0} \tilde{Z}_\wp = Z_a \ \big(= E^{W_a}[e^{-F}, \sigma_0 > 1] = e^{-\varphi_1(a)} \big).$$

It is now easy to see, for each $\Phi \in C_b(\mathcal{C})$, that as $|\wp| \to 0$

$$\begin{aligned}E^{\tilde{P}_{a,\wp}}[\Phi] &= \tilde{Z}_\wp^{-1} E^{W_a}[\Phi e^{-F_\wp(\pi_{\wp,1}x)}, \sigma_0 > 1] \\ &\quad \to Z_a^{-1} E^{W_a}[\Phi e^{-F(x)}, \sigma_0 > 1] = E^{P_a}[\Phi]\end{aligned}$$

by Lemma 3.1. This shows the weak convergence of $\tilde{P}_{a,\wp}$ to P_a as $|\wp| \to 0$.

To establish the weak convergence of $\mu_{a,\wp} \circ \pi_{\wp,2}^{-1}$ to P_a, we see that

$$E^{\mu_{a,\wp} \circ \pi_{\wp,2}^{-1}}[\Phi] = Z_\wp^{-1} E^{W_a}[\Phi(\pi_\wp x) e^{-F_\wp(\pi_{\wp,1}x)}, \min_k (\pi_{\wp,1}x)_k > 0].$$

However, since

$$\lim_{|\wp| \to 0} W_a\big(\min_k (\pi_{\wp,1}x)_k > 0\big) = W_a(\sigma_0 > 1), \tag{4.4}$$

we have for $\Phi \in C_b(\mathcal{C})$ that

$$\begin{aligned}\lim_{|\wp| \to 0} \Big| & E^{W_a}[\Phi(\pi_\wp x) e^{-F_\wp(\pi_{\wp,1}x)}, \min_k (\pi_{\wp,1}x)_k > 0] \\ & - E^{W_a}[\Phi(x) e^{-F_\wp(\pi_{\wp,1}x)}, \sigma_0 > 1] \Big| = 0.\end{aligned}$$

This with $\Phi \equiv 1$, in particular, shows that $|Z_\wp - \tilde{Z}_\wp| \to 0$ and accordingly $Z_\wp \to Z_a$ as $|\wp| \to 0$. We therefore obtain:

$$\lim_{|\wp|\to 0} \left| E^{\mu_{a,\wp}\circ\pi_{\wp,2}^{-1}}[\Phi] - E^{\tilde{P}_{a,\wp}}[\Phi] \right| = 0,$$

and this completes the proof of the lemma. To see (4.4), we may only prove

$$\limsup_{|\wp|\to 0} W_a\big(\min_k (\pi_{\wp,1}x)_k > 0\big) \le W_a(\sigma_{-\eta} > 1),$$

for every $\eta > 0$. This follows from the estimate (see Varadhan (1984)):

$$W_a(\|x - \pi_\wp x\|_\infty \ge \eta) \le \exp\{-C\eta^2/|\wp|\}$$

with some $C > 0$. □

The probability measure $\mu_{a,\wp}$ defined by (4.3) is supported on $(\mathbb{R}_+^\circ)^n$ and its log-density function $-F_\wp(\underline{x})$ has a singularity at the boundary. We extend it on $\mathbb{R}^n$ by introducing penalized convex potentials. Namely, for $\epsilon > 0$, let $\varphi_{1,\epsilon}$ and $\varphi_{2,\epsilon}$ be convex functions on $\mathbb{R}$, which coincide with φ_1 and φ_2 on the interval $[\epsilon, \infty)$, respectively, and satisfy that $\lim_{\epsilon\downarrow 0} \varphi_{i,\epsilon}(u) = \infty$ for all $u \le 0$ and $i = 1, 2$; recall that φ_1 and φ_2 are both convex on $\mathbb{R}_+^\circ$. With these extended potentials, we define $\mu_{a,\wp,\epsilon} \in \mathcal{P}(\mathbb{R}^n)$ as

$$d\mu_{a,\wp,\epsilon}(\underline{x}) = Z_{\wp,\epsilon}^{-1} \exp\{-F_{\wp,\epsilon}(\underline{x})\}\, dW_a \circ \pi_{\wp,1}^{-1}(\underline{x}), \tag{4.5}$$

where $Z_{\wp,\epsilon}$ is the normalizing constant and $F_{\wp,\epsilon}$ is the function $F_\wp$ with (φ_1, φ_2) replaced by $(\varphi_{1,\epsilon}, \varphi_{2,\epsilon})$.

Lemma 4.4. *If $a > 0$, $\mu_{a,\wp,\epsilon}$ weakly converges to $\mu_{a,\wp}$ on $\mathbb{R}^n$ as $\epsilon \downarrow 0$.*

Proof. We have $\lim_{\epsilon\downarrow 0} F_{\wp,\epsilon}(\underline{x}) = F_\wp(\underline{x})$ for $\underline{x} \in (\mathbb{R}_+^\circ)^n$ and $= \infty$ for $\underline{x} \in \mathbb{R}^n \setminus (\mathbb{R}_+^\circ)^n$, so that the proof is obvious. □

We are now in a position to state the Brascamp-Lieb inequality on a finite-dimensional Euclidean space $\mathbb{R}^n$. Let A be an $n \times n$ positive-definite symmetric matrix and let $\nu \in \mathcal{P}(\mathbb{R}^n)$ be the Gaussian measure with mean 0 and covariance matrix A^{-1}. Assume that $\mu \in \mathcal{P}(\mathbb{R}^n)$ is given and it is absolutely continuous with respect to ν with a log-concave Radon-Nikodym derivative $d\mu/d\nu$; namely, $-\log d\mu/d\nu$ is a convex function on $\mathbb{R}^n$. Then the celebrated Brascamp-Lieb inequality, in the form due to Caffarelli (2002), is formulated as follows; see also (1.2) in Giacomin (2003).

Lemma 4.5. *For every $v \in \mathbb{R}^n$ and every convex function ψ on $\mathbb{R}$ bounded below, we have*

$$E^\mu\big[\psi\big(v\cdot\underline{x} - E^\mu[v\cdot\underline{x}]\big)\big] \le E^\nu\big[\psi\big(v\cdot\underline{y}\big)\big],$$

where the random variables $\underline{x}$ and $\underline{y}$ are distributed as μ and ν, respectively, and $v\cdot\underline{x} = \sum_{k=1}^n v_k x_k$ denotes the inner product of $v = (v_k)_{k=1}^n$ and $\underline{x} = (x_k)_{k=1}^n$ in $\mathbb{R}^n$.

In order to pass to the limit (as $\epsilon \downarrow 0$, $|\wp| \to 0$ or $a \downarrow 0$) after applying Lemma 4.5, we prepare the following lemma, which guarantees the uniform integrability.

Lemma 4.6. *Let $\nu \in \mathcal{P}(\mathbb{R})$ be given. Then, for every non-negative convex function ψ on $\mathbb{R}$ such that $\int_{\mathbb{R}} \psi(a)\, d\nu(a) < \infty$, one can find another non-negative convex function $\tilde{\psi}$ on $\mathbb{R}$ satisfying $\int_{\mathbb{R}} \tilde{\psi}(a)\, d\nu(a) < \infty$ and $\lim_{|a|\to\infty} \tilde{\psi}(a)/\psi(a) = \infty$.*

Proof. Apply Theorem 22 on page 24-II in Dellacherie and Meyer (1978) with $\mathcal{H} = \{\psi\}$ and $\mathcal{L}^1 = L^1(\nu)$, noting that the function G appearing there is convex, increasing and non-negative. We may take $\tilde{\psi}(a) = G(\psi(a))$. □

After these preparations, we are now ready to complete the proof of Proposition 4.1.

Proof of Proposition 4.1 for the pair (X, B): Since $F_{\wp,\epsilon}(\underline{x})$ is a convex function on $\mathbb{R}^n$, for the pair $(\mu, \nu) =$ "$(\mu_{a,\wp,\epsilon}, W_a \circ \pi_{\wp,1}^{-1})$ shifted by a", one can apply Lemma 4.5 and get

$$E^{\mu_{a,\wp,\epsilon}}\big[\psi\big(v \cdot \underline{\hat{x}}\big)\big] \le E^{W_a \circ \pi_{\wp,1}^{-1}}\big[\psi\big(v \cdot \underline{\hat{y}}\big)\big], \tag{4.6}$$

for every $v \in \mathbb{R}^n$ and convex function ψ on $\mathbb{R}$ bounded below, where $\underline{\hat{x}}$ and $\underline{\hat{y}}$ indicate the centered random variables of $\underline{x}$ and $\underline{y}$, respectively. We let $\epsilon \downarrow 0$ in (4.6). In fact, since Lemma 4.6 taking ν as the distribution of $v \cdot \underline{\hat{y}}$ under $W_a \circ \pi_{\wp,1}^{-1}$ and (4.6) with $\tilde{\psi}$ (determined from $\psi \vee 0$) instead of ψ imply the uniform integrability of $\psi(v \cdot \underline{\hat{x}})$ under the family $\{\mu_{a,\wp,\epsilon}\}_{\epsilon>0}$, one can let $\epsilon \downarrow 0$ and Lemma 4.4 shows that

$$E^{\mu_{a,\wp}}\big[\psi\big(v \cdot \underline{\hat{x}}\big)\big] \le E^{W_a \circ \pi_{\wp,1}^{-1}}\big[\psi\big(v \cdot \underline{\hat{y}}\big)\big]. \tag{4.7}$$

This further implies

$$E^{\mu_{a,\wp} \circ \pi_{\wp,2}^{-1}}\big[\psi\big(I(f; \hat{x})\big)\big] \le E^{W_a}\big[\psi\big(I(f; \hat{x})\big)\big], \tag{4.8}$$

if the division points $\wp_f = \{\bar{t}_k\}_{k=1}^{\bar{n}}$ of the step function $f \in \mathcal{S}$ are contained in the partition $\wp$. For each $f \in \mathcal{S}$, we let $|\wp| \to 0$ in (4.8) in such a way that $\wp_f \subset \wp$ and then obtain from Lemmas 4.3 and 4.6 that

$$E^{P_a}\big[\psi\big(I(f; \hat{x})\big)\big] \le E^{W_a}\big[\psi\big(I(f; \hat{x})\big)\big] = E^{W_0}\big[\psi\big(I(f; \hat{x})\big)\big]. \tag{4.9}$$

This proves (4.1) for the pair (X, B) finally by letting $a \downarrow 0$, since P_a weakly converges to P_0. □

Proof of Proposition 4.1 for the pair (X_b, β_b): First, we introduce the polygonal approximations for $P_{a,b}$, and this can be carried out simultaneously for P_a. Indeed, for a finite partition $\wp = \{t_k\}_{k=0}^n$ of $[0, 1]$, we consider the polygonalization $\tilde{\pi}_\wp = \tilde{\pi}_{\wp,2} \circ \tilde{\pi}_{\wp,1} : \mathcal{C} \to \mathcal{C}$, where $\tilde{\pi}_{\wp,1} : \mathcal{C} \to \mathbb{R}^{n-1}$ is defined by $\tilde{\pi}_{\wp,1} x = (x(t_k))_{k=1}^{n-1}$ while $\tilde{\pi}_{\wp,2} : \mathbb{R}^{n-1} \to \mathcal{C}$ is the map which associates the polygon determined in (4.2) with $\underline{x} = (x_k)_{k=1}^{n-1} \in \mathbb{R}^{n-1}$ supplemented by $x_0 = a$ and $x_n = b$. We define $\mu_{a,b,\wp} \in \mathcal{P}(\mathbb{R}^{n-1})$ for $a, b > 0$ by

$$d\mu_{a,b,\wp}(\underline{x}) = Z_{a,b,\wp}^{-1} \exp\{-G_\wp(\underline{x})\}\, 1_{\{\min_k x_k > 0\}}\, dW_{a,b} \circ \tilde{\pi}_{\wp,1}^{-1}(\underline{x}),$$

with the normalizing constant $Z_{a,b,\wp}$, where

$$G_\wp(\underline{x}) = \sum_{k=1}^{n-1} \varphi_2(x_k)(t_k - t_{k-1}), \quad \underline{x} = (x_k)_{k=1}^{n-1} \in (\mathbb{R}_+^\circ)^{n-1}.$$

Then, if $a, b > 0$, one can prove similarly to Lemma 4.3 that $\mu_{a,b,\wp} \circ \tilde{\pi}_{\wp,2}^{-1}$ weakly converges to $P_{a,b}$ on $\mathcal{C}_+$ as $|\wp| \to 0$; note that (4.4) holds for $W_{a,b}$ in place of W_a.

The penalized measure $\mu_{a,b,\wp,\epsilon}$ of $\mu_{a,b,\wp}$ is introduced as in (4.5) with $G_{\wp,\epsilon}$, naturally defined from $G_\wp$, instead of $F_{\wp,\epsilon}$. Thus, since φ_2 is convex on $\mathbb{R}_+^\circ$, letting $\epsilon \downarrow 0$ and then $|\wp| \to 0$, we obtain the Brascamp-Lieb inequality for the pair $(P_{a,b}, W_{a,b})$:

$$E^{P_{a,b}}\big[\psi\big(I(f; \hat{x})\big)\big] \le E^{W_{a,b}}\big[\psi\big(I(f; \hat{x})\big)\big].$$

Letting $a \downarrow 0$ and $b \downarrow 0$ (when we discuss X_0), this shows (4.1) for the pair (X_b, β_b). □

Proof of Proposition 4.1 for the pair $(m^{\delta',\delta}, B)$: From (2.3) and (3.1), if $a > 0$, we have

$$dM_a^{\delta',\delta} = z_{a,\delta',\delta}^{-1} \exp\left\{-\left(\delta' \log x(1) + F^\delta(x)\right) + \varphi_1^\delta(a)\right\} 1_{\{\sigma_0 > 1\}}\, dW_a,$$

with F^δ determined by $(\varphi_1, \varphi_2) = (\varphi_1^\delta, \varphi_2^\delta)$, the functions given in Example 2.1-(1). In other words, comparing to the distribution P_a^δ of the BES(δ)-process, $\varphi_1^\delta(u)$ is replaced by $\varphi_1^\delta(u) + \delta' \log u$, which is still convex if $(\delta - 1)/2 \geq \delta'$. The rest of the proof is completely the same as that for the pair (X, B) under the condition that $\delta \geq 3$ and $0 < \delta' \leq (\delta - 1)/2$. □

We conclude this section with a comparison between the results obtained in this paper and our previous one Funaki, Hariya and Yor (2006). Theorem 4.2, especially, (4.1) with $\psi(a) = a^2$ implies the L^2-estimate:

$$E[I(f; \hat{x})^2] \leq \|f\|_2^2 \tag{4.10}$$

for the Wiener integrals $I(f; \hat{x})$ of $f \in L^2([0,1])$ relative to the centered processes $\hat{x}$ of $x = X$ and $m^{\delta',\delta}$ with $\delta \geq 3$ and $0 < \delta' \leq (\delta - 1)/2$. The right hand side of (4.10) is refined as $\|f\|_2^2 - \langle f, 1\rangle_2^2$ when $x = X_b$ as we have noticed after Proposition 4.1. Note that the (implicit) constant K in front of $\|f\|_2^2$ on the right hand side can be taken as $K = 1$, which is optimal. The estimate (4.10) implies the non-negativity of the quadratic forms $J(f, f) = \|f\|_2^2 - E[I(f; \hat{x})^2]$ when $x = X$ and $m^{\delta',\delta}$, or $J(f, f) = \|f\|_2^2 - \langle f, 1\rangle_2^2 - E[I(f; \hat{x})^2]$ when $x = X_b$. Direct proofs of the non-negativity $J(f, f) \geq 0$ for R^δ and r_0^δ are given in the first part Funaki, Hariya and Yor (2006) based on some explicit computations for the covariances of BES(δ)-processes and BES(δ)-bridges with $\delta \geq 1$. This covers a wider range of δ than those treatable by means of Theorem 4.2 as far as the L^2-estimates are concerned. The next section discusses this difference based on the Cameron-Martin formula.

One of the advantages of the Brascamp-Lieb inequality is that it yields a wide variety of accurate estimates. One can actually take $\psi(a) = a^2, |a|^p, e^a, e^{\epsilon a^2}$ for $p \geq 1$ and small $\epsilon > 0$ and others, see Funaki and Spohn (1997); Deuschel, Giacomin and Ioffe (2000) for some applications. For instance, the choice of $\psi(a) = e^a$ provides:

$$E\left[e^{I(f;\hat{x})}\right] \leq e^{\|f\|_2^2/2} \tag{4.11}$$

for $x = X, m^{\delta',\delta}$ (and also for X_b). The methods developed in Funaki, Hariya and Yor (2006) would require further efforts to derive such kind of estimates.

5. L^2-estimates via Cameron-Martin formula

This section shows that, in order to derive the L^2-estimates (4.10), it suffices to require the log-concavity of the partition functions $Z_h(\eta)$ defined by (5.1) at $\eta = 0$, and not necessarily for every $\eta \in \mathbb{R}$. As we shall see, the latter condition follows from the log-concavity of the density function $D(x)$ in $x \in \mathcal{C}$, which ensures that the Brascamp-Lieb inequality holds. We shall also give an expression for the quadratic form $J(f, f)$ in the present setting, see Proposition 5.4.

Recall that $W(\equiv W_a$ for some $a \in \mathbb{R})$ denotes the Wiener measure on $\mathcal{C}$. For a non-negative bounded function $D(x), x \in \mathcal{C}$ such that $E^W[D(x)] = 1$, we define a new probability measure P on $\mathcal{C}$ by

$$dP(x) = D(x)\,dW(x).$$

For $f \in L^2([0,1])$, we set $h(t) \equiv h_f(t) := \int_0^t f(u)\,du$. Then, clearly, h belongs to H, the Cameron-Martin subspace of $\mathcal{C}$:

$$H = \{k \in \mathcal{C}; k(0) = 0,\ k \text{ is absolutely continuous, } \|k\|_H^2 := \int_0^1 \dot{k}(t)^2\,dt < \infty\}.$$

Note that, by definition:

$$[h](x) = \int_0^1 f(t)\,dx(t),$$

with $f = \dot{h}$, where $[\cdot]$ denotes the Wiener integral.

For $\eta \in \mathbb{R}$, we define the partition functions $Z_h(\eta) \equiv Z_h^D(\eta)$ by

$$Z_h(\eta) = E^W[D(x + \eta h)]. \tag{5.1}$$

The purpose of this section is to point out the following proposition.

Proposition 5.1. *The following identity holds:*

$$E^P[[h](x)^2] - E^P[[h](x)]^2 = \|h\|_H^2 + Z_h''(0) - Z_h'(0)^2, \tag{5.2}$$

that is,

$$E^P[I(f;\hat{x})^2] = \|f\|_2^2 + \frac{d^2}{d\eta^2} \log Z_h(\eta)\Big|_{\eta=0}.$$

In particular, if

$$Z_h \textit{ is log-concave in a neighborhood of } \eta = 0, \tag{5.3}$$

then we have

$$E^P[I(f;\hat{x})^2] \le \|f\|_2^2. \tag{5.4}$$

Proof. By the Cameron-Martin relation,

$$Z_h(\eta) = E^W\left[D(x)\exp\left(\eta[h](x) - \frac{1}{2}\eta^2\|h\|_H^2\right)\right].$$

Differentiating both sides and taking $\eta = 0$, we get $Z_h'(0) = E^W[[h](x)D(x)]$, hence from the definition of P,

$$Z_h'(0) = E^P[[h](x)].$$

Similarly we have

$$Z_h''(0) = E^P[[h](x)^2] - \|h\|_H^2.$$

Combining these leads to (5.2). □

As we shall see, when $D(x)$ is log-concave in x (i.e., for all $x, y \in \mathcal{C}$ and $\lambda \in (0,1)$, $D(\lambda x + (1-\lambda)y) \ge D(x)^\lambda D(y)^{1-\lambda}$), then (5.3) holds; in fact, in that case, it may be seen that Z_h is log-concave on the whole of $\mathbb{R}$. To this end, we first recall the following lemma which asserts that the marginals of log-concave functions are still log-concave. This fact is due to Prékopa and Leindler, see Corollary 3.5 in Brascamp and Lieb (1976) and Theorem 13.2 in Simon (1979).

Lemma 5.2. *Let $G(\underline{x}, \underline{y}), \underline{x} \in \mathbb{R}^m, \underline{y} \in \mathbb{R}^n$, be log-concave on $\mathbb{R}^{m+n}$. Then the function*

$$\underline{x} \mapsto \int_{\mathbb{R}^n} G(\underline{x}, \underline{y})\, d\underline{y}$$

is log-concave.

Suppose that there exists a sequence $(D_n(x))_{n=1,2,\ldots}$ of log-concave functions on $\mathcal{C}$ such that each $D_n(x)$ only depends on $(x(t_k))_{k=1}^n$ for some sequence $(t_k)_{k=1}^n$ in $[0,1]$, and that $(D_n(x))_{n=1,2,\ldots}$ approximates $D(x)$. By the above lemma, it is clear that $E^W[D_n(x+\eta h)]$ is log-concave in η since the finite-dimensional distribution function of $(x(t_k))_{k=1}^n$ under W is log-concave. Therefore, for $\eta_1, \eta_2 \in \mathbb{R}$ and $\lambda \in (0,1)$,

$$E^W[D_n(x+\{\lambda\eta_1+(1-\lambda)\eta_2\}h)] \geq E^W[D_n(x+\eta_1 h)]^\lambda E^W[D_n(x+\eta_2 h)]^{1-\lambda},$$

and taking the limit $n \to \infty$ on both sides, we get

$$Z_h(\lambda\eta_1+(1-\lambda)\eta_2) \geq Z_h(\eta_1)^\lambda Z_h(\eta_2)^{1-\lambda},$$

which implies that Z_h is log-concave on the whole of $\mathbb{R}$.

Remark 5.3. *The distribution P_a of the process starting at $a > 0$ treated in Section 4 admits a log-concave density with respect to W_a. Therefore, from the above argument, the corresponding $Z_h(\eta)$ is log-concave in η and, in particular, the estimate* (5.4) *holds.*

In order to unify our discussion here with that in Funaki, Hariya and Yor (2006), it may be worthwhile presenting the expression of a quadratic form bearing on $f \in L^2([0,1])$ corresponding to the term $(d^2/d\eta^2)\log Z_h(\eta)|_{\eta=0}$ in Proposition 5.1. From now on, we assume that $D(x)$ is given in the form $D(x) = \exp(-F(x))$ and F is H-differentiable. We denote by $\nabla F(x)$ the H-derivative of F at $x \in \mathcal{C}$; that is, $\nabla F(x)$ is the unique element in H such that

$$\langle \nabla F(x), k\rangle_H = \frac{\partial}{\partial k} F(x) \quad \text{for all } k \in H.$$

Here $\partial/\partial k$ denotes the directional derivative along k. We write $\rho(\cdot,\cdot)$ for the covariance function of the Brownian motion: $\rho(s,t) = \min\{s,t\}$. Note that in the present setting (i.e., the setting of the classical Wiener space), $\nabla F(x)(s), s \in [0,1]$, may be given by

$$\frac{\partial}{\partial\rho(s,\cdot)} F(x), \quad s \in [0,1].$$

Proposition 5.4. *It holds that*

$$E^P[I(f;\hat{x})^2] = \|f\|_2^2 - J^P(f,f),$$

where

$$J^P(f,f) = 2\int_0^1 du\, f(u) \int_0^u ds\, f(s) \frac{d}{ds} E^P\left[\hat{x}(s) \frac{d}{du} \frac{\partial}{\partial\rho(u,\cdot)} F(x)\right].$$

Proof. Define the $\mathcal{F}_t \equiv \sigma\{x(s); s \leq t\}$-martingale $\{M(t); t \in [0,1]\}$ by $M(t) = E^W[D|\mathcal{F}_t]$. By the Clark-Ocone formula Ocone (1984),

$$M(t) = 1 + \int_0^t E^W\left[\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)} D \Big| \mathcal{F}_u\right](x)\, dx(u).$$

Thus, by Girsanov's formula, the process $\{B(t); t \in [0,1]\}$ defined by

$$B(t) = x(t) - \int_0^t \frac{E^W[\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}D\big|\mathcal{F}_u](x)}{M(u)}\,du$$

is the Brownian motion under P. Hence, denoting:

$$v_u(x) = \frac{E^W[\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}D\big|\mathcal{F}_u](x)}{M(u)},$$

the canonical process x under P is a semimartingale which decomposes as

$$x(t) = B(t) + \int_0^t v_u(x)\,du.$$

Thus, we obtain:

$$\begin{aligned} E^P[I(f;\hat{x})^2] &= \|f\|_2^2 + 2E^P\left[\int_0^1 du\, f(u)\hat{v}_u \int_0^u f(s)\,d\hat{x}(s)\right] \\ &= \|f\|_2^2 + 2\int_0^1 du\, f(u)\int_0^u ds\, f(s)\frac{d}{ds}E^P[\hat{x}(s)\hat{v}_u]. \end{aligned} \tag{5.5}$$

Note that, for $u > s$,

$$\begin{aligned} &E^P[\hat{x}(s)\hat{v}_u] \equiv E^P[\hat{x}(s)v_u] \\ &= E^W\left[(x(s) - E^P[x(s)])\frac{E^W[\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}D\big|\mathcal{F}_u](x)}{M(u)}D(x)\right] \\ &= E^W\left[(x(s) - E^P[x(s)])\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}D(x)\right] \\ &= -E^W\left[(x(s) - E^P[x(s)])\left\{\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}F(x)\right\}\exp(-F(x))\right] \\ &= -E^P\left[\hat{x}(s)\frac{d}{du}\frac{\partial}{\partial\rho(u,\cdot)}F(x)\right], \end{aligned}$$

where the third line follows by conditioning on $\mathcal{F}_u$. Combining this with (5.5), we obtain the conclusion of the proposition. □

6. Remarks on Stieltjes integrals for the mean $\bar{x}(t)$

The Wiener integrals $I(f;x)$ relative to $x(t)$ themselves may be defined by decomposing them into the sum

$$I(f;x) = I(f;\hat{x}) + I(f;\bar{x}). \tag{6.1}$$

We have extensively studied the first term on the right hand side, and in this section, we make some remarks about the second term.

Before doing this, we first recall Jeulin's lemma which we already mentioned in Section 1 of Funaki, Hariya and Yor (2006), see also p. 44 of Jeulin (1980) or Lemma 2 in Pitman and Yor (1986): let $Y = \{Y(t); t \in [0,1]\}$ be an $\mathbb{R}_+$-valued process such that $E[Y(t)] < \infty$ for $t \neq 0$ and, for some positive Borel function φ, the distribution of $Y(t)/\varphi(t)$ does not depend on t. Then, for every Borel measure μ on $(0,1]$, we have

$$\int_0^1 Y(t)\,\mu(dt) < \infty \; (a.s.) \quad \Longleftrightarrow \quad \int_0^1 \varphi(t)\,\mu(dt) < \infty.$$

For instance, since one can choose $\varphi(t) = 1/\sqrt{t}$ for $Y(t) = 1/R(t)$ by the scaling law of the BES(δ)-processes $R = R^\delta$, we have

$$\int_0^1 \frac{|f(t)|}{R(t)}\,dt < \infty \; (a.s.) \quad \Longleftrightarrow \quad \int_0^1 \frac{|f(t)|}{\sqrt{t}}\,dt < \infty.$$

Example 6.1. *The function* $f(t) = t^{-1/2}(\log \frac{2}{t})^{-\alpha}$ *with* $\frac{1}{2} < \alpha \le 1$ *satisfies* $f \in L^2([0,1])$, *but* $\int_0^1 |f(t)|/\sqrt{t}\,dt = \infty$.

We now go back to the definability of $I(f;\bar{x})$. For $\int_0^1 |f(t)|\,|d\bar{x}(t)|$ to be finite, for $\bar{R}(t) = \bar{R}^3(t)$, $\bar{r}_b(t) = \bar{r}_b^3(t)\,(b \neq 0)$ and $\bar{m}(t) = \bar{m}^{1,3}(t)$, Jeulin's condition is sufficient: $\int_0^1 |f(t)|/\sqrt{t}\,dt < \infty$, while for $\bar{r}_0(t) = \bar{r}_0^3(t)$, $\int_0^1 |f(t)|/\sqrt{t(1-t)}\,dt < \infty$ is sufficient. This can be seen from the following formulas (6.2)-(6.6) and (6.8)-(6.13).

$$\bar{R}(t) = \sqrt{\frac{8}{\pi}t}, \tag{6.2}$$

$$\bar{r}_0(t) = \sqrt{\frac{8}{\pi}t(1-t)}, \tag{6.3}$$

$$\bar{r}_b(t) = \frac{e^{b^2/2}}{bt^{3/2}} \int_0^\infty y^2 e^{-y^2/2t} \{p(1-t,y,b) - p(1-t,y,-b)\}\,dy, \tag{6.4}$$

$$\bar{m}(t) = \sqrt{\frac{2}{\pi}} \left\{ \sqrt{t(1-t)} + \frac{\pi}{2} - \arctan\sqrt{\frac{1-t}{t}} \right\} \Big(= \int_0^\infty \bar{r}_b(t)\, be^{-b^2/2}\,db \Big), \tag{6.5}$$

$$\dot{\bar{m}}(t) = \sqrt{\frac{2}{\pi}\Big(\frac{1}{t} - 1\Big)}, \tag{6.6}$$

where $p(t,y,z)$ is the heat kernel on $\mathbb{R}$. In fact, (6.2) follows from $E[R(t)] = \sqrt{t}E[R(1)]$ by the scaling property and

$$E[R(1)] = \sqrt{\frac{2}{\pi}} \int_0^\infty y^3 e^{-y^2/2}\,dy = \sqrt{\frac{8}{\pi}}.$$

To prove (6.3) and (6.4), we note the explicit formulas for the distributions of $r_b(t)$:

$$\frac{\ell_t(y) q_{1-t}(y,b)}{\ell_1(b)} dy \qquad \text{or} \qquad \sqrt{8\pi}\,\ell_t(y)\ell_{1-t}(y) dy$$

according as $b \neq 0$ or $b = 0$ for $y > 0$, where $\ell_t(y) = \frac{y}{t}p(t,0,y)$ and $q_t(x,y) = p(t,x,y) - p(t,x,-y)$, see p. 464 of Revuz and Yor (1999). In particular, we have

$$\begin{aligned} \bar{r}_0(t) &= \sqrt{\frac{2}{\pi}} t^{-3/2}(1-t)^{-3/2} \int_0^\infty y^3 e^{-y^2/2t(1-t)}\,dy \\ &= \sqrt{\frac{2}{\pi}} t^{-3/2}(1-t)^{-3/2} \cdot 2t^2(1-t)^2 \end{aligned}$$

and this yields (6.3). For (6.5) and (6.6), we first notice that, in its own filtration, $m(t)$ admits the semimartingale decomposition

$$m(t) = B(t) + \int_0^t \frac{ds}{\sqrt{1-s}} \Big(\frac{\Phi'}{\Phi}\Big)\Big(\frac{m(s)}{\sqrt{1-s}}\Big)\, ds, \tag{6.7}$$

where $B(t)$ is a Brownian motion and $\Phi(a) = \int_0^a e^{-y^2/2} dy$. This can be deduced from Imhof's relation:

$$dM_0^{1,3}\big|_{\mathcal{F}_t} = \frac{1}{x(t)} \Phi\Big(\frac{x(t)}{\sqrt{1-t}}\Big) dP_0^3\big|_{\mathcal{F}_t}$$

for $t \in [0,1]$ and $\mathcal{F}_t = \sigma\{x(s); s \le t\}$, and Girsanov's formula. Then, taking the expectation on both sides of (6.7) and applying Imhof's relation again, we obtain (6.5). The identity (6.6) follows from (6.5).

The formulas (6.2)-(6.6) imply the following asymptotic behaviors as $t \downarrow 0$ or $t \uparrow 1$:

$$\bar{R}(t),\, \bar{r}_0(t),\, \bar{r}_b(t),\, \bar{m}(t) \sim \sqrt{\frac{8}{\pi} t} \quad \text{as} \quad t \downarrow 0, \tag{6.8}$$

$$\dot{\bar{R}}(t),\, \dot{\bar{r}}_0(t),\, \dot{\bar{r}}_b(t),\, \dot{\bar{m}}(t) \sim \sqrt{\frac{2}{\pi t}} \quad \text{as} \quad t \downarrow 0, \tag{6.9}$$

$$\bar{R}(t) \to \sqrt{\frac{8}{\pi}},\ \dot{\bar{R}}(t) \to \sqrt{\frac{2}{\pi}} \quad \text{as} \quad t \uparrow 1, \tag{6.10}$$

$$\bar{r}_0(t) \to 0,\ \dot{\bar{r}}_0(t) \sim -\sqrt{\frac{2}{\pi(1-t)}} \quad \text{as} \quad t \uparrow 1, \tag{6.11}$$

$$\bar{r}_b(t) \to b,\ \dot{\bar{r}}_b(t) \to b - \frac{1}{b} \quad \text{as} \quad t \uparrow 1, \tag{6.12}$$

$$\bar{m}(t) \to \sqrt{\frac{\pi}{2}},\ \dot{\bar{m}}(t) \to 0 \quad \text{as} \quad t \uparrow 1, \tag{6.13}$$

where $\sim$ means that the ratio of both sides tends to 1. The details are omitted. Accordingly, if the function f satisfies "$f \in L^2([0,1])$ and Jeulin's condition", then the sum in (6.1) is definable for $x = R^3, r_b^3\,(b \ge 0)$ and $m^{1,3}$.

Recalling the SDE (2.1) (with $b(u) = (\delta-1)/2u$) for $R = R^\delta$, the Wiener integrals for the BES(δ)-processes R can be directly defined by

$$I(f;R) = I(f;B) + \frac{\delta-1}{2}\int_0^1 \frac{f(t)}{R(t)} dt.$$

The first term on the right is well defined for every $f \in L^2([0,1])$, while Jeulin's lemma shows that $\int_0^1 |f(t)|/R(t)\, dt < \infty$ ($a.s.$) if and only if $\int_0^1 |f(t)|/\sqrt{t}\, dt < \infty$, which coincides with the condition just required above.

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