

Asymptotic properties of certain diffusion ratchets with locally negative drift

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Abstract. We consider two reflecting diffusion processes $(X_t)_{t \geq 0}$ with a moving reflection boundary given by a non-decreasing pure jump Markov process $(R_t)_{t \geq 0}$. Between the jumps of the reflection boundary the diffusion part behaves as a reflecting Brownian motion with negative drift or as a reflecting Ornstein-Uhlenbeck process. In both cases at rate $\gamma(X_t - R_t)$ for some $\gamma \geq 0$ the reflection boundary jumps to a new value chosen uniformly in $[R_{t-}, X_t]$. Since after each jump of the reflection boundary the diffusions are reflected at a higher level we call the processes *Brownian ratchet* and *Ornstein-Uhlenbeck ratchet*. Such diffusion ratchets are biologically motivated by passive protein transport across membranes. The processes considered here are generalisations of the Brownian ratchet (without drift) studied in [Depperschmidt and Pfaffelhuber \(2010\)](#). For both processes we prove a law of large numbers, in particular each of the ratchets moves to infinity at a positive speed which can be computed explicitly, and a central limit theorem.

1. Introduction

Reflecting diffusion processes constitute an important class of stochastic processes that appear in various applications. We consider two particular examples of a *diffusion ratchet* variants of which were introduced in [Simon et al. \(1992\)](#) and [Peskin et al. \(1993\)](#) motivated by protein transport across cell membranes. Generally speaking a diffusion ratchet is a diffusion process reflected at a non-decreasing jump process. The name *ratchet* is justified by the fact that each jump prevents the diffusion from attaining lower values. In a sense a jump of the reflection boundary process can be thought of as a *click* of a ratchet. In [Budhiraja and Fricks](#)

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(2006) a diffusion ratchet (modelling a molecular motor) in which a particle moves according to a Brownian motion between equally spaced (deterministic) barriers is studied. The particle can cross such barriers from left to right but is reflected if it hits a barrier to its left. The two models that we consider in the present note are both generalisations of the diffusion ratchet studied in Depperschmidt and Pfaffelhuber (2010). In the model studied there a particle moves according to a reflecting Brownian motion and the reflection boundary jumps at a rate proportional to the distance between the particle and the current reflection boundary. At jump times the new reflection boundary is chosen uniformly between the old one and the current position of the particle.

Let us now introduce the models that we consider here, then briefly explain the biological motivation and finally state our results.

1.1. *The models.* Let $(\mathcal{X}, \mathcal{R}) := (X_t, R_t)_{t \geq 0}$ be a time-homogeneous Markov process starting in $(X_0, R_0) = (x_0, 0)$, for some $x_0 \geq 0$. Here $\mathcal{X} = (X_t)_{t \geq 0}$ is a diffusion process reflected at a non-decreasing jump process $\mathcal{R} = (R_t)_{t \geq 0}$. Given that $(\mathcal{X}, \mathcal{R})$ is in (X_t, R_t) at time t , the reflection boundary process jumps at rate $\gamma(X_t - R_t)$ for some $\gamma \geq 0$. If τ is a jump time of the reflection boundary then the new position is uniformly distributed on the interval $[R_{\tau-}, X_\tau]$. By this dynamics, $R_t \leq X_t$ for all $t \geq 0$, almost surely. In principle the above description works with any reflecting diffusion between the jumps. As mentioned earlier we study here two particular cases and to distinguish them we write $(\mathcal{X}, \mathcal{R})$ and $(\hat{\mathcal{X}}, \hat{\mathcal{R}})$ for the corresponding processes.

- (I) If for $\mu \geq 0$ the process $\mathcal{X} = (X_t)_{t \geq 0}$ is a Brownian motion with negative infinitesimal drift $-\mu$, unit variance (see Section 2.1) and reflection boundary process $\mathcal{R} = (R_t)_{t \geq 0}$ then we refer to the process $(\mathcal{X}, \mathcal{R})$ as a (γ, μ) -Brownian ratchet.
- (II) If for $\mu \geq 0$ the process $\hat{\mathcal{X}} = (\hat{X}_t)_{t \geq 0}$ is an Ornstein-Uhlenbeck process with infinitesimal drift $-\mu x$, unit variance (see Section 3.1) and reflection process $\hat{\mathcal{R}} = (\hat{R}_t)_{t \geq 0}$ then we refer to $(\hat{\mathcal{X}}, \hat{\mathcal{R}})$ as a (γ, μ) -Ornstein-Uhlenbeck ratchet.

Whenever we want to stress the dependence on the parameters, we write $(\mathcal{X}^{(\gamma, \mu)}, \mathcal{R}^{(\gamma, \mu)}) = (X_t^{(\gamma, \mu)}, R_t^{(\gamma, \mu)})_{t \geq 0}$ for the (γ, μ) -Brownian ratchet and $(\hat{\mathcal{X}}^{(\gamma, \mu)}, \hat{\mathcal{R}}^{(\gamma, \mu)}) = (\hat{X}_t^{(\gamma, \mu)}, \hat{R}_t^{(\gamma, \mu)})_{t \geq 0}$ for the (γ, μ) -Ornstein-Uhlenbeck ratchet.

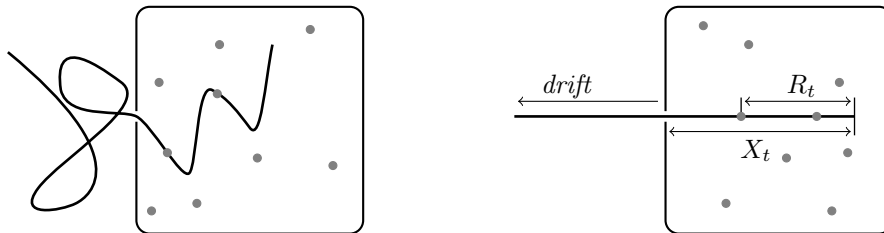


FIGURE 1.1. A diagram of the ratcheting mechanism for protein transport on the left and of a reflecting diffusion with negative local drift on the right.

1.2. *Biological motivation.* Inside a typical cell different proteins are involved in many processes. They usually need to be transported after or during the *translocation* (production) to various locations at which they are required. Depending on the protein and its functions there are different transport mechanisms. In the present paper we focus on the *passive* protein transport across membranes of e.g. endoplasmic reticulum (ER) or mitochondria for which ratcheting models were introduced by [Simon et al. \(1992\)](#) and [Peskin et al. \(1993\)](#). The main idea in these models is that due to thermal fluctuations the protein moves, say inside and outside the ER for definiteness, through a nanopore in the membrane according to a diffusion; see the left part of [Figure 1.1](#). Inside the ER, ratcheting molecules can bind to the protein at a certain rate. These ratcheting molecules are too big (in our model they are actually infinitesimally small but one can imagine that binding of the molecules leads to a deformation of the protein at the ratcheting sites) to pass through the nanopore and prevent the protein from diffusing outside the ER, i.e. the protein performs a reflected diffusion with jumping reflection boundary which is due to binding of new ratcheting molecules. In the last two decades such models have been studied extensively in biology, physics as well as in mathematics. For a detailed overview of the recent literature and for more biological motivation we refer to [Depperschmidt et al. \(2013\)](#) and references therein.

With this motivation in mind X_t (and \hat{X}_t) can be interpreted as the length of the protein inside ER at time t and R_t (and \hat{R}_t) as the distance between the “head” of the protein and the ratcheting molecule closest to the nanopore; see the right part of [Figure 1.1](#). Since typically proteins have to be unfolded during translocation into ER, the movement inside takes place against a force pointing outside which explains the locally negative drift of the ratchets.

1.3. *Results.* For both, the (γ, μ) -Brownian ratchet and the (γ, μ) -Ornstein-Uhlenbeck ratchet we prove a law of large numbers as well as a central limit theorem. Furthermore we compute the speed of the ratchets in terms of the Airy Ai -function in the case of Brownian ratchet and in terms of the Tricomi confluent hypergeometric function in the case of Ornstein-Uhlenbeck ratchet.

Theorem 1.1 (LLN and CLT for the Brownian ratchet). *Let $(\mathcal{X}, \mathcal{R}) = (X_t, R_t)_{t \geq 0}$ be the (γ, μ) -Brownian ratchet starting in $(x_0, 0)$ with $x_0 \geq 0$. If $\gamma, \mu \geq 0$ then*

$$\frac{X_t}{t} \xrightarrow{t \rightarrow \infty} v(\mu, \gamma) := -\frac{\gamma^{1/3}}{2^{2/3}} \frac{Ai'((2\gamma)^{-2/3} \mu^2)}{Ai((2\gamma)^{-2/3} \mu^2)} - \frac{1}{2} \mu \quad \text{almost surely,} \quad (1.1)$$

where $Ai(\cdot)$ is the Airy function. Furthermore in the case $\gamma > 0$ there is $\sigma = \sigma(\mu, \gamma) > 0$ such that

$$\frac{X_t - tv(\mu, \gamma)}{\sigma \sqrt{t}} \xrightarrow{t \rightarrow \infty} X.$$

Here “ \Rightarrow ” denotes convergence in distribution and X is a standard Gaussian random variable.

Note that though the result is formulated for $\mu \geq 0$ for the proof we only need to consider the case $\mu > 0$. In the case $\mu = 0$ the (γ, μ) -Brownian ratchet as well as the (γ, μ) -Ornstein-Uhlenbeck ratchet reduce to the process studied in [Depperschmidt and Pfaffelhuber \(2010\)](#).

Theorem 1.2 (LLN and CLT for the Ornstein-Uhlenbeck ratchet). *Assume $\mu > 0$ and $\gamma \geq 0$. Let $(\widehat{X}, \widehat{R}) = (\widehat{X}_t, \widehat{R}_t)_{t \geq 0}$ be the (γ, μ) -Ornstein-Uhlenbeck ratchet starting in $(x_0, 0)$ for $x_0 \geq 0$. For $x \geq 0$ set*

$$h_{\mu, \gamma}(x) := e^{-\gamma x / \mu - \mu x^2} U\left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3}, \frac{1}{2}, \left(\frac{\gamma}{\mu^{3/2}} + \sqrt{\mu x}\right)^2\right), \quad (1.2)$$

where U is the Tricomi confluent hypergeometric function (see (3.14) for a definition). Then

$$\frac{\widehat{X}_t}{t} \xrightarrow{t \rightarrow \infty} \widehat{v}(\mu, \gamma) := -\frac{h'_{\mu, \gamma}(0)}{2h_{\mu, \gamma}(0)} - \frac{\mu \int_0^\infty h_{\mu, \gamma}(x) dx}{h_{\mu, \gamma}(0)} \quad \text{almost surely.} \quad (1.3)$$

Furthermore in the case $\gamma > 0$ there is $\widehat{\sigma} = \widehat{\sigma}(\mu, \gamma) > 0$ such that

$$\frac{\widehat{X}_t - t\widehat{v}(\mu, \gamma)}{\widehat{\sigma}\sqrt{t}} \xrightarrow{t \rightarrow \infty} X$$

for a standard Gaussian random variable X .

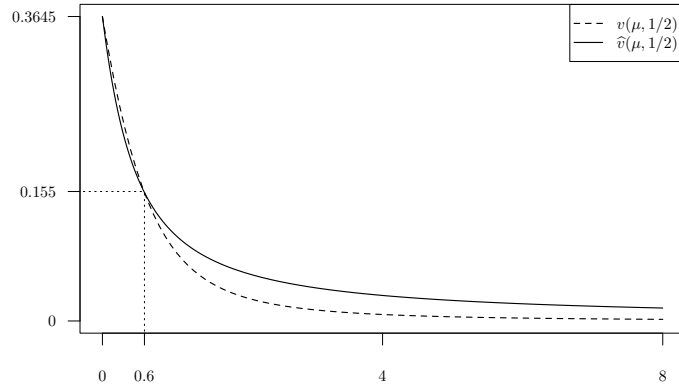


FIGURE 1.2. Comparison of the speed of the ratchets for $\gamma = \frac{1}{2}$ and $\mu \in [0, 8]$.

Remark 1.3 (Comparison of the ratchets). In Figure 1.2 we plot the speed of both ratchets in the interval $\mu \in [0, 8]$ in the case $\gamma = 1/2$. The plots are based on numerical computations using MATHEMATICA. In the neighbourhood of zero, here approximately in the interval $(0, 0.6)$, the Brownian ratchet is faster whereas outside that interval the Ornstein-Uhlenbeck ratchet has a higher speed. Heuristically this can be explained: If $X_t - R_t \approx \widehat{X}_t - \widehat{R}_t$ are large and μ small then the drift of X_t towards R_t is smaller than that of \widehat{X}_t towards \widehat{R}_t . Then in the Brownian case the reflection boundary jumps on average “earlier” and “higher” than in the Ornstein-Uhlenbeck case. Since both $X_t - R_t$ and $\widehat{X}_t - \widehat{R}_t$ are “shortened” at rate proportional to their values the described effect is not very pronounced and the speed of both ratchets is comparable in this region.

On the other hand if μ is large and $X_t - R_t \approx \widehat{X}_t - \widehat{R}_t$ are close to zero then \widehat{R}_t has a higher chance to jump “earlier” and “higher” than R_t because the drift X_t

towards R_t is constant and that of \widehat{X}_t towards \widehat{R}_t is proportional to their distance which is small in this case.

Note that the above heuristic arguments are similar in spirit to the following considerations in the case without jumping reflection boundaries ($\gamma = 0$, in that case the speed of both ratchets is zero). The invariant density of the reflected Brownian motion with negative drift $-\mu$ is $f(x) = 2\mu e^{-2\mu x}$, $x \geq 0$ (see e.g. Harrison, 1985, p. 94) and that of the reflected Ornstein-Uhlenbeck process with drift $-\mu x$ and unite variance is $g(x) = 2\sqrt{\mu/\pi}e^{-\mu x^2}$, $x \geq 0$ (this can be easily obtained from the invariant density of the Ornstein-Uhlenbeck process). For the expectations we have $\int_0^\infty xf(x)dx = 1/(2\mu)$ and $\int_0^\infty xg(x)dx = 1/\sqrt{\pi\mu}$. In particular, under the invariant distributions the expectation of the reflected Brownian motion is larger than that of the reflected Ornstein-Uhlenbeck process for $\mu < \pi/4$ whereas for $\mu > \pi/4$ the opposite inequality holds. \square

Outline. The rest of the paper is split in two sections in which Theorem 1.1 and Theorem 1.2 are proved. In Section 2 we deal with the Brownian ratchet. First, in Section 2.1 we recall an explicit construction of the reflecting Brownian motion with drift. It will be used in Section 2.2 to give a graphical construction of the Brownian ratchet. There we also prove a scaling property for the Brownian ratchet and show that the graphical construction can also be used to construct a coupling of Brownian ratchets with different initial conditions. Between the jump times the Brownian ratchet can be seen as a killed reflecting Brownian motion with drift. For that reason in Section 2.3 we compute the corresponding Green function and obtain several estimates on the moments of the killing time and the position at killing time. In Section 2.4 we study the Markov chain of the increments of the Brownian ratchet at jump times. We show that this Markov chain possesses a unique invariant distribution and compute the expectations under this distribution. These will be used later to compute the speed of the ratchet explicitly. In Section 2.5 we define a regeneration structure for the Brownian ratchet and show that the increments at these regeneration times have finite second moments. From that we obtain in Section 2.6 the assertion of Theorem 1.1. In Section 3, which has a similar structure to Section 2, we carry out the corresponding program for the Ornstein-Uhlenbeck ratchet.

2. Brownian ratchet with negative local drift

In this section we give a graphical construction of the Brownian ratchet with negative local drift from which we deduce a scaling property and show that the construction allows to couple two Brownian ratchets so that from some almost surely finite time on they have the same spatial as well as temporal increments. Then we study the Markov chain of the increments of the ratchet at the jump times of the boundary and show that it has a unique invariant distribution, which will allow to compute the speed of the ratchet explicitly. For the LLN and CLT we define regeneration times of the ratchet and show that the increments between these times have bounded second moments.

Before we start with the above schedule let us recall the definition and an explicit construction of the reflecting Brownian motion with drift.

2.1. *Reflecting Brownian motion with drift.* Though the definition given here is valid for any $\mu \in \mathbb{R}$ we will assume $\mu \geq 0$ because the case $\mu < 0$ is less interesting. For more information on reflecting Brownian motion with drift we refer to e.g. Harrison (1985); Graversen and Shiryaev (2000); Peskir (2006). A *reflecting Brownian motion* with infinitesimal drift $-\mu$ started in $x \geq 0$, which we denote by $RBM^x(-\mu)$, is a strong Markov process with continuous paths (i.e. a diffusion process) associated with the infinitesimal operator A^μ acting on

$$\mathcal{D}(A^\mu) := \{f \in C_b^2(\mathbb{R}_+) : f'(0+) = 0\}$$

as follows:

$$A^\mu f(y) := \frac{1}{2}f''(y) - \mu f'(y). \quad (2.1)$$

We shall omit the superscript x and write $RBM(-\mu)$ whenever the initial value is not important.

Let us also recall from Peskir (2006) an explicit construction of $RBM^x(-\mu)$ that will be useful for our purposes. Let $B = (B_t)_{t \geq 0}$ be a standard Brownian motion starting in 0. We define the Brownian motion with drift μ , denoted by B^μ , and its running maximum, denoted by M^μ , by

$$B_t^\mu = B_t + \mu t \quad \text{and} \quad M_t^\mu = \max_{0 \leq s \leq t} B_s^\mu, \quad \text{for } t \geq 0. \quad (2.2)$$

Furthermore we define $Z^{\mu,x} = (Z_t^{\mu,x})_{t \geq 0}$ by

$$Z_t^{\mu,x} = (x \vee M_t^\mu) - B_t^\mu. \quad (2.3)$$

Then in Peskir (2006, Thm. 2.1) it is shown that

$$RBM^x(-\mu) \stackrel{d}{=} Z^{\mu,x}. \quad (2.4)$$

2.2. *Graphical construction of the Brownian ratchet with negative local drift.* Assume $\mu \geq 0$ and let B^μ be as in (2.2). Furthermore let N^γ be an independent Poisson process on $\mathbb{R} \times [0, \infty)$ with intensity $\gamma \lambda^2(dx, dt)$ where λ^2 is Lebesgue measure on \mathbb{R}^2 . Using this two processes we now construct the Brownian ratchet with negative local drift, see Figure 2.3.

We define a sequence of jump times $(\tau_n)_{n=0,1,\dots}$ and a sequence $(S^{(n)})_{n=0,1,\dots}$ with $S^{(n)} = (S_t^{(n)})_{t \geq \tau_n}$ as follows:

$$\begin{aligned} \tau_0 &:= 0, \\ S_0^{(0)} &:= x_0, \quad S_t^{(0)} := \max\{S_0^{(0)}, \sup_{0 \leq s \leq t} \{B_s^\mu\}\}. \end{aligned}$$

Given τ_{n-1} and $S^{(n-1)}$ for some $n \geq 1$ we set

$$\tau_n := \inf\{t > \tau_{n-1} : N^\gamma \cap [B_t^\mu, S_t^{(n-1)}] \times \{t\} \neq \emptyset\}. \quad (2.5)$$

Furthermore we let $S_{\tau_n}^{(n)}$ be the space component of the almost surely unique element of $N^\gamma \cap [B_{\tau_n}^\mu, S_{\tau_n}^{(n-1)}] \times \{\tau_n\}$. For $t \geq \tau_n$ define

$$S_t^{(n)} := \max\{S_{\tau_n}^{(n)}, \sup_{\tau_n \leq s \leq t} \{B_s^\mu\}\}.$$

Finally we define $S := (S_t)_{t \geq 0}$ and $(\mathcal{X}, \mathcal{R}) := (X_t, R_t)_{t \geq 0}$ by setting

$$S_t = S_t^{(n)} \quad \text{and} \quad R_t = \sum_{i=1}^n \left(S_{\tau_i}^{(i-1)} - S_{\tau_i}^{(i)} \right) \quad \text{for } t \in [\tau_n, \tau_{n+1}) \quad (2.6)$$

$$\text{and} \quad X_t = R_t + S_t - B_t^\mu \quad \text{for } t \geq 0.$$

Note that S is the “running maximum” process that jumps down to Poisson points that are between the process itself and the Brownian motion with drift.

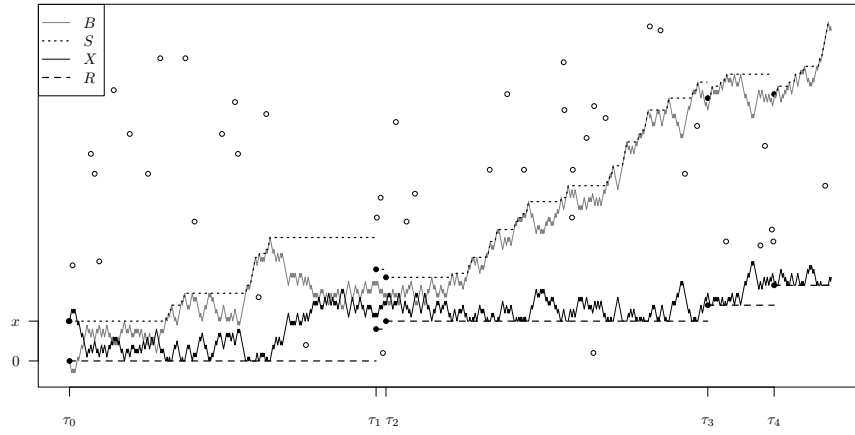


FIGURE 2.3. Graphical construction of the Brownian ratchet with locally negative drift.

In the following lemma we verify that $(\mathcal{X}, \mathcal{R})$ fits the description of (γ, μ) -Brownian ratchet given in Subsection 1.1.

Lemma 2.1. *The process $(\mathcal{X}, \mathcal{R})$ is (γ, μ) -Brownian ratchet started in $(x, 0)$.*

Proof: By construction $S_{\tau_i}^{(i-1)} \geq S_{\tau_i}^{(i)}$, so that \mathcal{R} is non-decreasing. Furthermore, $S_t \geq B_t$ implies $X_t \geq R_t$ for all $t \geq 0$. Between τ_n and τ_{n+1} the process \mathcal{X} is $RBM(-\mu)$ reflected at R_t starting at time τ_n in

$$\begin{aligned} R_{\tau_n} + S_{\tau_n} - B_{\tau_n}^\mu &= \sum_{i=1}^n (S_{\tau_i}^{(i-1)} - S_{\tau_i}^{(i)}) + S_{\tau_n}^{(n)} - B_{\tau_n}^\mu \\ &= \sum_{i=1}^{n-1} (S_{\tau_i}^{(i-1)} - S_{\tau_i}^{(i)}) + S_{\tau_n}^{(n-1)} - B_{\tau_n}^\mu = R_{\tau_{n-1}} + S_{\tau_n-} - B_{\tau_n}^\mu. \end{aligned}$$

Thus, $X_{\tau_n} = X_{\tau_n-}$ for all $n \geq 0$, and therefore the paths of \mathcal{X} are continuous.

Given the process up to time τ_n the jump rate of \mathcal{R} , i.e. the rate of at which τ_{n+1} occurs, is $\gamma(S_t - B_t^\mu) = \gamma(X_t - R_t)$. Then the reflection boundary jumps to $R_{\tau_{n+1}} = R_{\tau_n} + S_{\tau_{n+1}}^{(n)} - S_{\tau_{n+1}}^{(n+1)}$. By homogeneity of the Poisson process N^γ , $S_{\tau_{n+1}}^{(n+1)}$

is uniform on $[B_{\tau_{n+1}}^\mu, S_{\tau_{n+1}}^{(n)}]$. Thus, $R_{\tau_{n+1}}$ is uniform on $[R_{\tau_n}, X_{\tau_{n+1}}]$ because for some $U \sim U([0, 1])$ we have

$$\begin{aligned} R_{\tau_{n+1}} &= R_{\tau_n} + S_{\tau_{n+1}}^{(n)} - (B_{\tau_{n+1}}^\mu + U(S_{\tau_{n+1}}^{(n)} - B_{\tau_{n+1}}^\mu)) \\ &= R_{\tau_n} + (1 - U)(S_{\tau_{n+1}}^{(n)} - B_{\tau_{n+1}}^\mu) = R_{\tau_n} + (1 - U)(X_{\tau_{n+1}} - R_{\tau_n}). \end{aligned}$$

□

If we transform time and space in the graphical construction then we of course rescale the Brownian motion with drift and transform the Poisson process. There is only one such transformation that maps the Poisson process N^1 to N^γ and the Brownian motion with drift to a Brownian motion with another drift.

Lemma 2.2 (Scaling property). *For $\gamma > 0$ and $\mu \geq 0$ we have*

$$(X_t^{\gamma, \mu}, R_t^{\gamma, \mu})_{t \geq 0} \stackrel{d}{=} \gamma^{-1/3} (X_{\gamma^{2/3}t}^{1, \gamma^{-1/3}\mu}, R_{\gamma^{2/3}t}^{1, \gamma^{-1/3}\mu})_{t \geq 0}. \quad (2.7)$$

Proof: Assume that we construct $(X_t^{1, \gamma^{-1/3}\mu}, R_t^{1, \gamma^{-1/3}\mu})_{t \geq 0}$ starting in $(\gamma^{1/3}x_0, 0)$ using the Poisson process N^1 and the Brownian motion with drift $B^{\gamma^{-1/3}\mu}$. We define $g: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ by

$$g(t, x) = (\gamma^{-2/3}t, \gamma^{-1/3}x).$$

On the one hand, rescaling space and time using g we obtain the process on the right hand side of (2.7). On the other hand the rescaled graphical construction leads to the process on the left hand side of (2.7). For that we need to verify

$$g(N^1) \stackrel{d}{=} N^\gamma, \quad (2.8)$$

$$g((t, B_t^{\gamma^{-1/3}\mu})_{t \geq 0}) \stackrel{d}{=} (t, B_t^\mu)_{t \geq 0}. \quad (2.9)$$

Equation (2.8) is clear. For (2.9) we have using the scaling property of the Brownian motion

$$\begin{aligned} g((t, B_t^{\gamma^{-1/3}\mu})_{t \geq 0}) &= (\gamma^{-2/3}t, \gamma^{-1/3}B_t^{\gamma^{-1/3}\mu})_{t \geq 0} \\ &= (\gamma^{-2/3}t, \mu\gamma^{-2/3}t + \gamma^{-1/3}B_t^0)_{t \geq 0} \\ &\stackrel{d}{=} (\gamma^{-2/3}t, \mu\gamma^{-2/3}t + B_{\gamma^{-2/3}t}^0)_{t \geq 0} = (t, B_t^\mu)_{t \geq 0}. \end{aligned} \quad (2.10)$$

□

We now turn to the construction of a coupling of two (γ, μ) -Brownian ratchets starting in $((x, 0), (\tilde{x}, 0))$. Let $(B_t^\mu)_{t \geq 0}$ and N^γ be as before, and let $x, \tilde{x} \geq 0$ with $x \geq \tilde{x}$ without loss of generality. To construct the coupled Brownian ratchet

$$((X_t, R_t), (\tilde{X}_t, \tilde{R}_t))_{t \geq 0}, \quad (2.11)$$

set $\tau_0 = \tilde{\tau}_0 = 0$, $S_0^{(0)} = x$, $\tilde{S}_0^{(0)} = \tilde{x}$ and define as before the sequences $(\tau_n)_{n \geq 0}$, $(\tilde{\tau}_n)_{n \geq 0}$, $(S^{(n)})_{n \geq 0}$ and $(\tilde{S}^{(n)})_{n \geq 0}$. Furthermore define the corresponding processes S, \tilde{S} , $(X_t, R_t)_{t \geq 0}$ and $(\tilde{X}_t, \tilde{R}_t)_{t \geq 0}$ as in (2.6). Define the coupling time by

$$T_{\text{coupl}} = \inf\{t \geq 0 : S_t = \tilde{S}_t\}. \quad (2.12)$$

Note that, since we use the same Brownian motion and the same Poisson process for both ratchets we have $S_t = \tilde{S}_t$ for all $t \geq T_{\text{coupl}}$. Thus, on the event $\{T_{\text{coupl}} < \infty\}$, there are n, \tilde{n} such that for $k \geq 0$ we have almost surely

$$\begin{aligned} \tau_{n+k} &= \tilde{\tau}_{\tilde{n}+k}, \\ X_{\tau_{n+k+1}} - X_{\tau_{n+k}} &= \tilde{X}_{\tau_{n+k+1}} - \tilde{X}_{\tau_{n+k}}, \\ R_{\tau_{n+k+1}} - R_{\tau_{n+k}} &= \tilde{R}_{\tau_{n+k+1}} - \tilde{R}_{\tau_{n+k}}. \end{aligned} \quad (2.13)$$

The following lemma shows that T_{coupl} is almost surely finite, i.e., the coupling is successful.

Lemma 2.3 (Exponential moments of coupling time). *For $\mu > 0$ and $0 \leq \alpha \leq \mu^2/2$ we have*

$$\mathbb{E}[e^{\alpha T_{\text{coupl}}}] \leq e^{x(\mu - \sqrt{\mu^2 - 2\alpha})}.$$

Proof: At time T_{coupl} either both S and \tilde{S} use the same point of the Poisson process N^γ or the Brownian motion B^μ touches the maximum of S and \tilde{S} which is S by assumption $\tilde{x} \leq x$. Thus, we have $T_{\text{coupl}} \leq T$ for (recall M_t in (2.2))

$$T := \inf\{t > 0 : S_t = M_t\} = \inf\{t > 0 : S_t = B_t^\mu\}.$$

By construction S_t can increase only after this time T and decrease by jumping down when it uses points of the Poisson process. Ignoring this decrease by jumping down we obtain

$$T \leq H_x := \inf\{t > 0 : B_t^\mu = x\}.$$

It is well known that (see e.g. [Borodin and Salminen, 2002](#), p.295) for $\alpha \leq \mu^2/2$ we have

$$\mathbb{E}[e^{\alpha H_x}] = e^{x(\mu - \sqrt{\mu^2 - 2\alpha})}$$

and the result follows. \square

2.3. Green function of the killed reflected Brownian motion with drift. Since between the jumps the ratchet constructed in the previous section behaves as a killed reflected Brownian motion we will need in the sequel some functionals of that process, such as expected killing time or expected position at killing. To this end we need to compute the corresponding Green function.

Let us first give a short description on how the Green function of a killed diffusion can be computed; for details we refer to Chapter II in [Borodin and Salminen \(2002\)](#) or Chapter 4 in [Itô and McKean \(1974\)](#).

Remark 2.4 (Green function of a reflected diffusion with killing). Let $Y := (Y_t)_{t \geq 0}$ be a reflected diffusion process with killing on state-space $[0, \infty)$ associated with infinitesimal operator A acting on

$$\mathcal{D}(A) := \{f \in C_b^2(\mathbb{R}_+) : f'(0+) = 0\}$$

as follows

$$Af(y) := \frac{1}{2}f''(y) + b(x)f'(y) - c(x)f(x). \quad (2.14)$$

We consider in this paper the cases $b(x) = -\mu$ or $b(x) = -\mu x$ and $c(x) = \gamma x$. Since in both cases the killing time is almost surely finite the resulting diffusions are

transient. The speed and the killing measures of Y are given by (see e.g. [Borodin and Salminen, 2002](#), p. 17)

$$m(dx) = m(x) dx := 2e^{B(x)} dx \quad \text{and} \quad k(dx) = k(x) dx := 2c(x)e^{B(x)} dx, \quad (2.15)$$

where $B(x) := \int_0^x 2b(y) dy$. Let $p(\cdot; \cdot, \cdot)$ denote the transition density of Y with respect to the speed measure. Then the Green function of Y is defined by

$$G(x, y) := \int_0^\infty p(t; x, y) dt.$$

In the transient regular case (the latter means here that every point in $[0, \infty)$ can be reached with positive probability starting from any other point) the Green function of Y is positive and finite. It is obtained in terms of two independent solutions ϕ and ψ of the differential equation $Af = 0$ that are both unique up to a constant factor and satisfy the following conditions

- (i) ϕ is positive and strictly decreasing with $\phi(x) \rightarrow 0$ as $x \rightarrow \infty$,
- (ii) ψ is positive and strictly increasing,
- (iii) $\psi'(0+) = 0$ (this condition is for the reflecting boundary).

The Wronskian, defined by $w(\psi, \phi) := \psi'(x)\phi(x) - \psi(x)\psi'(x)$ is independent of x . Thus, the functions ϕ and ψ can be chosen so that their Wronskian equals one. Then (assuming $w(\psi, \phi) = 1$) the Green function of Y is given by

$$G(x, y) = \begin{cases} \phi(x)\psi(y) & : 0 \leq y \leq x, \\ \psi(x)\phi(y) & : 0 \leq x \leq y. \end{cases} \quad (2.16)$$

In our computations we will in principle not need the exact expressions for ϕ , ψ and G . In particular in the case of the Ornstein-Uhlenbeck process, where these solutions depend in a complicated manner on the model parameters, we will not compute the function ψ explicitly. Asymptotic bounds at infinity will suffice for our purposes.

In the following remark we collect some properties of the Airy functions that will be needed in the sequel. For further properties we refer to [Abramowitz and Stegun \(1992\)](#) (cf. also Remark 5.2 in [Depperschmidt and Pfaffelhuber \(2010\)](#)).

Remark 2.5 (Airy functions). The Airy functions Ai and Bi are two linearly independent solutions of the differential equation

$$u''(x) - xu(x) = 0. \quad (2.17)$$

We will only need the properties of the Airy functions on $[0, \infty)$. On that domain the functions are positive, Ai is decreasing with $Ai(x) \xrightarrow{x \rightarrow \infty} 0$ and Bi is increasing with $Bi(x) \xrightarrow{x \rightarrow \infty} \infty$.

The Wronskian is independent of x and is given by

$$w(Ai, Bi) = Bi'(0)Ai(0) - Ai'(0)Bi(0) = \frac{1}{\pi}. \quad (2.18)$$

The integral of Ai on \mathbb{R}_+ is

$$\int_0^\infty Ai(u) du = \frac{1}{3}. \quad (2.19)$$

We will also need the function

$$Gi(x) := Ai(x) \int_0^x Bi(y) dy + Bi(x) \int_x^\infty Ai(y) dy. \quad (2.20)$$

For fixed $\mu \geq 0$ and $C \in \mathbb{R}$ we define a function M by

$$\begin{aligned} M(x) := \pi \left\{ Ai(\mu^2 + x) \int_0^x (Bi(\mu^2 + y) + CAi(\mu^2 + y)) dy \right. \\ \left. + (Bi(\mu^2 + x) + CAi(\mu^2 + x)) \int_x^\infty Ai(\mu^2 + y) dy \right\}. \end{aligned} \quad (2.21)$$

Using (2.19), (2.20) and positivity of Ai and Bi on $[0, \infty)$ we have for $x \geq 0$

$$M(x) \leq \pi \left(Gi(\mu^2 + x) + \frac{|C|}{3} Ai(\mu^2 + x) \right). \quad (2.22)$$

Since the functions Gi and Ai are bounded on $[0, \infty)$ we may define

$$G^* := G^*(C, \mu) := \pi \cdot \max_{x \geq 0} \left\{ Gi(\mu^2 + x) + \frac{|C|}{3} Ai(\mu^2 + x) \right\}. \quad (2.23)$$

□

Definition 2.6 (Killed reflecting Brownian motion with drift). Let $Z := (Z_t)_{t \geq 0}$ denote a reflecting Brownian motion with drift $-\mu$ starting in $x \geq 0$ (see (2.3)) under the law \mathbb{P}_x and let \mathbb{E}_x denote the corresponding expectation. Furthermore using an exponentially distributed rate 1 random variable ξ independent of Z we define the *killing time* by

$$\tau = \inf\{t > 0 : \gamma \int_0^t Z_s ds \geq \xi\}.$$

Then *reflecting Brownian motion with infinitesimal drift $-\mu$ killed at rate γZ* is defined as the process $Z^K := (Z_t^K)_{t \geq 0}$ with $Z_t^K = Z_t$ for $t \in [0, \tau)$ and $Z_t^K = \Delta$ for $t \geq \tau$ for some $\Delta \notin \mathbb{R}$, often referred to as the *cemetery state*. The infinitesimal operator $A^{\mu, \gamma}$ corresponding to Z^K acts on C^2 functions $f : [0, \infty) \rightarrow \mathbb{R}$ satisfying $f'(0+) = 0$ as follows

$$A^{\mu, \gamma}(x) = \frac{1}{2} f''(x) - \mu f'(x) - \gamma x f(x). \quad (2.24)$$

In view of the scaling property it is enough to prove the results for the (γ, μ) -Brownian ratchet in a particular case. In what follows we assume $\gamma = \frac{1}{2}$ and fix $\mu \geq 0$. In this case the speed and killing measure corresponding to Z^K are given by

$$m(dx) = m(x) dx := 2e^{-2\mu x} dx \quad \text{and} \quad k(dx) = k(x) dx := xe^{-2\mu x} dx. \quad (2.25)$$

Lemma 2.7 (Green function of killed reflecting Brownian motion with drift). *The Green function of Z^K is given by*

$$G(x, y) := \begin{cases} \phi(x)\psi(y) & : 0 \leq y \leq x, \\ \psi(x)\phi(y) & : 0 \leq x \leq y, \end{cases} \quad (2.26)$$

where $\phi, \psi : [0, \infty) \rightarrow \mathbb{R}$ are defined by

$$\begin{aligned} \phi(x) &= \pi e^{\mu x} Ai(\mu^2 + x), \\ \psi(x) &= C e^{\mu x} Ai(\mu^2 + x) + e^{\mu x} Bi(\mu^2 + x), \end{aligned} \quad (2.27)$$

with

$$C := C(\mu) := -\frac{\mu Bi(\mu^2) + Bi'(\mu^2)}{\mu Ai(\mu^2) + Ai'(\mu^2)}. \quad (2.28)$$

Proof: As explained in Remark 2.4 the Green function G is obtained in terms of solutions of

$$A^{\mu,1/2}u(x) = 0, \quad x \geq 0, \quad (2.29)$$

where $A^{\mu,1/2}$ is defined in (2.24).

The functions ϕ and ψ defined in (2.27) are two independent solutions of (2.29). In Lemma 2.8 we show that they satisfy conditions (i) and (ii) from Remark 2.4, whereas (iii) holds by the choice of $C = C(\mu)$. It remains to show $w(\psi, \phi) = 1$. Using independence of the Wronskian of x and (2.18) we obtain

$$\begin{aligned} \psi'(0)\phi(0) - \psi(0)\phi'(0) &= \mu\psi(0)\phi(0) + (Bi'(\mu^2) + CAi'(\mu^2))\pi Ai(\mu^2) \\ &\quad - \mu\phi(0)\psi(0) - \pi Ai'(\mu^2)(Bi(\mu^2) + CAi(\mu^2)) \\ &= \pi (Bi'(\mu^2)Ai(\mu^2) - Bi(\mu^2)Ai'(\mu^2)) \\ &= \pi (Bi'(0)Ai(0) - Bi(0)Ai'(0)) = 1. \end{aligned}$$

In particular, $w(\psi, \phi)$ is also independent of μ . Altogether the assertion of the lemma follows. \square

Lemma 2.8 (Properties of ϕ and ψ). *Let ϕ and ψ be defined by (2.27). For any $\mu \geq 0$ the function ϕ is strictly decreasing and the function ψ is strictly increasing in x .*

Proof: Properties of the Airy function Ai (see Remark 2.5) imply that for any $\mu \geq 0$ the function ϕ is positive and that $\phi(x) \rightarrow 0$ as $x \rightarrow \infty$. We will show that $\phi'(x) < 0$. To this end, it is enough to show that for $\mu \geq 0, x \geq 0$

$$g(\mu, x) := \mu Ai(\mu^2 + x) + Ai'(\mu^2 + x) < 0.$$

First we show that $g_1(\mu) = g(\mu, 0) < 0$ for $\mu \geq 0$. The assertion is true for $\mu = 0$ and for $\mu \rightarrow \infty$ we have $g_1(\mu) \rightarrow 0$. So if $g_1(\mu)$ is positive on some interval then there is a local maximum in some μ_0 such that on the one hand we have $g_1(\mu_0) > 0$ and on the other hand

$$\begin{aligned} 0 &= g_1'(\mu_0) = Ai(\mu_0^2) + 2\mu_0^2 Ai'(\mu_0^2) + 2\mu_0 Ai''(\mu_0^2) \\ &= Ai(\mu_0^2) + 2\mu_0^2 (Ai'(\mu_0^2) + \mu_0 Ai(\mu_0^2)) \\ &= Ai(\mu_0^2) + 2\mu_0^2 g_1(\mu_0) > 0, \end{aligned}$$

leading to a contradiction.

Now we fix $\mu \geq 0$ and show $g(\mu, x) < 0$ for all $x \geq 0$. For $x = 0$ it is true by the above argument. As $x \rightarrow \infty$ we have $g(\mu, x) \rightarrow 0$. If $g(\mu, \cdot)$ has positive values in the interval $(0, \infty)$ then there is a local maximum x_0 such that $g(\mu, x_0) > 0$ and

$$\begin{aligned} 0 &= \frac{\partial}{\partial x} g(\mu, x_0) = \mu Ai'(\mu^2 + x_0) + Ai''(\mu^2 + x_0) \\ &= \mu Ai'(\mu^2 + x_0) + \mu^2 Ai(\mu^2 + x_0) + x_0^2 Ai(\mu^2 + x_0) \\ &= \mu g(\mu, x_0) + x_0^2 Ai(\mu^2 + x_0) > 0, \end{aligned}$$

leading again to a contradiction.

It remains to show that ψ is increasing. By the choice of C we have $\psi'(0) = 0$. Let $h(\mu, x) = \mu Bi(\mu^2 + x) + Bi'(\mu^2 + x)$. For all $x > 0$ and $\mu \geq 0$ we have

$$\psi'(x) = e^{\mu x} \left(h(\mu, x) - \frac{h(\mu, 0)}{g(\mu, 0)} g(\mu, x) \right) > 0.$$

To see the inequality note that, as we have shown above,

$$g(\mu, x)/g(\mu, 0) = |g(\mu, x)/g(\mu, 0)| < 1$$

and therefore

$$h(\mu, x) - \frac{h(\mu, 0)}{g(\mu, 0)} g(\mu, x) \geq h(\mu, x) - h(\mu, 0) > 0.$$

The last inequality follows from the fact that Bi and Bi' are increasing. \square

We set

$$\Phi(x) := e^{-2\mu x} \phi(x) = \pi e^{-\mu x} Ai(\mu^2 + x), \quad (2.30)$$

$$\Psi(x) := e^{-2\mu x} \psi(x) = e^{-\mu x} (Bi(\mu^2 + x) + CAi(\mu^2 + x)) \quad (2.31)$$

and note that Φ and Ψ solve the differential equation

$$u''(x) + 2\mu u'(x) - xu(x) = 0, \quad (2.32)$$

and that Φ is up to a constant factor the unique decreasing solution of that equation satisfying $\Phi(x) \rightarrow 0$ as $x \rightarrow \infty$. Furthermore we have

$$\Psi'(0) + 2\mu\Psi(0) = \psi'(0) = 0 \quad (2.33)$$

and simple calculation shows

$$\phi(x)\Psi'(x) - \psi(x)\Phi'(x) = w(\psi, \phi) = 1. \quad (2.34)$$

Remark 2.9 (Expected killing time and the density of the killing position). Using the Green function one can compute the mean killing time of the killed reflecting Brownian motion starting in $x \geq 0$. It is given by

$$\mathbb{E}_x[\tau] = \int_0^\infty G(x, y)m(y) dy = \int_0^\infty G(x, y)2e^{-2\mu y} dy \quad (2.35)$$

which can be written as

$$= 2 \left(\phi(x) \int_0^x \Psi(y) dy + \psi(x) \int_x^\infty \Phi(y) dy \right). \quad (2.36)$$

Furthermore the density of $Z_{\tau-}^K$, i.e. of the position at killing time, is given by (see [Borodin and Salminen, 2002](#), p. 14)

$$G(x, y)k(y) = G(x, y)ye^{-2\mu y}. \quad (2.37)$$

\square

Lemma 2.10 (Exponential moments of the killing position). *For $\alpha < \mu$ and any $x \geq 0$ we have*

$$\mathbb{E}_x[e^{\alpha Z_{\tau-}^K}] < \infty. \quad (2.38)$$

Proof: Set $y^* := \max_{y \geq 0} \{ye^{-(\mu-\alpha)y}\}$ and recall the function M (for C defined in (2.28)) and its bound G^* in (2.21) and (2.23). Then (2.38) follows from

$$\begin{aligned} \mathbb{E}_x[e^{\alpha Z_{\tau-}}] &= \int_0^\infty e^{\alpha y} G(x, y) y e^{-2\mu y} dy \\ &= \phi(x) \int_0^x y \psi(y) e^{-(2\mu-\alpha)y} dy + \psi(x) \int_x^\infty y \phi(y) e^{-(2\mu-\alpha)y} dy \\ &= e^{\mu x} \pi \left\{ Ai(\mu^2 + x) \int_0^x y e^{-(\mu-\alpha)y} (Bi(\mu^2 + y) + CAi(\mu^2 + y)) dy \right. \\ &\quad \left. + (Bi(\mu^2 + x) + CAi(\mu^2 + x)) \int_x^\infty y e^{-(\mu-\alpha)y} Ai(\mu^2 + y) dy \right\} \\ &\leq e^{\mu x} y^* M(x) \leq e^{\mu x} y^* G^*. \end{aligned}$$

□

Lemma 2.11 (Second moment of killing time). *There is a positive finite constant C^{kill} such that for all $x \geq 0$*

$$\mathbb{E}_x[\tau^2] < C^{\text{kill}}. \quad (2.39)$$

Proof: Since the killing time of the killed reflecting Brownian motion starting in $x \geq 0$ is bounded stochastically by the killing time of the killed reflecting Brownian motion starting in 0, we have

$$\mathbb{E}_x[\tau^2] \leq 1 + \mathbb{E}_0[\tau^2].$$

By the Kac's moment formula (see e.g. Fitzsimmons and Pitman, 1999, (5) on p. 119) we have

$$\mathbb{E}_0[\tau^2] = 2 \int_0^\infty G(0, x) m(x) \int_0^\infty G(x, y) m(y) dy dx.$$

Now using again (2.21) and (2.23) as in the proof of Lemma 2.10 we obtain

$$\mathbb{E}_0[\tau^2] \leq 4G^* \int_0^\infty G(0, x) m(x) e^{\mu x} dx = 8G^* \pi \psi(0) \int_0^\infty Ai(\mu^2 + x) dx.$$

Since Ai is decreasing we obtain

$$\mathbb{E}_0[\tau^2] \leq 8G^* \pi \psi(0) \int_0^\infty Ai(x) dx = \frac{8G^* \pi}{3} \psi(0),$$

where the last equality follows from (2.19). □

Lemma 2.12 (Bound on the expected killing position starting from x). *For any $x \geq 0$*

$$\mathbb{E}_x[Z_{\tau-}] \leq x + \phi(0)\psi(0) + \mu \mathbb{E}_0[\tau]. \quad (2.40)$$

Proof: By (2.37) and the definition of Φ and Ψ in (2.30) respectively (2.31) we have

$$\begin{aligned}\mathbb{E}_x[Z_{\tau-}] &= \int_0^\infty y^2 G(x, y) e^{-2\mu y} dy \\ &= \phi(x) \int_0^x y^2 \psi(y) e^{-2\mu y} dy + \psi(x) \int_x^\infty y^2 \phi(y) e^{-2\mu y} dy \\ &= \phi(x) \int_0^x y^2 \Psi(y) dy + \psi(x) \int_x^\infty y^2 \Phi(y) dy.\end{aligned}$$

Now using the fact that Φ and Ψ satisfy (2.32), integration by parts, (2.34) and the fact that $\Psi\phi = \psi\Phi$ we arrive at

$$\begin{aligned}\mathbb{E}_x[Z_{\tau-}] &= \phi(x) \int_0^x (y\Psi''(y) + 2\mu y\Psi'(y)) dy + \psi(x) \int_x^\infty (y\Phi''(y) + 2\mu y\Phi'(y)) dy \\ &= x[\phi(x)\Psi'(x) - \psi(x)\Phi'(x)] \\ &\quad + \phi(x) \left[-\Psi(x) + \Psi(0) + 2\mu x\Psi(x) - 2\mu \int_0^x \Psi(y) dy \right] \\ &\quad + \psi(x) \left[\Phi(x) - 2\mu x\Phi(x) - 2\mu \int_x^\infty \Phi(y) dy \right] \\ &= x + \phi(x)\Psi(0) - 2\mu \left(\phi(x) \int_0^x \Psi(y) dy + \psi(x) \int_x^\infty \Phi(y) dy \right) \\ &= x + \phi(x)\psi(0) - \mu \int_0^\infty G(x, y)m(y)dy \\ &= x + \phi(x)\psi(0) - \mu\mathbb{E}_x[\tau].\end{aligned}$$

Here the next to last equality follows from (2.36) and from $\psi(0) = \Psi(0)$. Since ϕ is decreasing and $\mathbb{E}_x[\tau] \leq \mathbb{E}_0[\tau]$ the assertion (2.40) follows. \square

2.4. Invariant distribution at jump times. In this subsection we consider the increments of the Brownian ratchet at the jump times of the boundary process. We show that they constitute a Markov chain with unique invariant distribution and compute the expected jump time and the expected killing position under the invariant distribution.

Definition 2.13 (Markov chain at jump times). Let $(\mathcal{X}, \mathcal{R})$ be (γ, μ) -Brownian ratchet with sequence of jump times of \mathcal{R} given by $(\tau_n)_{n \geq 0}$. We define the Markov chain $(\mathcal{Y}, \mathcal{W}, \eta) := (Y_n, W_n, \eta_n)_{n=1,2,\dots}$ of increments at jump times by

$$Y_n = X_{\tau_n} - R_{\tau_n}, \quad W_n = R_{\tau_n} - R_{\tau_{n-1}} \quad \text{and} \quad \eta_n = \tau_n - \tau_{n-1}. \quad (2.41)$$

Since for any k the law of $(Y_n, W_n, \eta_n)_{n=k+1, k+2, \dots}$ depends on $(Y_n, W_n, \eta_n)_{n=1, \dots, k}$ only through Y_k , $(\mathcal{Y}, \mathcal{W}, \eta)$ is indeed a Markov chain.

Proposition 2.14. *There exists a unique invariant distribution of the Markov chain $(\mathcal{Y}, \mathcal{W}, \eta)$.*

Proof: While uniqueness of an invariant distribution is guaranteed by the coupling result in Lemma 2.3, to prove existence we need to show that the moments of (Y_n, W_n, η_n) are bounded for all n . This implies then tightness of the sequence and also tightness of the Cesàro averages of the laws. Weak limits of subsequences

of the latter are invariant distributions of the Markov chain. Boundedness of the moments of η_n follows from Lemma 2.11 and that of the moments of Y_n (and W_n since it has the same distribution as Y_n) follows inductively from Lemma 2.12. For details we refer to the proof of Proposition 5.6 in Depperschmidt and Pfaffelhuber (2010). \square

Proposition 2.15. *Let ν be the invariant distribution of Y_1, Y_2, \dots and let \mathbb{E}_ν denote the expectation with respect to that distribution. Then there is a constant $K \in (0, \infty)$ so that*

$$\mathbb{E}_\nu[Y_1] = -\frac{1}{K}(\mu Ai(\mu^2) + Ai'(\mu^2)), \quad (2.42)$$

and

$$\mathbb{E}_\nu[\eta_1] = \frac{2Ai(\mu^2)}{K}. \quad (2.43)$$

Proof: Let f_ν denote the density of ν with respect to Lebesgue measure. Let $Z^{K,Y} := (Z_t^{K,Y})_{t \geq 0}$ be killed reflecting Brownian motion with drift starting in random value $Y \geq 0$ with increments independent of Y . Furthermore let U be uniformly distributed on $(0, 1)$. Invariance of ν implies that for killing time τ of $Z^{K,Y}$ we have

$$Y \stackrel{d}{=} U \cdot Z_{\tau-}^{K,Y}.$$

As in Depperschmidt and Pfaffelhuber (2010, Section 5.3) from that one can obtain the following recurrence equation

$$f_\nu(z) = \int_0^\infty f_\nu(x) \int_z^\infty e^{-2\mu u} G(x, u) du$$

and then compute

$$f'_\nu(z) = -\Psi(z) \int_z^\infty f_\nu(x) \phi(x) dx - \Phi(z) \int_0^z f_\nu(x) \psi(x) dx, \quad (2.44)$$

$$f''_\nu(z) = -\Psi'(z) \int_z^\infty f_\nu(x) \phi(x) dx - \Phi'(z) \int_0^z f_\nu(x) \psi(x) dx, \quad (2.45)$$

and

$$\begin{aligned} f'''_\nu(z) &= -\Psi''(z) \int_z^\infty f_\nu(x) \phi(x) dx + \psi'(z) f_\nu(z) \phi(z) \\ &\quad - \Phi(z) \int_0^z f_\nu(x) \psi(x) dx - \Phi'(z) f_\nu(z) \psi(z) \\ &= -2\mu f''_\nu(z) + z f'_\nu(z) + f_\nu(z), \end{aligned} \quad (2.46)$$

where for the last equality we used (2.32), equations for f''_ν , f'_ν and (2.34). Thus,

$$f'''_\nu(z) = -2\mu f''_\nu(z) + (z f_\nu(z))'.$$

Integrating we obtain

$$f''_\nu(z) = -2\mu f'_\nu(z) + z f_\nu(z). \quad (2.47)$$

The integration constant is zero because from (2.44), (2.45) and (2.33) we see that $f_\nu''(0) = -2\mu f_\nu'(0)$. By (2.44) the density f_ν must be strictly decreasing. Up to a constant factor the positive decreasing solution of (2.47) is Φ and it follows that

$$f_\nu(z) = \frac{1}{K}\Phi(z) \quad \text{with} \quad K = \int_0^\infty \Phi(x) dx. \quad (2.48)$$

From (2.47) it follows

$$\begin{aligned} \mathbb{E}_\nu[Y_1] &= \int_0^\infty x f_\nu(x) dx = \int_0^\infty (f_\nu''(x) + 2\mu f_\nu'(x)) dx \\ &= -f_\nu'(0) - 2\mu f_\nu(0) = \frac{\pi}{K}(\mu Ai(\mu^2) - Ai'(\mu^2) - 2\mu Ai(\mu^2)) \\ &= -\frac{\pi}{K}(\mu Ai(\mu^2) + Ai'(\mu^2)), \end{aligned}$$

(replace here K by $K\pi$ to get (2.42)) and

$$\begin{aligned} \mathbb{E}_\nu[\eta_1] &= 2 \int_0^\infty f_\nu(x) \int_0^\infty e^{-2\mu y} G(x, y) dy dx \\ &= 2 \int_0^\infty f_\nu(x) \left(\phi(x) \int_0^x \Psi(y) dy + \psi(x) \int_x^\infty \Phi(y) dy \right) dx. \end{aligned}$$

Now using Fubini's Theorem and then (2.44) we have

$$\begin{aligned} \mathbb{E}_\nu[\eta_1] &= 2 \int_0^\infty \left(\Psi(y) \int_y^\infty f_\nu(x) \phi(x) dx + \Phi(y) \int_0^y f_\nu(x) \psi(x) dx \right) dy \\ &= -2 \int_0^\infty f_\nu'(x) dx = 2f_\nu(0) = 2 \frac{\Phi(0)}{K} = 2 \frac{\pi Ai(\mu^2)}{K}. \end{aligned}$$

Again replacing K by $K\pi$ we obtain (2.43). \square

2.5. Regeneration structure. In this subsection we define a regeneration structure of the (γ, μ) -Brownian ratchet and show that the second moments of the regeneration times and of the corresponding spatial increments are finite.

Definition 2.16 (Brownian ratchet as a cumulative process). Given a Brownian ratchet $(\mathcal{X}, \mathcal{R}) = (X_t, R_t)_{t \geq 0}$ with $(X_0, R_0) = (x, 0)$, $x \geq 0$ we define a sequence of regeneration times as follows

$$\rho_0 := \inf\{t \geq 0 : X_t = R_t\}, \quad \tilde{\rho}_0 := \inf\{t \geq \rho_0 : R_{t-} \neq R_t\}, \quad (2.49)$$

where $R_{t-} = \lim_{s \rightarrow t, s < t} R_s$, and for $n \geq 1$ we set

$$\rho_n := \inf\{t \geq \tilde{\rho}_{n-1} : X_t = R_t\}, \quad \tilde{\rho}_n := \inf\{t \geq \rho_n : R_{t-} \neq R_t\}. \quad (2.50)$$

Then $\rho_0 < \tilde{\rho}_0 < \rho_1 < \tilde{\rho}_1 < \dots$ almost surely and we have $\rho_0 = 0$ in the case $x = 0$. Furthermore in this case the sequence $(X_{\rho_{n+1}} - X_{\rho_n}, \rho_{n+1} - \rho_n)_{n \geq 0}$ is iid (in the case $x > 0$ the sequence $(X_{\rho_{n+1}} - X_{\rho_n}, \rho_{n+1} - \rho_n)_{n \geq 1}$ is iid). We define

$$D_t := \min\{n : \rho_n > t\}, \quad S_n := \sum_{i=1}^n (X_{\rho_i} - X_{\rho_{i-1}}) \quad \text{and} \quad A_t := X_{\rho_0} + X_t - X_{\rho_{D_t}}. \quad (2.51)$$

Then we have

$$X_t = S_{D_t} + A_t, \quad (2.52)$$

that is, X_t is a type A cumulative process S_{D_t} with remainder A_t (see Roginsky, 1994).

It is well known (see e.g. Roginsky (1994); Smith (1955) cf. also Remark 6.1 in Depperschmidt and Pfaffelhuber (2010)) that to prove the law of large numbers and the central limit theorem for $(S_{D_t})_{t \geq 0}$ we need to show that the second moments of $\rho_1 - \rho_0$ and $X_{\rho_1} - X_{\rho_0}$ are bounded (this is done in Propositions 2.17 and 2.18). Then, to carry the results over to $(X_t)_{t \geq 0}$ we have to prove that the (scaled) remainder A_t/\sqrt{t} is asymptotically negligible as $t \rightarrow \infty$ (this is done in Proposition 2.19).

Proposition 2.17. *There exists a positive constant R^* such that for all $x \geq 0$*

$$\mathbb{E}_x[\rho_0^2] \leq R^* \quad \text{and} \quad \mathbb{E}_x[(\rho_1 - \rho_0)^2] \leq R^*. \quad (2.53)$$

Proof: In the case $x = 0$ we have $\rho_0 = 0$. Consider the case $x > 0$. Let H_1 be the hitting time of 0 of the Brownian motion with drift $-\mu$ started in 1. This hitting time has exponential moments (we used this fact already in the proof of Lemma 2.3). Furthermore set $T = \inf\{t \geq 0 : X_t - R_t \leq 1\}$ and note that $T \leq E_{1/2}$, where $E_{1/2}$ is independent exponential random variable with rate 1/2, because as long as $X_t - R_t > 1$ the rate at which the reflection boundary jumps into the interval $[X_t - 1, X_t]$ (and therefore T occurs) is 1/2. But T also occurs if X_t hits the interval $[R_t, R_t + 1]$. It follows that for any initial positions $x \geq 0$, ρ_0 is bounded stochastically by the sum of independent random variables $E_{1/2}$ and H_1 , both having exponential moments and not depending on x . Thus, $\mathbb{E}_x[\rho_0^2]$ is bounded by a constant not depending on x .

We write $\rho_1 - \rho_0 = (\rho_1 - \tilde{\rho}_0) + (\tilde{\rho}_0 - \rho_0)$ and argue that each of the terms in the brackets has bounded second moments. Note that $\tilde{\rho}_0 - \rho_0$ is the first jump time of the reflection boundary after ρ_0 . Thus, the finiteness of its second moment follows from Lemma 2.11 and that the bound there, also does not depend on x . The finiteness of the second moment of $(\rho_1 - \tilde{\rho}_0)$ follows by the same argument as the finiteness of the second moment of ρ_0 . \square

Proposition 2.18. *There exists a positive constant $R^{**} < \infty$ so that for any $x \geq 0$*

$$\mathbb{E}_x[X_{\rho_0}^2] \leq R^{**} \quad \text{and} \quad \mathbb{E}_x[(X_{\rho_1} - X_{\rho_0})^2] \leq R^{**}. \quad (2.54)$$

Proof: Recall that before touching the reflection boundary the process $(X_t)_{t \geq 0}$ behaves as a Brownian motion with drift $-\mu$ and is therefore bounded below by 0 and above by a Brownian motion without drift. Applying the second Wald identity and Proposition 2.17 (with R^* from that proposition) we get

$$\mathbb{E}_x[X_{\rho_0}^2] \leq \mathbb{E}_x[B_{\rho_0}^2] = \mathbb{E}_x[\rho_0] \leq \sqrt{\mathbb{E}_x[\rho_0^2]} \leq \sqrt{R^*}. \quad (2.55)$$

Now we write $X_{\rho_1} - X_{\rho_0} = (X_{\rho_1} - X_{\tilde{\rho}_0}) + (X_{\tilde{\rho}_0} - X_{\rho_0})$ and note that as in (2.55) the second moment of the first term is bounded by $\sqrt{R^*}$. The second moment of the second term is finite according to Lemma 2.10 with a bound independent of x . Now (2.54) follows with appropriately chosen R^{**} . \square

Proposition 2.19 (Asymptotics of A_t and $X_t^\mu - R_t$). *We have*

$$\frac{A_t}{\sqrt{t}} \rightarrow 0 \quad \text{and} \quad \frac{X_t^\mu - R_t}{\sqrt{t}} \rightarrow 0 \quad \text{a.s. as } t \rightarrow \infty.$$

We omit the proof here since it is almost the same as the proof of Proposition 6.7 in [Depperschmidt and Pfaffelhuber \(2010\)](#) where the corresponding result was shown for the Brownian ratchet without local drift. (Note that the definition of Y_n there should be $Y_n = \sup_{t \in [\rho_{n-1}, \rho_n]} |X_t - X_{\rho_{n-1}}|$. Also the denominator in the last two displays of that proof should be \sqrt{t} instead of t .)

2.6. Proof of Theorem 1.1. Here we only sketch the proof and refer for details to Section 7 in [Depperschmidt and Pfaffelhuber \(2010\)](#).

In the case $\gamma = 0$, the law of large numbers in Theorem 1.1 holds since $(X_t)_{t \geq 0}$ is then a reflecting Brownian motion with negative drift $-\mu$ bounded stochastically by a reflecting Brownian motion without drift. Therefore $X_t/t \rightarrow 0$ a.s. as $t \rightarrow \infty$.

Hence, we assume $\gamma > 0$ in the rest of the proof. We use the regeneration structure from Definition 2.16 and set

$$\begin{aligned} r &:= \mathbb{E}_x[\rho_1 - \rho_0], \\ m &:= \mathbb{E}_x[X_{\rho_1} - X_{\rho_0}], \\ \beta^2 &:= \text{Var}_x \left[X_{\rho_1} - X_{\rho_0} - \frac{(\rho_1 - \rho_0)m}{r} \right]. \end{aligned}$$

Here, r , m and β^2 are independent of x due to the regeneration structure. According to Propositions 2.17 and 2.18 the temporal and spatial increments $\rho_1 - \rho_0$ and $X_{\rho_1} - X_{\rho_0}$ have finite second moments. From that and Proposition 2.19 it follows

$$\frac{X_t}{t} \xrightarrow{t \rightarrow \infty} \frac{m}{r} \text{ a.s.}$$

Furthermore, using the CLT for cumulative processes (see e.g. [Smith, 1955](#); [Roginsky, 1994](#)) and Proposition 2.19 we obtain that for all $x \in \mathbb{R}$

$$\begin{aligned} \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{X_t - tm/r}{\beta(t/r)^{1/2}} \leq x \right) &= \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{A_t}{\beta(t/r)^{1/2}} + \frac{S_{D_t} - tm/r}{\beta(t/r)^{1/2}} \leq x \right) \\ &= \lim_{t \rightarrow \infty} \mathbb{P} \left(\frac{S_{D_t} - tm/r}{\beta(t/r)^{1/2}} \leq x \right) = \Phi(x), \end{aligned}$$

where Φ denotes the distribution function of the standard normal distribution. Hence, the central limit theorem holds for $\sigma = \beta/\sqrt{r}$.

It remains to compute m/r . To this end we use the ratio limit theorem for Harris recurrent Markov chains (see e.g. [Revuz, 1984](#)). Let ν denote the invariant distribution for $(\mathcal{Y}, \mathcal{W}, \eta)$. Using the ratio limit theorem we obtain that

$$\frac{m}{r} = \lim_{t \rightarrow \infty} \frac{X_t}{t} = \lim_{t \rightarrow \infty} \frac{R_t}{t} = \lim_{n \rightarrow \infty} \frac{R_{\tau_n}}{\tau_n} = \lim_{n \rightarrow \infty} \frac{\sum_{k=1}^n W_k}{\sum_{k=1}^n \eta_k} = \frac{\mathbb{E}_\nu[W_1]}{\mathbb{E}_\nu[\eta_1]},$$

where for the second equality we have used Proposition 2.19. We recall that $\mathbb{E}_\nu[W_1] = \mathbb{E}_\nu[Y_1]$.

Let $v : [0, \infty)^2 \rightarrow [0, \infty)$, $(\mu, \gamma) \mapsto v(\mu, \gamma)$, denote the speed m/r as a function of μ and γ . In the case $\gamma = \frac{1}{2}$ we obtain from Proposition 2.15

$$v \left(\mu, \frac{1}{2} \right) = \frac{\mathbb{E}_\nu[Y_1]}{\mathbb{E}_\nu[\tau_1]} = -\frac{\mu Ai(\mu^2) + Ai'(\mu^2)}{2Ai(\mu^2)} = -\frac{1}{2} \left(\frac{Ai'(\mu^2)}{Ai(\mu^2)} + \mu \right).$$

Now let $\mu \geq 0$ and $\gamma > 0$ be given. Using the scaling property (see Lemma 2.2) we obtain

$$(X_t^{\gamma, \mu}, R_t^{\gamma, \mu}) \stackrel{d}{=} (2\gamma)^{-1/3} \left(X_{(2\gamma)^{2/3}t}^{1/2, (2\gamma)^{-1/3}\mu}, R_{(2\gamma)^{2/3}t}^{1/2, (2\gamma)^{-1/3}\mu} \right).$$

Thus, we have

$$\begin{aligned} v(\mu, \gamma) &= \lim_{t \rightarrow \infty} \frac{\mathbb{E}[X_t^{\gamma, \mu}]}{t} = (2\gamma)^{1/3} \lim_{t \rightarrow \infty} \frac{\mathbb{E}\left[X_{(2\gamma)^{2/3}t}^{1/2, (2\gamma)^{-1/3}\mu}\right]}{(2\gamma)^{2/3}t} \\ &= (2\gamma)^{1/3} v\left((2\gamma)^{-1/3}\mu, \frac{1}{2}\right) \\ &= -\frac{(2\gamma)^{1/3}}{2} \left(\frac{Ai'((2\gamma)^{-2/3}\mu^2)}{Ai((2\gamma)^{-2/3}\mu^2)} + (2\gamma)^{-1/3}\mu \right) \end{aligned}$$

which concludes the proof of Theorem 1.1.

3. Ornstein-Uhlenbeck ratchet

In this section we carry out the same program for the Ornstein-Uhlenbeck ratchet as we did for the Brownian ratchet. The arguments in many of the proofs here are similar to the corresponding proofs in the previous section. Therefore some proofs in this section will be sketchy.

3.1. Graphical construction. Let us first recall the definition of the reflecting Ornstein-Uhlenbeck (OU) process with infinitesimal drift $-\mu x$ and unit variance. Suppose that $B = (B(t))_{t \geq 0}$ is a standard Brownian motion starting in 0 and let $x_0 \geq 0$. Then using the representation of the OU process as a time changed Brownian motion, we obtain that $\widehat{Z} := (\widehat{Z}_t)_{t \geq 0}$, defined by

$$\widehat{Z}_t := \left| x_0 e^{-\mu t} + \frac{1}{\sqrt{2\mu}} e^{-\mu t} B(e^{2\mu t} - 1) \right| \quad (3.1)$$

is a reflecting OU process with infinitesimal drift $-\mu x$ and unit variance starting in x_0 . It is a diffusion process on $[0, \infty)$ associated with infinitesimal operator \widehat{A}^μ acting on

$$\mathcal{D}(\widehat{A}^\mu) := \{f \in C_b^2(\mathbb{R}_+) : f'(0+) = 0\}$$

as follows:

$$\widehat{A}^\mu f(x) := \frac{1}{2} f''(x) - \mu x f'(x). \quad (3.2)$$

The graphical construction in the following definition (see Figure 3.4) is different from the graphical construction of the Brownian ratchet. Here we use a family of independent Brownian motions to construct reflected OU processes between the jumps of the ratchet. At any jump time a new reflected OU process starts in an initial value chosen uniformly between the state of the previous process and zero. Then we stick this “peaces” together to obtain the OU ratchet.

Definition 3.1 (Graphical construction of the Ornstein-Uhlenbeck ratchet). Let N^γ be a Poisson process and let $((B^{(i)}(t))_{t \geq 0})_{i=0,1,\dots}$ be independent standard Brownian motions. We define a sequence of stopping times $(\widehat{\tau}_n)_{n=0,1,\dots}$ and a

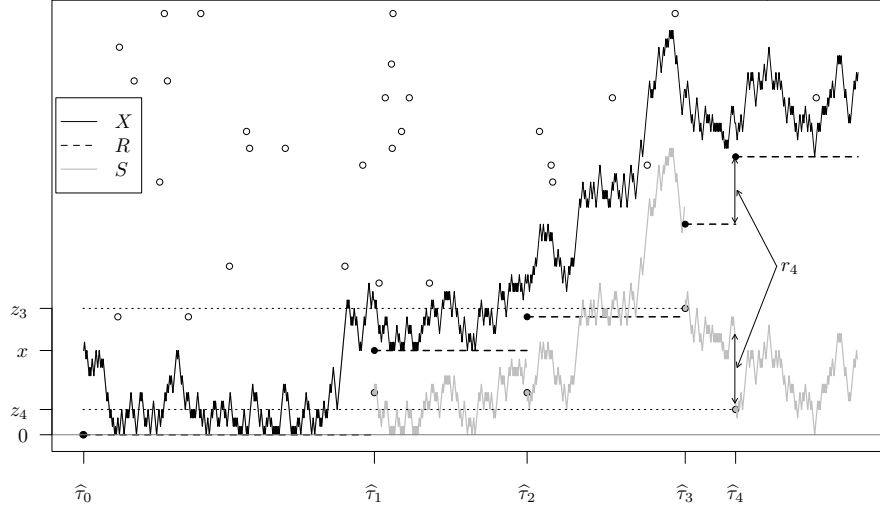


FIGURE 3.4. Graphical construction of the Ornstein-Uhlenbeck ratchet.

sequence of Ornstein-Uhlenbeck processes $(\widehat{S}^{(n)})_{n=0,1,\dots}$ with $\widehat{S}^{(n)} = (S_t^{(n)})_{t \geq \tau_n}$ reflecting at 0 as follows:

$$\widehat{\tau}_0 = 0, \quad (3.3)$$

$$\widehat{S}_t^{(0)} = \left| x_0 e^{-\mu t} + e^{-\mu t} \frac{1}{\sqrt{2\mu}} B^{(0)}(e^{2\mu t} - 1) \right|. \quad (3.4)$$

Given $\widehat{\tau}_{n-1}$ and $\widehat{S}^{(n-1)}$ for some $n \geq 1$ we set

$$\widehat{\tau}_n = \inf \{ t > \widehat{\tau}_{n-1} : N^\gamma \cap [0, \widehat{S}^{(n-1)}(t)] \times \{t\} \neq \emptyset \} \quad (3.5)$$

and let z_n be the space component of the almost surely unique element of $N^\gamma \cap [0, \widehat{S}^{(n-1)}] \times \{\widehat{\tau}_n\}$. Furthermore we set $r_n = \widehat{S}_{\widehat{\tau}_n}^{(n-1)} - z_n$ and for $t \geq \widehat{\tau}_n$ we define

$$\widehat{S}_t^{(n)} = \left| z_n e^{-\mu(t-\widehat{\tau}_n)} + e^{-\mu(t-\widehat{\tau}_n)} \frac{1}{\sqrt{2\mu}} B^{(n)}(e^{2\mu(t-\widehat{\tau}_n)} - 1) \right|. \quad (3.6)$$

Finally we set for $t \in [\widehat{\tau}_n, \widehat{\tau}_{n+1})$

$$\widehat{R}_t = \sum_{i \leq n} r_i \quad \text{and} \quad \widehat{S}_t = \widehat{S}_t^{(n)} \quad (3.7)$$

and for $t \geq 0$

$$\widehat{X}_t = \widehat{R}_t + \widehat{S}_t. \quad (3.8)$$

Note that by construction we have

$$\widehat{S} \leq \widehat{S}^{(0)} \quad (3.9)$$

stochastically. \square

The following lemma is the analogue of Lemma 2.1. Though the graphical construction there is somewhat different the proof is similar and will be omitted here.

Lemma 3.2. *The process $(\widehat{\mathcal{X}}, \widehat{\mathcal{R}}) := (\widehat{X}_t, \widehat{R}_t)_{t \geq 0}$ is a (γ, μ) -Ornstein-Uhlenbeck ratchet starting in $(x_0, 0)$.*

Now we construct a coupling of two Ornstein-Uhlenbeck ratchets starting in $((x_0, 0), (x'_0, 0))$, where we assume $x_0 \geq x'_0 \geq 0$ without loss of generality. Let the Poisson process N^γ and a sequence $((B^{(i)}(t))_{t \geq 0})_{i=0,1,\dots}$ of independent standard Brownian motions be given as before. To construct the coupling

$$((\widehat{X}_t, \widehat{R}_t), (\widehat{X}'_t, \widehat{R}'_t))_{t \geq 0},$$

set $\widehat{\tau}_0 = \widehat{\tau}'_0 = 0$, $\widehat{S}_0^{(0)} = x$, $\widehat{S}'_0^{(0)} = x'$ and define the sequences $(\widehat{\tau}_n)_{n \geq 0}$ and $(\widehat{\tau}'_n)_{n \geq 0}$ and the processes $(\widehat{S}^{(n)})_{n \geq 0}$ and $(\widehat{S}'^{(n)})_{n \geq 0}$ as in Definition 3.1. Furthermore define \widehat{S} , \widehat{S}' , $(\widehat{X}_t, \widehat{R}_t)_{t \geq 0}$ and $(\widehat{X}'_t, \widehat{R}'_t)_{t \geq 0}$ as in (3.7). Then define the coupling time by

$$\widehat{T}_{\text{coupl}} := \inf\{t > 0 : \widehat{S}_t = \widehat{S}'_t\}.$$

Since we use the same Brownian motions and the same Poisson process for both ratchets we have $\widehat{S}_t = \widehat{S}'_t$ for all $t \geq \widehat{T}_{\text{coupl}}$. Thus, on the event $\{\widehat{T}_{\text{coupl}} < \infty\}$, there are $\widehat{n}, \widehat{n}'$ such that for $k \geq 0$

$$\begin{aligned} \widehat{\tau}_{n+k} &= \widehat{\tau}'_{n+k}, \\ \widehat{X}_{\widehat{\tau}_{n+k+1}} - \widehat{X}_{\widehat{\tau}_{n+k}} &= \widehat{X}'_{\widehat{\tau}'_{n+k+1}} - \widehat{X}'_{\widehat{\tau}'_{n+k}}, \\ \widehat{R}_{\widehat{\tau}_{n+k+1}} - \widehat{R}_{\widehat{\tau}_{n+k}} &= \widehat{R}'_{\widehat{\tau}'_{n+k+1}} - \widehat{R}'_{\widehat{\tau}'_{n+k}}. \end{aligned}$$

The following lemma shows that the coupling is successful with probability one.

Lemma 3.3 (Exponential moments of coupling time). *For any $\mu > 0$ there is $\alpha > 0$ so that*

$$\mathbb{E}[e^{\alpha \widehat{T}_{\text{coupl}}}] < \infty.$$

Proof: We only need to consider the case $x_0 > x'_0$. In that case $\widehat{S}_t \geq \widehat{S}'_t$ for all t . Furthermore, from (3.9) it follows that \widehat{S} is stochastically dominated by the reflected Ornstein-Uhlenbeck process $\widehat{S}^{(0)}$. Thus, $\widehat{T}_{\text{coupl}}$ is stochastically bounded by the hitting time of 0, say H_0 , by the process $\widehat{S}^{(0)}$. For H_0 we have (see Borodin and Salminen, 2002, p. 542)

$$\mathbb{P}_{x_0}[H_0 > t] = \text{Erf}(x_0 / \sqrt{2(e^{2\mu t} - 1)}),$$

where $\text{Erf}(x) = 2\pi^{-1/2} \int_0^x e^{-u^2} du = 2\pi^{-1/2} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(2n+1)}$. Thus, as $t \rightarrow \infty$ we obtain

$$\mathbb{P}_{x_0}[H_0 > t] = \frac{2}{\sqrt{\pi}} \frac{x_0}{\sqrt{2(e^{2\mu t} - 1)}} + o(e^{-3\mu t}) \leq x_0 C e^{-\mu t}$$

for suitably chosen positive C . □

3.2. Green function of the killed reflected Ornstein-Uhlenbeck process. In this subsection we carry out analogous computations to those in Subsection 2.3; recall in particular Remark 2.4.

Definition 3.4 (Killed reflecting Ornstein-Uhlenbeck process). Let $\widehat{Z} := (\widehat{Z}_t)_{t \geq 0}$ denote reflecting Ornstein-Uhlenbeck process with infinitesimal drift $-\mu x$ and unit infinitesimal variance starting in $x \geq 0$ and let \mathbb{P}_x denote the corresponding law on the paths space and \mathbb{E}_x the expectation under this law. Furthermore using an exponentially distributed rate 1 random variable ξ independent of \widehat{Z} we define the *killing time* by

$$\widehat{\tau} = \inf\{t > 0 : \gamma \int_0^t \widehat{Z}_s ds \geq \xi\}.$$

The *reflecting Ornstein-Uhlenbeck process killed at rate $\gamma \widehat{Z}$* is defined as the process $\widehat{Z}^K := (\widehat{Z}_t^K)_{t \geq 0}$ with $\widehat{Z}_t^K = \widehat{Z}_t$ for $t \in [0, \tau)$ and $\widehat{Z}_t^K = \Delta$ for $t \geq \tau$ where $\Delta \notin \mathbb{R}$ is the *cemetery state*. The infinitesimal operator $\widehat{A}^{\mu, \gamma}$ corresponding to \widehat{Z}^K acts on C^2 functions $f : [0, \infty) \rightarrow \mathbb{R}$ satisfying $f'(0+) = 0$ as follows

$$\widehat{A}^{\mu, \gamma} f(x) = \frac{1}{2} f''(x) - \mu x f'(x) - \gamma x f(x). \tag{3.10}$$

The speed measure and the killing measure corresponding to the killed Ornstein-Uhlenbeck process are given by

$$m(dx) = m(x) dx := 2e^{-\mu x^2} dx \quad \text{and} \quad k(dx) = k(x) dx := 2\gamma x e^{-\mu x^2} dx. \tag{3.11}$$

In the case of Ornstein-Uhlenbeck ratchet the *confluent hypergeometric functions* play a similar role as the Airy functions in the case of Brownian ratchet. In the following remark we collect some of their properties that will be needed in the sequel. We refer to [Olver et al. \(2010, Ch. 13\)](#) for most of the properties and for more information on confluent hypergeometric functions.

Remark 3.5 (Confluent hypergeometric functions). Assume that $b \notin \mathbb{Z}$ and consider the Kummer equation (also known as the confluent hypergeometric equation)

$$xz''(x) + (b - x)z'(x) - az(x) = 0. \tag{3.12}$$

Note that solutions also exist in the case $b \in \mathbb{Z}$ but some of relations that we are going to recall in this remark and use in the following may not hold in this case.

(1) *Standard solutions*: One standard solution of that equation is given by the function $x \mapsto M(a, b, x)$ which is defined by

$$M(a, b, x) := \sum_{n=0}^{\infty} \frac{(a)_n x^n}{(b)_n n!}, \tag{3.13}$$

where $(a)_n$ is the Pochhammer's symbol defined by $(a)_0 = 1$ and $(a)_n = a(a + 1) \cdots (a + n - 1)$ for $n \geq 1$. Another standard solution is given by the function $x \mapsto U(a, b, x)$ which can be defined (in the case $b \notin \mathbb{Z}$) by

$$U(a, b, x) := \frac{\Gamma(1 - b)}{\Gamma(a - b + 1)} M(a, b, x) + \frac{\Gamma(b - 1)}{\Gamma(a)} x^{1-b} M(1 + a - b, 2 - b, x), \tag{3.14}$$

where Γ denotes the Gamma function. If $a \neq 0, -1, -2, \dots$, then U and M are independent solutions of (3.12). For $a = 0, -1, \dots$ we have $|\Gamma(a)| = \infty$ and the second summand on the right hand side of (3.14) vanishes. Thus, as is easily seen from (3.13) and (3.14), in that case both U and M are polynomials which are equal up to a multiplicative constant. The system of independent solutions in the case $a = 0, -1, \dots$ is given by

$$U(a, b, z) \quad \text{and} \quad z^{1-b} M(a - b + 1, 2 - b, z). \tag{3.15}$$

(2) *Behaviour in the neighbourhood of zero:* For any $a, b \in \mathbb{R}$

$$M(a, b, x) = 1 + \mathcal{O}(x) \quad \text{as } x \downarrow 0. \quad (3.16)$$

For $b \in (0, 1)$

$$U(a, b, x) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \mathcal{O}(x^{1-b}) \quad \text{as } x \downarrow 0. \quad (3.17)$$

For $b \in (1, 2)$

$$U(a, b, x) = \frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} + \frac{\Gamma(1-b)}{\Gamma(a-b+1)} + \mathcal{O}(x^{2-b}) \quad \text{as } x \downarrow 0. \quad (3.18)$$

(3) *Behaviour at infinity:* For $a \neq -1, -2, \dots$ and $b > 0$

$$M(a, b, x) \sim \frac{x^{a-b}}{\Gamma(a)} e^x \quad \text{as } x \rightarrow \infty. \quad (3.19)$$

Here, as usual, we write $g(x) \sim h(x)$ if $g(x)/h(x) \rightarrow 1$ as $x \rightarrow \infty$. Furthermore

$$U(a, b, x) \sim x^{-a} \quad \text{as } x \rightarrow \infty. \quad (3.20)$$

Note also that U is uniquely determined by this property.

(4) *Bounds for positive zeros of U and M :* If $a, b \geq 0$ then M has no zeros on $[0, \infty)$. Let $P(a, b)$ be the number of positive zeros of $U(a, b, x)$. First we note that using the Kummer transformation

$$U(a, b, x) = x^{1-b} U(a-b+1, 2-b, x) \quad (3.21)$$

we get

$$P(a, b) = P(a-b+1, 2-b). \quad (3.22)$$

If a, b and $a+b-1$ are non-integers, $b < 1$ and $a+1 \geq b$ then $P(a, b) = 0$. Let $a < 0$ and $1 \leq b \leq 2$. If x_0 is a positive zero of $U(a, b, x)$ then by (2.19) in [Gatteschi \(1990\)](#)

$$x_0 < 4\left(\frac{b}{2} - a\right). \quad (3.23)$$

(5) *Differentiation formulas:* We have

$$\frac{d}{dx} U(a, b, x) = -aU(a+1, b+1, x) \quad (3.24)$$

and

$$\frac{d}{dx} M(a, b, x) = \frac{a}{b} M(a+1, b+1, x). \quad (3.25)$$

(6) *Recurrence relation:* There are many recurrence relations for U and M . We will need the following (it follows from (13.4.25) in [Abramowitz and Stegun \(1992\)](#) and (3.24))

$$U(a, b+1, x) = U(a, b, x) + aU(a+1, b+1, x). \quad (3.26)$$

□

By straightforward computation one can show that if a function $f_{\gamma,\mu}(x)$ is a solution of the Kummer equation (3.12) with $a = -\frac{\gamma^2}{4\mu^3}$ and $b = \frac{1}{2}$, then

$$F_{\gamma,\mu}(x) = e^{-\gamma x/\mu} f_{\gamma,\mu} \left(\left(\frac{\gamma}{\mu^{3/2}} + \sqrt{\mu}x \right)^2 \right) \quad (3.27)$$

is a solution of

$$\widehat{A}^{\mu,\gamma} f = 0. \quad (3.28)$$

As in the case of killed Brownian motion with negative drift there are positive solutions $\widehat{\phi}$ and $\widehat{\psi}$ of (3.28) on $[0, \infty)$ with Wronskian $w(\widehat{\psi}, \widehat{\phi}) := \widehat{\psi}'(x)\widehat{\phi}(x) - \widehat{\psi}(x)\widehat{\phi}'(x) = 1$ such that

$$\begin{cases} \widehat{\phi} \text{ is decreasing and } \widehat{\phi}(x) \rightarrow 0 \text{ as } x \rightarrow \infty, \\ \widehat{\psi} \text{ is increasing and } \widehat{\psi}'(0) = 0. \end{cases} \quad (3.29)$$

Then the Green function of the killed Ornstein-Uhlenbeck process is given by

$$G(x, y) := \begin{cases} \widehat{\phi}(x)\widehat{\psi}(y) & : 0 \leq y \leq x, \\ \widehat{\psi}(x)\widehat{\phi}(y) & : 0 \leq x \leq y. \end{cases} \quad (3.30)$$

We define the function $x \mapsto p(x)$ by

$$p(x) = \frac{\gamma}{\mu^{3/2}} + \sqrt{\mu}x. \quad (3.31)$$

For general γ and μ the solutions of (3.12) with $a = -\frac{\gamma^2}{4\mu^3}$ and $b = \frac{1}{2}$ with $a = -\frac{\gamma^2}{4\mu^3}$ are hard to deal with. Thus, we will not formulate an analogue of Lemma 2.7 for the killed reflecting Ornstein-Uhlenbeck process. In the following lemma we will identify the decreasing solution $\widehat{\phi}$. For $\widehat{\psi}$ we will assume that for any given $\mu, \gamma > 0$ there is a suitable linear combination, say $\widehat{M}(x)$, of

$$M \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \text{ and } U \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \quad (3.32)$$

if $\frac{\gamma^2}{4\mu^3} \neq 1, 2, \dots$, and

$$p(x)M \left(-\frac{\gamma^2}{4\mu^3} + \frac{1}{2}, \frac{3}{2}, p^2(x) \right) \text{ and } U \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \quad (3.33)$$

if $\frac{\gamma^2}{4\mu^3} = 1, 2, \dots$, so that the function $\widehat{\psi} : [0, \infty) \rightarrow [0, \infty)$ defined by

$$\widehat{\psi}(x) = e^{-\gamma x/\mu} \widehat{M}(x) \quad (3.34)$$

satisfies the condition in (3.29).

Lemma 3.6. *The function $x \mapsto \widehat{\phi}(x)$ defined on $[0, \infty)$ by*

$$\widehat{\phi}(x) = e^{-\gamma x/\mu} U \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \quad (3.35)$$

is a positive decreasing solution of (3.28) with $\widehat{\phi}(x) \rightarrow 0$ as $x \rightarrow \infty$.

Proof: Since $\widehat{\phi}$ is of the form (3.27) it is a solution of (3.28). By (3.20) it is clear that $\widehat{\phi}(x) > 0$ for large x and $\widehat{\phi}(x) \rightarrow 0$ as $x \rightarrow \infty$. If we show that $\widehat{\phi}''(x) > 0$ for $x > 0$, i.e. that $\widehat{\phi}$ is convex, then it follows that $\widehat{\phi}$ is positive and decreasing on $(0, \infty)$. We have

$$\begin{aligned}\widehat{\phi}''(x) &= 2\mu x \widehat{\phi}'(x) + 2\gamma x \widehat{\phi}(x) \\ &= 2\mu x e^{-\frac{\gamma x}{\mu}} 2\sqrt{\mu} p(x) U' \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \\ &= 4\mu^{3/2} x e^{-\frac{\gamma x}{\mu}} p(x) \frac{\gamma^2}{4\mu^3} U \left(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right),\end{aligned}$$

where the last step follows by (3.24). Again by (3.20) it is clear that $\widehat{\phi}''(x) > 0$ for large x . Thus, to show that $\widehat{\phi}''(x) > 0$ for $x > 0$ it is enough to show that this function has no positive zeros.

Using (3.14) and the definition of M one can easily compute that $U(\frac{1}{2}, \frac{3}{2}, p^2(x)) = \frac{1}{p(x)}$ and $U(0, \frac{3}{2}, p^2(x)) = 1$. Thus in the case $\frac{\gamma^2}{4\mu^3} \in \{\frac{1}{2}, 1\}$ we have $\widehat{\phi}''(x) > 0$ for $x \geq 0$.

Let us now consider the case $\frac{\gamma^2}{4\mu^3} \in (0, 1) \setminus \{\frac{1}{2}\}$. Then by (3.22) the number of positive zeros of $x \mapsto U(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, x)$ equals the number of positive zeros of $x \mapsto U(\frac{1}{2} - \frac{\gamma^2}{4\mu^3}, \frac{1}{2}, x)$. Since

$$\frac{1}{2} - \frac{\gamma^2}{4\mu^3} + 1 \geq \frac{1}{2},$$

by Remark 3.5(iv), the function $x \mapsto U(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, x)$ has no positive zeros, which implies that $\widehat{\phi}''(x) > 0$ for $x > 0$.

If $\frac{\gamma^2}{4\mu^3} > 1$ then, by (3.23) all positive zeros of $x \mapsto U(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, x)$ are bounded by $\gamma^2/\mu^3 - 1$. Since the set $\{x \geq 0 : p^2(x) < \gamma^2/\mu^3 - 1\}$ is empty, the function $\widehat{\phi}''$ is positive on $(0, \infty)$. \square

In the following example we compute the function $\widehat{\psi}$ in a special case.

Example 3.7 ($\widehat{\psi}(x)$ in a special case). Assume $0 < \frac{\gamma^2}{4\mu^3} < \frac{1}{2}$, let $\widehat{\phi}$ be as in (3.35) and define

$$\widehat{\psi}_1(x) := -e^{-\gamma x/\mu} M \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right)$$

and

$$\widehat{\psi}(x) := \widehat{\psi}_1(x) + C \widehat{\phi}(x), \quad \text{with } C = -\frac{\widehat{\psi}'_1(0)}{\widehat{\phi}'(0)}.$$

By the choice of C we have $\widehat{\psi}'(0) = 0$ as required. Furthermore, Remark 3.5(iii) implies that $\widehat{\psi}(x) \rightarrow \infty$ for $x \rightarrow \infty$. To show that $\widehat{\psi}$ satisfies the conditions from (3.29) we need to show convexity of $\widehat{\psi}$, i.e. positivity of $\widehat{\psi}''$ on $[0, \infty)$ and $\widehat{\psi}(0) > 0$.

Using (3.25) we obtain

$$\begin{aligned} \frac{1}{2}\widehat{\psi}_1''(x) &= \mu x \widehat{\psi}_1'(x) + \gamma x \widehat{\psi}_1(x) = -2\mu^{3/2} x e^{-\frac{\gamma x}{\mu}} p(x) M' \left(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \\ &= \frac{\gamma^2}{\mu^{3/2}} x e^{-\frac{\gamma x}{\mu}} p(x) M \left(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right). \end{aligned} \quad (3.36)$$

By assumption we have $1 - \frac{\gamma^2}{4\mu^3} \geq 0$ and therefore Remark 3.5(iv) implies that the function $x \mapsto M(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x))$ has no zeros on $(0, \infty)$. Since by Remark 3.5(iii) this function is positive for large x we obtain $\widehat{\psi}_1''(x) > 0$ and $\widehat{\psi}_1'(x) > 0$ for $x > 0$.

In Lemma 3.6 we have shown that $\widehat{\phi}''$ is positive and $\widehat{\phi}'$ is negative on $(0, \infty)$. Together with positivity of $\widehat{\psi}_1''$ and $\widehat{\psi}_1'$ it follows

$$\widehat{\psi}''(x) = \widehat{\psi}_1''(x) - \frac{\widehat{\psi}_1'(0)}{\widehat{\phi}'(0)} \widehat{\phi}''(x) > 0, \quad x > 0.$$

Finally we have

$$\widehat{\psi}(0) = \widehat{\psi}_1(0) - \frac{\widehat{\psi}_1'(0)}{\widehat{\phi}'(0)} \widehat{\phi}(0) = -\frac{\widehat{\psi}_1'(0)\widehat{\phi}(0) - \widehat{\psi}_1(0)\widehat{\phi}'(0)}{\widehat{\phi}'(0)} = -\frac{w(\widehat{\psi}_1, \widehat{\phi})}{\widehat{\phi}'(0)}.$$

Here $w(\widehat{\psi}_1, \widehat{\phi}) = \widehat{\psi}_1'(x)\widehat{\phi}(x) - \widehat{\psi}_1(x)\widehat{\phi}'(x)$ is a constant independent of x , and since $\widehat{\psi}_1'(x)$, $\widehat{\phi}(x)$ and $\widehat{\psi}_1(x)\widehat{\phi}'(x)$ are positive for large x and $\widehat{\phi}'(x)$ is negative this constant must be positive. Note also that, as is easily computed, $w(\widehat{\psi}_1, \widehat{\phi}) = w(\widehat{\psi}, \widehat{\phi})$. In order this to be one, we would need to normalize $\widehat{\psi}$ by that value. Together with $\widehat{\phi}'(0) < 0$ it follows $\widehat{\psi}(0) > 0$ as required. \square

Lemma 3.8 (Upper bounds of $\widehat{\phi}(x)$ and $\widehat{\psi}(x)$). *Let $\widehat{\psi}$ and $\widehat{\phi}$ be as defined in (3.34) and (3.35). For any positive γ and μ there are finite positive K_1 and K_2 such that*

$$\widehat{\psi}(x) \leq K_1 e^{\gamma x/\mu + \mu x^2} \quad \text{and} \quad \widehat{\phi}(x) \leq K_2 e^{-\gamma x/\mu}. \quad (3.37)$$

Proof: Assume first that $\frac{\gamma^2}{4\mu^3} \neq 1, 2, \dots$. Then two independent solutions of the equation (3.12) are given by (3.32). In view of (3.19), (3.20), (3.16) and (3.17) it is clear that any linear combination of the functions from (3.32) multiplied by $e^{-\gamma x/\mu}$ is bounded on any interval of the form $[0, t]$ for $t < \infty$. Furthermore, as $x \rightarrow \infty$, it grows at most as a constant times $e^{-\gamma x/\mu} e^{p^2(x)} = e^{\gamma^2/\mu^3 + \gamma x/\mu + \mu x^2}$ from which the first part of the assertion follows. If $\frac{\gamma^2}{4\mu^3} = 1, 2, \dots$, then two independent solutions of the equation (3.12) are given by the functions in (3.33). Using similar arguments as above one can show the first part of the assertion also in this case.

The second part of the assertion follows from the definition of $\widehat{\phi}$ and (3.20). \square

We set

$$\widehat{\Phi}(x) = e^{-\mu x^2} \widehat{\phi}(x), \quad (3.38)$$

$$\widehat{\Psi}(x) = e^{-\mu x^2} \widehat{\psi}(x), \quad (3.39)$$

and note that $\widehat{\Phi}$ and $\widehat{\Psi}$ solve the differential equation

$$\frac{1}{2}u''(x) = (\gamma x - \mu)u(x) - \mu x u'(x). \quad (3.40)$$

From (3.37) it follows

$$\widehat{\Psi}(x) \leq K_1 e^{\gamma x/\mu} \quad \text{and} \quad \widehat{\Phi}(x) \leq K_2 e^{-\gamma x/\mu - \mu x^2}. \quad (3.41)$$

Furthermore,

$$\widehat{\phi}(x)\widehat{\Psi}'(x) - \widehat{\psi}(x)\widehat{\Phi}'(x) = \widehat{\phi}(x)\widehat{\psi}'(x) - \widehat{\psi}(x)\widehat{\phi}'(x) = 1 \quad (3.42)$$

and

$$\widehat{\Psi}'(0) = 0. \quad (3.43)$$

Remark 3.9 (Density of the killing position and expected killing time). As in Remark 2.9 in the Brownian case, the density of the position at killing time of the killed Ornstein-Uhlenbeck process starting in x is given by

$$f_x(y) = G(x, y)k(y) = G(x, y)2\gamma y e^{-\mu y^2}. \quad (3.44)$$

The expected killing time is given by

$$\begin{aligned} \mathbb{E}_x[\widehat{\tau}] &= \int_0^\infty G(x, y)m(y) dy = \int_0^\infty G(x, y)2e^{-\mu y^2} dy \\ &= 2 \left(\widehat{\phi}(x) \int_0^x \widehat{\Psi}(y) dy + \widehat{\psi}(x) \int_x^\infty \widehat{\Phi}(y) dy \right). \end{aligned} \quad (3.45)$$

Lemma 3.10 (Exponential moments of the killing position). *For $\alpha < \gamma/\mu$ and any $x \geq 0$ we have*

$$\mathbb{E}_x[e^{\alpha X_{\widehat{\tau}-}}] < \infty. \quad (3.46)$$

Proof: We have

$$\begin{aligned} \mathbb{E}_x[e^{\alpha X_{\widehat{\tau}-}}] &= \int_0^\infty e^{\alpha y} G(x, y) 2\gamma y e^{-2\mu y^2} dy \\ &= 2\gamma \left[\widehat{\phi}(x) \int_0^x y e^{\alpha y} \widehat{\Psi}(y) dy + \widehat{\psi}(x) \int_x^\infty y e^{\alpha y} \widehat{\Phi}(y) dy \right]. \end{aligned}$$

Using (3.37) and (3.41) we see that the term in the brackets is bounded by

$$K_1 K_2 \left[e^{-\gamma x/\mu} \int_0^x y e^{\gamma y/\mu + \alpha y} dy + e^{\gamma x/\mu + \mu x^2} \int_x^\infty y e^{-\gamma y/\mu - \mu y^2 + \alpha y} dy \right]$$

which itself can be bounded by $\widetilde{C}(\gamma, \mu) x e^{\alpha x}$ for some constant $\widetilde{C}(\gamma, \mu)$ independent of x . \square

Lemma 3.11 (Expected killing position starting from x). *There is a positive finite constant \widehat{c} such that for all $x \geq 0$*

$$\mathbb{E}_x[\widehat{Z}_{\widehat{\tau}-}] \leq x + \widehat{c}. \quad (3.47)$$

Proof: We have

$$\begin{aligned} \mathbb{E}_x[\widehat{Z}_{\widehat{\tau}-}] &= 2\gamma \int_0^\infty y^2 G(x, y) e^{-\mu y^2} dy \\ &= \widehat{\phi}(x) \int_0^x 2\gamma y^2 \widehat{\Psi}(y) dy + \widehat{\psi}(x) \int_x^\infty 2\gamma y^2 \widehat{\Phi}(y) dy. \end{aligned}$$

Let us first consider the two integrals. The functions $\widehat{\Phi}$ and $\widehat{\Psi}$ satisfy (3.40), which can be rewritten as

$$2\gamma x u(x) = u''(x) + 2\mu x u'(x) + 2\mu u(x). \quad (3.48)$$

Using this (twice in each computation) and partial integration we obtain

$$\begin{aligned} \int_0^x 2\gamma y^2 \widehat{\Psi}(y) dy &= \int_0^x y \left(\widehat{\Psi}''(y) + 2\mu y \widehat{\Psi}'(y) + 2\mu \widehat{\Psi}(y) \right) dy \\ &= x \widehat{\Psi}'(x) - \widehat{\Psi}(x) + \widehat{\Psi}(0) + 2\mu x^2 \widehat{\Psi}(x) - \frac{\mu}{\gamma} \widehat{\Psi}'(x) - \frac{2\mu^2}{\gamma} x \widehat{\Psi}(x) \end{aligned}$$

and

$$\begin{aligned} \int_x^\infty 2\gamma y^2 \widehat{\Phi}(y) dy &= \int_x^\infty y \left(\widehat{\Phi}''(y) + 2\mu y \widehat{\Phi}'(y) + 2\mu \widehat{\Phi}(y) \right) dy \\ &= -x \widehat{\Phi}'(x) + \widehat{\Phi}(x) - 2\mu x^2 \widehat{\Phi}(x) + \frac{\mu}{\gamma} \widehat{\Phi}'(x) + \frac{2\mu^2}{\gamma} x \widehat{\Phi}(x). \end{aligned}$$

Now from $\widehat{\psi}\widehat{\Phi} = \widehat{\phi}\widehat{\Psi}$, (3.42) and (3.43) it follows

$$\begin{aligned} \mathbb{E}_x[Z_{\widehat{\tau}-}] &= \left(x - \frac{\mu}{\gamma}\right) \left(\widehat{\phi}(x)\widehat{\Psi}'(x) - \widehat{\psi}(x)\widehat{\Phi}'(x)\right) + \widehat{\phi}(x)\widehat{\Psi}(0) + \frac{\mu}{\gamma}\widehat{\phi}(x)\widehat{\Psi}'(0) \\ &= x - \frac{\mu}{\gamma} + \widehat{\phi}(x)\widehat{\Psi}(0) \leq x + \frac{\mu}{\gamma} + \widehat{\phi}(0)\widehat{\Psi}(0) \end{aligned}$$

where the last inequality follows because $\widehat{\phi}$ is decreasing. This concludes the proof. \square

Lemma 3.12 (Second moment of killing time). *For all $x \geq 0$ we have $\mathbb{E}_x[\widehat{\tau}^2] < \infty$.*

Proof: As in the proof of Lemma 2.11 it is enough to show that $\mathbb{E}_0[\widehat{\tau}^2]$ is finite. We have

$$\mathbb{E}_0[\widehat{\tau}^2] = 2 \int_0^\infty G(0, x) m(x) \int_0^\infty G(x, y) m(y) dy dx. \quad (3.49)$$

Using Lemma 3.8 we obtain (recall m in (3.11))

$$\begin{aligned} \int_0^\infty G(x, y) m(y) dy &= \widehat{\phi}(x) \int_0^x \widehat{\psi}(y) m(y) dy + \widehat{\psi}(x) \int_x^\infty \widehat{\phi}(y) m(y) dy \\ &\leq 2\widehat{\phi}(x) K_1 \int_0^x e^{\gamma y/\mu} dy + 2\widehat{\psi}(x) K_2 \int_x^\infty e^{-\gamma y/\mu - \mu y^2} dy \\ &\leq 2\widehat{\phi}(x) K_1 \int_0^x e^{\gamma y/\mu} dy + 2\widehat{\psi}(x) K_2 e^{-\gamma x/\mu} \int_x^\infty e^{-\mu y^2} dy. \end{aligned}$$

Now using an estimate for $x \geq 0$

$$\int_x^\infty e^{-\mu y^2} dy \leq \frac{1}{\mu x + \sqrt{\mu} \sqrt{\mu x^2 + 4/\pi}} e^{-\mu x^2} \leq \frac{\sqrt{\pi}}{2\sqrt{\mu}} e^{-\mu x^2}$$

(this can be deduced e.g. from Abramowitz and Stegun (1992, 7.1.13)), and (3.37) we obtain after some simple calculations

$$\begin{aligned} \int_0^\infty G(x, y) m(y) dy &\leq 2K_1 K_2 (e^{-\gamma x/\mu} \frac{\mu}{\gamma} (e^{\gamma x/\mu} - 1) + \frac{\sqrt{\pi}}{2\sqrt{\mu}} e^{\gamma x/\mu + \mu x^2} e^{-\gamma x/\mu - \mu x^2}) \\ &\leq C(\gamma, \mu), \end{aligned}$$

for suitably chosen constant $C(\gamma, \mu)$ depending on γ and μ but independent of x . Putting this into equation (3.49) we get

$$\mathbb{E}_0[\widehat{\tau}^2] \leq 2C^2(\gamma, \mu)$$

and the proof is completed. \square

3.3. *Invariant Distribution at jump times.* Also for the Ornstein-Uhlenbeck ratchet we consider the Markov chain of increments at jump times.

Definition 3.13 (Markov chain at jump times). Let $(\widehat{\mathcal{X}}, \widehat{\mathcal{R}})$ be a (γ, μ) -Ornstein-Uhlenbeck ratchet with sequence of jump times of $\widehat{\mathcal{R}}$ given by $(\widehat{\tau}_n)_{n \geq 0}$. At jump times we define the Markov chain $(\widehat{\mathcal{Y}}, \widehat{\mathcal{W}}, \widehat{\eta}) := (\widehat{Y}_n, \widehat{W}_n, \widehat{\eta}_n)_{n \geq 1}$ by

$$\widehat{Y}_n = \widehat{X}_{\widehat{\tau}_n} - \widehat{R}_{\widehat{\tau}_n}, \quad \widehat{W}_n = \widehat{R}_{\widehat{\tau}_n} - \widehat{R}_{\widehat{\tau}_{n-1}} \quad \text{and} \quad \widehat{\eta}_n = \widehat{\tau}_n - \widehat{\tau}_{n-1}. \quad (3.50)$$

Now, using the moment bounds in Lemma 3.3, Lemma 3.11 and Lemma 3.12 the next result follows as in the case of the Brownian ratchet.

Proposition 3.14. *There exists a unique invariant distribution of the Markov chain $(\mathcal{Y}, \mathcal{W}, \eta)$.*

Our next aim is to compute the moments under this invariant distribution. First we show that the invariant density of the \widehat{Y} component satisfies a differential equation.

Lemma 3.15 (ODE for the density of invariant distribution). *Let ν be the invariant distribution of $\widehat{Y}_1, \widehat{Y}_2, \dots$ and let \widehat{f}_ν be the corresponding density. Then \widehat{f}_ν is a positive decreasing solution of*

$$\frac{1}{2} \widehat{f}_\nu''(x) = -\mu x \widehat{f}_\nu'(x) + \gamma x \widehat{f}_\nu(x) \quad (3.51)$$

satisfying $\widehat{f}_\nu''(0) = 0$ and $\int_0^\infty \widehat{f}_\nu(x) dx = 1$.

Proof: Similar to the case of killed Brownian motion with negative drift one can write down a recurrence equation for \widehat{f}_ν . We have

$$\widehat{f}_\nu(z) = \int_0^\infty \widehat{f}_\nu(x) \int_z^\infty G(x, u) du dx \quad (3.52)$$

and

$$\widehat{f}_\nu'(z) = 2\gamma \left(-\widehat{\Psi}(z) \int_z^\infty \widehat{f}_\nu(x) \widehat{\phi}(x) dx - \widehat{\Phi}(z) \int_0^z \widehat{f}_\nu(x) \widehat{\psi}(x) dx \right) \quad (3.53)$$

$$\widehat{f}_\nu''(z) = 2\gamma \left(-\widehat{\Psi}'(z) \int_z^\infty \widehat{f}_\nu(x) \widehat{\phi}(x) dx - \widehat{\Phi}'(z) \int_0^z \widehat{f}_\nu(x) \widehat{\psi}(x) dx \right) \quad (3.54)$$

$$\widehat{f}_\nu'''(z) = 2\gamma (x \widehat{f}_\nu(x))' - 2\mu (x \widehat{f}_\nu'(x))'. \quad (3.55)$$

The equations (3.52)–(3.54) are analogous to the case of the Brownian ratchet with negative drift. We only give some details on how we obtain (3.55).

Differentiating (3.54) and using the fact that $\widehat{\Phi}$ and $\widehat{\Psi}$ solve (3.40) we obtain

$$\begin{aligned}
\frac{1}{2\gamma}\widehat{f}_\nu'''(x) &= -\widehat{\Psi}''(x) \int_x^\infty \widehat{f}_\nu(z)\widehat{\phi}(z) dz + \widehat{\Psi}'(x)\widehat{f}_\nu(x)\widehat{\phi}(x) \\
&\quad - \widehat{\Phi}''(x) \int_0^x \widehat{f}_\nu(z)\widehat{\psi}(z) dz - \widehat{\Phi}'(x)\widehat{f}_\nu(x)\widehat{\psi}(x) \\
&= -2((\gamma x - \mu)\widehat{\Psi}(x) - \mu x\widehat{\Psi}'(x)) \int_x^\infty \widehat{f}_\nu(z)\widehat{\phi}(z) dz + \widehat{\Psi}'(x)\widehat{f}_\nu(x)\widehat{\phi}(x) \\
&\quad - 2((\gamma x - \mu)\widehat{\Phi}(x) - \mu x\widehat{\Phi}'(x)) \int_0^x \widehat{f}_\nu(z)\widehat{\psi}(z) dz - \widehat{\Phi}'(x)\widehat{f}_\nu(x)\widehat{\psi}(x) \\
&= \widehat{f}_\nu(x)(\widehat{\Psi}'(x)\widehat{\phi}(x) - \widehat{\Phi}'(x)\widehat{\psi}(x)) \\
&\quad + 2(\gamma x - \mu) \left(-\widehat{\Psi}(x) \int_x^\infty \widehat{f}_\nu(z)\widehat{\phi}(z) dz - \widehat{\Phi}(x) \int_0^x \widehat{f}_\nu(z)\widehat{\psi}(z) dz \right) \\
&\quad + 2\mu x \left(\widehat{\Psi}'(x) \int_x^\infty \widehat{f}_\nu(z)\widehat{\phi}(z) dz + \widehat{\Phi}'(x) \int_0^x \widehat{f}_\nu(z)\widehat{\psi}(z) dz \right).
\end{aligned}$$

Thus, using (3.42), (3.53) and (3.54) we obtain

$$\begin{aligned}
\widehat{f}_\nu'''(x) &= 2\gamma\widehat{f}_\nu(x) + 2(\gamma x - \mu)\widehat{f}_\nu'(x) - 2\mu x\widehat{f}_\nu''(x) \\
&= 2\gamma(x\widehat{f}_\nu'(x) + \widehat{f}_\nu(x)) - 2\mu(x\widehat{f}_\nu''(x) + \widehat{f}_\nu'(x)) \\
&= 2\gamma(x\widehat{f}_\nu(x))' - 2\mu(x\widehat{f}_\nu'(x))',
\end{aligned}$$

which shows (3.55).

Now integrating $\widehat{f}_\nu'''(x)$ we obtain

$$\widehat{f}_\nu''(x) = 2\gamma x\widehat{f}_\nu(x) - 2\mu x\widehat{f}_\nu'(x). \quad (3.56)$$

Here, the integration constant is zero because by (3.54) and (3.43) it follows that $\widehat{f}_\nu''(0) = 0$. By (3.53) $x \mapsto \widehat{f}_\nu(x)$ is decreasing. This concludes the proof. \square

Recall $p(x)$ in (3.31). One can check that the general solution of (3.51) is given by $e^{-\gamma x/\mu - \mu x^2}$ multiplied by a linear combination of

$$M \left(-\frac{\gamma^2}{4\mu^3} + \frac{1}{2}, \frac{1}{2}, p^2(x) \right) \quad \text{and} \quad U \left(-\frac{\gamma^2}{4\mu^3} + \frac{1}{2}, \frac{1}{2}, p^2(x) \right) \quad (3.57)$$

if $\frac{\gamma^2}{4\mu^3} - \frac{1}{2} \neq 1, 2, \dots$, and

$$p(x)M \left(-\frac{\gamma^2}{4\mu^3} + 1, \frac{3}{2}, p^2(x) \right) \quad \text{and} \quad U \left(-\frac{\gamma^2}{4\mu^3} + \frac{1}{2}, \frac{1}{2}, p^2(x) \right) \quad (3.58)$$

if $\frac{\gamma^2}{4\mu^3} - \frac{1}{2} = 1, 2, \dots$. In view of (3.19) the modulus of any solution containing a non-zero proportion of M in the above cases behaves (up to a polynomial factor) as $e^{\gamma x/\mu}$ as $x \rightarrow \infty$ and therefore cannot converge to 0 for $x \rightarrow \infty$. Thus, the density of the invariant distribution of the \widehat{Y} component of the Markov chain $(\widehat{\mathcal{Y}}, \widehat{\mathcal{W}}, \widehat{\eta})$ is given by

$$\widehat{f}_\nu(x) = \frac{1}{\int_0^\infty h_{\mu,\gamma}(y) dy} h_{\mu,\gamma}(x), \quad (3.59)$$

where

$$h_{\mu,\gamma}(x) := e^{-\gamma x/\mu - \mu x^2} U\left(-\frac{\gamma^2}{4\mu^3} + \frac{1}{2}, \frac{1}{2}, p^2(x)\right). \quad (3.60)$$

The following result shows that \widehat{f}_ν defined in (3.59) satisfies conditions stated in Lemma 3.15.

Lemma 3.16. *We have*

- (1) $h_{\mu,\gamma}$ is positive on $[0, \infty)$,
- (2) $h_{\mu,\gamma}''(0) = 0$,
- (3) $\int_0^\infty h_{\mu,\gamma}(x) dx < \infty$,
- (4) $h_{\mu,\gamma}'(x) < 0$ for $x > 0$.

Proof: Let $\gamma, \mu > 0$ be given. Throughout the proof we write h for $h_{\mu,\gamma}$. The assertions (ii) and (iii) hold because h is a solution of (3.51) and because by (3.20) $h(x) \sim e^{-\gamma x/\mu - \mu x^2}$ as $x \rightarrow \infty$.

To show (i) we first observe that by (3.20) $h(x) > 0$ for sufficiently large x . Thus, it is enough to show that h has no positive zeros. By the Kummer transformation (3.21) the function h has positive zeros if and only if the function $x \mapsto U(1 - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x))$ has positive zeros which is not the case as we have seen in the proof of Lemma 3.6.

To show (iv) we show that h is convex, i.e. $h''(x) > 0$ for $x > 0$. Then (iv) follows from (i) and from $h(x) \rightarrow 0$ as $x \rightarrow \infty$. For $x > 0$ we have

$$\begin{aligned} \frac{1}{2x} h''(x) &= -\mu h'(x) + \gamma h(x) \\ &= -\mu \left(\left(-\frac{\gamma}{\mu} - 2\mu x \right) h(x) \right. \\ &\quad \left. - e^{-\gamma x/\mu - \mu x^2} 2\sqrt{\mu} \left(\frac{\gamma}{\mu^{3/2}} + \sqrt{\mu} x \right) \left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3} \right) U\left(\frac{3}{2} - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right) \right) \\ &\quad + \gamma h(x) \\ &= 2(\gamma + \mu^2 x) \left(h(x) + e^{-\gamma x/\mu - \mu x^2} \left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3} \right) U\left(\frac{3}{2} - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right) \right) \\ &= 2(\gamma + \mu^2 x) e^{-\gamma x/\mu - \mu x^2} \left(U\left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x) \right) \right. \\ &\quad \left. + \left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3} \right) U\left(\frac{3}{2} - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right) \right) \\ &= 2(\gamma + \mu^2 x) e^{-\gamma x/\mu - \mu x^2} U\left(\frac{1}{2} - \frac{\gamma^2}{4\mu^3}, \frac{3}{2}, p^2(x) \right), \end{aligned}$$

where the last equality follows from (3.26). Now again by the Kummer transformation h'' is positive on $(0, \infty)$ if and only if the function $x \mapsto U(-\frac{\gamma^2}{4\mu^3}, \frac{1}{2}, p^2(x))$ is positive on that interval. This was shown in Lemma 3.6. \square

3.4. Proof of Theorem 1.2. The proof in the case of the Ornstein-Uhlenbeck ratchet is almost the same as in the case of the Brownian ratchet with negative drift. That is, we can again define a sequence of regeneration times and show that the temporal and spatial increments of the ratchet between this regeneration times have finite

second moments. All the ingredients needed for the proof of that have been provided in the previous subsections. We content ourselves with computation of the speed of the ratchet. To this end we need (as in Proposition 2.15) to compute the expectation of \widehat{Y}_1 and of $\widehat{\eta}_1$ under the invariant distribution ν .

Using (3.51) we obtain

$$\begin{aligned}\mathbb{E}_\nu[\widehat{Y}_1] &= \int_0^\infty x \widehat{f}_\nu(x) dx = \frac{1}{2\gamma} \int_0^\infty \left(\widehat{f}_\nu''(x) + 2\mu x \widehat{f}_\nu'(x) \right) dx \\ &= \frac{1}{2\gamma} \left(-\widehat{f}_\nu'(0) - 2\mu \int_0^\infty \widehat{f}_\nu(x) dx \right) = \frac{1}{2\gamma} \left(-\widehat{f}_\nu'(0) - 2\mu \right).\end{aligned}$$

Furthermore, recalling (3.45), we have using Fubini's Theorem in the second equality and (3.53) in the third

$$\begin{aligned}\mathbb{E}_\nu[\widehat{\eta}_1] &= 2 \int_0^\infty \widehat{f}_\nu(x) \left(\widehat{\phi}(x) \int_0^x \widehat{\Psi}(y) dy + \widehat{\psi}(x) \int_x^\infty \widehat{\Phi}(y) dy \right) dx \\ &= 2 \int_0^\infty \left(\widehat{\Psi}(y) \int_y^\infty \widehat{f}_\nu(x) \widehat{\phi}(x) dx + \widehat{\Phi}(y) \int_0^y \widehat{f}_\nu(x) \widehat{\psi}(x) dx \right) dy \\ &= -\frac{1}{\gamma} \int_0^\infty \widehat{f}_\nu'(y) dy = \frac{\widehat{f}_\nu(0)}{\gamma}.\end{aligned}$$

Now the speed of the Ornstein-Uhlenbeck ratchet is given by

$$\widehat{v}(\mu, \gamma) := \frac{\mathbb{E}_\nu[\widehat{Y}_1]}{\mathbb{E}_\nu[\widehat{\eta}_1]} = -\frac{\widehat{f}_\nu'(0) + 2\mu}{2\widehat{f}_\nu(0)} = -\frac{h'_{\mu,\gamma}(0)}{2h_{\mu,\gamma}(0)} - \frac{\mu \int_0^\infty h_{\mu,\gamma}(x) dx}{h_{\mu,\gamma}(0)}.$$

□

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