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Record process on the Continuum Random Tree

Romain Abraham and Jean-François Delmas

Laboratoire MAPMO, CNRS, UMR 7349, Fédération Denis Poisson, FR 2964, Université d'Orléans, B.P. 6759, 45067 Orléans cedex 2, France.
E-mail address: romain.abraham@univ-orleans.fr
URL: http://www.univ-orleans.fr/mapmo/membres/abraham/

Jean-François Delmas, Université Paris-Est, École des Ponts, CERMICS, 6-8 av. Blaise Pascal, Champs-sur-Marne, 77455 Marne La Vallée, France.
E-mail address: delmas@cermics.enpc.fr
URL: http://cermics.enpc.fr/~delmas/

Abstract. By considering a continuous pruning procedure on Aldous's Brownian tree, we construct a random variable Θ which is distributed, conditionally given the tree, according to the probability law introduced by Janson as the limit distribution of the number of cuts needed to isolate the root in a critical Galton-Watson tree. We also prove that this random variable can be obtained as the a.s. limit of the number of cuts needed to cut down the subtree of the continuum tree spanned by n leaves.

1. Introduction

The problem of randomly cutting a rooted tree arises first in Meir and Moon (1970). Given a rooted tree T_n with n edges, select an edge uniformly at random and delete the subtree not containing the root attached to this edge. On the remaining tree, iterate this procedure until only the edge attached to the root is left. We denote by X_n the number of edge-removals needed to isolate the root. The problem is then to study asymptotics of this random number X_n, depending on the law of the initial tree T_n.

In the original paper, Meir and Moon (1970) considered Cayley trees and obtained asymptotics for the first two moments of X_n. Limits in distribution were then obtained by Panholzer (2006) for some simply generated trees, by Drmota et al. (2009) for random recursive trees, by Holmgren (2010) for binary search trees, by Bertoin (2012) for Cayley trees and by Janson (2006) for conditioned Galton-Watson trees. The main result of Janson (2006) states that, if the offspring

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distribution of the Galton-Watson process is critical (that is with mean equal to 1) with finite variance, which we take equal to 1 for simplicity, then the following convergence in distribution of the conditional laws (specified by their moments) holds:

$$\mathcal{L}(X_n/\sqrt{n}\,|\,T_n/\sqrt{n}) \xrightarrow[n\to+\infty]{(d)} \mathcal{L}(Z_\mathcal{T}\,|\,\mathcal{T}) \tag{1.1}$$

where $\mathcal{T}$ is the so-called continuum random tree (CRT) introduced by Aldous (1991, 1993) and can be seen as the limit in distribution of $T_n/\sqrt{n}$ (see Aldous (1993)). Furthermore, the random variable $Z_\mathcal{T}$ has (unconditional) Rayleigh distribution with density $x\,\mathrm{e}^{-x^2/2}\,\mathbf{1}_{\{x>0\}}$. However, there is no constructive description of $Z_\mathcal{T}$ conditionally on $\mathcal{T}$.

The first goal of the paper is to give a continuous pruning procedure of the CRT that leads to a random variable that is indeed distributed, conditionally given the tree, as $Z_\mathcal{T}$. In order to better understand the intuitive idea of the record process on the CRT, let us first consider the pruning of the simple tree consisting in the segment $[0,1]$ divided into n segments of equal length, rooted at 0. Select an edge at random and discard what is located on the right of this edge. Then chose again an edge at random on the remaining segments and iterate the procedure until the segment attached to 0 is chosen. It is clear that the continuous analogue of this procedure (when the number n of segments tends to $+\infty$) is the so-called stick-breaking scheme: consider a uniform random variable U_1 on $[0,1]$, then conditionally given U_1, consider a uniform random variable U_2 on $[0,U_1]$ and so on. The sequence $(U_n)_{n\geq 0}$ corresponds to the successive cuts of the interval $[0,1]$ in the continuous pruning. Moreover, this sequence can be obtained as the records of a Poisson point process. More precisely, if we consider a Poisson point measure $\sum_{i\in I}\delta_{(x_i,t_i)}$ on $[0,1]\times[0,+\infty)$ with intensity the Lebesgue measure, then the sequence (U_n) is distributed as the sequence of jumps of the record process

$$\theta(x) = \inf\{t_i, x_i \in [0,x]\}.$$

In our case, the limiting object is Aldous's CRT (instead of the segment $[0,1]$). More precisely, we consider a real tree $\mathcal{T}$ associated with the branching mechanism $\psi(u) = \alpha u^2$ under the excursion measure $\mathbb{N}$. This tree is coded by the height process $\sqrt{2/\alpha}B_{ex}$ where B_{ex} is a positive Brownian excursion. This tree is endowed with two measures: the length measure $\ell(dx)$ which corresponds to the Lebesgue measure on the skeleton of the tree, and the mass measure $m^\mathcal{T}(dx)$ which is uniform on the leaves of the tree. Let $\sigma = m^\mathcal{T}(\mathcal{T})$ be the total mass of $\mathcal{T}$. Aldous's CRT corresponds to the distribution of the tree $\mathcal{T}$ conditioned on the total mass $\sigma = 1$, with $\alpha = 1/2$. We then add cut points on $\mathcal{T}$ as above thanks to a Poisson point measure on $\mathcal{T}\times[0,+\infty)$ with intensity

$$\alpha\ell(dx)d\theta$$

in the same spirit as in Aldous and Pitman (1998) (see also Abraham and Serlet (2002) for a direct construction, and Abraham et al. (2010) for the pruning of a general Lévy tree). We denote by (x_i,q_i) the atoms of this point measure, x_i represents the location of the cut point and q_i represents the time at which it appears. For $x\in\mathcal{T}$, we denote by

$$\theta(x) = \inf\{q_i,\ x_i \in [\![\emptyset, x]\!]\}$$

where $[\![\emptyset, x]\!] \subset \mathcal{T}$ denotes the path between x and the root. When a mark appears, we cut the tree on this mark and discard the subtree not containing the root. Then $\theta(x)$ represents the time at which x is separated from the root. Then we define

$$\Theta = \int_{\mathcal{T}} \theta(x) m^{\mathcal{T}}(dx) \qquad \text{and} \qquad Z = \sqrt{\frac{2\alpha}{\sigma}}\Theta.$$

We prove (see Theorem 3.2) that, conditionally on $\mathcal{T}$, Z and $Z_{\mathcal{T}}$ have indeed the same law. The proof of this result relies on another representation of Θ in terms of the mass of the pruned tree (a similar result also appears in Addario-Berry et al. (2012)). More precisely, if we set

$$\sigma_q = \int_{\mathcal{T}} \mathbf{1}_{\{\theta(x)\geq q\}} m^{\mathcal{T}}(dx)$$

the mass of the remaining tree at time q, then we have

$$\Theta = \int_0^{+\infty} \sigma_q \, dq.$$

Using this framework, we can extend in some sense Janson's result by obtaining an a.s. convergence in a special case. We consider, conditionally given $\mathcal{T}$, n leaves uniformly chosen (i.e. sampled according to the mass measure $m^{\mathcal{T}}$) and we denote by T_n the sub-tree of $\mathcal{T}$ spanned by these n leaves and the root. The tree T_n is distributed under $\mathbb{N}[\,\cdot \mid \sigma = 1]$ as a uniform ordered binary tree with n leaves (and hence $2n-1$ edges) with random edge lengths. We denote by T_n^* the tree obtained by removing from T_n the edge attached to the root, and by X_n^* the number of discontinuities of the process $(\theta(x), x \in T_n^*)$. The quantity $X_n^* + 1$ represents the number of cuts needed to reduce the binary tree T_n to a single branch attached to the root. Notice that in our framework, several cuts may appear on the same branch, so X_n^* looks like X_{2n-1} for uniform ordered binary trees but is not exactly the same. Then, we prove in Theorem 4.2 that $\mathbb{N}$-a.e. or $\mathbb{N}[\,\cdot \mid \sigma = 1]$-a.s.:

$$\lim_{n\to+\infty} \frac{X_n^*}{\sqrt{2n}} = Z.$$

This result can be extended by studying the fluctuations of the quantity $X_n^*/\sqrt{2n}$ around its limit, this is the purpose of Hoscheit (2012). In this setting the fluctuations come from the approximation of the record process by its intensity, whereas there is no contribution from the approximation of $\mathcal{T}$ by T_n.

Using the second representation of Θ and results from Abraham et al. (2013a) on the pruning of general Lévy trees, we also derive a.s. asymptotics on the sizes $(\sigma_i, i \in \mathcal{I})$ of the removed sub-trees during the cutting procedure. According to Propositions 4.4 and 4.5, we have $\mathbb{N}$-a.e.

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \sum_{i\in\mathcal{I}} \mathbf{1}_{\{\sigma^i \geq 1/n\}} = \lim_{n\to+\infty} \sqrt{n} \sum_{i\in\mathcal{I}} \sigma^i \mathbf{1}_{\{\sigma^i \leq 1/n\}} = 2\sqrt{\frac{\alpha}{\pi}}\Theta.$$

This result is extended to general Lévy trees in Abraham and Delmas (2013).

The paper is organized as follows. In Section 2, we introduce the frameworks of discrete trees and real trees and define rigorously Aldous's CRT, the mark process and the record process on the tree. Section 3 is devoted to the identification of the law of Θ conditionally given the tree. In Section 4, we prove the a.s. convergence of X_n^* as well as the convergence results on the masses of the removed subtrees.

Finally, we gathered in Section 5 several technical lemmas that are needed in the proofs but are not the heart of the paper.

2. The continuum random tree and the mark process

2.1. *Real trees.* We recall here the definition and basic properties of real trees. We refer to Evans (2008) Saint Flour lectures for more details on the subject.

Definition 2.1. A real tree is a metric space $(\mathcal{T}, d)$ satisfying the following two properties for every $x, y \in \mathcal{T}$:

- (Unique geodesic) There is a unique isometric map $f_{x,y}$ from $[0, d(x, y)]$ into $\mathcal{T}$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x, y)) = y$.
- (No loop) If φ is a continuous injective map from $[0, 1]$ into $\mathcal{T}$ such that $\varphi(0) = x$ and $\varphi(1) = y$, then

$$\varphi([0, 1]) = f_{x,y}([0, d(x, y)]).$$

A rooted real tree is a real tree with a distinguished vertex denoted $\emptyset$ and called the root.

We denote by $[\![x, y]\!] = f_{x,y}([0, d(x, y)])$ the range of the mapping $f_{x,y}$, which is the unique continuous injective path between x and y in the tree, and $[\![x, y[\![= [\![x, y]\!]\setminus\{y\}$. A point $x \in \mathcal{T}$ is said to be a leaf if the set $\mathcal{T} \setminus \{x\}$ remains connected. We denote by $\mathrm{Lf}(\mathcal{T})$ the set of leaves of $\mathcal{T}$. The skeleton of the tree is the set of non-leaves points $\mathcal{T} \setminus \mathrm{Lf}(\mathcal{T})$. As the trace of the Borel σ-field on the skeleton of $\mathcal{T}$ is generated by the intervals $[\![x, y]\!]$, one can define a length measure denoted by $\ell(dx)$ on a real tree by:

$$\ell([\![x, y]\!]) = d(x, y).$$

We will consider here only compact real trees and these trees can be coded by some continuous function (see Le Gall (2006) or Duquesne (2008)). We consider a continuous function $\zeta : [0, +\infty) \to [0, +\infty)$ with compact support $[0, \sigma]$ and such that $\zeta(0) = \zeta(\sigma) = 0$. This function ζ will be called in the following the height function. For every $s, t \geq 0$, we set

$$m_\zeta(s, t) = \inf_{r\in[s\wedge t, s\vee t]} \zeta(r),$$

and

$$d(s, t) = \zeta(s) + \zeta(t) - m_\zeta(s, t).$$

We then define the equivalence relation $s \sim t$ iff $d(s, t) = 0$. We set $\mathcal{T}$ the quotient space

$$\mathcal{T} = [0, +\infty)/\sim .$$

The pseudo-distance d induces a distance on $\mathcal{T}$ and we keep notation d for this distance. We denote by p the canonical projection from $[0, +\infty)$ onto $\mathcal{T}$. The metric space $(\mathcal{T}, d)$ is a compact real tree which can be viewed as a rooted real tree by setting $\emptyset = p(0)$.

On such a compact real tree, we define another measure : the mass measure $m^{\mathcal{T}}$ defined as the push-forward of the Lebesgue measure by the projection p. It is a finite measure supported by the leaves of $\mathcal{T}$ and its total mass is

$$m^{\mathcal{T}}(\mathcal{T}) = \sigma.$$

This coding is very useful to define random real trees. For instance, Aldous's CRT is the random real tree coded by $2B_{ex}$ where B_{ex} denotes a normalized Brownian excursion (i.e. a positive Brownian excursion with duration 1). Here, we will work under the σ-finite measure $\mathbb{N}$ which denotes the law of a real tree coded by an excursion away from 0 of $\sqrt{\frac{2}{\alpha}}|B|$ where $|B|$ is a standard reflected Brownian motion. The tree $\mathcal{T}$ under $\mathbb{N}$ is then the genealogical tree of a continuous state branching process with branching mechanism $\psi(u) = \alpha u^2$ under its canonical measure. In particular, under $\mathbb{N}$, σ has density on $(0, +\infty)$:

$$\frac{dr}{2\sqrt{\alpha\pi}\, r^{3/2}}. \tag{2.1}$$

We keep parameter α in order to stay in the framework of Abraham and Delmas (2012), and give the result in the setting of Aldous's CRT ($\alpha = 1/2$) or of Brownian excursion ($\alpha = 2$).

Using the scaling property of the Brownian motion, there exists a regular version of the measure $\mathbb{N}$ conditioned on the length of the height process ζ. We write $\mathbb{N}^{(r)}$ for the probability measure $\mathbb{N}[\,\cdot\,|\sigma = r]$. In particular, we handle Aldous's CRT if we work under $\mathbb{N}^{(1)}$ with $\alpha = 1/2$.

If $x_1, \dots, x_n \in \mathcal{T}$, we denote by $\mathcal{T}(x_1, \dots, x_n)$ the subtree spanned by $\emptyset, x_1, \dots, x_n$, i.e. the smallest connected subset of $\mathcal{T}$ that contains $x_1, \dots, x_n$ and the root. In other words, we have

$$\mathcal{T}(x_1, \dots, x_n) = \bigcup_{i=1}^{n} [\![\emptyset, x_i]\!].$$

With an abuse of notation, we write for every $t_1, \dots, t_n \geq 0$, $\mathcal{T}(t_1, \dots, t_n)$ for the subtree $\mathcal{T}(p(t_1), \dots, p(t_n))$.

2.2. *The mark process.* We define now a mark process M on the tree $\mathcal{T}$. Conditionally given $\mathcal{T}$, let $M(dx, dq)$ be a Poisson point measure on $\mathcal{T} \times [0, +\infty)$ with intensity $2\alpha\ell(dx)dq$. An atom (x_i, q_i) of this random measure represents a mark on the tree $\mathcal{T}$, x_i is the location of this mark whereas q_i denotes the time at which the mark appears.

Remark 2.2. The coefficient 2α in the intensity is added to have the same intensity as in the pruning procedures of Abraham and Serlet (2002); Abraham et al. (2010, 2013a) but, as we shall see, it does not appear in the law of the number of records.

In fact we will sometimes work with the restriction of M to $\mathcal{T} \times [0, a]$ for some $a > 0$. To simplify the notations, we will always denote by M the mark process (even the restricted one) and will write $\mathbb{M}_a^{\mathcal{T}}$ for the law of M restricted to $\mathcal{T} \times [0, a]$, conditionally given $\mathcal{T}$. We also write $\mathbb{N}_a[d\mathcal{T}\, dM] = \mathbb{N}[d\mathcal{T}]\mathbb{M}_a^{\mathcal{T}}[dM]$, and $\mathbb{N}_a^{(r)}[d\mathcal{T}\, dM] = \mathbb{N}^{(r)}[d\mathcal{T}]\mathbb{M}_a^{\mathcal{T}}[dM]$.

We set for every $q \geq 0$ and $x \in \mathcal{T}$:

$$\theta(x) = \inf\{q > 0,\ M([\![\emptyset, x]\!] \times [0, q]) > 0\} \quad \text{and} \quad \mathcal{T}_q = \{x \in \mathcal{T}; \theta(x) \geq q\}, \tag{2.2}$$

respectively the first time a mark appears between the root and x, and the tree obtained by pruning the original tree at the marks present at time q. We also define the mass of the tree $\mathcal{T}_q$:

$$\sigma_q = m^{\mathcal{T}}(\mathcal{T}_q).$$

According to Abraham et al. (2010), $\mathcal{T}_q$ is distributed under $\mathbb{N}_\infty$ as a Lévy tree with branching mechanism

$$\psi_q(u) = \psi(u+q) - \psi(q) = \alpha u^2 + 2\alpha q u.$$

We will denote by $\mathbb{N}^{\psi_q}$ the distribution of $\mathcal{T}_q$ under $\mathbb{N}$. Moreover, thanks to Girsanov formula (Abraham and Delmas (2012), Lemma 6.2), we have, for every nonnegative Borel function F

$$\mathbb{N}^{\psi_q}[F(\mathcal{T})] = \mathbb{N}[F(\mathcal{T}_q)] = \mathbb{N}\left[F(\mathcal{T})\,\mathrm{e}^{-\alpha q^2 \sigma}\right]. \tag{2.3}$$

With the same abuse of notation as for the spanned subtree, we write for every $t \in \mathbb{R}_+$, $\theta(t)$ instead of $\theta(p(t))$.

2.3. *Discrete trees.* We recall here the definition of a discrete ordered rooted tree according to Neveu's formalism Neveu (1986).

We consider $\mathcal{U} = \displaystyle\bigcup_{n=0}^{+\infty} (\mathbb{N}^*)^n$ the set of finite sequences of positive integers. The empty sequence $\emptyset$ belongs to $\mathcal{U}$. If $u, v \in \mathcal{U}$, we denote by uv the sequence obtained by juxtaposing the sequences u and v.

A discrete ordered rooted tree T is a subset of $\mathcal{U}$ satisfying the three following properties

- $\emptyset \in T$. $\emptyset$ is called the root of T.
- For every $u \in \mathcal{U}$ and $i \in \mathbb{N}^*$, if $ui \in T$ then $u \in T$.
- For every $u \in T$, there exists an integer $k_u(T)$ such that

$$ui \in T \iff 1 \le i \le k_u(T).$$

The integer $k_u(T)$ is the number of offsprings of the vertex u. The leaves of the tree are the $u \in T$ such that $k_u(T) = 0$. We will consider here only binary trees i.e, discrete trees such that $k_u(T) = 0$ or 2.

We can add edge lengths to a discrete tree by considering weighted trees. A weighted tree is defined by a discrete ordered rooted tree T and a weight $h_u \in [0, +\infty)$ for every $u \in T$. The elements $u \in T$ must be viewed as the edges of the tree and h_u is the length of the edge u. Obviously, such a weighted tree can be viewed as a real tree and we will always make the confusion between a discrete weighted tree and the associated real tree.

3. Janson's random variable

Let $\mathcal{T}$ be a compact real tree and let M be a mark process on $\mathcal{T}$. We set

$$\Theta = \int_{\mathcal{T}} \theta(x) m^{\mathcal{T}}(dx).$$

Remark that this can be re-written using the coding by $\Theta = \int_0^\sigma \theta(s)ds$.

Using the tree-valued process $(\mathcal{T}_q, q \ge 0)$, we can give another expression for Θ. Let $(\theta_i, i \in \mathcal{I})$ be the set of jumping times of $(\sigma_q, q \ge 0)$. We set:

$$\mathcal{T}^i = \{x \in \mathcal{T}; \theta(x) = \theta_i\} \quad \text{and} \quad \sigma^i = m^{\mathcal{T}}(\mathcal{T}^i) = \sigma_{\theta_i -} - \sigma_{\theta_i}. \tag{3.1}$$

According to Abraham and Delmas (2012), we have that $\mathbb{M}_\infty^{\mathcal{T}}$-a.s. $\mathcal{T}^i$ is a real tree for all $i \in \mathcal{I}$. Then the following result is straightforward as by definition $\Theta = \sum_{i \in \mathcal{I}} \theta_i \sigma^i$ and $\sigma_q = \sum_{\theta_i \ge q} \sigma^i$.

Proposition 3.1. *We have $\mathbb{M}_\infty^{\mathcal{T}}$-a.s.:*

$$\Theta = \int_0^{+\infty} \sigma_q \, dq.$$

The main result of this section is the following theorem that identifies Θ as Janson's random variable whose distribution is characterized by its moments.

Theorem 3.2. *For every positive integer r, we have*

$$\mathbb{M}_\infty^{\mathcal{T}}[\Theta^r] = \frac{r!}{(2\alpha)^r} \int_{\mathcal{T}^r} \frac{m^{\mathcal{T}}(dx_1)\dots m^{\mathcal{T}}(dx_r)}{\prod_{i=1}^r \ell(\mathcal{T}(x_1,\dots,x_i))}.$$

Proof: Using the expression of Proposition 3.1 for Θ, we have

$$\begin{aligned}
\mathbb{M}_\infty^{\mathcal{T}}[\Theta^r] &= r! \; \mathbb{M}_\infty^{\mathcal{T}} \left[\int_{0 \le q_1 < q_2 < \dots < q_r} dq_1 \dots dq_r \, \sigma_{q_1} \dots \sigma_{q_r} \right] \\
&= r! \, \mathbb{M}_\infty^{\mathcal{T}} \left[\int_{0 \le q_1 < q_2 < \dots < q_r} dq_1 \dots dq_r \prod_{k=1}^r \int_{\mathcal{T}} m^{\mathcal{T}}(dx_k) \mathbf{1}_{\{x_k \in \mathcal{T}_{q_k}\}} \right] \\
&= r! \int_{\mathcal{T}^r} m^{\mathcal{T}}(dx_1) \dots m^{\mathcal{T}}(dx_r) \int_{0 \le q_1 < q_2 < \dots < q_r} dq_1 \dots dq_r \\
&\qquad\qquad \mathbb{M}_\infty^{\mathcal{T}}[x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_r}].
\end{aligned}$$

To evaluate the probability that appears in the last equation, let us remark that, if $y \in \mathcal{T}_q$, then $y \in \mathcal{T}_{q'}$ for every $q' < q$. Therefore, we have

$$\begin{aligned}
&\mathbb{M}_\infty^{\mathcal{T}}[x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_r}] \\
&\qquad = \mathbb{M}_\infty^{\mathcal{T}}[x_2 \in \mathcal{T}_{q_2}, \dots, x_r \in \mathcal{T}_{q_r} \mid x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_1}] \\
&\qquad\qquad \times \mathbb{M}_\infty^{\mathcal{T}}[x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_1}].
\end{aligned}$$

On one hand, we have

$$\begin{aligned}
\mathbb{M}_\infty^{\mathcal{T}}[x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_1}] &= \mathbb{M}_\infty^{\mathcal{T}} \left[M(\mathcal{T}(x_1,\dots,x_r) \times [0,q_1]) = 0 \right] \\
&= \exp\bigl(-2\alpha q_1 \ell(\mathcal{T}(x_1,\dots,x_r))\bigr).
\end{aligned}$$

On the other hand, by standard properties of Poisson point measures, we have

$$\begin{aligned}
&\mathbb{M}_\infty^{\mathcal{T}}[x_2 \in \mathcal{T}_{q_2}, \dots, x_r \in \mathcal{T}_{q_r} \mid x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_1}] \\
&\qquad\qquad = \mathbb{M}_\infty^{\mathcal{T}}[x_2 \in \mathcal{T}_{q_2 - q_1}, \dots, x_r \in \mathcal{T}_{q_r - q_1}].
\end{aligned}$$

We finally obtain by induction, with the convention $q_0 = 0$:

$$\mathbb{M}_\infty^{\mathcal{T}}[x_1 \in \mathcal{T}_{q_1}, \dots, x_r \in \mathcal{T}_{q_r}] = \prod_{k=1}^r \mathrm{e}^{-2\alpha(q_k - q_{k-1})\ell(\mathcal{T}(x_k,\dots,x_r))}.$$

Plugging this expression in the integral gives, after an obvious change of variables

$$\begin{aligned}\mathbb{M}_\infty^{\mathcal{T}}[\Theta^r] &= r! \int_{\mathcal{T}^r} m^{\mathcal{T}}(dx_1)\dots m^{\mathcal{T}}(dx_r) \\ &\qquad \int_{0\leq q_1<\dots<q_r} dq_1\dots dq_r \prod_{k=1}^r \mathrm{e}^{-2\alpha(q_k-q_{k-1})\ell(\mathcal{T}(x_k,\dots,x_r))} \\ &= r! \int_{\mathcal{T}^r} m^{\mathcal{T}}(dx_1)\dots m^{\mathcal{T}}(dx_r) \prod_{k=1}^r \int_0^{+\infty} da_k\, \mathrm{e}^{-2\alpha a_k \ell(\mathcal{T}(x_k,\dots,x_r))} \\ &= \frac{r!}{(2\alpha)^r} \int_{\mathcal{T}^r} \frac{m^{\mathcal{T}}(dx_1)\dots m^{\mathcal{T}}(dx_r)}{\prod_{k=1}^r \ell(\mathcal{T}(x_k,\dots,x_r))}\cdot\end{aligned}$$

□

We can then deduce from the results of Janson (2006), that for $\alpha = 2$, under $\mathbb{N}_\infty^{(1)}$, Θ has Rayleigh distribution. Using then scaling argument in r and α or directly Corollary 5.3 in the Appendix, we get the following result.

Corollary 3.3. *For all $r > 0$, the random variable $Z = \sqrt{\frac{2\alpha}{r}}\Theta$ is distributed under $\mathbb{N}_\infty^{(r)}$ according to a Rayleigh distribution with density $x\,\mathrm{e}^{-x^2/2}\,\mathbf{1}_{\{x\geq 0\}}$.*

In particular, we have easily the first moments of Θ:

$$\mathbb{N}_\infty^{(r)}[\Theta] = \frac{1}{2}\sqrt{\frac{\pi r}{\alpha}} \quad \text{and} \quad \mathbb{N}_\infty^{(r)}\left[\Theta^2\right] = \frac{r}{\alpha}\cdot \tag{3.2}$$

4. A.s. convergence

4.1. *Statement of the main result.* Let $r \geq 0$ and let $\mathcal{T}$ be a tree distributed according to $\mathbb{N}^{(r)}$. Let $(U_1,\dots,U_n)$ be n points uniformly chosen at random on $[0,r]$, independent of $\mathcal{T}$. We denote by T_n the random tree spanned by these n points i.e.

$$T_n = \mathcal{T}(U_1,\dots,U_n)$$

viewed as a discrete ordered weighted tree. Notice that T_n has $2n-1$ edges. Let $(h_1,\dots,h_{2n-1})$ be the lengths of the edges given in lexicographic order. We consider the total length of T_n:

$$L_n = \ell(T_n) = \sum_{k=1}^{2n-1} h_k.$$

We define m_n as the first branching point of T_n, i.e.

$$\bigcap_{k=1}^n [\![\emptyset, p(U_k)]\!] = [\![\emptyset, m_n]\!] \tag{4.1}$$

and we consider the length of the edge of T_n attached to the root

$$h_{\emptyset,n} := d(\emptyset, m_n) = \ell([\![\emptyset, m_n]\!]) = h_1. \tag{4.2}$$

Let T_n^* be the sub-tree of T_n where we remove the edge $[\![\emptyset, m_n[\![$:

$$T_n^* = T_n \setminus [\![\emptyset, m_n[\![,$$

and L_n^* its total length *i.e.* $L_n^* = L_n - h_{\emptyset,n}$.

We set $\theta(x-) = \inf\{\theta(y), y \in [\![\emptyset, x[\![\}$ and X_n^* the number of records on the tree T_n^*:

$$X_n^* = \sum_{x \in T_n^*} \mathbf{1}_{\{\theta(x-) > \theta(x)\}}.$$

Remark 4.1. The introduction of the tree T_n^* is motivated by the fact that the number

$$\sum_{x \in T_n} \mathbf{1}_{\{\theta(x-) > \theta(x)\}}$$

of records on the whole tree is $\mathbb{N}_\infty$-a.e. infinite. Moreover, $X_n^* + 1$ represents the number of cuts that appears on the reduced tree T_n until a mark appears on the branch attached to the root which reduces the tree to a trivial one consisting of the root and a single branch attached to it. Hence it is the analogue of the discrete quantity X_n and is the right quantity to be studied.

We can then state the main result of this section which will be proven in Section 4.5.

Theorem 4.2. *We have that, for all $r > 0$, $\mathbb{N}_\infty^{(r)}$-a.s.:*

$$\lim_{n \to +\infty} \frac{X_n^*}{\sqrt{2n}} = \sqrt{\frac{\alpha}{2r}}\Theta = Z.$$

Remark 4.3. Notice that the binary tree T_n has $2n-1$ vertices; and it corresponds to a critical Galton-Watson tree with reproduction law taking values in $\{0, 2\}$ and with variance 1 conditionally on its number of edges being $2n-1$. This and Theorem 1.6 in Janson (2006) for $\alpha = 1/2$ and $r = 1$, imply that the number of edges with more than one cut is of order less that $\sqrt{n}$.

We deduce from Theorem 4.2 and Corollary 4.9 that for all $r > 0$, $\mathbb{N}_\infty^{(r)}$-a.s.:

$$\lim_{n \to +\infty} \frac{X_n^*}{L_n} = \alpha \frac{\Theta}{\sigma}\cdot$$

In the left hand-side, we have the average of the number of records on T_n^* (as $\ell(T_n^*)$ is of the same order as L_n) and in the right hand-side, the ratio Θ/σ appears as the value of $\theta(U)$ for a leaf chosen uniformly according to the normalized mass measure $m^{\mathcal{T}}/\sigma$ and α is a constant related to the branching mechanism. This result is then natural as intuitively the normalized mass measure is the weak limit of the normalized length measure on T_n.

4.2. *Other a.s. convergence results.* Recall the definition (2.2) of the pruned subtree $\mathcal{T}_q$. Let $\mathbb{T}$ be the set of trees with their mass measure (see Abraham et al. (2013b)). We define the backward filtration $\mathcal{G} = (\mathcal{G}_q, q \geq 0)$ with $\mathcal{G}_q = \sigma(\mathcal{T}_r, r \geq q)$. Following Abraham et al. (2013a), we get that the random measure:

$$\mathcal{N}(d\mathcal{T}', dq) = \sum_{i \in \mathcal{I}} \delta_{\mathcal{T}^i, \theta_i}(d\mathcal{T}', dq)$$

is under $\mathbb{N}_\infty$ a point measure on $\mathbb{T} \times \mathbb{R}$ with intensity:

$$\mathbf{1}_{\{q>0\}} 2\alpha\sigma_q \, \mathbb{N}^q \left[d\mathcal{T}'\right] \, dq.$$

This means that for every non-negative predictable process $(Y(\mathcal{T}',q), q\in\mathbb{R}_+, \mathcal{T}'\in\mathbb{T})$ with respect to the backward filtration $\mathcal{G}$,

$$\mathbb{N}_\infty\left[\int Y(\mathcal{T}',q)\mathcal{N}(d\mathcal{T}',dq)\right] = \mathbb{N}_\infty\left[\int \mathcal{Y}_q\, \mathbf{1}_{\{q>0\}} 2\alpha\sigma_q\, dq\right], \tag{4.3}$$

where $(\mathcal{Y}_q=\int Y(\mathcal{T}',q)\mathbb{N}^q[d\mathcal{T}'], q\in\mathbb{R}_+)$ is predictable with respect to the backward filtration $\mathcal{G}$. We refer to Daley and Vere-Jones (2003, 2008) for the general theory of random point measures.

Recall $\sigma^i = m^{\mathcal{T}}(\mathcal{T}^i)$.

Proposition 4.4. *We have $\mathbb{N}_\infty$-a.e.:*

$$\lim_{n\to+\infty}\frac{1}{\sqrt{n}}\sum_{i\in\mathcal{I}}\mathbf{1}_{\{\sigma^i\geq 1/n\}} = 2\sqrt{\frac{\alpha}{\pi}}\Theta = \sqrt{\frac{2\sigma}{\pi}}\, Z.$$

Proof: Let $K>0$ be large. We consider the $\mathcal{G}$-stopping time $\tau_K=\inf\{q;\sigma_q<K/2\alpha\}$. We define for every $\theta>0$ and every positive integer n,

$$Q_n(\theta)=\sum_{i\in\mathcal{I}}\mathbf{1}_{\{\sigma^i\geq 1/n\}}\mathbf{1}_{\{\theta_i>\theta\}}.$$

We have $Q_n(\tau_K)=\sum_{i\in I}\mathbf{1}_{\{\sigma^i\geq 1/n\}}\mathbf{1}_{\{\sigma_{\theta_i+}<K/2\alpha\}}$ so that:

$$\begin{aligned}
\mathbb{N}_\infty\left[Q_n(\tau_K)\right] &= \mathbb{N}_\infty\left[\int_{\tau_K}^{+\infty} dq\;\sigma_q\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\right]\\
&\leq \mathbb{N}_\infty\left[\int_0^{+\infty} dq\;\min\left(\sigma_q,\frac{K}{2\alpha}\right)\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\right]\\
&= \int_0^{+\infty} dq\;\mathbb{N}\left[\min\left(\sigma,\frac{K}{2\alpha}\right)\mathrm{e}^{-\alpha q^2\sigma}\right]\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\\
&= \frac{1}{4\alpha\pi}\int_0^{+\infty} dq\int_0^{+\infty}\frac{du}{u^{3/2}}\min\left(u,\frac{K}{2\alpha}\right)\mathrm{e}^{-\alpha q^2 u}\int_{1/n}^{+\infty}\frac{dr}{r^{3/2}}\,\mathrm{e}^{-\alpha q^2 r}\\
&= \frac{1}{8\alpha^{3/2}\sqrt{\pi}}\int_{\mathbb{R}_+^2}\frac{du}{u^{3/2}}\,\frac{dr}{r^{3/2}}\,\min\left(u,\frac{K}{2\alpha}\right)\frac{1}{\sqrt{u+r}}\mathbf{1}_{\{r>1/n\}},
\end{aligned}$$

where we used (4.3) for the first equality, Girsanov formula (2.3), and the density (2.1) of the distribution of σ under $\mathbb{N}$. Elementary computations yields there exists a finite constant c which depends on K but not on n such that:

$$\mathbb{N}_\infty\left[Q_n(\tau_K)\right] = \mathbb{N}_\infty\left[\int_{\tau_K}^{+\infty} dq\;\sigma_q\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\right] \leq c\sqrt{n}(1+\log(n)). \tag{4.4}$$

Classical results on random point measures imply that the process $(N_n(\theta\vee\tau_K),\theta\geq 0)$, with:

$$N_n(\theta)=Q_n(\theta)-2\alpha\int_\theta^{+\infty} dq\;\sigma_q\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]$$

is a backward martingale with respect to $\mathcal{G}$. Moreover, since $(Q_n(\theta),\theta\geq 0)$ is a pure jump process with jumps of size 1, the process $(M_n(\theta\vee\tau_K),\theta\geq 0)$, with:

$$M_n(\theta)=N_n(\theta)^2-2\alpha\int_\theta^{+\infty} dq\;\sigma_q\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]$$

is also a backward martingale with respect to $\mathcal{G}$.

Using (4.4), we get that $\mathbb{N}_\infty\left[\left(N_{n^4}(\tau_K)/n^2\right)^2\right]$ is less than a constant times $n^{-3/2}$; therefore

$$\sum_{n=1}^{+\infty}\left(\frac{N_{n^4}(\tau_K)}{n^2}\right)^2$$

is finite in $L^1(\mathbb{N}_\infty)$ and thus is $\mathbb{N}_\infty$-a.e. finite. This implies that $\mathbb{N}_\infty$-a.e.:

$$\lim_{n\to+\infty}\frac{N_{n^4}(\tau_K)}{n^2}=0.$$

Moreover, we have by monotone convergence:

$$\begin{aligned}\frac{2\alpha}{\sqrt{n}}\int_{\tau_K}^{+\infty}dq\ \sigma_q\mathbb{N}\left[\mathbf{1}_{\{\sigma\geq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]&=2\alpha\int_{\tau_K}^{+\infty}dq\ \sigma_q\int_1^{+\infty}\frac{dr}{2\sqrt{\alpha\pi}r^{3/2}}\,\mathrm{e}^{-\alpha q^2\frac{r}{n}}\\&\xrightarrow[n\to\infty]{\mathbb{N}_\infty\text{-a.e.}}\ 2\sqrt{\frac{\alpha}{\pi}}\int_{\tau_K}^{+\infty}dq\sigma_q.\end{aligned}$$

We get that the sequence $(Q_{n^4}(\tau_K)/n^2, n\geq 1)$ converges toward $2\sqrt{\frac{\alpha}{\pi}}\int_{\tau_K}^{+\infty}dq\ \sigma_q$ $\mathbb{N}_\infty$-a.e. . Since $(Q_n(\theta), n\geq 1)$ is non-decreasing, we deduce that $\mathbb{N}_\infty$-a.e.:

$$\lim_{n\to+\infty}\frac{Q_n(\tau_K)}{\sqrt{n}}=2\sqrt{\frac{\alpha}{\pi}}\int_{\tau_K}^{+\infty}dq\ \sigma_q.$$

Since σ is finite $\mathbb{N}_\infty$-a.e., we get that $\mathbb{N}_\infty$-a.e. $\tau_K=0$ for K large enough. This gives the result. □

Proposition 4.5. *We have $\mathbb{N}_\infty$-a.e.:*

$$\lim_{n\to+\infty}\sqrt{n}\sum_{i\in\mathcal{I}}\sigma^i\mathbf{1}_{\{\sigma^i\leq 1/n\}}=2\sqrt{\frac{\alpha}{\pi}}\Theta=\sqrt{\frac{2\sigma}{\pi}}\ Z.$$

Proof: The proof is very similar to the proof of Proposition 4.4. We set:

$$Q_n(\theta)=\sum_{i\in\mathcal{I}}\sigma^i\mathbf{1}_{\{\sigma^i\leq 1/n\}}\mathbf{1}_{\{\theta_i\geq\theta\}}.$$

Mimicking the proof of Proposition 4.4, we have for some finite constant c which depends on K but not on n:

$$\begin{aligned}\mathbb{N}_\infty\left[Q_n(\tau_K)\right]&=\mathbb{N}_\infty\left[\int_{\tau_K}^{+\infty}dq\ \sigma_q\mathbb{N}\left[\sigma\mathbf{1}_{\{\sigma\leq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\right]\\&\leq\frac{1}{8\alpha^{3/2}\sqrt{\pi}}\int_{\mathbb{R}_+^2}\frac{du}{u^{3/2}}\frac{dr}{r^{3/2}}\min\left(u,\frac{K}{2\alpha}\right)\frac{1}{\sqrt{u+r}}\ r\mathbf{1}_{\{r\leq 1/n\}}\\&\leq cn^{-1/2}(1+\log(n))<+\infty,\end{aligned}$$

as well as:

$$\begin{aligned}&\mathbb{N}_\infty\left[\int_{\tau_K}^{+\infty}dq\ \sigma_q\mathbb{N}\left[\sigma^2\mathbf{1}_{\{\sigma\leq 1/n\}}\,\mathrm{e}^{-\alpha q^2\sigma}\right]\right]\\&\qquad\leq\frac{1}{8\alpha^{3/2}\sqrt{\pi}}\int_{\mathbb{R}_+^2}\frac{du}{u^{3/2}}\frac{dr}{r^{3/2}}\min\left(u,\frac{K}{2\alpha}\right)\frac{1}{\sqrt{u+r}}\ r^2\mathbf{1}_{\{r\leq 1/n\}}\\&\qquad\leq cn^{-3/2}(1+\log(n)).\end{aligned}$$

Classical results on random point measures imply that the processes $(N_n(\theta \vee \tau_K), \theta \geq 0)$ and $(M_n(\theta \vee \tau_K), \theta \geq 0)$, with:

$$N_n(\theta) = Q_n(\theta) - 2\alpha \int_\theta^{+\infty} dq\ \sigma_q \mathbb{N}\left[\sigma \mathbf{1}_{\{\sigma \leq 1/n\}}\, \mathrm{e}^{-\alpha q^2 \sigma}\right]$$

$$M_n(\theta) = N_n(\theta)^2 - 2\alpha \int_\theta^{+\infty} dq\ \sigma_q \mathbb{N}\left[\sigma^2 \mathbf{1}_{\{\sigma \leq 1/n\}}\, \mathrm{e}^{-\alpha q^2 \sigma}\right]$$

are backward martingales with respect to $\mathcal{G}$. We get that $\mathbb{N}_\infty\left[\left(n^2 N_{n^4}(\tau_K)\right)^2\right]$ is less than a constant times $n^{-3/2}$. Following the proof of Proposition 4.4, we deduce that $\mathbb{N}_\infty$-a.e. $\lim_{n\to+\infty} n^2 N_{n^4}(\tau_K) = 0$. Furthermore, we have:

$$\begin{aligned} 2\alpha\sqrt{n} \int_{\tau_K}^{+\infty} dq\ \sigma_q \mathbb{N}\left[\sigma \mathbf{1}_{\{\sigma \leq 1/n\}}\, \mathrm{e}^{-\alpha q^2 \sigma}\right] &= 2\alpha\sqrt{n} \int_{\tau_K}^{+\infty} dq\ \sigma_q \int_0^{\frac{1}{n}} \frac{dr}{2\sqrt{\alpha\pi r}}\, \mathrm{e}^{-\alpha q^2 r} \\ &= 2\alpha \int_{\tau_K}^{+\infty} dq\ \sigma_q \int_0^1 \frac{dr}{2\sqrt{\alpha\pi r}}\, \mathrm{e}^{-\alpha q^2 \frac{r}{n}} \\ &\to 2\sqrt{\frac{\alpha}{\pi}} \int_{\tau_K}^{+\infty} dq\ \sigma_q. \end{aligned}$$

We conclude the proof as in the proof of Proposition 4.4. □

4.3. *The record process on the real half-line.* We consider here the half-line $[0, +\infty)$ instead of a real-tree $\mathcal{T}$ (the half-line is in fact a real tree that we supposed rooted at 0). We define the mark process M under $\mathbb{M}_a$ (we omit the $\mathcal{T} = [0, +\infty)$ in the notation), it is a Poisson point measure on $[0, +\infty)^2$ with intensity $2\alpha \mathbf{1}_{\{x\geq 0,\ 0\leq q\leq a\}} dx\, dq$ and we set for every $x \geq 0$

$$\theta(x) = \min(a, \inf\{q_i; x_i \leq x\}) \quad \text{and} \quad X(x) = X(0) + \sum_{0<y\leq x} \mathbf{1}_{\{\theta(y-)>\theta(y)\}}.$$

Remark 4.6. Let us denote by $1 \geq x_1 > x_2 > \cdots$ the jumping times of the process $(\theta(x), 0 \leq x \leq 1)$ under $\mathbb{M}_\infty$. By standard arguments on Poisson point measure, the random variable x_1 is uniformly distributed on $[0, 1]$. Conditionally given x_1, the random variable x_2 is uniformly distributed on $[0, x_1]$ and so on. We are thus considering the standard stick breaking scheme and the random vector $(1 - x_1, x_1 - x_2, \ldots)$ is distributed according to the Poisson-Dirichlet distribution with parameter $(0, 1)$.

For fixed x, $\theta(x)$ represents the first time a mark arrives between x and 0 (if it arrives before time a that is if $\theta(x) < a$); and $X(x) - X(0)$ denotes the number of (decreasing) records of the process $(\theta(u),\ u \in [0, x])$. It is also the number of cuts that appear between x and 0 in the stick-breaking scheme before time a.

By construction θ and (θ, X) are Markov processes. Notice that θ is non-increasing and X is non-decreasing, and $\mathbb{M}_\infty$-a.s. $X(x) = +\infty$ for every $x > 0$.

As most of our further proofs will be based on martingale arguments, let us first compute the infinitesimal generator of the former Markov processes. Notice first that $\inf\{q_i; x_i \leq x\}$ is distributed under $\mathbb{M}_\infty$ as an exponential random variable with parameter $2\alpha x$. Let g be a bounded measurable function defined on $[0, +\infty]$. For every $q \in [0, +\infty]$ and $x > 0$, we have

$$\mathbb{M}_q[g(\theta(x))] = \mathbb{M}_\infty[g(\min(q, Y_x))] = \mathrm{e}^{-2\alpha q x}\, g(q) + \int_0^q g(u)\, 2\alpha x\, \mathrm{e}^{-2\alpha x u}\ du,$$

where Y_x is exponentially distributed with parameter $2\alpha x$. Notice that if g belongs to $\mathcal{C}^1(\mathbb{R}^+)$ with g' bounded on $\mathbb{R}_+$, we have by an obvious integration by parts that, for $q \in [0,+\infty]$ and $x > 0$,

$$\mathbb{M}_q[g(\theta(x))] = g(0) + \int_0^q g'(u)\ \mathrm{e}^{-2\alpha x u}\ du.$$

We can then compute the infinitesimal generator of θ denoted by $\mathcal{L}$. Let g be a bounded measurable function defined on $[0,+\infty]$ such that $g - g(+\infty)$ is integrable with respect to the Lebesgue measure on $\mathbb{R}^+$. For $q \in [0,+\infty]$, we have:

$$\begin{aligned}\mathcal{L}(g)(q) &= \lim_{x\to 0} \frac{\mathbb{M}_q[g(\theta(x))] - g(q)}{x} \\ &= \lim_{x\to 0} -g(q)\frac{1-\mathrm{e}^{-2\alpha q x}}{x} + \int_0^q 2\alpha g(u)\,\mathrm{e}^{-2\alpha x u}\ du \\ &= 2\alpha \int_0^q (g(u) - g(q))\ du.\end{aligned}$$

Therefore, we get that the process $M^g = (M^g_x, x \geq 0)$ is a martingale under $\mathbb{M}_q$, where M^g is defined by:

$$M^g_x = g(\theta(x)) + 2\alpha \int_0^x dy \int_0^{\theta(y)} \Big(g(\theta(u)) - g(y)\Big)\ du. \tag{4.5}$$

Remark 4.7. If furthermore g belongs to $\mathcal{C}^1(\mathbb{R}^+)$ and if $x \mapsto x g'(x)$ is integrable with respect to the Lebesgue measure on $\mathbb{R}^+$, then we have for $q \in [0,+\infty]$:

$$\mathcal{L}(g)(q) = -2\alpha \int_0^q x g'(x)\ dx.$$

Similarly, we can also compute the infinitesimal generator of (θ, X), which we still denote by $\mathcal{L}$. This quantity is of interest only for $\theta(0)$ finite. Let g be a bounded measurable function defined on $\mathbb{R}^+ \times \mathbb{N}$. For $(q,k) \in \mathbb{R}^+ \times \mathbb{N}$, we denote by $\mathbb{M}_{(q,k)}$ the law of the process (θ, X) starting from (q,k). Standard computations on birth and death processes yield that for $(q,k) \in \mathbb{R}^+ \times \mathbb{N}$:

$$\begin{aligned}\mathcal{L}(g)(q,k) &= \lim_{x\to 0} \frac{\mathbb{M}_{(q,k)}[g(\theta(x), X(x))] - g(q,k)}{x} \\ &= \lim_{x\to 0} -g(q,k)\frac{1-\mathrm{e}^{-2\alpha q x}}{x} + \int_0^q 2\alpha g(u,k+1)\ \mathrm{e}^{-2\alpha x u}\ du + o(1) \\ &= 2\alpha \int_0^q (g(u,k+1) - g(q,k))\ du.\end{aligned}$$

In that case, we get that the process $M^g = (M^g_x, x \geq 0)$ defined by:

$$M^g_x = g(\theta(x), X(x)) - 2\alpha \int_0^x dy \int_0^{\theta(y)} \Big(g(u, X(y)+1) - g(\theta(y), X(y))\Big)\ du, \tag{4.6}$$

is a bounded martingale under $\mathbb{M}_{(q,k)}$.

Finally, let us exhibit some martingales associated with the process X which show that this process can be viewed as a Poisson process with stochastic intensity $2\alpha\theta(u)du$. Let $n \in \mathbb{N}$. Taking $g(q,k) = k \wedge n$ in (4.6), we deduce that the process $N^{(n)} = (N^{(n)}_x, x \geq 0)$ defined for $x \geq 0$ by:

$$N^{(n)}_x = X(x) \wedge n - 2\alpha \int_0^x \theta(u) \mathbf{1}_{\{X(u)<n\}}\ du$$

is a bounded martingale under $\mathbb{M}_{(q,k)}$ (for $q < +\infty$). Notice that for $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$, we have:

$$\begin{aligned}\mathbb{M}_{(q,k)}[|N_x^{(n)}|] &\leq \mathbb{M}_{(q,k)}[X(x) \wedge n] + 2\alpha \int_0^x \mathbb{E}_{(q,k)}[\theta(u)]\, du \\ &= k \wedge n + 2\alpha \int_0^x \mathbb{E}_{(q,k)}[\theta(u)\mathbf{1}_{\{X(u)<n\}}]\, du + 2\alpha \int_0^x \mathbb{E}_{(q,k)}[\theta(u)]\, du \\ &\leq k + 4\alpha q x,\end{aligned}$$

where we used that X is non-negative in the first equality, that $N^{(n)}$ is a martingale in the second one, and that θ is non-increasing in the last one. As $(N^{(n)}, n \in \mathbb{N})$ converges a.s. to the process $N = (N_x, x \geq 0)$ defined for $x \in \mathbb{R}^+$ by:

$$N_x = X(x) - 2\alpha \int_0^x \theta(u)\, du, \tag{4.7}$$

we deduce that N is a martingale under $\mathbb{M}_{(q,k)}$ for every $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$.

By taking $g(q, k) = k^2$ in (4.6) and using elementary stochastic calculus and similar arguments as above, we also get that the process $M = (M_x, x \geq q0)$ defined for $x \geq 0$ by:

$$M_x = N_x^2 - 2\alpha \int_0^x \theta(u)\, du \tag{4.8}$$

is a martingale under $\mathbb{M}_{(q,k)}$ for every $(q, k) \in \mathbb{R}^+ \times \mathbb{N}$.

4.4. *Sub-tree spanned by n leaves.* We recall here some properties of the sub-tree spanned by n leaves uniformly chosen.

We first recall the density of $(h_1, \ldots, h_{2n-1})$ under $\mathbb{N}^{(r)}$, see Aldous (1993) or Pitman (2006) (Theorem 7.9), see also Duquesne and Le Gall (2005). We denote by L_n the total length of T_n:

$$L_n = \sum_{k=1}^{2n-1} h_k.$$

Lemma 4.8. *Under $\mathbb{N}^{(r)}$, $(h_1, \ldots, h_{2n-1})$ has density:*

$$f_n^{(r)}(h_1, \ldots, h_{2n-1}) = 2\, \frac{(2n-2)!}{(n-1)!}\, \frac{\alpha^n}{r^n}\, L_n\, \mathrm{e}^{-\alpha L_n^2/r}\, \mathbf{1}_{\{h_1>0,\ldots,h_{2n-1}>0\}}.$$

The random variable L_n^2, is distributed under $\mathbb{N}^{(r)}$ as $r\Gamma_n/\alpha$ where Γ_n is a $\gamma(1, n)$ random variable with density $\mathbf{1}_{\{x>0\}}\, x^{n-1}\, \mathrm{e}^{-x} / (n-1)!$.

Corollary 4.9. *We have that $\mathbb{N}^{(r)}$-a.s.*

$$\lim_{n\to+\infty} L_n/\sqrt{n} = \sqrt{r/\alpha}.$$

Proof: Using Lemma 4.8, we compute

$$\mathbb{N}^{(r)}\left[\sum_{n=1}^{+\infty} \left(\frac{L_n^2}{n} - \frac{r}{\alpha}\right)^4\right] = \frac{r}{\alpha} \sum_{n=1}^{+\infty} \mathbb{E}\left[\left(\frac{\Gamma_n}{n} - 1\right)^4\right] = \frac{r}{\alpha} \sum_{n=1}^{+\infty} \frac{1}{n^2}\left(3 + \frac{1}{n}\right) < +\infty.$$

This implies that $\mathbb{N}^{(r)}$-a.s. $\sum_{n=1}^{+\infty} \left(\frac{L_n^2}{n} - \frac{r}{\alpha}\right)^4$ is finite which proves the corollary. $\square$

We end this section by studying the edge attached to the root defined in (4.1) whose length is denoted $h_{\emptyset,n}$, see (4.2).

Proposition 4.10. *The sequence* $(\sqrt{n}h_{\emptyset,n}, n \geq 1)$ *converges in distribution under* $\mathbb{N}^{(r)}$ *to* $\sqrt{r/\alpha}\, E_1/2$, *where* E_1 *is an exponential random variable with mean 1.*

Proof: Let $k \in (-1, +\infty)$. We set $H_k = (\alpha/r)^{k/2}\mathbb{N}^{(r)}[h^k_{\emptyset,n}]$. We have using Lemma 4.8,

$$H_k = 2\frac{(2n-2)!}{(n-1)!}\frac{\alpha^{n+k/2}}{r^{n+k/2}}\int_{\mathbb{R}_+^{2n-1}} dh_1 \dots dh_{2n-1} h_1^k\, L_n\, \mathrm{e}^{-\alpha L_n^2/r}\,.$$

Consider the change of variables:

$$u_1 = \sqrt{\frac{\alpha}{r}}h_1,\ \cdots,\ u_{2n-2} = \sqrt{\frac{\alpha}{r}}h_{2n-2},\ x = \sqrt{\frac{\alpha}{r}}L_n,$$

with Jacobian equal to $\left(\frac{\alpha}{r}\right)^{n-\frac{1}{2}}$.We get:

$$\begin{aligned}
H_k &= 2\frac{(2n-2)!}{(n-1)!}\frac{\alpha^{n+k/2}}{r^{n+k/2}}\int_{\mathbb{R}_+^{2n-1}} \left(\frac{r}{\alpha}\right)^{k/2} u_1^k \left(\frac{r}{\alpha}\right)^{1/2} x\, \mathrm{e}^{-\alpha x^2/r} \\
&\qquad\qquad \mathbf{1}_{\{u_1+\cdots+u_{2n-2}\leq x\}} \left(\frac{r}{\alpha}\right)^{n-\frac{1}{2}} du_1 \cdots du_{2n-2}\, dx \\
&= 2\frac{(2n-2)!}{(n-1)!}\int\int_{\mathbb{R}_+^2} du_1\, dx\, \mathbf{1}_{\{u_1\leq x\}}\, u_1^k x\, \mathrm{e}^{-x^2} \\
&\qquad\qquad \int\int_{\mathbb{R}_+^{2n-3}} du_2 \dots du_{2n-2}\mathbf{1}_{\{u_2+\cdots+u_{2n-2}\leq x-u_1\}} \\
&= 2\frac{(2n-2)!}{(n-1)!}\frac{1}{(2n-3)!}\int_0^{+\infty} dx\, x\, \mathrm{e}^{-x^2}\int_0^x dh\, h^k (x-h)^{2n-3}.
\end{aligned}$$

Set $y = x^2$, to get:

$$\begin{aligned}
H_k &= 2\ \frac{(2n-2)!}{(n-1)!}\ \frac{1}{(2n-3)!}\beta(k+1, 2n-2)\int_0^{+\infty} dx\ x^{2n+k-1}\, \mathrm{e}^{-x^2} \\
&= \frac{(2n-2)!}{(n-1)!}\ \frac{1}{(2n-3)!}\beta(k+1, 2n-2)\int_0^{+\infty} dy\ y^{n+\frac{k}{2}-1}\, \mathrm{e}^{-r} \\
&= \frac{(2n-2)!}{(n-1)!}\ \frac{1}{(2n-3)!}\ \frac{\Gamma(k+1)(2n-3)!}{\Gamma(2n+k-1)}\ \Gamma(n+\frac{k}{2}) \\
&= \frac{\Gamma(k+1)}{2^k}\ \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+\frac{k}{2}-\frac{1}{2})},
\end{aligned}$$

where, for the last equality, we used twice the duplication formula:

$$\frac{\Gamma(2n-1)}{\Gamma(n)} = \frac{2^{2n-2}\Gamma(n-1/2)}{\sqrt{\pi}}. \tag{4.9}$$

We observe that $\lim_{n\to+\infty}\mathbb{N}^{(r)}[n^{k/2}h^k_{\emptyset,n}] = \frac{k!}{2^k}\left(\frac{r}{\alpha}\right)^{k/2} = \mathbb{E}[(\sqrt{r}E_1/(2\sqrt{\alpha}))^k]$. This gives the result, as the exponential distribution is characterized by its moments. $\square$

From the proof of Proposition 4.10, we also get the following result.

Lemma 4.11. *For all* $k \in (-1, +\infty)$, *we have, when* n *goes to infinity:*

$$\mathbb{N}^{(r)}[h^k_{\emptyset,n}] = \left(\frac{r}{\alpha}\right)^{k/2}\frac{\Gamma(k+1)}{2^k}\ \frac{\Gamma(n-\frac{1}{2})}{\Gamma(n+\frac{k}{2}-\frac{1}{2})} \sim (r/\alpha)^{k/2}n^{-k/2}2^{-k}\Gamma(k+1).$$

4.5. *Proof of Theorem 4.2.* We first want to show that, as for a standard Poisson process, the record counting process on each branch behaves like its (stochastic) intensity when the number of jumps tends to $+\infty$.

In other words, we set:

$$\Delta_n = \frac{X_n^*}{\sqrt{n}} - \frac{2\alpha}{\sqrt{n}} \int_{T_n^*} \theta(x)\, \ell(dx)$$

and we want to prove that Δ_n tends *a.s.* to 0 (at least along some subsequence).

Using the martingale of Equation (4.8), we have that:

$$\mathbb{N}_\infty^{(r)}\left[\Delta_n^2 \,\middle|\, T_n\right] = \frac{2\alpha}{\sqrt{n}} \mathbb{N}_\infty^{(r)}\left[\frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x)\, \ell(dx) \,\middle|\, T_n\right]. \tag{4.10}$$

Then we use the following lemma whose proof is postponed to Section 4.6.

Lemma 4.12. *Let $r > 0$. There exists a non-negative sequence $(R'_n, n \geq 1)$ of random variables adapted to the the filtration $(\sigma(T_n), n \geq 1)$ and which converges $\mathbb{N}_\infty^{(r)}$-a.s. to 0 such that, for all $n \geq 1$, $\mathbb{N}^{(r)}$-a.s.:*

$$r\mathbb{N}_\infty^{(r)}\left[\frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x)\, \ell(dx) \,\middle|\, T_n\right] \leq \frac{L_n}{\sqrt{n}} \mathbb{N}_\infty^{(r)}\left[\Theta \,\middle|\, T_n\right] + R'_n. \tag{4.11}$$

With this lemma, we have:

$$\begin{aligned}
\mathbb{N}_\infty^{(r)}\left[\sum_{n\geq 1} \Delta_{n^4}^2 \mathbf{1}_{\{R'_{n^4}\leq 1\}}\right] &= \sum_{n\geq 1} \mathbb{N}_\infty^{(r)}\left[\mathbb{N}_\infty^{(r)}\left[\Delta_{n^4}^2 \middle|\, T_{n^4}\right] \mathbf{1}_{\{R'_{n^4}\leq 1\}}\right] \\
&\leq \sum_{n\geq 1} \frac{2\alpha}{n^2 r} \mathbb{N}_\infty^{(r)}\left[\left(\frac{L_{n^4}}{n^2}\mathbb{N}_\infty^{(r)}\left[\Theta \,\middle|\, T_{n^4}\right] + R'_{n^4}\right) \mathbf{1}_{\{R'_{n^4}\leq 1\}}\right] \\
&\leq \sum_{n\geq 1} \frac{2\alpha}{n^2 r}\left(\frac{1}{n^2}\mathbb{N}_\infty^{(r)}\left[L_{n^4}^2\right]^{1/2} \mathbb{N}_\infty^{(r)}\left[\Theta^2\right]^{1/2} + 1\right) \\
&< +\infty,
\end{aligned}$$

where we used (4.10) and (4.11) for the first inequality, Cauchy-Schwartz inequality for the second one, and Lemma 4.8 as well as (3.2) for the last one. This result implies that $\mathbb{N}_\infty^{(r)}$-a.s. $\lim_{n\to+\infty} \Delta_{n^4}\mathbf{1}_{\{R'_{n^4}\leq 1\}} = 0$ and thus $\mathbb{N}_\infty^{(r)}$-a.s. $\lim_{n\to+\infty} \Delta_{n^4} = 0$ as the sequence $(R'_n, n \geq 1)$ converges $\mathbb{N}^{(r)}$-a.s. to 0.

In order to conclude, it remains to study the asymptotic behavior of

$$\frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x)\, \ell(dx)$$

which is the purpose of the next proposition which will also be proven in Section 4.6.

Proposition 4.13. *We have that, for all $r > 0$, $\mathbb{N}_\infty^{(r)}$-a.s.:*

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \int_{T_n^*} \theta(x)\, \ell(dx) = \frac{1}{\sqrt{r\alpha}}\, \Theta. \tag{4.12}$$

We deduce from (4.12), that $\mathbb{N}_\infty^{(r)}$-a.s. the sequence $(X^*_{n^4}/n^2, n \geq 1)$ converges to $2\sqrt{\frac{r}{\alpha}}\Theta$. Then using that $(X^*_n, n \geq 1)$ is increasing, we get for $k \in \mathbb{N}$, such that $n^4 < k \leq (n+1)^4$, that:

$$\frac{n^2}{(n+1)^2}\,\frac{X^*_{n^4}}{n^2} \leq \frac{X^*_k}{\sqrt{k}} \leq \frac{(n+1)^2}{n^2}\,\frac{X^*_{(n+1)^4}}{(n+1)^2}\cdot$$

Thus, we get that $\mathbb{N}_\infty^{(r)}$-a.s. the sequence $(X^*_k/\sqrt{k}, k \geq 1)$ converges to $2\sqrt{\frac{\alpha}{r}}\Theta$.

4.6. *Proof of Proposition 4.13 and Lemma 4.12.* First, let us remark that, as $L_n/\sqrt{n}$ tends $\mathbb{N}_\infty^{(r)}$-a.s. to $\sqrt{r/\alpha}$ by Corollary 4.9 and is $\sigma(T_n)$-measurable, it suffices to study the limit of

$$\frac{1}{L_n}\int_{T_n^*} \theta(x)\,\ell(dx).$$

Let us exhibit a martingale that converges to Θ. Let $\mathcal{F}_n$ be the σ-field generated by T_n and $(\theta(x), x \in T_n)$. The filtration $(\mathcal{F}_n, n \geq 1)$ is increasing towards $\vee_{n\geq 1}\mathcal{F}_n = \mathcal{F}$, the σ-field generated b y $\mathcal{T}$ and $(\theta(s), s \in [0,\sigma]) = (\theta(x), x \in \mathcal{T})$.

We consider the process $(M_n, n \geq 1)$ defined by, for $q \in [0, +\infty]$:

$$M_n = \mathbb{N}_q^{(r)}\left[\Theta \;\middle|\; \mathcal{F}_n\right].$$

Thanks to (3.2), we get that:

$$\mathbb{N}_q^{(r)}[M_n^2] \leq \mathbb{N}_q^{(r)}\left[\Theta^2\right] \leq \mathbb{N}_\infty^{(r)}\left[\Theta^2\right] = \frac{r}{\alpha}\cdot$$

Therefore $(M_n, n \geq 1)$ is (a well defined) square integrable non-negative martingale. In particular it converges $\mathbb{N}_q^{(r)}$-a.s. (and in $L^2(\mathbb{N}_q^{(r)})$) to Θ as the increasing σ-fields $\mathcal{F}_n$ increase to $\mathcal{F}$.

In the next lemma whose proof is given in Section 4.7, we compare

$$\frac{1}{L_n}\int_{T_n^*} \theta(x)\,\ell(dx)$$

to M_n.

Lemma 4.14. *We have, for $n \geq 1$,*

$$-R_n \leq M_n - \frac{r}{L_n}\int_{T_n^*} \theta(x)\,\ell(dx) \leq V_n, \tag{4.13}$$

where $(R_n, n \geq 1)$ and $(V_n, n \geq 1)$ are non-negative sequences which converge $\mathbb{N}_\infty^{(r)}$-a.s. to 0. Furthermore the non-negative sequence $(R'_n, n \geq 1)$, with $R'_n = \mathbb{N}_\infty^{(r)}[R_n|T_n]\,L_n/\sqrt{n}$, converges $\mathbb{N}_\infty^{(r)}$-a.s. to 0.

This lemma ends the proof of Proposition 4.13. Moreover, as $\mathbb{N}_\infty^{(r)}[M_n \mid T_n] = \mathbb{N}_\infty^{(r)}[\Theta \mid T_n]$, it also proves Lemma 4.12.

4.7. *Proof of Lemma 4.14.* In order to first give a description of the marked tree conditionally on $\mathcal{F}_n$, we consider the sub-trees that are grafted on T_n. For $x, y \in \mathcal{T}$, we define an equivalence relation by setting

$$x \sim_{T_n} y \iff [\![\emptyset, x]\!] \cap T_n = [\![\emptyset, y]\!] \cap T_n$$

and we set $(\mathcal{T}_i, i \in I_n)$ for the different equivalent classes. The set $\mathcal{T}_i$ can be viewed as a rooted real tree with root $x_i = \mathcal{T}_i \cap T_n$. Notice that x_i represents the point of

T_n at which the tree $\mathcal{T}_i$ is grafted on T_n. Finally, we set $\theta_i = \theta(x_i)$ and $\sigma_i = m^{\mathcal{T}}(\mathcal{T}_i)$ which corresponds to the length of the height process of $\mathcal{T}_i$.

Using Theorem 3 of Le Gall (1993) (combined with the spatial motion θ), we get the following result.

Lemma 4.15. *Under $\mathbb{N}_q$ conditionally on $\mathcal{F}_n$, the point measure*

$$\sum_{i\in I_n} \delta_{(\mathcal{T}_i,\theta_i,x_i)}(d\mathcal{T}, dq', dx)$$

is a Poisson point measure with intensity

$$2\alpha \mathbf{1}_{T_n}(x)\ell(dx)\ \mathbb{N}[d\mathcal{T}]\ \delta_{\theta(x)}(dq').$$

We deduce from that Lemma the next result.

Lemma 4.16. *Under $\mathbb{N}_q^{(r)}$ and conditionally on $\mathcal{F}_n$, the point measure*

$$\mathcal{N}_n(d\sigma, dq', dx) = \sum_{i\in I_n} \delta_{(\sigma_i,\theta_i,x_i)}(d\sigma, dq', dx)$$

is distributed as a Poisson point measure:

$$\tilde{\mathcal{N}}(d\sigma, dq', dx) = \sum_{j\in J} \delta_{(\tilde{\sigma}_j,\theta_j,x_j)}(d\sigma, dq', dx)$$

with intensity

$$2\alpha \mathbf{1}_{T_n}(x)\ell(dx)\ \frac{d\sigma}{2\sqrt{\alpha\pi}\ \sigma^{3/2}}\mathbf{1}_{\{\sigma>0\}}\ \delta_{\theta(x)}(dq')$$

conditioned on $\{\sum_{j\in J}\tilde{\sigma}_j = r\}$.

We can compute some elementary functionals of $\mathcal{N}_n$.

Lemma 4.17. *Under $\mathbb{N}_q^{(r)}$ and conditionally on $\mathcal{F}_n$, the point measure $\mathcal{N}_n$ has intensity:*

$$2\alpha \mathbf{1}_{T_n}(x)\ell(dx)\ \mathbb{E}^{(r),L_n}[d\sigma]\ \delta_{\theta(x)}(dq),$$

where $\mathbb{E}^{(r),L_n}$ satisfies, for any non-negative measurable function F:

$$2\alpha \int_{T_n} \ell(dx)\ \mathbb{E}^{(r),L_n}[F(x,\sigma)] = \mathbb{E}\left[\sum_{j\in J} F(s_j,\tilde{\sigma}_j)\ \Big|\ \sum_{j\in J}\tilde{\sigma}_j = r\right].$$

We also have:

$$\mathbb{E}^{(r),L_n}[\sigma] = \frac{r}{2\alpha L_n} \quad \text{and} \quad \mathbb{E}^{(r),L_n}[\sigma^{3/2}] \leq \frac{2}{\sqrt{\alpha\pi}}\,\frac{1}{L_n}\, r^2\, \mathrm{e}^{-\alpha L_n^2/r}. \tag{4.14}$$

Proof: The first part of the Lemma is a consequence of the exchangeability of $(\sigma_i, i\in I_n)$. With $F(q,r') = r'$, we get:

$$2\alpha L_n \mathbb{E}^{(r),L_n}[\sigma] = 2\alpha \int_{T_n} \ell(dx)\ \mathbb{E}^{(r),L_n}[\sigma] = \mathbb{E}\left[\sum_{j\in J}\tilde{\sigma}_j \mid \sum_{j\in J}\tilde{\sigma}_j = r\right] = r.$$

This gives the first equality of (4.14). Recall that:

$$\mathbb{N}\left[1-\mathrm{e}^{-\mu\sigma}\right] = \int_0^\infty \frac{dr}{2\sqrt{\alpha\pi}\ r^{3/2}}\ \left(1-\mathrm{e}^{-\mu r}\right) = \sqrt{\mu/\alpha}.$$

We have, using the Palm formula for Poisson point measures, for $a > 1/2$:

$$\begin{aligned}\mathbb{E}\left[\sum_{j\in J}\tilde{\sigma}_j^a\,\mathrm{e}^{-\mu\sum_{i\in J}\tilde{\sigma}_i}\right] &= \mathbb{E}\left[\sum_{j\in J}\tilde{\sigma}_j^a\,\mathrm{e}^{-\mu\tilde{\sigma}_j}\,\mathrm{e}^{-\mu\sum_{i\in J, i\neq j}\tilde{\sigma}_i}\right]\\ &= 2\alpha L_n\mathbb{N}\left[\sigma^a\,\mathrm{e}^{-\mu\sigma}\right]\exp\left(-2\alpha L_n\mathbb{N}\left[1-\mathrm{e}^{-\mu\sigma}\right]\right)\\ &= 2\alpha L_n\mathbb{N}\left[\sigma^a\,\mathrm{e}^{-\mu\sigma}\right]\mathrm{e}^{-2L_n\sqrt{\alpha\mu}}.\end{aligned}$$

Moreover, we have:

$$\mathbb{N}\left[\sigma^a\,\mathrm{e}^{-\mu\sigma}\right] = \int_0^\infty \frac{dr}{2\sqrt{\alpha\pi}\,r^{3/2}}\,r^a\,\mathrm{e}^{-\mu r} = \frac{1}{2\sqrt{\alpha\pi}}\Gamma(a-1/2)\mu^{1/2-a}.$$

We deduce that:

$$\begin{aligned}&\mathbb{E}\left[\sum_{j\in J}\left(2\sqrt{\alpha}L_n\tilde{\sigma}_j^{3/2}+\frac{1}{\Gamma(3/2)}\,\tilde{\sigma}_j^2\right)\mathrm{e}^{-\mu\sum_{i\in J}\tilde{\sigma}_i}\right]\\ &\qquad= 2\alpha L_n\,\mathrm{e}^{-2L_n\sqrt{\alpha\mu}}\left(2\sqrt{\alpha}L_n\mathbb{N}[\sigma^{3/2}\,\mathrm{e}^{-\mu\sigma}]+\frac{1}{\Gamma(3/2)}\,\mathbb{N}[\sigma^2\,\mathrm{e}^{-\mu\sigma}]\right)\\ &\qquad= 2\alpha L_n\,\mathrm{e}^{-2L_n\sqrt{\alpha\mu}}\,\frac{1}{2\sqrt{\alpha\pi}}\left(\frac{2\sqrt{\alpha}L_n}{\mu}+\frac{1}{\mu^{3/2}}\right)\\ &\qquad= \frac{2}{\sqrt{\pi}}\frac{\partial^2}{\partial\mu^2}\,\mathrm{e}^{-2L_n\sqrt{\mu\alpha}}.\end{aligned}$$

Let us recall the Laplace transform for the density of a stable subordinator of index $1/2$: for $a > 0$ and $\mu \geq 0$,

$$a\int_0^{+\infty}\frac{dr}{\sqrt{2\pi r^3}}\,\mathrm{e}^{-\mu r-a^2/(2r)} = \mathrm{e}^{-a\sqrt{2\mu}}.$$

From that formula, we have

$$\begin{aligned}\frac{\partial^2}{\partial\mu^2}\,\mathrm{e}^{-2L_n\sqrt{\mu\alpha}} &= \frac{\partial^2}{\partial\mu^2}\,\frac{1}{\sqrt{\pi}}\int_0^{+\infty}\frac{dx}{x^{3/2}}\,\mathrm{e}^{-1/x}\,\mathrm{e}^{-\alpha L_n^2\mu x}\\ &= \frac{1}{\sqrt{\pi}}\left(\alpha L_n^2\right)^2\int_0^{+\infty}dx\,\sqrt{x}\,\mathrm{e}^{-1/x}\,\mathrm{e}^{-\alpha L_n^2\mu x}\\ &= \frac{L_n\sqrt{\alpha}}{\sqrt{\pi}}\int_0^{+\infty}dr\,\sqrt{r}\,\mathrm{e}^{-\alpha L_n^2/r}\,\mathrm{e}^{-\mu r}\\ &= 2\alpha L_n\int_0^{+\infty}\frac{dr}{2\sqrt{\alpha\pi}\,r^{3/2}}\,r^2\,\mathrm{e}^{-\alpha L_n^2/r}\,\mathrm{e}^{-\mu r}.\end{aligned}$$

We deduce that:

$$\mathbb{E}\left[\sum_{j\in J}\left(2\sqrt{\alpha}L_n\tilde{\sigma}_j^{3/2}+\frac{1}{\Gamma(3/2)}\,\tilde{\sigma}_j^2\right)\,\Big|\,\sum_{i\in J}\tilde{\sigma}_i = r\right] = \frac{4\alpha L_n}{\sqrt{\pi}}r^2\,\mathrm{e}^{-\alpha L_n^2/r}.$$

Then, using the first part of Lemma 4.17 with $F(s,\sigma) = 2\sqrt{\alpha}L_n\sigma^{3/2}+\frac{1}{\Gamma(3/2)}\,\sigma^2$, we get the second inequality of (4.14). $\square$

Now we prove Lemma 4.14.

We consider the set $I_n^* = \{i \in I_n,\ x_i \geq m_n\}$ of indexes such that $\mathcal{T}_i$ is not grafted on the edge $[\![\emptyset, m_n]\!]$. We set:

$$A_n = \{s \geq 0; [\![\emptyset, s]\!] \cap T_n^* \neq \emptyset\} = \overline{\bigcup_{i\in I_n^*} T^i}, \quad M_n^* = \mathbb{N}_q^{(r)}\left[\int_{A_n} \theta(s)\ ds \,\Big|\, \mathcal{F}_n\right]$$
$$\text{and } V_n = M_n - M_n^*.$$

Notice that the sequence $(A_n, n \in \mathbb{N}^*)$ is non-decreasing and that $\bigcap_{n\in\mathbb{N}^*} A_n^c = \emptyset$, as there is no tree grafted on the root. By dominated convergence, this implies that $\mathbb{N}_q^{(r)}$-a.s.:

$$\lim_{n\to+\infty} \int_{A_n^c} \theta(s)\ ds = 0.$$

As:

$$V_{n+m} = \mathbb{N}_q^{(r)}\left[\int_{A_{n+m}^c} \theta(s)\ ds \,\Big|\, \mathcal{F}_{n+m}\right] \leq \mathbb{N}_q^{(r)}\left[\int_{A_n^c} \theta(s)\ ds \,\Big|\, \mathcal{F}_{n+m}\right],$$

and as $\mathcal{F}_{n+m}$ increases to $\mathcal{F}$, we get that $\limsup_{m\to+\infty} V_{n+m} \leq \int_{A_n^c} \theta(s)\ ds$ and thus $\mathbb{N}_q^{(r)}$-a.s.

$$\lim_{n\to+\infty} V_n = 0. \tag{4.15}$$

We define the function H_q (see Proposition 5.5 for a closed formula) by:

$$H_q(r) = \mathbb{N}_q^{(r)}[\Theta].$$

We have, with $\Theta_i = \Theta(\mathcal{T}_i) = \int_{\mathcal{T}_i} \theta(x)\ m^{\mathcal{T}}(dx)$:

$$\begin{aligned} M_n^* = \mathbb{N}_q^{(r)}\left[\int_{A_n} \theta(s)\ ds \,\Big|\, \mathcal{F}_n\right] = \mathbb{N}_q^{(r)}\left[\sum_{i\in I_n^*} \Theta_i \,\Big|\, \mathcal{F}_n\right] &= \mathbb{N}_q^{(r)}\left[\sum_{i\in I_n^*} \mathbb{N}_{\theta(x_i)}^{(\sigma_i)}[\Theta] \,\Big|\, \mathcal{F}_n\right] \\ &= \mathbb{N}_q^{(r)}\left[\sum_{i\in I_n^*} H_{\theta(x_i)}(\sigma_i) \,\Big|\, \mathcal{F}_n\right]. \end{aligned}$$

Since $H_q(r) \leq qr$, see (5.14) in Proposition 5.5, we get using the first equality of (4.14) in Lemma 4.17:

$$\begin{aligned} M_n^* &= 2\alpha \int_{T_n^*} \ell(dx)\ \mathbb{E}^{(r),L_n}[H_{\theta(x)}(\sigma)] \\ &\leq 2\alpha \int_{T_n^*} \ell(dx)\ \theta(x)\mathbb{E}^{(r),L_n}[\sigma] = r\,\frac{1}{L_n}\int_{T_n^*} \ell(dx)\ \theta(x). \end{aligned}$$

This gives the upper bound of (4.13).

We shall now prove the lower bound of (4.13). Since $H_q(r) \geq qr - \frac{1}{2}\sqrt{\alpha\pi}\ q^2 r^{3/2}$, see (5.14) in Proposition 5.5, we also get using the second equality of (4.14) in Lemma 4.17:

$$\begin{aligned} M_n \geq M_n^* &\geq r\,\frac{1}{L_n}\int_{T_n^*} \ell(dx)\ \theta(x) - \frac{1}{2}\sqrt{\alpha\pi}\ \mathbb{E}^{(r),L_n}[\sigma^{3/2}]\int_{T_n^*} \ell(dx)\ \theta(x)^2 \\ &\geq r\,\frac{1}{L_n}\int_{T_n^*} \ell(dx)\ \theta(x) - \frac{1}{2}\, r^2\, \mathrm{e}^{-\alpha L_n^2/r}\ \theta_{\emptyset,n}^2 \end{aligned}$$

where $\theta_{\emptyset,n} = \theta(m_n)$. This proves the lower bound of (4.13) with:

$$R_n = \frac{1}{2}\, r^2\, \mathrm{e}^{-\alpha L_n^2/r}\, \theta_{\emptyset,n}^2. \tag{4.16}$$

It remains to prove that this quantity tends to 0. First, we have:

$$\mathbb{N}_\infty^{(r)}[h_{\emptyset,n}^2 \theta_{\emptyset,n}^2] = \mathbb{N}_\infty^{(r)}[h_{\emptyset,n}^2 \mathbb{N}_\infty^{(r)}[\theta_{\emptyset,n}^2 \mid h_{\emptyset,n}]] = \frac{1}{(2\alpha)^2},$$

where we used that $\theta_{\emptyset,n}$ is exponentially distributed conditionally given $h_{\emptyset,n}$ for the second equality. We deduce that:

$$\mathbb{N}_\infty^{(r)}\left[\sum_{n=1}^{+\infty} \frac{h_{\emptyset,n}^2 \theta_{\emptyset,n}^2}{n^2}\right] < \infty$$

and hence $\mathbb{N}_\infty^{(r)}$-a.s.:

$$\sum_{n=1}^{+\infty} \frac{h_{\emptyset,n}^2 \theta_{\emptyset,n}^2}{n^2} < \infty.$$

This implies that, $\mathbb{N}_\infty^{(r)}$-a.s., for some finite $\mathcal{F}_n$-measurable random variable C_1:

$$h_{\emptyset,n}^2 \theta_{\emptyset,n}^2 \leq C_1 n^2.$$

Using Lemma 4.11, we have $\mathbb{N}_\infty^{(r)}[h_{\emptyset,n}^{-1/2}] \sim n^{1/4}\sqrt{\alpha\pi/2r}$, which implies by similar arguments that, $\mathbb{N}_\infty^{(r)}$-a.s., for some finite $\sigma(T_n)$-measurable random variable C_2:

$$h_{\emptyset,n}^{-1/2} \leq C_2 n^{3/2}. \tag{4.17}$$

Finally, using Formula (4.16) for R_n, we have $\mathbb{N}_\infty^{(r)}$-a.s.:

$$R_n \leq C_1 C_2^4\; n^8 r^2\; \mathrm{e}^{-\alpha L_n^2/r}\,.$$

As $\mathbb{N}_\infty^{(r)}$-a.s. $\lim_{n\to+\infty} L_n/\sqrt{n} = \sqrt{r/\alpha}$, we deduce that $\lim_{n\to+\infty} R_n = 0$.

Using (4.17), we deduce that:

$$R'_n = \frac{L_n}{\sqrt{n}} \mathbb{N}_\infty^{(r)}[R_n \mid T_n] = \frac{L_n}{\sqrt{n}}\, \frac{r^2}{2}\, \mathrm{e}^{-\alpha L_n^2/r}\, \frac{1}{4\alpha^2}\, \frac{1}{h_{\emptyset,n}^2} \leq C_2^4\, \frac{r^2}{8\alpha^2}\, n^{11/2} L_n\, \mathrm{e}^{-\alpha L_n^2/r}\,.$$

Thus, we get that the non-negative sequence $(R'_n, n \geq 1)$, converges $\mathbb{N}_\infty^{(r)}$-a.s. to 0, which ends the proof.

5. Appendix

5.1. *Computations on Rayleigh distributions.* Let Z be a Rayleigh random variable.

Lemma 5.1. *Let $\mu > 0$, $c \geq 0$. We have:*

$$\frac{1}{\sqrt{\pi}} \int_0^{+\infty} \frac{dr}{\sqrt{r}}\; \mathrm{e}^{-\mu r}\, \mathbb{E}\left[\mathrm{e}^{-\sqrt{2r}\, cZ'}\right] = \frac{1}{c + \sqrt{\mu}}\,\cdot \tag{5.1}$$

Proof: We set

$$J = \sqrt{\frac{\mu}{2}} \int_0^\infty \frac{dr}{\sqrt{r}}\; \mathrm{e}^{-\mu r} \int_0^\infty dx\; x\, \mathrm{e}^{-x^2/2}\, \mathrm{e}^{-c\sqrt{2r}\; x}\,.$$

With the change of variable $t^2 = 2\mu r$ and with $\rho = c/\sqrt{\mu}$, we get:

$$\begin{aligned}
J &= \int_{[0,+\infty)^2} dtdx\ x\ \exp(-(t^2+x^2+2\rho tx)/2) \\
&= \int_{[0,+\infty)^2} dtdx\ (x+\rho t)\ \mathrm{e}^{-(t^2+x^2+2\rho tx)/2} - \rho \int_{[0,+\infty)^2} dtdx\ t\ \mathrm{e}^{-(t^2+x^2+2\rho tx)/2} \\
&= \int_0^\infty dt\ \left[-\exp(-(t^2+x^2+2\rho tx)/2)\right]_{x=0}^{x=+\infty} - \rho J \\
&= \int_0^\infty dt\ \mathrm{e}^{-t^2/2} - \rho J \\
&= \sqrt{\pi/2} - \rho J.
\end{aligned}$$

This implies that $J = \dfrac{\sqrt{\pi/2}}{\rho+1} = \sqrt{\dfrac{\mu}{2}}\dfrac{\sqrt{\pi}}{c+\sqrt{\mu}}$, which is exactly what we needed. $\square$

5.2. *Joint law of* (Θ,σ). Notice that the joint law of (Θ,σ) under $\mathbb{N}_\infty$ is given in Corollary 3.3. However, we shall need the joint distribution under $\mathbb{N}_q$. For this reason, we compute the Laplace transform of (Θ,σ) using the theory of superprocess.

Let $\lambda > 0$, $\mu \geq 0$. We set for $q \in [0,+\infty]$:

$$F(q) = \mathbb{N}_q\left[1 - \mathrm{e}^{-\lambda\Theta-\mu\sigma}\right]. \tag{5.2}$$

We define the function:

$$G(x) = \left(\sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha}\right)\mathrm{e}^{\frac{2\alpha}{\lambda}\left(x-\sqrt{\mu/\alpha}\right)} - x - \frac{\lambda}{2\alpha}. \tag{5.3}$$

The function G is one-to-one from $[\sqrt{\mu/\alpha},+\infty)$ to $[0,+\infty)$, is increasing and is of class $\mathcal{C}^\infty$.

Lemma 5.2. *Let* $\lambda > 0$, $\mu \geq 0$. *The function* F *is of class* $\mathcal{C}^1$ *on* $[0,+\infty)$ *and solves the following equation on* $[0,+\infty)$*:*

$$\alpha F(q)^2 + 2\alpha\int_0^q xF'(x)\,dx = \lambda q + \mu. \tag{5.4}$$

Furthermore, we have $F = G^{-1}$.

Proof: The first part of the Lemma is a well known result from Laplace transform of superprocess Dynkin (1993) (Theorem 1.8) or equivalently of Brownian snake Le Gall (1999) (Theorem 4). We set $f(x) = \lambda x + \mu$. We introduce the function $u_t(q)$ defined for $t \geq 0$ and $q \geq 0$ by:

$$u_t(q) = \mathbb{N}_q\left[1 - \mathrm{e}^{-\int_0^\sigma f(\hat\theta_s)\mathbf{1}_{\{\zeta_s\ leq t\}}\,ds}\right].$$

We deduce from Theorem II.5.11 of Perkins (2002) that u is the unique non-negative solution of:

$$u_t(q) + \mathbb{E}_q\left[\int_0^t \alpha u_{t-s}(\theta(s))^2\,ds\right] = \mathbb{E}_q\left[\int_0^t f(\theta(s))\,ds\right].$$

Using the Markov property of θ, we get that for $t \geq r \geq 0$:

$$u_t(q) + \mathbb{E}_q\left[\int_0^r \alpha u_{t-s}(\theta(s))^2\,ds\right] = \mathbb{E}_q\left[\int_0^r f(\theta(s))\,ds\right] + \mathbb{E}_q[u_{t-r}(\theta(r))]. \tag{5.5}$$

Notice that $\lim_{t\to+\infty} u_t(q) = F(q)$. And we have:

$$u_t(q) \le F(q) \le \mathbb{N}\left[1 - \mathrm{e}^{-(q+\mu)\sigma}\right] = \sqrt{(q+\mu)/\alpha}.$$

By monotone convergence, we deduce from (5.5) that:

$$F(q) + \mathbb{E}_q\left[\int_0^r \alpha F(\theta(s))^2\, ds\right] = \mathbb{E}_q\left[\int_0^r f(\theta(s))\, ds\right] + \mathbb{E}_q[F(\theta(r))].$$

This implies that the process $N = (N_t, t \ge 0)$ defined by:

$$N_t = F(\theta(t)) + \int_0^t \left(f(\theta(s)) - \alpha F(\theta(s))^2\right)\, ds,$$

is a martingale under $\mathbb{E}_q$, for $q < +\infty$. We deduce from (4.5) (with $g = F$) that:

$$\int_0^t \left(f(\theta(s)) - \alpha F(\theta(s))^2 - 2\alpha \int_0^{\theta(s)} (F(x) - F(q))\, dx \right)\, ds$$

is a martingale. Since it is predictable, it is a.s. constant. We get that a.e. for $q \ge 0$:

$$f(q) - \alpha F(q)^2 + 2\alpha q F(q) - 2\alpha \int_0^q F(x)\, dx = 0,$$

that is a.e.:

$$F(q) = \sqrt{q^2 - 2\int_0^q F(x)\, dx + (f(q)/\alpha)} + q.$$

Since by construction F is non-decreasing, we get that F is continuous and then of class $\mathcal{C}^1$. An obvious integration by parts gives (5.4).

We now prove the second part of the Lemma. Notice that $F(0) = \mathbb{N}_0\left[1 - \mathrm{e}^{-\mu\sigma}\right] = \sqrt{\mu/\alpha}$. By differentiating (5.4) we have:

$$2\alpha F'(q)(F(q) + q) = \lambda. \tag{5.6}$$

This implies that $F' > 0$ and thus F is one-to-one from $[0, +\infty)$ to $[\sqrt{\mu/\alpha}, +\infty)$. Moreover, F^{-1} solves the differential equation

$$g'(x) = \frac{2\alpha}{\lambda}(g(x) + x). \tag{5.7}$$

Elementary computations give that the unique solution to (5.7) with the initial condition $g(\sqrt{\mu/\alpha}) = 0$ is G. Thus, we get by uniqueness $F^{-1} = G$. $\square$

Notice that $F(+\infty) = +\infty$ which doesn't able us to compute directly the Laplace transform of (Θ, σ). However, we deduce easily the following result, which gives an alternative proof of Corollary 3.3.

Corollary 5.3. *Let $\lambda > 0$, $\mu \ge 0$. We have:*

$$\mathbb{N}_\infty\left[\sigma\, \mathrm{e}^{-\mu\sigma - \lambda\Theta}\right] = \frac{1}{2\sqrt{\alpha\mu} + \lambda}\cdot \tag{5.8}$$

In particular, under $\mathbb{N}_\infty$, conditionally on σ, $\sqrt{\frac{2\alpha}{\sigma}}\,\Theta$ is distributed as a Rayleigh random variable Z independent of σ.

Proof: We have for $q \in [0, +\infty)$:

$$\partial_\mu F(q) = \mathbb{N}_q \left[\sigma \, \mathrm{e}^{-\lambda\Theta - \mu\sigma}\right]. \tag{5.9}$$

Since $G(F(q)) = q$ we get:

$$(\partial_\mu G)(F(q)) + G'(F(q)) \, \partial_\mu F(q) = 0.$$

We have:

$$\partial_\mu G(x) = -\frac{1}{\lambda} \, \mathrm{e}^{\frac{2\alpha}{\lambda}(x - \sqrt{\mu/\alpha})} = -\frac{1}{\lambda} \, \frac{1}{2\sqrt{\alpha\mu} + \lambda} (2\alpha G(x) + 2\alpha x + \lambda).$$

Notice that $G'(F(q)) = 1/F'(q)$. We deduce from (5.6) that:

$$\begin{aligned}\partial_\mu F(q) &= \frac{1}{2\alpha(F(q) + q)} \, \frac{1}{2\sqrt{\alpha\mu} + \lambda} (2\alpha q + 2\alpha F(q) + \lambda) \\ &= \frac{1}{2\sqrt{\alpha\mu} + \lambda} \left(1 + \frac{\lambda}{2\alpha(F(q) + q)}\right).\end{aligned}$$

Letting q go to infinity gives the first part of the Corollary.

For the last part, use Lemma 5.1 and the distribution of σ under $\mathbb{N}$ given in (2.1) to conclude. □

The last part of the Section is devoted to the computation of the first moment of Θ under $\mathbb{N}_q^{(r)}$, with $q < +\infty$. We first give the asymptotic expansion of F with respect to small λ. We write $O(\lambda^k)$ for any function g of q, μ and λ such that for any $q > 0$, $\mu > 0$ and $\varepsilon > 0$ there exists a finite constant C (depending on q, μ and ε) such that for all $\lambda \in [0, \varepsilon]$, $|g(q, \mu, \lambda)| \leq C\lambda^k$. Notice that $O(\lambda^k)$ is not uniform in q or μ.

Lemma 5.4. *Let $q \in (0, +\infty)$. We set $z = q\sqrt{\frac{\alpha}{\mu}}$. We have:*

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log(1 + z) - \frac{\lambda^2}{4\alpha^{3/2}\mu^{1/2}} \, \frac{z - \log(1 + z)}{1 + z} + O(\lambda^3). \tag{5.10}$$

In particular, we deduce that:

$$\partial_\lambda F(q)_{|\lambda=0} = \frac{1}{2\alpha} \log(1 + z) \quad \textit{and} \quad \partial^2_\lambda F(q)_{|\lambda=0} = -\frac{1}{2\alpha^{3/2}\mu^{1/2}} \, \frac{z - \log(1 + z)}{1 + z}. \tag{5.11}$$

Proof: Using the second part of Lemma 5.2 and (5.3), we get:

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log\left(\frac{2\alpha q + 2\alpha F(q) + \lambda}{2\sqrt{\alpha\mu} + \lambda}\right). \tag{5.12}$$

Using (5.2), we get that $F(q)$ decreases to $\sqrt{\mu/\alpha}$ when λ goes down to 0, that is $F(q) = \sqrt{\mu/\alpha} + O(1)$. Plugging this in the right-hand side of (5.12), we get:

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + O(\lambda).$$

Plugging this in the right-hand side of (5.12), we get:

$$F(q) = \sqrt{\frac{\mu}{\alpha}} + \frac{\lambda}{2\alpha} \log(1 + z) + O(\lambda^2).$$

Plugging this again in the right-hand side of (5.12), we get (5.10). This readily implies (5.11). □

We can then compute the first moment of Θ under $\mathbb{N}_q^{(r)}$.

Proposition 5.5. *Let $H_q(r) = \mathbb{N}_q^{(r)}[\Theta]$. We have, for $r > 0$ and $q \in [0, +\infty)$:*

$$H_q(r) = \sqrt{\frac{r}{2\alpha}} \int_0^{q\sqrt{2\alpha r}} dy\, \mathbb{E}\left[\mathrm{e}^{-yZ}\right]. \tag{5.13}$$

and

$$0 \le qr - H_q(r) \le \frac{1}{2}\sqrt{\pi\alpha}\, q^2 r^{3/2}. \tag{5.14}$$

Proof: By the change of variable $y = q\sqrt{2\alpha z}$, we have

$$H_q(r) = \frac{q\sqrt{r}}{2} \int_0^r \frac{dz}{\sqrt{z}} \int_0^{+\infty} dx\, x\, \exp\left(-\frac{x^2}{2} - q\sqrt{2\alpha z}\, x\right).$$

Then we compute for $\mu > 0$,

$$\begin{aligned}
&\int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi r}}\, \mathrm{e}^{-\mu r}\, H_q(r)\\
&\quad = \frac{q}{4\sqrt{\pi\alpha}} \int_0^{+\infty} dr\, \mathrm{e}^{-\mu r} \int_0^r \frac{dz}{\sqrt{z}} \int_0^{+\infty} dx\, x\, \exp\left(-\frac{x^2}{2} - q\sqrt{2\alpha z}\, x\right)\\
&\quad = \frac{q}{4\sqrt{\pi\alpha}} \int_0^{+\infty} \frac{dz}{\sqrt{z}} \int_z^{+\infty} dr\, \mathrm{e}^{-\mu r} \int_0^{+\infty} dx\, x\, \exp\left(-\frac{x^2}{2} - q\sqrt{2\alpha z}\, x\right)\\
&\quad = \frac{q}{4\sqrt{\pi\alpha}} \frac{1}{\mu} \int_0^{+\infty} \frac{dz}{\sqrt{z}}\, \mathrm{e}^{-\mu z} \int_0^{+\infty} dx\, x\, \exp\left(-\frac{x^2}{2} - q\sqrt{2\alpha z}\, x\right)\\
&\quad = \frac{q}{4\sqrt{\alpha}} \frac{1}{\mu} \frac{1}{\sqrt{\mu} + q\sqrt{\alpha}},
\end{aligned}$$

where we used equality (5.1) for the last equality.

On the other hand, we have:

$$\begin{aligned}
\int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi r}}\, \mathrm{e}^{-\mu r}\, \mathbb{N}_q^{(r)}[\Theta] &= -\partial_\mu \int_0^{+\infty} \frac{dr}{2\sqrt{\alpha\pi}\, r^{3/2}}\, \mathrm{e}^{-\mu r}\, \mathbb{N}_q^{(r)}[\Theta]\\
&= -\partial_\mu \mathbb{N}_q\left[\mathrm{e}^{-\mu\sigma}\,\Theta\right]\\
&= -\partial_\mu\left[\partial_\lambda F(q)_{|\lambda=0}\right]\\
&= -\frac{1}{2\alpha}\,\partial_\mu \log\left(1 + q\sqrt{\frac{\alpha}{\mu}}\right)\\
&= \frac{q}{4\sqrt{\alpha}}\, \frac{1}{\mu}\, \frac{1}{\sqrt{\mu} + q\sqrt{\alpha}},
\end{aligned}$$

where we used Definition (5.2) of F for the third equality an d (5.11) for the fourth one. Therefore, we have that dr-a.e. $\mathbb{N}_q^{(r)}[\Theta] = H_q(r)$. Then the equality holds for all $r > 0$ by continuity (using again a scaling argument).

Then, use $0 \le 1 - \mathrm{e}^{-z} \le z$ for $z \ge 0$, to get (5.14). □

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