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On the easiest way to connect k points in the Random Interlacements process

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Abstract. We consider the random interlacements process with intensity u on $\mathbb{Z}^d$, $d \geq 5$ (call it $\mathcal{I}^u$), built from a Poisson point process on the space of doubly infinite nearest neighbor trajectories on $\mathbb{Z}^d$. For $k \geq 3$ we want to determine the minimal number of trajectories from the point process that is needed to link together k points in $\mathcal{I}^u$. Let

$$n(k,d) := \lceil \frac{d}{2}(k-1) \rceil - (k-2).$$

We prove that almost surely given any k points $x_1, ..., x_k \in \mathcal{I}^u$, there is a sequence of $n(k,d)$ trajectories $\gamma^1, ..., \gamma^{n(k,d)}$ from the underlying Poisson point process such that the union of their traces $\bigcup_{i=1}^{n(k,d)} \mathrm{Tr}(\gamma^i)$ is a connected set containing $x_1, \dots, x_k$. Moreover we show that this result is sharp, *i.e.* that a.s. one can find $x_1, ..., x_k \in \mathcal{I}^u$ that cannot be linked together by $n(k,d) - 1$ trajectories.

1. Introduction

The random interlacement set is the trace left by a Poisson point process on the space of doubly infinite nearest neighbor trajectories modulo time shift on $\mathbb{Z}^d$. The intensity measure of the Poisson process is given by $u\nu$, where $u > 0$ and ν is a measure on the space of doubly infinite trajectories which was constructed by Sznitman (2010), see (2.9) below. This measure essentially makes the trajectories

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in the Poisson point process look like double sided simple random walk paths. The interlacement set is a site percolation model that exhibits polynomially decaying infinite-range dependence which sometimes complicates analysis.

One of the motivations for introducing the random interlacements model was to use it as a tool for the study of the behavior of simple random walks on large but finite graphs. For instance, random interlacements describe the local picture left by the trace of a simple random walk on a discrete torus or a discrete cylinder, see Windisch (2008) and Sznitman (2009b) respectively. Recent works that have used random interlacements to obtain results about simple random walks on large graphs are for example Sznitman (2009c,a) and Teixeira and Windisch (2011).

It is known that the interlacement set is always a connected set, see Corollary (2.3) in Sznitman (2010). Recently, in Ráth and Sapozhnikov (2010) and Procaccia and Tykesson (2011) a stronger result was shown: given any two points x and y in the interlacement set, one can find a path between x and y using the trace of at most $\lceil d/2 \rceil$ trajectories. The proofs in Ráth and Sapozhnikov (2010) and Procaccia and Tykesson (2011) are very different; in Procaccia and Tykesson (2011) the concept of stochastic dimension from Benjamini et al. (2004) is used, while in Ráth and Sapozhnikov (2010) the approach of the problem is based on estimating capacities of random sets constructed using random walks.

The result we present in this paper completes these works, giving a full picture of how a finite number of points are connected together within the interlacement set. Fix $k \geq 2$, and $d \geq 5$, given a realization $\mathcal{I}^u$ of the random interlacement of intensity u constructed from the Poisson point process ω_u on the space of doubly infinite trajectories (see the next section for formal definition), a.s. for any sequence of points $x_1, ..., x_k \in \mathcal{I}^u$, there is a sequence of $n(k,d)$ trajectories $\gamma^1, \dots, \gamma^{n(k,d)} \in \omega_u$ such that

(a) $\bigcup_{i=1}^{n(k,d)} \mathrm{Tr}(\gamma^{n(k.d)})$ is a connected set (where Tr denote the trace or image of a doubly infinite trajectory $\gamma : \mathbb{Z} \to \mathbb{Z}^d$),

(b) $x_j \in \bigcup_{i=1}^{n(k,d)} \mathrm{Tr}(\gamma^{n(k.d)}) \quad \forall j \in [1,k]$.

In addition, this result is sharp: of course the $n(k,d)$ trajectories are not always needed to link the k points (e.g. $x_1, \dots, x_k$ might all lie on the trace of a common trajectory) but with probability one, there exist $y_1, ..., y_k \in \mathcal{I}^u$ such that there are no sequences of $n(k,d) - 1$ trajectories satisfying the two conditions (a) and (b) above.

This result and its proof give detailed geometric information about the random interlacement process and thus on the local structure of a simple random-walk on a torus or a cylinder: this tells us that in order to connect together three trajectories A, B, and C together in the random interlacement by using a minimal number of extra trajectories, the best strategy when d is even is linking A to B and B to C , whereas when d is odd one can use one trajectory less by connecting A, B and C using a three-branch star scheme. To link 4 points or more, the best strategy can always be obtained by combining the strategy for 2 and 3 points. Our result somehow completes the information given by the recent paper of Černý and Popov (2012), which provides sharp estimates for the ratio between the graph distance and the Euclidean distance in the interlacement process.

The main results from Ráth and Sapozhnikov (2010) and Procaccia and Tykesson (2011) correspond to the case $k = 2$. The proof of the upper bound for $n(k,d)$ pushes the techniques developed in Ráth and Sapozhnikov (2010) further, while the proof of the lower bound uses a more novel approach based on diagrammatic sums. Replacing multi-index sums by diagrams to make computations tractable is an idea that is first due to Feynman (see e.g. Feynman (1949)). This method has since been used a lot in mathematical physics in both rigorous and non-rigorous fashion. A prototypical example of extensive rigorous use of diagrammatic sums is the theory of Lace Expansion developed by Brydges and Spencer (see Slade (2006)).

We conclude this section by stating the convention for the use of constants throughout the paper: The letters c, c', C, C' etc. denote finite positive constants which are allowed to depend only on the dimension d and the intensity u. Their values might change from line to line. Numbered constants c_i are finite positive, and supposed to be the same inside a certain neighborhood (for example a proof). They are defined where they first appear. Dependence of additional quantities will be indicated, for example $c(\delta)$ denotes a constant that might depend on d, u and δ.

In the next section we give a rigorous definition of the random interlacement process and state our result in full detail.

2. Notation and results

2.1. *Definition and construction of random interlacements.* We consider the trajectory spaces W and W_+ of doubly infinite and infinite transient nearest neighbor trajectories in $\mathbb{Z}^d$ (and $\mathcal{W}, \mathcal{W}_+$ the usual sigma algebras associated to them):

$$W := \{\gamma : \mathbb{Z} \to \mathbb{Z}^d;\ |\gamma(n) - \gamma(n+1)| = 1,\ \forall n \in \mathbb{Z};\ |\{n;\ \gamma(n) = y\}| < \infty,\ \forall y \in \mathbb{Z}^d\},$$

$$W_+ := \{\gamma : \mathbb{N} \to \mathbb{Z}^d;\ |\gamma(n) - \gamma(n+1)| = 1,\ \forall n \in \mathbb{Z};\ |\{n;\ \gamma(n) = y\}| < \infty,\ \forall y \in \mathbb{Z}^d\},$$

where we use the convention that $\mathbb{N}$ includes 0. For $\gamma \in W$, we define the trace of γ, $\mathrm{Tr}(\gamma) = \{\gamma(n),\ n \in \mathbb{Z}\}$. For trajectories $\gamma, \gamma' \in W$, we write $\gamma \sim \gamma'$ if for some $k \in \mathbb{Z}$ we have $\gamma(\cdot) = \gamma'(\cdot + k)$. The space of trajectories in W modulo time shift will be denoted by W^* and is defined as follows:

$$W^* := W/\sim .$$

As the trace is invariant modulo time-shift we can naturally extend the notion of trace to W^*.

For $K \subset \mathbb{Z}^d$ and $\gamma \in W_+$, we let $H_K(\gamma)$, $\tilde{H}_K(\gamma)$ and $T_K(\gamma)$ denote the entrance time, hitting time and exit time of K by γ:

$$H_K(\gamma) := \inf\{n \geq 0 : \gamma(n) \in K\}, \tag{2.1}$$

$$\tilde{H}_K(\gamma) := \inf\{n \geq 1 : \gamma(n) \in K\}, \tag{2.2}$$

$$T_K(\gamma) := \inf\{n \geq 0 : \gamma(n) \notin K\}. \tag{2.3}$$

For $x \in \mathbb{Z}^d$, set $H_x := H_{\{x\}}$. Let P_x be the law on W_+ which corresponds to a simple (i.e. nearest-neighbor symmetric) random walk on $\mathbb{Z}^d$ started at x. For $K \subset \mathbb{Z}^d$, let P_x^K be the law of simple random walk started at x conditioned on the event that the walk does not hit K:

$$P_x^K[\cdot] := P_x[\cdot | \tilde{H}_K = \infty].$$

For a finite $K \subset \mathbb{Z}^d$, we define the equilibrium measure

$$e_K(x) := \begin{cases} P_x[\tilde{H}_K = \infty], & x \in K \\ 0, & x \notin K. \end{cases} \tag{2.4}$$

The capacity of a finite set $K \subset \mathbb{Z}^d$ is defined as

$$\operatorname{cap}(K) := \sum_{x \in \mathbb{Z}^d} e_K(x). \tag{2.5}$$

and the normalized equilibrium measure of K is given by

$$\tilde{e}_K(\cdot) := e_K(\cdot)/\operatorname{cap}(K). \tag{2.6}$$

For $x, y \in \mathbb{Z}^d$ we let $|x-y| := \|x-y\|_1$ denote the l_1 distance (which corresponds to the graph distance on $\mathbb{Z}^d$) between x and y. The following bounds of hitting-probabilities are well-known, see Theorem 4.3.1 in Lawler and Limic (2010). For any $x, y \in \mathbb{Z}^d$ with $x \neq y$,

$$c|x-y|^{-(d-2)} \leq P_x[\tilde{H}_y < \infty] \leq c'|x-y|^{-(d-2)}. \tag{2.7}$$

We are now ready to introduce a Poisson point process on $W^* \times \mathbb{R}_+$. For $K \subset \mathbb{Z}^d$, let

$$W_K := \{\gamma \in W \,:\, \gamma(\mathbb{Z}) \cap K \neq \emptyset\}.$$

Let π^* be the projection from W to W^* and let $W_K^* := \pi^*(W_K)$ be the set of trajectories in W^* that enter K. We denote by Q_K the finite measure on W_K such that for $A, B \in \mathcal{W}_+$ and $x \in \mathbb{Z}^d$,

$$Q_K[(X_{-n})_{n\geq 0} \in A, X_0 = x, (X_n)_{n \geq 0} \in B] = P_x^K[A] e_K(x) P_x[B]. \tag{2.8}$$

We let the measure ν be the unique σ-finite measure such that

$$\mathbb{1}_{W_K^*}\nu = \pi^* \circ Q_K, \text{ for all finite } K \subset \mathbb{Z}^d. \tag{2.9}$$

Sznitman proved the existence and uniqueness of ν in Theorem 1.1 of Sznitman (2010). We introduce the space of locally finite point measures in $W^* \times \mathbb{R}_+$:

$$\begin{aligned} \Omega := \Big\{ \omega = \sum_{i=1}^{\infty} \delta_{(\gamma_i, u_i)};\ \gamma_i \in W^*, u_i > 0, \\ \omega(W_K^* \times [0,u]) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \text{ and } u > 0 \Big\}, \end{aligned} \tag{2.10}$$

as well as the space of locally finite point measures on W^*:

$$\tilde{\Omega} := \left\{ \sigma = \sum_{i=1}^{\infty} \delta_{\gamma_i};\ \gamma_i \in W^*,\ \sigma(W_K^*) < \infty, \text{ for every finite } K \subset \mathbb{Z}^d \right\}. \tag{2.11}$$

For $0 \leq u' \leq u$ the map $\omega_{u',u}$ from Ω into $\tilde{\Omega}$ is defined as

$$\omega_{u',u} := \sum_{i=1}^{\infty} \delta_{\gamma_i} \mathbb{1}\{u' < u_i \leq u\}, \text{ for } \omega = \sum_{i=1}^{\infty} \delta_{(\gamma_i, u_i)} \in \Omega. \tag{2.12}$$

If $u' = 0$, we use the short-hand notation ω_u. For convenience reasons we often improperly consider ω_u as a set of trajectories instead of a point measure.

On Ω we consider $\mathbb{P}$, the law of a Poisson point process with intensity measure $\nu(d\gamma)dx$ (see Equation (1.42) in Sznitman (2010) for a characterization of $\mathbb{P}$). It is easy to see that under $\mathbb{P}$, the point process $\omega_{u,u'}$ is a Poisson point process on $\tilde{\Omega}$

with intensity measure $(u-u')\nu(dw^*)$. Given $\sigma \in \tilde{\Omega}$, the set of points in $\mathbb{Z}^d$ that is visited by at least one trajectory in σ is denoted by

$$\mathcal{I}(\sigma) := \bigcup_{\gamma \in \sigma} \mathrm{Tr}(\gamma). \tag{2.13}$$

For $0 \le u' \le u$, we define the *random interlacement set* between intensities u' and u as

$$\mathcal{I}^{u',u} := \mathcal{I}(\omega_{u',u}). \tag{2.14}$$

In case $u' = 0$, we use the short-hand notation $\mathcal{I}^u$. For a point process σ on Ω or $\tilde{\Omega}$ we let $\sigma|_A$ denote the restriction of σ to $A \subset W^*$.

When needed we will identify trajectory $\gamma \in W^*$, with a canonical element of its equivalence class $(\gamma_n)_{n \ge 0}$.

2.2. *Main result.* We say that the sequence of trajectories $(\gamma^i)_{i=1}^n$ connects the sequence of points $(x_i)_{i=1}^k$ if the union of their traces (or images) includes a connected subset that contains $x_1, ..., x_k$. We say that $(\gamma^i)_{i=1}^n$ connects strictly $(x_i)_{i=1}^k$ if it connects it and there is no strict subsequence of $(\gamma^i)_{i=1}^n$ that does. Note that if a sequence of trajectories connects points, one can extract from it a subsequence that connects them strictly.

Theorem 2.1. *For every $k \ge 2$, for every $u > 0$, and for $\mathbb{P}-$almost every realization of the Poisson process ω_u, the two following properties are satisfied:*

(i) *Given a sequence of k points $(x_i)_{i=1}^k$ in $(\mathcal{I}^u)^k$, it is possible to find a sequence $(\gamma^i)_{i=1}^{n(k,d)}$ in $(\omega_u)^{n(k,d)}$ that connects it.*
(ii) *It is possible to find $(x_i)_{i=1}^k$ in $(\mathcal{I}^u)^k$ such that there exists no sequence $(\gamma^i)_{i=1}^{n(k,d)-1} \in (\omega_u)^{n(k,d)-1}$ that connects it.*

Remark 2.2. The result is restricted to $d \ge 5$ but this is not in fact a true restriction. Indeed if $d = 3$ or 4 the trace of each trajectory in ω_u intersect the trace of all the others, so that Theorem 2.1 trivially holds with $n(k,3) = n(k,4) = k$.

The proofs of (i) and (ii) are quite independent and are found in Section 3 and Section 4 respectively. In what follows we say that a sequence of points $(x_i)_{i=1}^k$ is n-connected (in $(\mathcal{I}^u)$) if (i) occurs with $n(k,d)$ replaced by n.

3. Proof of (i) of Theorem 2.1

As will be seen later in this section, in order to prove that $n(k,d)$ trajectories are sufficient to connect k points, it is essentially sufficient to prove this in the case $k = 2$ and $k = 3$. The case $k = 2$ having been proved in Ráth and Sapozhnikov (2010) and Procaccia and Tykesson (2011), we can focus on the case $k = 3$.

The first step is to reformulate the result.

Proposition 3.1. *Let $d \ge 5$ and suppose $x_1, x_2, x_3 \in \mathbb{Z}^d$. Let X^1, X^2 and X^3 be three independent simple random walks on $\mathbb{Z}^d$ with starting points x_1, x_2 and x_3 respectively. Consider also a random interlacement process ω_u independent of X^1, X^2 and X^3.*

For any choice of x_1, x_2 and x_3 and for every $u > 0$, almost surely one can find $d-4$ trajectories $(\gamma^i)_{i=1}^{d-4}$ in $(\omega_u)^{d-4}$ such that the union of the traces of the γ^is forms a connected subset that intersects the traces of X^1, X^2 and X^3.

We also need a similar result for the case of two trajectories, which is proved in Ráth and Sapozhnikov (2010, Section 4) with a slightly different formulation. The reader can check that Proposition 3.2 can also be proved using the same line of proof (simplified) that for Proposition 3.1.

Proposition 3.2. *Let $d \geq 5$ and suppose $x_1, x_2 \in \mathbb{Z}^d$. Let X^1, X^2 be two independent simple random walks on $\mathbb{Z}^d$ with starting points x_1, x_2 respectively. Consider also a random interlacement process ω_u which is independent of the walks X^1 and X^2.*

For every choice of x_1 and x_2 and for every $u > 0$, almost surely one can find $\lceil d/2 \rceil - 2$ trajectories $(\gamma^i)_{i=1}^{\lceil d/2 \rceil - 2}$ in $(\omega_u)^{\lceil d/2 \rceil - 2}$ such that the union of the traces of the γ^is forms a connected subset that intersects the traces of X^1 and X^2.

Remark 3.3. Notice that when d is even, Proposition 3.1 can easily be deduced from Proposition 3.2. Hence in what follows, we will only care about the case d odd.

Proof of Theorem 2.1 (i) from Proposition 3.1 and 3.2: The first step of the proof is to reformulate the conclusion of Theorem 2.1 into a statement that is easier to prove, see (3.1) below. For this purpose, we need to introduce some definition.

Let $x_1, \dots, x_k$ in $\mathbb{Z}^d$. We say that the sequence of points $(x_1, \dots, x_k)$ is *well behaved* for ω_u, and we will write WB, if each point of the sequence belongs to the interlacement set and if there exists a sequence $0 < t_1 < \cdots < t_k \leq u$ such that for all i there exists $(\gamma^i, t_i) \in \omega$ with $x_i \in \gamma^i$.

An equivalent formulation of (i) from Theorem 2.1 is

$$\text{For all } k \text{ and for all } (x_i)_{i=1}^k \in (\mathbb{Z}^d)^k \text{ we have that}$$
$$\mathbf{P}\left[\exists (\gamma^i)_{i=1}^{n(k,d)}, (\gamma^i)_{i=1}^{n(k,d)} \text{ connects } (x_i)_{i=1}^k \mid (x_i)_{i=1}^k \ WB\right] = 1, \tag{3.1}$$

or alternatively

$$\text{For all } k,\ \mathbf{P}\left[\forall (x_i)_{i=1}^k, (x_i)_{i=1}^k \ WB \Rightarrow \exists (\gamma^i)_{i=1}^{n(k,d)}, (\gamma^i)_{i=1}^{n(k,d)} \text{ connects } (x_i)_{i=1}^k\right] = 1. \tag{3.2}$$

Indeed clearly, if (i) of Theorem 2.1 holds, so does (3.1). We prove the other implication by contradiction: if (i) from Theorem 2.1 is violated, with positive probability one can find k points in $\mathcal{I}^u$ that cannot be connected by $n(k,d)$ trajectories in ω_u. As these points are in $\mathcal{I}^u$, one can by definition find a sequence $(\gamma^i)_{i=1}^k$ of trajectories in ω_u such that $x_i \in \gamma^i$ for all i. If all the γ^i are distinct, a.s. after an eventual reordering of the sequence we get that $(x_1, \dots, x_k)$ is well behaved so that (3.1) cannot hold. On the other hand, if there are repetitions in $(\gamma^i)_{i=1}^k$, one extracts a well behaved subsequence $(x'_i)_{i=1}^{k'}$ of $(x_i)_{i=1}^k$ by deleting the points x_i for which

$$\exists j < i, \gamma_i = \gamma_j, \tag{3.3}$$

and reordering the remaining subsequence. Then if (3.1) holds then one can a.s. connect $(x'_i)_{i=1}^{k'}$ with $n(k',d)$ trajectories. Then using the definition (3.3) one can link all the points $(x_i)_{i=1}^k$ together with $n(k',d) + (k - k') \leq n(k,d)$ trajectories (just by using the γ_j corresponding to the $k' - k$ remaining points if necessary in addition to the trajectories that connect $(x'_i)_{i=1}^{k'}$) which yields a contradiction. Hence we can focus on proving (3.1).

We want to use Proposition 3.1 and hence our task is to isolate (using conditioning) some mutually independent random walks and a random interlacement process independent of them from the larger process we have.

Let $\tau_0 := 0$ and for $i = 1, ..., k$ let recursively

$$\tau_i := \min\{s > \tau_{i-1} \mid x_i \in \mathcal{I}^s\}. \tag{3.4}$$

Note that by definition of ω_u, in $\omega_{\tau_{i-1},\tau_i}$, with probability one, there exists a unique trajectory γ^i which has x_i in its trace.

Furthermore, by the strong Markov property for Poisson processes, the law of γ^i is independent of that of τ_i (and the trajectories $(\gamma^i)_{i=1}^k$ are independent) and if we parametrize the oriented trajectory γ^i as $(\gamma^i_n)_{n\in\mathbb{Z}}$ such that 0 is the first time that γ^i visits x_i is 0, then from the definition of the random interlacement process (recall (2.8)),

$$(\gamma^i_n)_{n\geq 0} \text{ is a simple random walk on } \mathbb{Z}^d \text{ started at } x_i. \tag{3.5}$$

Set $\mathcal{T} := \max_{i\in[1,k]} \tau_i$. The event $\{(x_i)_{i=1}^k$ is well behaved$\}$ is equal to $\{\mathcal{T} \leq u\}$, and up to an event of probability 0, it coincides with $\{\mathcal{T} < u\}$. Note that conditioned on $\mathcal{T}$, the process $\omega_{\mathcal{T},u}$ is independent of $\mathcal{T}$ and of the γ^is. We now deal with the cases k odd and k even separately.

Case $k = 2p + 1$ is odd

Setting $X^i := (\gamma^i_n)_{n\geq 0}$, and using conditional independence of $\omega_{\mathcal{T},u}$, we can apply Proposition 3.1 and for every $j = 1, ..., p$ find a sequence of $(d-4)$ trajectories $(\gamma^i)_{i=k+(j-1)(d-4)+1}^{k+j(d-4)}$ in $\omega_{\mathcal{T},u}$ that connects together the traces of X^{2j-1}, X^{2j} and X^{2j+1}.

One can then conclude by observing that $k + p(d-4) = n(k, d)$ for k odd and that $(\gamma^i)_{i=1}^{k+p(d-4)}$ is a set of trajectories in ω_u that connects $(x_i)_{i=1}^k$.

Case $k = 2p$ is even

We use Proposition 3.1 for $i = 1, ..., p-1$ to connect together $X^1, \ldots, X^{2p-1}$ and Proposition 3.2 to connect X_{2p-1} and X_{2p} with the trajectories $(\gamma^i)_{i=k+(p-1)(d-4)+1}^{k+(p-1)(d-4)+\lceil d/2\rceil-2}$ from $\omega_{\mathcal{T},u}$, and conclude in a similar manner.

□

Before the proof of Proposition 3.1 for d odd, (in what follows we always consider that d is odd) we must introduce additional notation in order to reformulate the statement. Introduce the number

$$k_d := \lceil d/2\rceil - 2 = \frac{d-3}{2}. \tag{3.6}$$

For a finite set $A \subset \mathbb{Z}^d$ and $\sigma \in \tilde{\Omega}$, let $N_A(\sigma)$ be the number of trajectories in σ that intersect A. Let $\gamma^1, ..., \gamma^{N_A(\sigma)}$ be the trajectories from σ that intersect A, parameterized so that $\gamma^i_0 \in A$ and $\gamma^i_n \notin A$ for all $n < 0$ and all $i \in \{1, ..., N_A(\sigma)\}$. For $\sigma \in \tilde{\Omega}$, $A \subset \mathbb{Z}^d$ and $R \in \mathbb{Z}_+$ we define the random set of vertices $\Psi(\sigma, A, R)$ as

$$\Psi(\sigma, A, R) = \bigcup_{i=1}^{N_A(\sigma)} \left(\{\gamma_i(t) \,:\, 1 \leq t \leq R^2/8\} \cap B(\gamma_i(0), R/2)\right) \tag{3.7}$$

Definition 3.4. Let $r, R \in \mathbb{R}_+ \cup \{\infty\}$ with $r < R$. For $\sigma \in \tilde{\Omega}$, let σ_R be the restriction of σ to the trajectories that intersect $B(R)$. Let $\sigma_{r,R}$ be the restriction of σ_R to the set of trajectories that do not intersect $B(r)$.

Observe that σ_r and $\sigma_{r,R}$ are supported on disjoint sets of trajectories and that

$$\sigma = \sigma_r + \sigma_{r,\infty}. \tag{3.8}$$

Let $(\sigma^{(i,j)})_{1\le i\le 4,\, 1\le j\le k_d}$ in $\tilde{\Omega}$ be a family of i.i.d. random interlacement processes with parameter $\bar{u} := u/4k_d$ defined by

$$\sigma^{(i,j)} := \omega_{\bar{u}((i-1)k_d+(j-1)),\bar{u}((i-1)k_d+j)}, \tag{3.9}$$

and let $(X^i)_{i=1}^3$ be three independent simple random walks starting from x_1, x_2 and x_3 respectively. Given R, let $T^i(B(R))$ be the first exit time of X^i from $B(R)$ and $Y_i := (Y_n^{i,R})_{n\ge 0} = (X^i_{n+T^i(B(R))})_{n\ge 0}$ (and $Y^i = X^i$ when $R = \infty$). We call $\mathbf{P}$ the probability measure governing all these processes.

We define sequences of random subsets of $\mathbb{Z}^d$. For $0 \le r < R \le \infty$, and $i = 1, 2, 3$ set

$$A_i^{(1)}(r,R) = A_i^{(1)}(R) := \left\{ Y_n^{i,R} \,:\, 1 \le n \le R^2/8 \right\} \cap B(Y_0^i, R/2),\ 1 \le i \le 3. \tag{3.10}$$

Then recursively, for $2 \le j \le k_d$ and with r, R, i as above, define

$$A_i^{(j)}(r,R) := \Psi\left(\sigma^{(i,j)}_{r,\infty}, A_i^{(j-1)}(r,R), R\right) = \Psi\left(\sigma^{(i,j)}_{r,jR}, A_i^{(j-1)}(r,R), R\right). \tag{3.11}$$

We simply write $A_i^{(j)}$ when $r = 0$ and $R = \infty$.

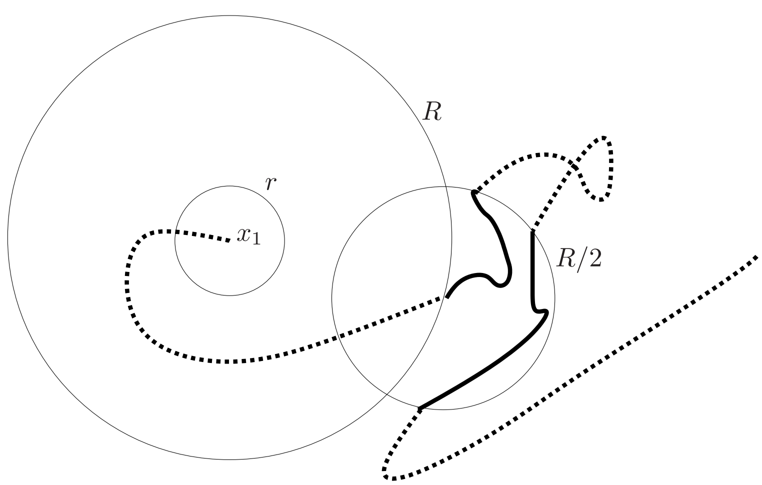

Figure 3.1. This figure shows how $A_1^{(1)}(r,R)$ is created from a simple random walk started at x_1. The set $A_1^{(1)}(r,R)$ consists of the solid thick lines.

Note that by construction if $y \in A_i^{(j)}(r,R)$ then there exists a sequence of $k_d - 1$ trajectories in $\mathcal{I}^u$ linking it to the trace of X^i. Thus to prove Proposition 3.1, it is in fact sufficient to prove (recall that $2(k_d-1)+1 = d-4$),

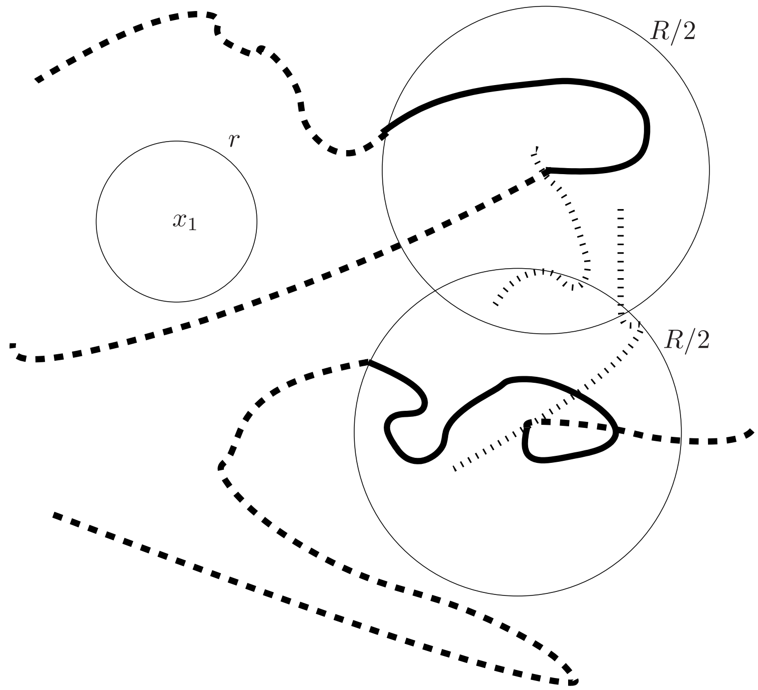

FIGURE 3.2. This figure shows how, given $A_1^{(1)}(r,R)$ from Figure (3.1), the set $A_1^{(2)}(r,R)$ is created using trajectories from $\sigma^{(1,2)}(r,\infty)$. Here $A_1^{(2)}(r,R)$ is given by the solid thick lines, and $A_1^{(1)}(r,R)$ is represented by the thin dotted lines.

Lemma 3.5. *With probability one, one can find $\gamma \in \sigma^{(4,1)}$ that connects $A_1^{(k_d)}$, $A_2^{(k_d)}$ and $A_3^{(1)}$ together.*

Inspired by Ráth and Sapozhnikov (2010), we prove Lemma 3.5 by combining Borel's Lemma and

Lemma 3.6. *Let $d \geq 5$ be odd and let $x_1, x_2, x_3 \in \mathbb{Z}^d$. Let R and r be integers, such that $R > \max(|x_1|, |x_2|, |x_3|)$. There exist constants $c(u,d) > 0$, $R_0(u,d) < \infty$ and $\varepsilon(u,d) > 0$, such that for any r and R with $R > R_0$ and $\varepsilon R \geq r^{d-2}$,*

$$\mathbf{P}\left[\exists \gamma \in \sigma_{r,2R}^{(4,1)} \,:\, \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R)\right] \geq c. \tag{3.12}$$

We prove Lemma 3.6 by using a method based on the control of the capacity of the sets $A_i^{(j)}(r,R)$ at the end of the section.

Proof of Lemma 3.5 from Lemma 3.6: For real numbers $r < R$ such that $x_1, x_2, x_3 \in B(R)$, set

$$D(r,R) := \{\exists \gamma \in \sigma^{(4,1)} \,:\, \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R)\}. \tag{3.13}$$

We choose ϵ so that Lemma (3.6) applies. Let $r_0 = \max(|x_1|, |x_2|, |x_3|)$ and $R_0 = \epsilon^{-1} r_0^{d-2}$. For $k \geq 1$, we define recursively

$$r_k = dR_{k-1}^2 \text{ and } R_k = \epsilon^{-1} r_k^{d-2}. \tag{3.14}$$

We write $D_k = D_k(X^1, X^2, X^3)$ (reasons for underlining only the dependence in X^i will become clear later) for $D(r_k, R_k)$ and write ι_k for $\mathbb{1}_{D_k}$, the indicator function of D_k. We want to show that

$$\mathbf{P}\left[D_k \text{ occurs for infinitely many values of } k\right] = 1, \tag{3.15}$$

which implies Lemma 3.5.

We will be done using Borel's Lemma (it is cited as in Lemma 4.12 in Ráth and Sapozhnikov (2010)) if we can show that there is some c such that for all $k \geq 1$ we have almost surely

$$\mathbf{P}[D_k | \iota_1, ..., \iota_{k-1}] \geq c. \tag{3.16}$$

Let $I_k := (\iota_i)_{i=1}^k$. Then I_k is measurable with respect to the σ-algebra generated by the following random objects: $(\{X_n^i : 1 \leq n \leq T_{B(R_{k-1})} + R_{k-1}^2/8\})_{i \leq 3}$ and $(\sigma^{(i,j)}_{R_{k-1}(1+k_d)})_{i \leq 4, j \leq k_d}$.

On the other hand, the event D_k depends on $(\sigma^{(i,j)})_{i \leq 4, j \leq k_d}$ only through $(\sigma^{(i,j)}_{r_k,\infty})_{i \leq 4, j \leq k_d}$. Since $R_{k-1}(1+k_d) < r_k$, the point measures $(\sigma^{(i,j)}_{R_{k-1}(1+k_d)})_{i \leq 4, j \leq k_d}$ and $(\sigma^{(i,j)}_{r_k,\infty})_{i \leq 4, j \leq k_d}$ are independent.

Let $\tilde{X}^i$ be defined by $\tilde{X}^i_n = X^i_{n+B(R_{k-1})+R_{k-1}^2/8}$. By the strong Markov property, conditionally on $\tilde{X}^i_0$, $\tilde{X}^i$ is independent of X^i (and its law is the one of a simple random walk). Furthermore, as $R_{k-1} + R_{k-1}^2/8 < r_k$, D_k depends on X^i only through $\tilde{X}^i$.

Thus by conditional independence

$$\mathbf{P}[D_k \mid I_k, (\tilde{X}^i_0)_{i=1}^3] = \mathbf{P}[D_k(\tilde{X}^1, \tilde{X}^2, \tilde{X}^3)] \mid (\tilde{X}^i_0)_{i=1}^3] \geq c, \tag{3.17}$$

where the last inequality follows from Lemma 3.6, with (x_1, x_2, x_3) replaced by $(\tilde{X}^i_0)_{i=1}^3$.

□

We can now focus on the proof of Lemma 3.6. Before starting we cite results from Ráth and Sapozhnikov (2010) that give estimates on the capacities of the sets $A_i^{(j)}(r, R)$.

Lemma 3.7. *Ráth and Sapozhnikov (2010, Lemmata 4.7, 4.8) Let $d \geq 5$ and let j be a positive integer. There exist constants $C_j = C_j(u, d)$ and $\epsilon_j = \epsilon_j(u, d)$ such that for any positive integers r and R with $r^{d-2} \leq \epsilon_j R$ and if $x_1, x_2, x_3 \in B(R)$, we have*

$$\mathbf{E}[\text{cap}(A_i^{(j)}(r, R))] \geq C_j R^{\min(d-2, 2j)}. \tag{3.18}$$

Moreover, under the same condition there exist positive finite constants $c_j = c_j(u, d)$, such that,

$$\mathbf{E}[\text{cap}(A_i^{(j)}(r, R))] \leq c_j R^{\min(d-2, 2j)}, \tag{3.19}$$

and

$$\mathbf{E}[\text{cap}(A_i^{(j)}(r, R))^2] \leq c_j R^{2\min(d-2, 2j)}. \tag{3.20}$$

As a consequence (using Chebychev inequality and changing the value of c_j if needed),

$$\mathbf{P}[\mathrm{cap}(A_i^{(j)}(r,R)) \geq c_j R^{\min(d-2,2j)}] \geq c_j. \tag{3.21}$$

Proof of Lemma 3.6: We choose the constants ϵ_s from Lemma 3.7 and assume that r and R are such that Lemma 3.7 applies. We consider the two following events

$$\begin{aligned} E_1 &:= \{\exists \gamma \in \sigma_{2R}^{(4,1)} : \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R)\}, \\ E_2 &:= \{\exists \gamma \in \sigma_r^{(4,1)} : \gamma \text{ intersects } A_3^{(1)}(r,R)\}. \end{aligned} \tag{3.22}$$

Note that

$$\{\exists \gamma \in \sigma_{r,2R}^{(4,1)} : \gamma \text{ connects } A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R), \text{ and } A_3^{(1)}(r,R)\} \supset E_1 \setminus E_2. \tag{3.23}$$

Let $\mathbf{P}^{(4,1)}$ denote the law of $\sigma^{(4,1)}$. Our main task is to prove that there exists a universal constant c such that

$$\mathbf{P}^{(4,1)}[E_1] \geq 1 - \exp(-cR^{4-2d}\,\mathrm{cap}(A_1^{(k_d)}(r,R))\,\mathrm{cap}(A_2^{(k_d)}(r,R))\,\mathrm{cap}(A_3^{(1)}(r,R))), \tag{3.24}$$

and

$$\mathbf{P}^{(4,1)}[E_2] \leq u\,\mathrm{cap}(A_3^{(1)}(r,R))(r/(R-r))^{d-2}. \tag{3.25}$$

According to (3.21) (and independence), choosing c small enough one has with positive probability larger than c

$$\begin{aligned} \mathrm{cap}(A_1^{(k_d)}(r,R)) &\geq cR^{2k_d}, \\ \mathrm{cap}(A_2^{(k_d)}(r,R)) &\geq cR^{2k_d}, \\ \mathrm{cap}(A_3^{(1)}(r,R)) &\geq cR^2. \end{aligned} \tag{3.26}$$

Hence (3.24), (3.25) and (3.19) imply (recall that $2k_d = d-3$) that

$$\mathbf{P}\left[E_1\right] \geq c(1-\exp(-c^4)) \text{ and } \mathbf{P}\left[E_2\right] \leq \mathbf{P}\left[E_1\right]/2, \tag{3.27}$$

provided that R is large enough. This together with (3.23) is enough to conclude. From now on, we write A_1, A_2 and A_3 for $A_1^{(k_d)}(r,R), A_2^{(k_d)}(r,R)$ and $A_3^{(1)}(r,R)$. In order to prove (3.24) and (3.25) one considers the following construction of $\sigma^{(1,4)}|_{W^*_{A_3}}$:

- Let $\mathcal{N}$ be a Poisson variable of mean $\bar{u}\,\mathrm{cap}(A_3)$.
- Conditionally on $\mathcal{N}$, let $(\gamma^i)_{i=1}^{\mathcal{N}}$ be a sequence of independent (and independent of $\mathcal{N}$) of $\mathcal{N}$ doubly-infinite trajectory with distribution $\pi^* \circ \bar{Q}_{A_3}$, where $\bar{Q}_{A_3}(\cdot) = Q_{A_3}(\cdot)/Q_{A_3}(W_{A_3})$ is the renormalized version of the measure defined in (2.8).

Note that from this construction one has

$$\begin{aligned} \mathbf{P}^{(4,1)}[E_1 \mid \mathcal{N}] &= 1 - [1 - \bar{Q}_{A_3}(\gamma \text{ hits } A_1 \text{ and } A_2)]^{\mathcal{N}}, \\ \mathbf{P}^{(4,1)}[E_2 \mid \mathcal{N}] &= 1 - [1 - \bar{Q}_{A_3}(\gamma \text{ hits } B(r))]^{\mathcal{N}}, \end{aligned} \tag{3.28}$$

where $(\gamma_n)_{n\in\mathbb{Z}}$ is a trajectory distributed according to $\bar{Q}_{A_3}$. Let P_x be the law of the simple random walk Y starting from x and T_1 and T_2 the hitting times of A_1

and A_2 respectively. From the definition of $\bar{Q}_{A_3}$ we have

$$\bar{Q}_{A_3}(\gamma \text{ hits } A_1 \text{ and } A_2) \geq \bar{Q}_{A_3}(\exists n_2 \geq n_1 \geq 0, \gamma_{n_1} \in A_1, \gamma_{n_2} \in A_2) \geq \min_{x \in A_3} P_x(T_1 \leq T_2 < \infty). \tag{3.29}$$

Moreover using the strong Markov property and the identity

$$P_x[T_1 < \infty] = \sum_{z \in A_1} g(x,z) e_{A_1}(z), \tag{3.30}$$

we get

$$\begin{aligned} P_x(T_1 \leq T_2 < \infty) &\geq \left(\sum_{z \in A_1} g(x,z) e_{A_1}(z) \right) \left(\inf_{y \in A_1} \sum_{z \in A_2} g(y,z) e_{A_2}(z) \right) \\ &\geq \left(\min_{y,z \in B((k_d+1)R)} g(y,z) \right)^2 \left(\sum_{z \in A_1} e_{A_1}(z) \right) \left(\sum_{z \in A_2} e_{A_2}(z) \right) \\ &\geq c R^{4-2d} \operatorname{cap}(A_1) \operatorname{cap}(A_2), \end{aligned} \tag{3.31}$$

(to get the last inequality recall (2.5) and (2.7)). Hence

$$\bar{Q}_{A_3}(\gamma \text{ hits } A_1 \text{ and } A_2) \geq c R^{4-2d} \operatorname{cap}(A_1) \operatorname{cap}(A_2). \tag{3.32}$$

Together with the first line of (3.28) and averaging with respect to $\mathcal{N}$, this proves (3.24).

Let us now get (3.25). We note that $\pi^* \circ \bar{Q}_{A_3}$ is invariant under change of orientation of the trajectories (see Theorem 1.1 of Sznitman (2010)) so that if $\tilde{T} := \max\{n | \gamma_n \in A_3\}$, then $(\gamma_n)_{n \geq 0}$ and $(\gamma_{\tilde{T}-n})_{n \geq 0}$ have the same law. Hence

$$\bar{Q}_{A_3}(\gamma \text{ hits } B(r)) \leq 2 \bar{Q}_{A_3}\left((\gamma_n)_{n \geq 0} \text{ hits } B(r)\right). \tag{3.33}$$

Moreover (recall (3.30))

$$\begin{aligned} \bar{Q}_{A_3}((\gamma_n)_{n \geq 0} \text{ hits } B(r)) &\leq \max_{|x| \geq R/2} P_x(H_{B_r} < \infty) \\ &= \max_{|x| \geq R/2} \sum_{z \in A_1} g(x,z) e_{B(r)}(z) \leq C(r/(R-r))^{d-2}. \end{aligned} \tag{3.34}$$

All of this combined gives

$$\bar{Q}_{A_3}(\gamma \text{ hits } B(r)) \leq C(r/(R-r))^{2-d}. \tag{3.35}$$

Combining with (3.28) and averaging with respect to $\mathcal{N}$ gives

$$\mathbf{P}^{(4,1)}[E_2] \leq u \operatorname{cap}(A_3)(r/(R-r))^{2-d}. \tag{3.36}$$

□

4. Proof of (ii) in Theorem 2.1

The aim of this Section is to prove that if one selects k points very distant from each other in the random interlacement, they are really unlikely to be connected by less than $n(k,d)$ trajectories (together with a quantitative upper-bound on the probability).

Proposition 4.1. *Given $\varepsilon > 0$, for any $x_1, \ \dots \ , x_k \in \mathbb{Z}^d$ and for any $n < n(k,d)$ one has*

$$\mathbb{P}[(x_i)_{i=1}^k \text{ is } n\text{-connected }] \le C(d,k,\varepsilon) \max_{i\neq j} |x_i - x_j|^{-1+\varepsilon}. \tag{4.1}$$

Whereas it is quite intuitive that Proposition 4.1 implies the second half of Theorem 2.1, the proof is not completely straight-forward so we write it in full detail.

Proof of Theorem 2.1 (ii) from Proposition 4.1: Set $n < n(k,d)$. For $i = 1, \dots, k$ denote by B_R^i the Euclidean ball of center $ie^R\mathbf{e}_1$ (with $\mathbf{e}_1 = (1,0,\dots,0) \in \mathbb{Z}^d$) and of radius R. We want to show that the probability of the event

$$\mathcal{A}_R := \left\{ \exists (x_i)_{i=1}^k \in \prod_{i=1}^k B_R^i,\ (x_i)_{i=1}^k \text{ is not } n\text{-connected }, \forall i \in [1,k], x_i \in \mathcal{I}_u \right\}. \tag{4.2}$$

tends to one when R tends to infinity, so that $\mathbf{P}\left[\bigcup_{R\ge 1} \mathcal{A}_R\right] = 1$ (which implies Theorem 2.1 (*ii*)). According to Proposition 4.1, using a union bound, one has for R large enough

$$\mathbf{P}[E_R^1] := \mathbf{P}\left[\exists (x_i)_{i=1}^k \in \prod_{i=1}^k B_R^i, (x_i)_{i=1}^k \text{ is } n\text{-connected} \right] \le CR^{kd}e^{-R/2}. \tag{4.3}$$

Moreover from the definition of random interlacements (in particular of the measure ν in equation (2.9)) we have

$$\mathbf{P}[E_R^2] := \mathbf{P}\left[\exists i \in [1,k], \mathcal{I}_u \cap B_R^i = \emptyset\right] \le ke^{-u\,\mathrm{cap}(B_1^R)} \le ke^{-cR^{d-2}}. \tag{4.4}$$

Hence we conclude that the probability of $\mathcal{A}_R \subset (E_R^1 \cup E_R^2)^c$ tends to one as $R \to \infty$. □

We prove Proposition 4.1 by induction on k. The strategy that we use is the following: first we encode the way the k points are connected by some tree scheme $\mathcal{T}$. This is done in Proposition 4.2. Then we bound from above the probability that k points are connected together using a given scheme by a diagrammatic sum (Lemma 4.4). Finally we prove an upper-bound on this sum (Proposition 4.5). For some tree-schemes the multi-index sum given by Lemma 4.4 is infinite and those are to be treated separately. However they are easily dealt with by using the induction hypothesis.

Proposition 4.2. *Assume there is a sequence of distinct trajectories $(\gamma^i)_{i=1}^n$ ($\gamma^i \neq \gamma^j$ for $i \neq j$) that connects strictly $(x_i)_{i=1}^k$.*

Then one can construct:

(a) *a sequence $(y_i)_{i=1}^m \in (\mathbb{Z}^d)^m$, with $m = n + k - 1$ and $y_i = x_i$ for $i \le k$,*
(b) *a tree $\mathcal{T}$ with m labeled vertices $A_1, \dots, A_m$, and $m-1$ oriented edges whose set we call $\mathcal{E}$,*
(c) *a function $t : \mathcal{E} \to \{1, \dots, n\}$, that to each edge associates a **type**,*

that satisfies the following properties:

(i) *The set of oriented edges that share the same label forms an (oriented) path in the tree.*

(ii) *For all indices $i \leq k$, all the edges connected to the vertex A_i (ignoring their orientation) are all of the same type (hence those vertices have at most degree 2). For $i \geq k+1$ the edges connected to the vertex A_i are of two different types (exactly).*

(iii) *If $A_{a_1}A_{a_2}\dots A_{a_l}$, $l \geq 2$ is the path of vertices linked by edges of type h and $(\gamma^h_n)_{n\in\mathbb{Z}}$ is a time parametrization of γ^h, then there exists a non-decreasing sequence $b_1,\dots,b_l$ in $\mathbb{Z}$ such that $\gamma_{b_i} = y_{a_i}$ for all $i \in [1,l]$.*

Given $(\mathcal{T},\mathcal{E},t)$ satisfying $(i)-(ii)$ we say that $(x_i)_{i=1}^k$ is connected with scheme $(\mathcal{T},\mathcal{E},t)$ (or $\mathcal{T}$ to simplify notation), if there exists $(y_i)_{i=k+1}^m$ in $(\mathbb{Z}^d)^{m-k}$, $(\gamma^i)_{i=1}^n \in (\omega_u)^n$, $\gamma^i \neq \gamma^j$ for $i \neq j$, such that (iii) holds. Furthermore if this holds with $(y_i)_{i=k+1}^m$ fixed, we say that $(x_i)_{i=1}^k$ is connected with scheme $(\mathcal{T},\mathcal{E},t)$ using $(y_i)_{i=k+1}^m$.

Remark 4.3. Remark that we allow repetition in the sequence $y_1,\dots,y_m$ and that the choice of the tree may not be unique. Moreover it can easily be checked by the reader that if a sequence of points is connected with scheme $(\mathcal{T},\mathcal{E},t)$, then the sequence is n-connected. An example for the construction of $\mathcal{T}$ together with the type function is given in figure 4.3.

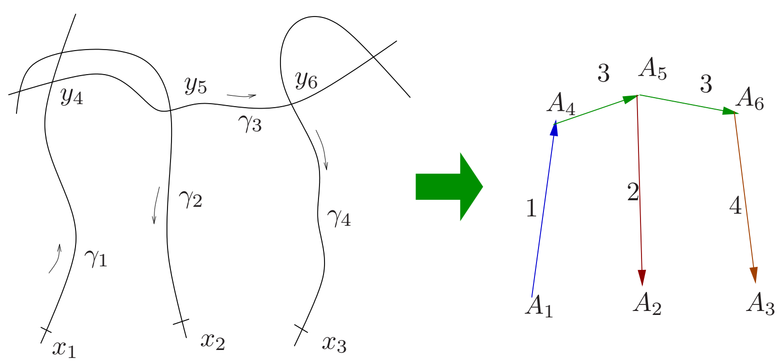

FIGURE 4.3. Examples of the process of tree creation when $k = 3$ and $n_1 = 4$. On the left, the n_1 oriented trajectories are represented together with the xs and the points of intersection of the trajectories. On the right this is encoded in the corresponding tree.

Proof: We prove the statement by induction on k. If $k = 2$ and x_1 and x_2 are strictly connected by $(\gamma^i)_{i=1}^n$, then it is possible (changing the order of the γ^i if necessary) to find $(y_i)_{3\leq i\leq n+1}$ such that $y_i \in \gamma^{i-2}\cap\gamma^{i-1}$ and $x_1 \in \gamma^1$, $x_2 \in \gamma^n$. Then the tree $\mathcal{T}$ is just the paths $A_1A_3A_4\dots A_{n+1}A_2$ and the edge A_iA_{i+1} has type $i-1$ (A_1A_3 is of type 1 and $A_{n+1}A_2$ is of type n). Orientation of the edges can then be chosen to satisfy (iii).

For $k \geq 3$, we remark that if $(\gamma_i)_{i=1}^n$ strictly connects $(x_i)_{i=1}^k$, then it connects $(x_i)_{i=1}^{k-1}$. Thus one can find a subsequence of trajectories that strictly connects $(x_i)_{i=1}^{k-1}$. Hence after reordering of the indices, one may assume that $(\gamma_i)_{i=1}^{n'}$ for $n' \leq n$ strictly connects $(x_i)_{i=1}^{k-1}$.

Using the induction hypothesis one can find a tree $\mathcal{T}'$ with $k + n' - 2$ vertices $(A_i)_{i\in[1,n'+k-1]\setminus\{k\}}$ and a sequence of $\mathbb{Z}^d$ vertices $(y_i)_{i\in[0,n'+k-1]\setminus\{k\}}$, that satisfies $(i) - (iii)$ (the label k is not used here for a reason that will become clear soon).

Assume for the rest of the proof that $n' < n$ (the case $n' = n$ is treated briefly at the end). Note that since $(\gamma_i)_{i=1}^n$ strictly connects $(x_i)_{i=1}^k$, one can find $y_{n'+k}$ in the trace of one of the trajectories $(\gamma_i)_{i\leq n'}$ (without loss of generality we can assume it belongs to $\mathrm{Tr}(\gamma_{n'})$), such that $y_{n'+k}$ and x_k are strictly connected by $(\gamma_i)_{i=n'+1}^k$.

We are now ready to construct the tree $\mathcal{T}$. First we construct a path

$$A_{n'+k}A_{n'+k+1}\dots A_{n+k+1}A_k$$

composed of $n - n'$ edges of different types ($n' + 1$ to n), just as one did for the $k = 2$ case.

Then one plugs $A_{n'+k}$ into the old tree $\mathcal{T}'$ as follows. Let $A_{a_1}, \dots, A_{a_l}$, $l \geq 2$ be the path of vertices linked by edges of type n'. By (iii) of the induction hypothesis, there exists a non-decreasing sequence in $\mathbb{Z}$, $b_1, \dots, b_l$ such that $\gamma_{b_i}^{n'} = y_{a_i}$ for all $i \in [1, l]$. By definition $y_{n'+k} = \gamma_b^{n'}$ for some $b \in \mathbb{Z}$.

One then constructs $\mathcal{T}$ from $\mathcal{T}'$ by adding a new edge of type n_2 to include A_{k+n_2} in the tree in the following manner.

- (a) if $b \leq b_1$, one adds an edge $A_{k+n'}A_{a_1}$ (and the path $A_{n'+k}A_{n'+k+1}\dots \dots A_{n+k-1}A_k$ previously constructed),
- (b) if $b \in (b_i, b_{i+1}]$ then one replaces the edge $A_{a_i}A_{a_{i+1}}$ by two edges $A_{a_i}A_{n_2+k}$ and $A_{n_2+k}A_{a_{i+1}}$,
- (c) if $b > b_l$ then one adds an edge $A_{a_l}A_{n_2+k}$.

When $n' = n$ the procedure is exactly the same except that $y_{n'+k}$ is replaced by x_k (and $A_{n'+k}$ by A_k) and that only the second stage is needed (the paths to be plugged is only the single point A_k in this case). We let the reader check that assumptions $(i) - (iii)$ are satisfied by $\mathcal{T}$. □

According to Proposition 4.2, one has

$$\{(x_i)_{i=1}^k \text{ is } n\text{-connected }\} = \cup_{\mathcal{T}\in\mathbb{T}_n}\{(x_i)_{i=1}^k \text{ is connected with scheme } \mathcal{T}\}, \quad (4.5)$$

where $\mathbb{T}_n$ denotes the (finite) set of all schemes $\mathcal{T}$ with less than $n+k-1$ vertices. Thus, to prove Proposition 4.1, we only need to prove that for every $\mathcal{T} \in \mathbb{T}_n$,

$$\mathbb{P}\left[(x_i)_{i=1}^k \text{ is connected with scheme } \mathcal{T}\right] \leq C \max_{i\neq j} |x_i - x_j|^{-1+\varepsilon}. \quad (4.6)$$

For this purpose we will use the following Lemma that estimates the l.h.s. of (4.6).

Lemma 4.4. *Let $\mathcal{E}$ denote the set of edges of $\mathcal{T}$, a tree with $n + k - 1$ vertices. Then*

$$\mathbb{P}\left[(x_i)_{i=1}^k \textit{ is connected with scheme } \mathcal{T}\right] \leq C \sum_{(y_i)_{i=k+1}^{n+k-1}\in(\mathbb{Z}^d)^{n-1}} \prod_{A_iA_j\in\mathcal{E}} (|y_i - y_j| + 1)^{2-d}. \quad (4.7)$$

Proof: By a simple union bound it is sufficient to prove that

$$\mathbb{P}\left[x_1,\dots,x_k \text{ are connected with scheme } \mathcal{T} \text{ using } (y_i)_{i=k+1}^m\right] \le C \prod_{A_iA_j\in\mathcal{E}} (|y_i-y_j|+1)^{2-d}. \quad (4.8)$$

We prove equation (4.8) in two steps. First we show that given subsets $E_1,\dots,E_n$ of W^* with finite ν-measure, one has

$$\mathbb{P}[\exists(\gamma^i)_{i=1}^n \in (\omega_u)^n,\ \forall i\neq j,\ \gamma^i\neq\gamma^j,\ \forall i,\ \gamma^i\in E_i] \le u^n \prod_{i=1}^n \nu(E_i). \quad (4.9)$$

Indeed let $\omega_{\mathrm{d}t}=\omega_{t,t+\mathrm{d}t}$ denote infinitesimal division of the Poisson process. One has

$$\mathbb{P}[\exists(\gamma^i)_{i=1}^n \in (\omega_u)^n,\ \forall i\neq j,\ \gamma^i\neq\gamma^j,\ \forall i,\ \gamma^i\in E_i] \le \int_{\{(t_i)_{i=1}^n\in[0,u]^n \ \mid\ \forall i\neq j\ t_i\neq t_j\}} \mathbb{P}[\forall i\in[1,n]\ \omega_{\mathrm{d}t_i}\cap E_i\neq\emptyset]. \quad (4.10)$$

Indeed the integral in the second line is the expected value of the number of n-tuple of distinct trajectories $(\gamma^i)_{i=1}^n$ that satisfies $\gamma^i\in E_i$ for all $i=1,\dots,n$. From the definition of a Poisson point process and indepence of the increments $\omega_{\mathrm{d}t_i}$ this is equal to

$$\int_{\{(t_i)_{i=1}^n\in[0,u]^n \ \mid\ \forall i\neq j\ t_i\neq t_j\}} \prod_{i=1}^n \nu(E_i)\mathrm{d}t_i = u^n\prod_{i=1}^n \nu(E_i). \quad (4.11)$$

Secondly we show that for any choice of points $(z_i)_{i=1}^m$ one has

$$\nu(\{\gamma : \gamma \text{ visits } z_1,z_2,\dots,z_m \text{ in that order }\}) \le C_m \prod_{i=1}^{m-1}\frac{1}{(|z_{i+1}-z_i|+1)^{d-2}}. \quad (4.12)$$

Parameterizing $\gamma=(\gamma_n)_{n\ge 0}$ so that 0 is the first time of visit of z_1 and using the definition of ν given by (2.8)-(2.9) one has

$$\begin{aligned}\nu(\{\gamma : \gamma \text{ visits } z_1,z_2,\dots,z_m \text{ in that order }\})\\ &= P_0(\tilde H_0=\infty)P_{z_1}(\exists n_2\le n_3\le\dots\le n_m, \forall i\in[2,m],\ X_{n_i}=z_i)\\ &= P_0(\tilde H_0=\infty)\prod_{i=1}^{m-1} P_{z_i}(H_{z_{i+1}}<\infty), \quad (4.13)\end{aligned}$$

where the last inequality follows by multiple application of the Markov property at the successive stopping times H_{z_i}. Then (4.12) is deduced by using (2.7).

Combining (4.12) with (4.9) used for the events $E_i:=\{\gamma^i$ visits successively $y_{a_1^i}$ $\dots,y_{a_m^i}\}$ where $A_{a_1^i}A_{a_2^i}\dots A_{a_m^i}$ are the paths corresponding to oriented edges of type i in $\mathcal{T}$ we get (4.8).

□

Our problem is that for some schemes in $\mathbb{T}_n$, the r.h.s of (4.7) diverges. Therefore, we must first identify which are the bad trees for which that happens and prove (4.6) for them without using (4.7). Afterwards, we use the following proposition that gives an upper bound for the r.h.s. of (4.7) for the good trees, and allows us to conclude.

Proposition 4.5. *Given a labeled tree T with k leafs $A_1, \dots, A_k$ and m nodes $A_{k+1}, \dots, A_{k+m}$ and edges E, we associate to each edge a length $l(e) \in [0, d)$. Suppose that the lengths of the edges are such that:*

(i) *The total length of the tree $l(T) = \sum_{e \in E} l(e)$ is strictly smaller than $d(k-1)$.*
(ii) *The length of any (strict) subtree containing at least k_1 of the original leafs A_i is at least $d(k_1 - 1)$.*

Then for any $\varepsilon > 0$ there exists a constant C_ε such that, for every $x_1, \dots, x_k$

$$\sum_{(y_i)_{i=k+1}^{k+m} \in (\mathbb{Z}^d)^m} \prod_{A_i A_j \in E} (|y_i - y_j| + 1)^{l(A_i A_j) - d} \leq C_\varepsilon \max_{i \neq j} |x_i - x_j|^{d(k-1) - l(T) + \varepsilon}. \quad (4.14)$$

where we use the convention that $y_i = x_i$ for $i \leq k$.

The proof is postponed to the end of the section.

Proof of Proposition 4.1: The statement is proved by induction on k. The case $k = 2$ can easily be proved using Proposition 4.5. In that case, the tree is a segment of n edges in series linking to leaves. So we only need to focus on the induction step. It is necessary to prove (4.6) for all trees with $k + n - 1$ vertices.

First consider the trees where there exists $i \leq k$ such that A_i is not a leaf (after permutation of the indices we can consider that A_1 is not a leaf). In that case A_1 has degree two and the tree $\mathcal{T}$ can be split into two trees, each of them linking k_1 and k_2 of the A_is together, and using respectively n_1 and n_2 types of edges respectively, with $k_1 + k_2 = k + 1$ and $n_1 + n_2 = n + 1$ (recall that the two edges getting out of A_1 are of the same type).

As $n < n(k, d)$, one has either $n_1 < n(k_1, d)$ or $n_2 < n(k_2, d)$. Suppose without loss of generality that $n_1 < n(k_1, d)$. In that case a subset of $k_1 < k$ vertices is connected by $n_1 < n(k_1, d)$ trajectories (see Remark 4.3) and one can use the induction hypothesis to get (4.6). In the rest of the proof we consider only trees for which all the A_is, $i \leq k$ are leafs.

A connected subgraph of $\mathcal{T}$ which is a tree and whose leafs are leafs of $\mathcal{T}$ is said to be a proper subtree of $\mathcal{T}$. We consider now the trees $\mathcal{T}$ with $k + n - 1$ vertices that have a proper subtree with k_1 vertices and that uses only edges of n_1 different types with $n_1 < n(k_1, d)$. Then according to Remark 4.3, a subset of $k_1 < k$ vertices is connected by $n_1 < n(k_1, d)$ trajectories and again one can prove (4.6) using the induction hypothesis.

Now suppose that $\mathcal{T}$ is a tree for which all subtrees with $k_1 < k$ vertices use at least $n(k_1, d)$ type of edges. To each edge of the tree, we associate an edge-length 2, and apply Proposition 4.5 to conclude. Assumption (i) of the proposition is satisfied since $n < n(k, d)$ and the total number of edges $n + k - 2$ is given by Proposition 4.2. Assumption (ii) is satisfied because of our assumption on proper subtrees, indeed the reader can check that if a proper subtree with k_1 vertices uses n_1 type of edges, it must have at least $n_1 + k_1 - 2$ edges: this is because vertices in the tree have degree at most 4 and that on vertices of degree 3 two of the incident edges have the same type, and on vertices of degree 4, one has two pairs of incident edges with the same type (by (ii) of Proposition 4.2). □

Proof of Proposition 4.5: We perform the proof by induction on k. When $k=2$, it is easy to show that the sum is equal to

$$O(|x_1-x_2|^{l(T)-d}(\log|x_1-x_2|)^{\#\{\text{edges of length } 0\}})$$

where $l(T)$ is the length of the tree.

When $k\geq 3$ our strategy is to bound the r.h.s of (4.14) by sums corresponding to trees with $k-1$ vertices and then conclude by using the induction hypothesis.

We remark that if T includes two edges e and e' linked to a common vertex of degree two, one can replace it by a unique edge of length $l(e')+l(e)+\delta$ (see Figure 4.4). Indeed as long as $l(e')+l(e)<d$ we have

$$\sum_{y\in\mathbb{Z}^d}(|x-y|+1)^{l(e)-d}(|y-z|+1)^{l(e')-d}=O((1+|x-z|)^{l(e)+l(e')+\delta-d}). \quad (4.15)$$

So if one calls T_1 the tree obtained after this change (relabeling the vertices of T_1 from $A_1,\dots,A_{k+m-1}$, calling E_1 the corresponding edge set and for simplicity denote by l the length of the edges on the new tree) one get that there exists a constant C such that

$$\sum_{(y_i)_{i=k+1}^{k+m}\in(\mathbb{Z}^d)^m}\prod_{A_iA_j\in E}(|y_i-y_j|+1)^{l(A_iA_j)-d}$$
$$\leq C\sum_{(y_i)_{i=k+1}^{k+m-1}\in(\mathbb{Z}^d)^m}\prod_{A_iA_j\in E_1}(|y_i-y_j|+1)^{l(A_iA_j)-d}. \quad (4.16)$$

Note that adding the δ is only necessary if one of the edges has length zero in order to avoid having a log term. Also note that one can choose the δ small enough so that after this transformation $l(T_1)\leq d(k-1)$. In particular, this implies that all the edges are still of length smaller than d.

Then after having reduced all consecutive edge in this manner we obtain (what we call the first stage of the reduction) a tree T' with $k+m'$ vertices ($m'\leq m$) and k leaves, no vertices of degree 2, and satisfying

$$\sum_{(y_i)_{i=k+1}^{k+m}\in(\mathbb{Z}^d)^m}\prod_{A_iA_j\in E}(|y_i-y_j|+1)^{l(A_iA_j)-d}$$
$$\leq C\sum_{(y_i)_{i=k+1}^{k+m'}\in(\mathbb{Z}^d)^m}\prod_{A_iA_j\in E'}(|y_i-y_j|+1)^{l(A_iA_j)-d}. \quad (4.17)$$

We can chose the δ small enough so that $l(T_1)\leq l(T)+\varepsilon/2$.

After the first stage of the reduction, it is possible to find in T' two leafs at graph distance 2 of each another (i.e. separated by only two edges): say without loss of generality that A_k and A_{k-1} are linked to A_{k+1} with edges A_kA_{k+1} and $A_{k+1}A_{k-1}$ of length l_1 resp. l_2. We consider the inequality

$$(x_k-y_{k+1})^{l_1-d}(x_{k-1}-y_{k+1})^{l_2-d}\leq(x_k-y_{k+1})^{l_1+l_2-2d}+(x_{k-1}-y_{k+1})^{l_1+l_2-2d}. \quad (4.18)$$

Let T_1'' and T_2'' be trees with $k-1$ leafs, obtained by replacing the edges e and e' in T' by a unique edge e'' of length $l_1+l_2-d\geq 0$ linking A_{k+1} and A_k resp.

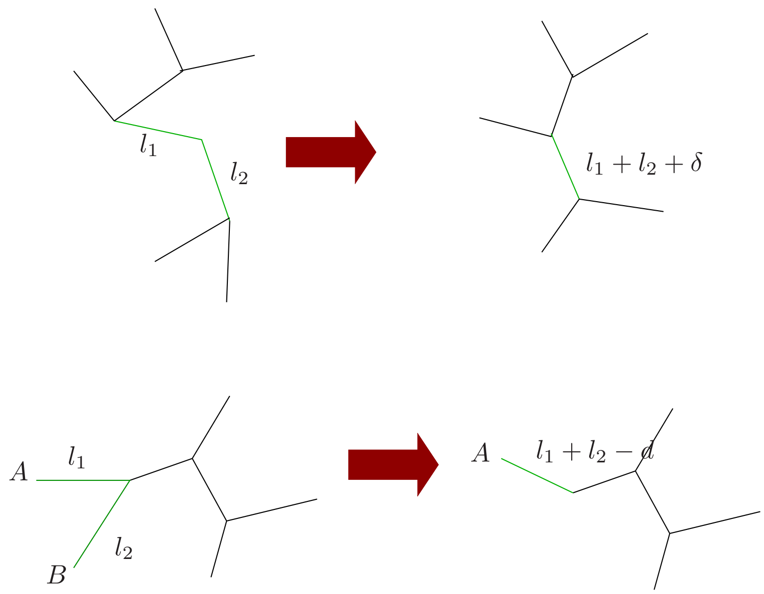

FIGURE 4.4. Illustration of the two stages of the tree reduction procedure

A_{k+1} and A_{k-1} and deleting the vertex left alone (A_{k-1} resp. A_k). Indeed using (4.18) one gets that

$$\sum_{(y_i)_{i=k+1}^{k+m'} \in (\mathbb{Z}^d)^{m'}} \prod_{A_iA_j \in E'} (|y_i - y_j| + 1)^{l(A_iA_j)-d}$$
$$\leq \sum_{(y_i)_{i=k+1}^{k+m'} \in (\mathbb{Z}^d)^{m'}} \prod_{A_iA_j \in E_1''} (|y_i - y_j| + 1)^{l(A_iA_j)-d}$$
$$+ \sum_{(y_i)_{i=k+1}^{k+m'} \in (\mathbb{Z}^d)^{m'}} \prod_{A_iA_j \in E_1''} (|y_i - y_j| + 1)^{l(A_iA_j)-d}. \quad (4.19)$$

where E_1'' and E_2'' denote the edge sets of T_1'' and T_2'' respectively.

Note that for $i = 1, 2$, $l(T_i'') = l(T') - d \leq l(T) - d + \varepsilon/2$ so that condition (i) is satisfied if ε is small enough (the new tree has one less leaf). Note that any proper subtree of T'' that does not contain e'' is also a proper subtree of T' and any proper subtree τ of T'' that contains e'' can be associated to a subtree τ' of T' by replacing e'' by e and e' (the inverse of the above transformation) such that $l(\tau') = l(\tau) + d$ and τ' has one more leaf than τ. Hence if condition (ii) is satisfied for T' it is also satisfied for T'' so that one can apply the induction hypothesis (with $\varepsilon/2$) on the trees T_1'' and T_2'' to conclude.

□

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