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Fluctuations for the number of records on subtrees of the Continuum Random Tree

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Abstract. We study the asymptotic behavior of the number of cuts $X(T_n)$ needed to isolate the root in a rooted binary random tree T_n with n leaves. We focus on the case of subtrees of the Continuum Random Tree generated by uniform sampling of leaves. We elaborate on a recent result by Abraham and Delmas, who showed that $X(T_n)/\sqrt{2n}$ converges a.s. towards a Rayleigh-distributed random variable Θ, which gives a continuous analog to an earlier result by Janson on conditioned, finite-variance Galton-Watson trees. We prove a convergence in distribution of $n^{-1/4}(X(T_n)-\sqrt{2n}\Theta)$ towards a random mixture of Gaussian variables. The proofs use martingale limit theor y for random processes defined on the CRT, related to the theory of records of Poisson point processes.

1. Introduction

The Continuum Random Tree (CRT) is a random metric measure space, introduced by Aldous (Aldous (1991)) as a scaling limit of various discrete random tree models. In particular, if we consider μ, a critical probability measure on $\mathbb{N}$, with variance $0 < \sigma^2 < \infty$ and if we consider a random Galton-Watson tree $\mathcal{T}_n$ with offspring distribution μ, conditioned on having n vertices, then we have the following convergence in distribution:

$$\lim_{n\to\infty} \frac{\sigma}{\sqrt{n}} \mathcal{T}_n = \mathscr{T}, \tag{1.1}$$

in the sense of Gromov-Hausdorff convergence of compact metric spaces (see for instance Duquesne and Le Gall (2005) for more information about the Gromov-Hausdorff topology), where $\mathscr{T}$ is a CRT. The family of conditioned Galton-Watson trees turns out to be quite large, since it contains for instance uniform rooted

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planar binary trees (take $\mu(0) = \mu(2) = 1/2$) or uniform rooted labelled trees (Cayley trees, take $\mu(k) = e^1/k!,\ k \geq 0$). There is a combinatorial characterization of conditioned Galton-Watson trees: they correspond to the class of so-called *simply generated trees* (see Janson (2012) for a detailed survey).

In their 1970 paper (Meir and Moon (1970)), Meir and Moon considered the problem of isolating the root through uniform cuts in random Cayley trees. The problem is as follows: start with a rooted discrete tree $\mathcal{T}_n$, having exactly n edges (in our context, *rooted* means that, among the $n+1$ vertices of $\mathcal{T}_n$, one has been distinguished). At each step, remove an edge, selected uniformly among all edges, then discard the connected component not containing the root. This procedure is iterated on the remaining tree until the root is the only remaining vertex. The number $X(\mathcal{T}_n)$ of cuts that is needed to isolate the root is random, with values in $\{1, \dots, n\}$.

Meir and Moon showed that when $\mathcal{T}_n$ is a uniform Cayley tree with n edges,

$$\mathbb{E}[X(\mathcal{T}_n)] \sim \sqrt{\pi n/2} \quad \text{and} \quad \mathrm{Var}(X(\mathcal{T}_n)) \sim (2 - 1/\pi)n.$$

Later, the limiting distribution was found to be the Rayleigh distribution (the distribution on $[0, \infty)$ with density $x \exp(-x^2/2)dx$) by Panholzer f or (a subset of) the class of simply generated trees (Panholzer (2006)) and, using a different proof, by Janson for the class of critical, finite-variance, conditioned Galton-Watson trees (Janson (2006)).

In Janson (2006), the distribution of the limiting Rayleigh variable was obtained using a moment problem, but the question arose whether it had a connection with the convergence (1.1) above. Indeed, it is well-known that the distance from the root to a uniform leaf of the CRT is Rayleigh-distributed. As a consequence, several approaches were used to describe a cutting procedure on the CRT that could account for the convergence of $X(\mathcal{T}_n)/\sqrt{n}$. All these works are relying on the Aldous-Pitman fragmentation of the CRT, first described in (Aldous and Pitman (1998)). We will give below a brief description of this procedure, as it will be central in this work. Using an extension of the Aldous-Broder algorithm, Addario-Berry, Broutin and Holmgren described a fragmentation-reconstruction procedure for Cayley trees and its analog for the CRT. The invariance they prove shows that the limiting random variable in Janson's result can indeed be realized as the height of a uniform leaf in a CRT. However, it is not the same CRT as the one arising from the scaling limit of $\mathcal{T}_n/\sqrt{n}$. Indeed, the random variables $n^{-1/2}\mathcal{T}_n$ and $n^{-1/2}X(\mathcal{T}_n)$ do *not* converge jointly to a CRT $\mathscr{T}$ and the height of a random leaf $H(\mathscr{T})$. Bertoin and Miermont (Bertoin and Miermont (2012)) describe the so-called *cut-tree* $\mathrm{cut}(\mathscr{T})$ of a given CRT $\mathscr{T}$ following the genealogy of fragments in the Aldous-Pitman fragmentation. The limiting variable can then be described as the height of a uniform leaf in $\mathrm{cut}(\mathscr{T})$, which is again a CRT, thus recovering Rayleigh distribution.

Following Abraham and Delmas (Abraham and Delmas (2013)), we will use a different point of view, based on the theory of records of Poisson point processes. We will now review some of their results, in order to set the notations and to describe the framework.

1.1. *The Brownian CRT.* In this section, we will recall some basic facts about the Brownian CRT. For details, see Aldous (1991); Duquesne and Le Gall (2005). We will write $\mathbb{T}$ for the set of (pointed isometry classes of) compact, rooted real trees endowed with a finite Borel measure. Recall that real trees are metric spaces (X, d) such that

(i) For every $s, t \in X$, there is a unique isometric map $f_{s,t}$ from $[0, d(s,t)]$ to X such that $f_{s,t}(0) = s$ and $f_{s,t}(d(s,t)) = t$. The image of $f_{s,t}$ is noted $[\![s,t]\!]$.

(ii) For every $s, t \in X$, if q is a continuous injective map from $[0,1]$ to X such that $q(0) = s$ and $q(1) = t$, then $q([0,1]) = f_{s,t}([0, d(s,t)])$.

There exists a metric on $\mathbb{T}$ that makes it a Polish metric space, but we will not attempt to describe it here. For more details, see Abraham et al. (2013).

The Brownian CRT (or Aldous's CRT) is a random element of $\mathbb{T}$, defined using the so-called *contour process* description: if f is a continuous nonnegative map $f : [0,\sigma] \to \mathbb{R}_+$, such that $f(0) = f(\sigma) = 0$, then the real tree encoded by f is defined by $\mathscr{T}_f = [0,\sigma]_{/\sim_f}$, where $\sim_f$ is the equivalence relation

$$x \sim_f y \;\Leftrightarrow\; f(x) = f(y) = \min_{u \in [x \wedge y, x \vee y]} f(u), \quad x, y \in [0,\sigma].$$

The metric on $\mathscr{T}_f$ is defined by

$$d_f(x,y) = f(x) + f(y) - 2 \min_{u \in [x \wedge y, x \vee y]} f(u), \quad x, y \in [0,\sigma],$$

so that $d_f(x,y) = 0$ if and only if $x \sim_f y$. Hence, d_f is definite-positive on $\mathscr{T}_f$ and defines a true metric. It can be checked (see Duquesne and Le Gall (2005)) that $(\mathscr{T}_f, d_f)$ is indeed a real tree. We define the *mass-measure* $\mathbf{m}^{\mathscr{T}_f}$ on $\mathscr{T}_f$ as the image of Lebesgue measure on $[0,\sigma]$ by the canonical projection $[0,\sigma] \to \mathscr{T}_f$. Thus, $\mathbf{m}^{\mathscr{T}_f}$ is a finite measure on $\mathscr{T}_f$, with total mass $\mathbf{m}^{\mathscr{T}_f}(\mathscr{T}_f) = \sigma$. When the context is clear, we will usually drop the reference to the tree and write $\mathbf{m}$ for the mass-measure $\mathbf{m}^{\mathscr{T}}$.

Now, the Brownian Continuum Random Tree (CRT) corresponds to the real tree encoded by $f = 2B^{\mathrm{ex}}$, twice the normalized Brownian excursion. Since the length of the normalized Brownian excursion is 1 a.s., the CRT has total mass 1, *i.e.* the mass measure $\mathbf{m}$ is a probability measure. The distribution of the CRT will be noted $\mathbb{P}$, or sometimes $\mathbb{P}^{(1)}$ if we want to emphasize the fact that $\mathbf{m}$ has mass 1. Sometimes, we will consider *scaled* versions of the CRT. If $r > 0$, we consider the scaled Brownian excursion

$$B_t^{\mathrm{ex},r} = \sqrt{r} B_{t/r}^{\mathrm{ex}}, \; t \in [0,r]$$

and the associated real tree $\mathscr{T}_{2B^{\mathrm{ex},r}}$, whose distribution will be noted $\mathbb{P}^{(r)}$. Note that the transformation above corresponds to rescaling all the distances in a $\mathbb{P}^{(1)}$-distributed tree by a factor $\sqrt{r}$.

The measure $\mathbf{m}$ is supported by the set of *leaves* of $\mathscr{T}$, which are the points $\mathrm{x} \in \mathscr{T}$ such that $\mathscr{T} \setminus \{\mathrm{x}\}$ is connected. There is another natural measure ℓ defined on the CRT, called *length measure*, which is σ-finite and such that $\ell([\![x,y]\!]) = d(x,y)$. Also, the CRT is rooted at one particular vertex $\emptyset$, which is the equivalence class of 0, but it can be shown (see Proposition 4.8 in Duquesne and Le Gall (2005)) that if x is chosen according to $\mathbf{m}$, then, if $\mathscr{T}^{\mathrm{x}}$ is the tree $\mathscr{T}$ re-rooted at x, $(\mathscr{T}, \mathrm{x})$ has same distribution as $(\mathscr{T}^{\mathrm{x}}, \emptyset)$.

When we consider the real tree $\mathscr{T}$ encoded by $2B$, where B is an *excursion* of Brownian motion, distributed under the (σ-finite) excursion measure $\mathbb{N}$, we get that $\mathscr{T}$ is a compact metric space, with a length measure ℓ and with a finite measure

$\mathbf{m}$. We will write σ for the (random) total mass of $\mathbf{m}$. Under $\mathbb{N}$, σ is distributed as the length of a random excursion of Brownian Motion, that is

$$\mathbb{N}[\sigma \geq t] = \sqrt{\frac{2}{\pi t}}.$$

The Brownian CRT can be seen as a conditioned version of the tree distributed as $\mathbb{N}[d\mathscr{T}]$, in the sense that, if F is some nonnegative measurable functional defined on the tree space $\mathbb{T}$, then

$$\mathbb{N}[F(\mathscr{T})] = \int_0^\infty \frac{d\sigma}{\sqrt{2\pi}\ \sigma^{3/2}} \mathbb{E}^{(\sigma)}[F(\mathscr{T})].$$

In the sequel, we will make use of this disintegration of $\mathbb{N}$, since some computations are easier to do under $\mathbb{N}$ (see Lemma 2.4).

1.2. *The Aldous-Pitman fragmentation.* Given a CRT $\mathscr{T}$, we consider a Poisson point process

$$\mathcal{N}(ds, dt) = \sum_{i \in I} \delta_{(s_i, t_i)}(ds, dt)$$

on $\mathscr{T} \times \mathbb{R}_+$, with intensity $\ell(ds) \otimes dt$. We will sometimes refer to $\mathcal{N}$ as the *fragmentation measure*. If (s_i, t_i) is an atom of $\mathcal{N}$, we will say that the point s_i was *marked* at time t_i. For $t \geq 0$, we can consider the connected components of $\mathscr{T}$ separated by the atoms of $\mathcal{N}(\cdot \times [0,t])$. They define a random forest $\mathcal{F}_t$ of subtrees of $\mathscr{T}$. Aldous and Pitman proved that if we consider the trees $(\mathscr{T}_k(t), k \geq 1)$ composing $\mathcal{F}_t$, ranked by decreasing order of their mass, then the process

$$((\mathbf{m}(\mathscr{T}_1(t)), \mathbf{m}(\mathscr{T}_2(t)), \dots),\ t \geq 0)$$

is a binary, self-similar fragmentation process, with index 1/2 and erosion coefficient 0, according to the terminology later framed by Bertoin.

1.3. *Separation times.* In order to give a continuous analogue to the cutting procedure on discrete trees described above, we will use the Aldous-Pitman fragmentation on the CRT. Given a CRT $\mathscr{T}$ and a fragmentation measure $\mathcal{N}$, we will define, for any $s \in \mathscr{T}$, the *separation time* from the root $\emptyset$ by

$$\theta(s) = \inf\ \{t \geq 0,\ \mathcal{N}([\![\emptyset, s]\!] \times [0,t]) \geq 1\},$$

with the convention $\inf \emptyset = +\infty$. This separation process will be our main object of study. Note that, under the definition above, conditionally on $\mathscr{T}$, $\theta(\emptyset) = \infty$ a.s., and $\theta(s) < \infty$ a.s. for all $s \neq \emptyset$, since $\theta(s)$ is then exponentially distributed with parameter $\ell([\![\emptyset, s]\!]) = d(\emptyset, s)$. Note also that $\theta(s) \to \infty$ when $s \to \emptyset$, which justifies our convention for $\theta(\emptyset)$.

It is also possible to define the separation process started from any $q \geq 0$, rather than from infinity. In order to do this, we consider only the marks whose t-component is smaller than q:

$$\theta(s) = \inf\ \{0 \leq t \leq q,\ \mathcal{N}([\![\emptyset, s]\!] \times [0,t]) \geq 1\}, \tag{1.2}$$

with the convention $\inf \emptyset = q$. Note that, under this definition, we always have $\theta(\emptyset) = q$, as well as $\lim \theta(s)_{s \to \emptyset} = q$ a.s. In the case where $q = \infty$, we recover the same distribution as the separation process defined earlier. The (quenched) distribution of the separation process started at $q \in [0, \infty]$ on a given CRT $\mathscr{T}$ will be noted $\mathbb{P}_q^{\mathscr{T}}$.

We will also note $\mathbb{P}_q^{(r)}$ the (annealed) distribution of the process $(\theta(s), s \in \mathscr{T})$ started at $q \in [0, \infty]$, when $\mathscr{T}$ is distributed as a Brownian CRT with mass $r > 0$:

$$\mathbb{P}_q^{(r)} = \int_{\mathbb{T}} \mathbb{P}^{(r)}(d\mathscr{T}) \, \mathbb{P}_q^{\mathscr{T}}.$$

Again, to keep things simple, we will usually work under $\mathbb{P}_\infty = \mathbb{P}_\infty^{(1)}$. The jump points of the separation process correspond to points s marked by the fragmentation measure at a time t where they belong to the connected component of the root. This implies that they accumulate in the neighbourhood of the root if $q = \infty$. If T is a subtree of $\mathscr{T}$, we note $X(T)$ the number of jumps of the separation process on T. This number can be finite or infinite, according to whether T contains the root or not, in the case $q = \infty$.

1.4. *Linear record process.* One can consider the record process on the real line (*i.e.* when $\mathscr{T} = \mathbb{R}_+$), defined using a Poisson point measure with intensity $ds \otimes dt$. We get, for any $q \in (0, \infty]$, a random process $(\theta(s),\ s \geq 1)$ such that $\theta(0) = q$, $\mathbb{P}_q^{\mathbb{R}_+}$-a.s. The distribution of this process will be noted $\mathbf{P}_q = \mathbb{P}_q^{\mathbb{R}_+}$. We can consider the jump process

$$X_t = \sum_{s \in [0,t]} \mathbf{1}_{\{\theta(s-) > \theta(s)\}},$$

counting the number of jumps of θ on $[0, t]$. It should be noted that if $q = \infty$, then θ jumps infinitely often in the neighbourhood of the root, so that a.s. $X_t = \infty$ for any $t > 0$. It is easy to check that, for any bounded, measurable functional g defined on $[0, q]$, we have

$$\mathbf{E}_q \left[g(\theta(s)) \right] = \mathrm{e}^{-qs} \, g(q) + \int_0^q g(x) s \, \mathrm{e}^{-sx} \, dx.$$

In particular,

$$\mathbf{E}_q \left[\theta(s) \right] = \frac{1 - \mathrm{e}^{-qs}}{s}. \tag{1.3}$$

When $q < \infty$, if $t \geq 0$, and conditionally on $\theta(t) = q'$, the next jump of θ can be seen to be equal to $\inf \{s \geq t,\ \mathcal{N}([0, q'], [t, s]) \geq 1\}$, which is exponentially distributed, with parameter q'. Thus, X is the counting process of a point measure on $\mathbb{R}_+$ with intensity $\theta(s) ds$. Elementary properties of counting processes of point measures (see Abraham and Delmas (2013) for more details) then show that, for any $q \in (0, \infty)$, the processes

$$\left(N_t = X_t - \int_0^t \theta(s) \; ds,\ t \geq 0 \right) \tag{1.4}$$

$$\left(N_t^2 - \int_0^t \theta(s) \; ds,\ t \geq 0 \right) \tag{1.5}$$

$$\left(N_t^4 - 3 \left(\int_0^t \theta(s) \; ds \right)^2 - \int_0^t \theta(s) \; ds,\ t \geq 0 \right) \tag{1.6}$$

are $\mathbf{P}_q$-martingales in the natural filtration of θ.

1.5. *Number of records on subtrees.* Given a CRT $\mathscr{T}$, let $(\mathrm{x}_n,\ n \geq 1)$ be an iid sequence of leaves of $\mathscr{T}$, sampled according to $\mathbf{m}$. If $n \geq 1$, we consider T_n, the subtree spanned by the leaves $(\emptyset, \mathrm{x}_1, \dots, \mathrm{x}_n)$. The tree T_n is a random rooted binary tree with edge-lengths, whose distribution is explicitly known (see Aldous (1993)). Its length $L_n = \ell(\mathsf{T}_n)$ is known to be distributed according to the Chi$(2n)$-distribution, that is

$$\mathbb{P}(L_n \in dx) = \frac{2^{1-n}}{(n-1)!} x^{2n-1} \exp(-x^2/2) \mathbf{1}_{\{x>0\}}\, dx. \tag{1.7}$$

Note that the case $n = 1$ gives a Rayleigh distribution, as was mentioned earlier. It is proven in Abraham and Delmas (2013) that, a.s.:

$$\lim_{n\to\infty} \frac{L_n}{\sqrt{2n}} = 1. \tag{1.8}$$

The tree T_n has exactly $2n-1$ edges. The edge adjacent to the root will be noted $[\![\emptyset, s_{\emptyset,n}]\!]$, where $s_{\emptyset,n}$ is the first branching point in T_n; the height of $s_{\emptyset,n}$ is noted $h_{\emptyset,n} = \ell([\![\emptyset, s_{\emptyset,n}]\!])$. Recall from Proposition 4.10 in Abraham and Delmas (2013) that $\sqrt{n}h_{\emptyset,n}$ converges in distribution to a nondegenerate random variable, and that we have the following moment computation, for $\alpha > -1$:

$$\mathbb{E}\left[h_{\emptyset,n}^{\alpha}\right] = \frac{\Gamma(\alpha+1)}{2^{\alpha/2}} \frac{\Gamma(n-1/2)}{\Gamma(n+\alpha/2-1/2)} \sim_{n\to\infty} \Gamma(\alpha+1) 2^{-\alpha/2} n^{-\alpha/2}. \tag{1.9}$$

We will also use the notation $\mathsf{T}_n^* = (\mathsf{T}_n \setminus [\![\emptyset, s_{\emptyset,n}]\!]) \cup \{s_{\emptyset,n}\}$ for the subtree above the lowest branching point in T_n. When a new leaf x_n is sampled, it gets attached to the tree T_{n-1} through a new edge, that connects to T_{n-1} at the vertex $s_n \in \mathsf{T}_{n-1}$. We write

$$\mathrm{B}_n = (\mathsf{T}_n \setminus \mathsf{T}_{n-1}) \cup \{s_n\} = [\![s_n, \mathsf{x}_n]\!].$$

The quantity X_n^* is the continuum counterpart of the edge-cutting number $X(T_n)$ that can be found in the literature. Indeed, as soon as a jump appears on the first edge $[\![\emptyset, s_{\emptyset,n}]\!]$, all subsequent jumps will be on this edge, even closer to the root. Thus, X_n^* can be seen as the number of cuts before the first cut on $[\![\emptyset, s_{\emptyset,n}]\!]$ was made. In some sense, the first mark appearing on $[\![\emptyset, s_{\emptyset,n}]\!]$ is analog to the last cut needed to isolate the root in the discrete case.

The following theorem is the analog of the convergence (in distribution) that can be found in Janson (2006) $X(\mathcal{T}_n)/\sqrt{n} \to \mathcal{R}$, where $\mathcal{R}$ is Rayleigh-distributed. We will write Θ for the mean separation time $\int_{\mathscr{T}} \theta(ds)\mathbf{m}(ds)$.

Theorem (Abraham and Delmas (2013)). We have $\mathbb{P}_\infty$-a.s:

$$\lim_{n\to\infty} \frac{X_n^*}{\sqrt{2n}} = \Theta. \tag{1.10}$$

Furthermore, under $\mathbb{P}_\infty$, Θ has Rayleigh distribution.

Note that T_n^* has $2n-2$ edges, so that the rescaling is $\sqrt{2n}$. In comparison, Janson considers random trees with n edges, which explains the difference between the two results. It should be noted that Abraham and Delmas show a slightly more general result, since they consider scaled versions of the CRT, proving the result under all the measures $\mathbb{P}_\infty^{(r)}$, $r > 0$. While our main result, Theorem 1.1 below is still true in these cases, we restrict ourselves to the case of Aldous's tree ($r = 1$) for convenience.

The purpose of this work is to investigate the fluctuations of $X_n^*/\sqrt{2n}$ around its limit Θ. It is shown in Theorem 1.1, which is the main result of this work, that these fluctuations are typically of the order $n^{1/4}$.

Theorem 1.1. *Under $\mathbb{P}_\infty$, we have the following convergence in distribution:*

$$\lim_{n\to\infty} n^{1/4}\left(\frac{X_n^*}{\sqrt{2n}} - \Theta\right) = Z, \tag{1.11}$$

where Z is a random variable which is, conditionally on Θ, distributed according to

$$\mathbb{E}_\infty\left[e^{itZ}\middle|\,\Theta\right] = e^{-t^2\Theta/\sqrt{2}}. \tag{1.12}$$

In other words, Z is distributed as $2^{1/4}\sqrt{\Theta}G$, where G is an independent standard normal random variable. As Θ is Rayleigh-distributed under $\mathbb{P}_\infty$, the Laplace transform (1.12) can be explicitly computed, but does not correspond to any known distribution.

The proof of Theorem 1.1 will be carried out in two steps: we write

$$\left(\frac{X_n^*}{\sqrt{2n}} - \Theta\right) = \frac{1}{\sqrt{2n}}\left(X_n^* - \int_{\mathsf{T}_n^*}\theta(s)\ell(ds)\right) + \left(\frac{1}{\sqrt{2n}}\int_{\mathsf{T}_n^*}\theta(s)\ell(ds) - \Theta\right). \tag{1.13}$$

In Section 2, we will show that, when averaging over $\mathscr{T}$, the variance arising from the random choice of the leaves $(\mathrm{x}_n,\ n\geq 1)$ does not bring any significant contribution to (1.11). We prove this by decomposing $\mathscr{T}$ conditionally on its subtree T_n and by proving a general disintegration formula (Lemma 2.4). Therefore, the second term in (1.13) converges to 0 when suitably renormalized.

In Section 3, we prove Theorem 1.1 by showing that, when properly rescaled, the difference $(X_n^* - \int_{\mathsf{T}_n^*}\theta(s)\mathbf{m}(ds))$ is asymptotically normally distributed (Proposition 2.3). This is a consequence of the classical martingale convergence theorems of Hall and Heyde (1980).

In the Appendix, we collect several technical lemmas.

2. Variance in the weak convergence of length measure to mass measure

The main result of this section is Proposition 2.1.

Proposition 2.1. *As $n\to\infty$, we have the following convergence in probability:*

$$\lim_{n\to\infty} n^{1/4}\left(\int_{\mathsf{T}_n^*}\theta(s)\frac{\ell(ds)}{\sqrt{2n}} - \Theta\right) = 0. \tag{2.1}$$

Recall that, conditionally on $\mathscr{T}$, we sample independent leaves $(\mathrm{x}_n,\ n\geq 1)$ with common distribution $\mathbf{m}(d\mathrm{x})$. We will consider the filtration $(\mathcal{F}_n,\ n\geq 1)$ defined by

$$\mathcal{F}_n = \sigma\left(\{(\mathrm{T}_1,\ldots,\mathrm{T}_n),\ (\theta(s),\ s\in\mathsf{T}_n)\}\right),\quad n\geq 1.$$

A key step in the proof of the a.s. convergence of $X_n^*/\sqrt{2n}$ to Θ in Abraham and Delmas (2013) is the convergence of $M_n = \mathbb{E}_\infty[\Theta|\mathcal{F}_n]$. Since $(M_n,\ n\geq 1)$ is a closed L^2 martingale, it converges $\mathbb{P}_\infty^{(1)}$-a.s. (and in L^2) towards $M_\infty = \Theta$ (notice that Θ is indeed $\mathcal{F}_\infty$-measurable, since $\cup_{n\geq 1}\mathsf{T}_n$ is dense in $\mathscr{T}$, and since θ is continuous $\mathbf{m}$-almost everywhere). The proof of Proposition 2.1 will be divided in two. First, we prove the next proposition:

Proposition 2.2. *We have the following convergence in probability:*

$$\lim_{n\to\infty} n^{1/4}\left(\frac{1}{\sqrt{2n}}\int_{\mathsf{T}_n^*}\theta(s)\ell(ds) - \mathbb{E}_\infty[\Theta|\mathcal{F}_n]\right) = 0. \tag{2.2}$$

Then, we prove a more precise statement than the convergence of $\mathbb{E}_\infty[\Theta|\mathcal{F}_n]$ towards Θ.

Proposition 2.3. *We have*

$$\lim_{n\to\infty} n^{1/4}\left(\mathbb{E}_\infty[\Theta|\mathcal{F}_n] - \Theta\right) = 0, \tag{2.3}$$

in probability, as $n\to\infty$.

Of course, Propositions 2.2 and 2.3 imply Proposition 2.1. Before we can prove Proposition 2.2, we need to describe more precisely how the marked tree $(\mathscr{T},\theta)$ is distributed conditionally on $\mathcal{F}_n$.

2.1. *Subtree decomposition.* Given the subtree T_n, the set $\mathscr{T}\setminus\mathsf{T}_n$ is a random forest; let $(\mathcal{X}_i,\ i\in I_n)$ be the collection of its connected components. For any connected component $\mathcal{X}_i$ of $\mathscr{T}\setminus\mathsf{T}_n$, there is a unique point $s_i\in\mathsf{T}_n$ such that

$$\bigcap_{x\in\mathcal{X}_i} [\![\emptyset, x]\!] = [\![\emptyset, s_i]\!].$$

For any $i\in I_n$, we will write $\mathscr{T}_i$ for the tree $\mathcal{X}_i\cup\{s_i\}$, rooted at $s_i\in\mathsf{T}_n$. We will sometimes use the notation

$$\begin{aligned}\mathscr{E}_n =&\{s\in\mathscr{T},\ [\![\emptyset,s]\!]\cap\mathsf{T}_n^* = \emptyset\} \\ =&([\![\emptyset,s_{\emptyset,n}]\!]\setminus\{s_{\emptyset,n}\})\cup\bigcup_{i\in I_n,\ s_i\in[\![\emptyset,s_{\emptyset,n}]\!]}\mathcal{X}_i\end{aligned} \tag{2.4}$$

for the set of all vertices in the tree such that the unique path linking them to the root intersects T_n on $[\![\emptyset,s_{\emptyset,n}]\!]$. Many things are known about the distribution of the forest $(\mathscr{T}_i,\ i\in I_n)$. For instance, Pitman pointed out (see Dong et al. (2006)) that the stickbreaking construction of the CRT in Aldous (1991) implied that the sequence $(\mathbf{m}(\mathscr{T}_i),\ i\in I_n)$, ranked in decreasing order, is distributed according to the Poisson-Dirichlet distribution with parameters $\alpha=1/2$ and $\theta=n-1/2$ (for more background on Poisson-Dirichlet distributions, see Pitman (2006)). We will give another description, focusing on the tree structure of $\mathscr{T}$ conditionally on $\mathcal{F}_n$. This description can be seen as a conditioned version of Theorem 3 in Le Gall (1993).

Lemma 2.4. *Let F be a nonnegative functional on $\mathbb{T}\times\mathsf{T}_n$. Then*

$$\mathbb{E}_\infty\left[\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\middle|\mathcal{F}_n\right] = \int_0^1 \frac{\mathrm{e}^{-L_n^2 v/(2-2v)}}{\sqrt{2\pi}v^{3/2}(1-v)^{3/2}}\,dv\int_{\mathsf{T}_n}\ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]. \tag{2.5}$$

Proof: Let Y be a $\mathcal{F}_n$-measurable random variable; let us compute the quantity $\mathbb{E}^{(1)}_{\infty}\left[Y\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\right]$. In order to do this computation, we will perform a disintegration with respect to σ in the following expression: for $\mu\geq 0$,

$$\begin{aligned}
I(\mu)&=\mathbb{N}_{\infty}\left[Y\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\,\mathrm{e}^{-\mu\sigma}\right]\\
&=\mathbb{N}_{\infty}\left[Y\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\,\mathrm{e}^{-\mu\sigma_i}\,\mathrm{e}^{-\mu\sum_{j\neq i}\sigma_j}\right].
\end{aligned}$$

Using a Palm formula, we get:

$$\begin{aligned}
&=\mathbb{N}_{\infty}\Bigg[Y\int_{\mathsf{T}_n}\ell(ds)\mathbb{N}_{\theta(s)}[F(\mathscr{T},s)\,\mathrm{e}^{-\mu\sigma}]\\
&\qquad\qquad\times\exp\left(-\int_{\mathsf{T}_n}\ell(ds)\int_0^{\infty}\frac{du}{\sqrt{2\pi}u^{3/2}}(1-\mathrm{e}^{-\mu u})\right)\Bigg]\\
&=\mathbb{N}_{\infty}\left[Y\int_{\mathsf{T}_n}\ell(ds)\mathbb{N}_{\theta(s)}[F(\mathscr{T},s)\,\mathrm{e}^{-\mu\sigma}]\,\mathrm{e}^{-L_n\sqrt{2\mu}}\right],
\end{aligned}$$

since $\mathbb{N}[1-\exp(-\mu\sigma)]=\sqrt{2\mu}$. We can disintegrate the σ-finite measure $\mathbb{N}_{\theta(s)}$ according to the total mass σ:

$$\begin{aligned}
I(\mu)&=\mathbb{N}_{\infty}\left[Y\int_{\mathsf{T}_n}\ell(ds)\int_0^{\infty}\frac{dv}{\sqrt{2\pi}v^{3/2}}\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)\,\mathrm{e}^{-\mu\sigma}]\,\mathrm{e}^{-L_n\sqrt{2\mu}}\right]\\
&=\mathbb{N}_{\infty}\Bigg[Y\int_{\mathsf{T}_n}\ell(ds)\int_0^{\infty}\frac{dv}{\sqrt{2\pi}v^{3/2}}\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]\,\mathrm{e}^{-\mu v}\\
&\qquad\qquad\times\int_0^{\infty}L_n\frac{dr}{\sqrt{2\pi r^3}}\,\mathrm{e}^{-\mu r-L_n^2/(2r)}\Bigg],
\end{aligned}$$

using the well-known formula

$$\mathrm{e}^{a\sqrt{2s}}=\int_0^{\infty}\mathrm{e}^{-sr}\,\frac{a}{\sqrt{2\pi r^3}}\,\mathrm{e}^{-a^2/2r}\;\,dr,$$

for the Laplace transform of the density of the 1/2-stable subordinator (see for instance Chapter III, Proposition (3.7) in Revuz and Yor (1999)). By the Fubini-Tonelli theorem, we then get:

$$\begin{aligned}
I(\mu)&=\mathbb{N}_{\infty}\Bigg[Y\int_{\mathsf{T}_n}\ell(ds)\int_0^{\infty}\frac{dv}{\sqrt{2\pi}v^{3/2}}\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]\,\mathrm{e}^{-\mu v}\\
&\qquad\qquad\times\int_v^{\infty}\frac{L_n\,\mathrm{e}^{-\mu(t-v)}\;dt}{\sqrt{2\pi}(t-v)^{3/2}}\,\mathrm{e}^{-L_n^2/(2t-2v)}\Bigg]\\
&=\int_0^{\infty}\frac{\mathrm{e}^{-\mu t}\,dt}{\sqrt{2\pi}t^{3/2}}\mathbb{N}_{\infty}\Bigg[Y\int_{\mathsf{T}_n}\ell(ds)\int_0^{t}\frac{L_nt^{3/2}\;dv}{\sqrt{2\pi}v^{3/2}(t-v)^{3/2}}\\
&\qquad\qquad\times\mathrm{e}^{-L_n^2/(2t-2v)}\,\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]\Bigg].
\end{aligned}$$

Now, we can use the scaling property of the marked tree $(\mathscr{T},\theta)$ under $\mathbb{N}_\infty$ and the fact that the total mass σ has density $dt/(\sqrt{2\pi}t^{3/2})$ under $\mathbb{N}_\infty$, to get that, for any $\mathcal{F}_n$-measurable random variable Y,

$$\mathbb{E}_\infty\left[Y\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\right]=\mathbb{N}_\infty\left[Y\frac{1}{\sqrt{2\pi}}\int_0^1\frac{L_n\,dv}{v^{3/2}(1-v)^{3/2}}\,\mathrm{e}^{-L_n^2/(2-2v)}\right.$$
$$\left.\times\int_{\mathsf{T}_n}\ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]\right].$$

Now, recall the absolute continuity relation the distribution of T_n under $\mathbb{N}_\infty$ and under $\mathbb{E}_\infty$ (Corollary 4 in Le Gall (1993)): for any measurable bounded G,

$$\mathbb{E}_\infty[G(\mathsf{T}_n)]=\mathbb{N}_\infty\left[\ell(\mathsf{T}_n)\,\mathrm{e}^{-\ell(\mathsf{T}_n)^2/2}\,G(\mathsf{T}_n)\right].$$

Since $\exp(-L_n^2/(2-2v))=\exp(-L_n^2/2)\cdot\exp(-L_n^2v/(2-2v))$, we get:

$$\mathbb{E}_\infty\left[Y\sum_{i\in I_n}F(\mathscr{T}_i,s_i)\right]=\mathbb{E}_\infty\left[Y\frac{1}{\sqrt{2\pi}}\int_0^1\frac{dv}{v^{3/2}(1-v)^{3/2}}\,\mathrm{e}^{-L_n^2v/(2-2v)}\right.$$
$$\left.\times\int_{\mathsf{T}_n}\ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[F(\mathscr{T},s)]\right].$$

Taking conditional expectations with respect to $\mathcal{F}_n$ gives the desired result. □

Remark 2.5. Notice that if $F(\mathscr{T},s)=\mathbf{m}(\mathscr{T})$, we find the striking identity

$$\frac{1}{\sqrt{2\pi}}\int_0^1\frac{L_n\,\mathrm{e}^{-L_n^2v/(2-2v)}}{v^{1/2}(1-v)^{3/2}}dv=1. \tag{2.6}$$

In other words, the function $f_a(v)=a\,\mathrm{e}^{-a^2v/(2-2v)}/(\sqrt{2\pi}v^{1/2}(1-v)^{3/2}))$ is a probability density on $(0,1)$ for any $a>0$. This probability distribution has already been described in the context of the Aldous-Pitman fragmentation: if $a>0$, Aldous and Pitman show that it is the distribution of the size of the fragment containing the root at time a. We refer to Aldous and Pitman (1998); Bertoin (2006) for more information on the "tagged fragment" process in self-similar fragmentations.

2.2. *Proof of Proposition 2.2.* We now have everything we need to prove Proposition 2.2.

Proof of Proposition 2.2: We will start from Lemma 4.14 in Abraham and Delmas (2013): we have a.s. for $n\geq 1$

$$-R_n\leq\mathbb{E}_\infty[\Theta|\mathcal{F}_n]-\frac{1}{L_n}\int_{\mathsf{T}_n^*}\theta(s)\;\ell(ds)\leq V_n, \tag{2.7}$$

where we noted $V_n=\mathbb{E}_\infty[\int_{\mathscr{E}_n}\theta(s)\mathbf{m}(ds)|\mathcal{F}_n]$ (recall the definition of $\mathscr{E}_n$ in (2.4)) and where $R_n=\exp(-L_n^2/4)\theta(h_{\emptyset,n})^2/4$. Furthermore, there $\mathbb{P}_\infty$-a.s. exists a constant $C>0$ such that

$$R_n\leq Cn^8\,\mathrm{e}^{-L_n^2/8}.$$

Thus, considering that $L_n/\sqrt{2n}$ converges a.s. to 1 (1.8), we get that $n^{1/4}R_n$ converges a.s. to 0. Therefore, we needn't worry about the left-hand side of (2.7) and the only thing we need to prove is that $n^{1/4}V_n$ converges in distribution to 0 as $n \to \infty$. The proof in Abraham and Delmas (2013) uses a dominated convergence argument to show that V_n a.s. converges to 0, but we will need a more precise estimate for V_n. By definition, using the notation

$$\Theta_i^{(n)} = \int_{\mathscr{T}_i} \theta(s)\ \mathbf{m}(ds), \quad i \in I_n,$$

we have

$$V_n = \mathbb{E}_\infty \left[\int_{\mathscr{E}_n} \theta(s)\ \mathbf{m}(ds) \Big| \mathcal{F}_n \right] = \mathbb{E}_\infty \left[\sum_{i\in I_n} \Theta_i^{(n)} \mathbf{1}_{\{s_i \in [\![\emptyset,\ xrac]\!]\}} \Big| \mathcal{F}_n \right].$$

Using the disintegration formula from Lemma 2.4, we get:

$$V_n = \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{dv}{v^{3/2}(1-v)^{3/2}}\ \mathrm{e}^{-L_n^2 v/(2-2v)} \int_{[\![\emptyset, s_{\emptyset,n}]\!]} \mathbb{E}_{\theta(s)}^{(v)}[\Theta]\ \ell(ds).$$

Using the fact that $\theta(s)$ is, conditionally on T_n, exponentially distributed with parameter s, we get:

$$\begin{aligned}
\mathbb{E}_\infty \left[V_n | \mathsf{T}_n \right] &= \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{dv}{v^{3/2}(1-v)^{3/2}}\ \mathrm{e}^{-L_n^2 v/(2-2v)} \int_0^{h_{\emptyset,n}} ds \int_0^\infty s\,\mathrm{e}^{-st}\, \mathbb{E}_t^{(v)}[\Theta]\ dt \\
&\le \frac{1}{2} \int_0^1 \frac{dv}{v^{3/2}(1-v)^{3/2}}\ \mathrm{e}^{-L_n^2 v/(2-2v)} \\
&\qquad \times \int_0^{h_{\emptyset,n}} ds \left(\int_0^{v^{-1/2}} stv\,\mathrm{e}^{-st}\ dt + \int_{v^{-1/2}}^\infty s\sqrt{v}\,\mathrm{e}^{-st}\ dt \right),
\end{aligned}$$

using the domination $\mathbb{E}_q^{(v)}[\Theta] \le \sqrt{\pi/2}\min(qv, \sqrt{v})$ (Lemma A.4). For technical reasons, we will restrict ourselves to the event $\{h_{\emptyset,n} < 1/2\}$, but this will not be too restrictive, since $h_{\emptyset,n} \to 0$ a.s. Computing the integrals, we eventually get that $\mathbb{E}_\infty[V_n|\mathsf{T}_n]\mathbf{1}_{\{h_{\emptyset,n}<1/2\}}$ is dominated by

$$W_n = \left(\frac{1}{2} \int_0^1 \frac{dv}{v^{1/2}(1-v)^{3/2}}\ \mathrm{e}^{-L_n^2 v/(2-2v)} \int_0^{h_{\emptyset,n}} \frac{1-\mathrm{e}^{-s/\sqrt{v}}}{s}\ ds \right) \mathbf{1}_{\{h_{\emptyset,n}<1/2\}}.$$

We will use the domination $(1-\exp(-s))/s \le \mathbf{1}_{[0,1]}(s) + 2/(s+1)\mathbf{1}_{(1,\infty)}(s)$, which gives:

$$\begin{aligned}
W_n &\le \frac{1}{2} \int_0^1 \frac{\mathrm{e}^{-L_n^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}}\ dv \left(\frac{h_{\emptyset,n}}{\sqrt{v}} \mathbf{1}_{\{h_{\emptyset,n}/\sqrt{v}\le 1\}} \right. \\
&\qquad \left. + \left(1 + 2\log\left(\frac{h_{\emptyset,n}/\sqrt{v}+1}{2} \right) \right) \mathbf{1}_{\{h_{\emptyset,n}/\sqrt{v}\ge 1\}} \right) \mathbf{1}_{\{h_{\emptyset,n}<1/2\}} \\
&= \left(\frac{1}{2} \int_0^{h_{\emptyset,n}^2} \frac{\mathrm{e}^{-L_n^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}} \left(1 - 2\log 2 + 2\log\left(\frac{h_{\emptyset,n}}{\sqrt{v}} + 1 \right) \right)\ dv \right) \mathbf{1}_{\{h_{\emptyset,n}<1/2\}} \qquad (2.8) \\
&\qquad + \left(\frac{1}{2} \int_{h_{\emptyset,n}^2}^1 \frac{\mathrm{e}^{-L_n^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}} \frac{h_{\emptyset,n}}{\sqrt{v}}\ dv \right) \mathbf{1}_{\{h_{\emptyset,n}<1/2\}}. \qquad (2.9)
\end{aligned}$$

As far as (2.8) is concerned, we can dominate $\exp(-\alpha L_n^2 v/(1-v))$ by 1 as well as $(1-v)^{-3/2}$ by its value at $h_{\emptyset,n}^2$, *i.e.* $(1-h_{\emptyset,n}^2)^{-3/2} < (3/4)^{-3/2}$ to get:

$$\begin{aligned}(2.8) &\leq \frac{1}{2(3/4)^{3/2}} \int_0^{h_{\emptyset,n}^2} \frac{dv}{\sqrt{v}} \left(1 - 2\log 2 + 2\log\left(\frac{h_{\emptyset,n}}{\sqrt{v}} + 1\right)\right) \mathbf{1}_{\{h_{\emptyset,n}<1/2\}} \\ &= C \cdot h_{\emptyset,n} \mathbf{1}_{\{h_{\emptyset,n}<1/2\}},\end{aligned}$$

where C is some deterministic constant. Concerning (2.9), we can bound $1/\sqrt{v}$ by $1/h_{\emptyset,n}$, to get:

$$\begin{aligned}(2.9) &\leq \left(\frac{1}{L_n}\int_{h_{\emptyset,n}^2}^1 \frac{1}{2}\frac{L_n\, \mathrm{e}^{-L_n^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}}\, dv\right)\mathbf{1}_{\{h_{\emptyset,n}<1/2\}} \\ &\leq \left(\frac{1}{L_n}\int_0^1 \frac{1}{2}\frac{L_n\, \mathrm{e}^{-L_n^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}}\, dv\right)\mathbf{1}_{\{h_{\emptyset,n}<1/2\}} = \frac{\sqrt{\pi}}{\sqrt{2}L_n}\mathbf{1}_{\{h_{\emptyset,n}<1/2\}},\end{aligned}$$

by equation (2.6). Putting things together, we get that $\mathbb{P}_\infty$-a.s.

$$\mathbb{E}_\infty\left[V_n|\mathsf{T}_n\right]\mathbf{1}_{\{h_{\emptyset,n}<1/2\}} \leq C\cdot h_{\emptyset,n}\mathbf{1}_{\{h_{\emptyset,n}<1/2\}} + \frac{\sqrt{\pi}}{\sqrt{2}L_n}. \tag{2.10}$$

Now, $n^{1/4}h_{\emptyset,n}\mathbf{1}_{\{h_{\emptyset,n}<1/2\}}$ converges in L^1 to 0 thanks to (1.9). Similarly, an easy moment computation using (1.7) for the density of L_n shows that $n^{1/4}/L_n$ also converges in L^1 to 0, so that the same is true for $n^{1/4}V_n\mathbf{1}_{\{h_{\emptyset,n}<1/2\}}$. Hence, $n^{1/4}V_n\mathbf{1}_{\{h_{\emptyset,n}<1/2\}}$ converges to 0 in probability. Since a.s. there is a (random) $n_0 \geq 1$ such that $h_{\emptyset,n} < 1/2$ for any $n \geq n_0$, we also get that $n^{1/4}V_n$ converges to 0 in probability. Combining this with the a.s. convergence to 0 for $n^{1/4}R_n$, we indeed get a convergence in probability:

$$\lim_{n\to\infty} n^{1/4}\left(\mathbb{E}_\infty[\Theta|\mathcal{F}_n] - \frac{1}{L_n}\int_{\mathsf{T}_n^*}\theta(s)\ \ell(ds)\right) = 0. \tag{2.11}$$

To get the announced result, we still have to prove that

$$\lim_{n\to\infty} n^{1/4}\left(\frac{1}{L_n} - \frac{1}{\sqrt{2n}}\right)\int_{\mathsf{T}_n^*}\theta(s)\ \ell(ds) = 0. \tag{2.12}$$

This is not difficult: simply write

$$n^{1/4}\left(\frac{1}{L_n} - \frac{1}{\sqrt{2n}}\right)\int_{\mathsf{T}_n^*}\theta(s)\ell(ds) = n^{1/4}\left(1 - \frac{L_n}{\sqrt{2n}}\right)\left(\frac{1}{L_n}\int_{\mathsf{T}_n^*}\theta(s)\ell(ds)\right).$$

Now, recall that $1/L_n\int_{\mathsf{T}_n^*}\theta(s)\ell(ds)$ converges to Θ $\mathbb{P}_\infty$-a.s., hence in probability. Furthermore, we can compute

$$n^{1/2}\mathbb{E}_\infty\left[\left(1-\frac{L_n}{\sqrt{2n}}\right)^2\right] = n^{1/2}\mathbb{E}_\infty\left(1 + \frac{L_n^2}{2n} - 2\frac{L_n}{\sqrt{2n}}\right),$$

Using the density (1.7) of L_n, we easily get that

$$\mathbb{E}_\infty[L_n] = \sqrt{2}\frac{\Gamma(n+1/2)}{\Gamma(n)} \quad ; \quad \mathbb{E}_\infty\left[L_n^2\right] = 2n.$$

Therefore, after computations, we get $n^{1/2}\mathbb{E}_\infty[(1-L_n/\sqrt{2n})^2] \sim 1/(8\sqrt{n})$, so that in the end, $n^{1/4}(1-L_n/\sqrt{2n})$ converges to 0 in L^2. This implies convergence in probability, hence the convergence of (2.12). □

2.3. *Rate of convergence in the Martingale Convergence Theorem.* Before we can move on to Proposition 2.3, we are going to state a lemma that will be needed in the proof.

Lemma 2.6. *If $1<\alpha<2$, then, the sequence $\int_{\mathsf{T}_n^*}\theta(s)^\alpha \ell(ds)/L_n$ is bounded in $L^1(\mathbb{P}_\infty)$.*

Proof: The main idea is that the measure $\ell(ds)/L_n$ converges a.s. to the mass measure $\mathbf{m}(ds)$, in the sense of weak convergence of probability measures on $\mathscr{T}$. Since the function θ is neither continuous nor bounded on $\mathscr{T}$, we cannot use this fact directly, but it will be the inspiration for the proof. We will compute the first moment of $Z_n=\int_{\mathsf{T}_n^*}\theta(s)^\alpha\ell(ds)/L_n^*$, using the notation $L_n^*=\ell(\mathsf{T}_n^*)$. Since $\theta(s)$ is, conditionally on $\mathscr{T}$, exponentially distributed with parameter $\ell([\![\emptyset,s]\!])$, we get

$$\begin{aligned}\mathbb{E}_\infty[Z_n] &= \mathbb{E}_\infty\left[\int_{\mathsf{T}_n^*}\ell([\![\emptyset,s]\!])^{-\alpha}\ \frac{\ell(ds)}{L_n^*}\right]\\ &= \mathbb{E}_\infty\left[\int_{\mathscr{T}}(d(\emptyset,s)-d(s,\mathsf{T}_n^*))^{-\alpha}\mathbf{1}_{\mathscr{T}\setminus\mathcal{E}_n}(s)\ \mathbf{m}(ds)\right],\end{aligned}$$

where $d(s,\mathsf{T}_n^*)$ is the distance from the leaf s to the closed subtree T_n^* of $\mathscr{T}$. The last equality comes from the fact that if s is a leaf of $\mathscr{T}$ selected uniformly (according to $\mathbf{m}(ds)$) among all leaves of $\mathscr{T}\setminus\mathcal{E}_n$, then its projection $\pi(s,\mathsf{T}_n)$ on T_n is uniformly distributed (according to length measure) on T_n^*. We will rewrite the last expression so as to make the leaves $\emptyset,\mathsf{x}_1,\ldots,\mathsf{x}_n$ apparent. The set $\mathscr{T}\setminus\mathcal{E}_n$ can be written as

$$\mathscr{T}\setminus\mathcal{E}_n=\{s\in\mathscr{T},\ [\![\emptyset,\pi(\emptyset,\mathsf{T}_n^*)]\!]\cap[\![s,\pi(s,\mathsf{T}_n^*)]\!]=\emptyset\}, \tag{2.13}$$

since $\pi(\emptyset,\mathsf{T}_n^*)=s_{\emptyset,n}$. Note that T_n^* is actually the subtree spanned by the n leaves $\mathsf{x}_1,\ldots,\mathsf{x}_n$ and that its definition does not depend on $\emptyset$ or on s.

We then apply the fundamental *re-rooting invariance* of the Brownian CRT, which implies, in this context, that when re-rooting $\mathscr{T}$ at s, the re-rooted tree $\mathscr{T}^s$ is distributed as a CRT, and the sequence $(\emptyset,\mathsf{x}_1,\ldots,\mathsf{x}_n)$ is distributed as a sample of $n+1$ uniform leaves in $\mathscr{T}^s$. Thus,

$$\begin{aligned}\mathbb{E}_\infty[Z_n] &= \mathbb{E}_\infty\left[\int_{\mathscr{T}}(d(\emptyset,s)-d(s,\mathsf{T}_n^*))^{-\alpha}\mathbf{1}_{\{[\![\emptyset,\pi(\emptyset,\mathsf{T}_n^*)]\!]\cap[\![s,\pi(s,\mathsf{T}_n^*)]\!]=\emptyset\}}(s)\ \mathbf{m}(ds)\right]\\ &= \mathbb{E}_\infty\left[\int_{\mathscr{T}}(d(\emptyset,s)-h_{\emptyset,n})^{-\alpha}\mathbf{1}_{\{[\![\emptyset,\pi(\emptyset,\mathsf{T}_n^*)]\!]\cap[\![s,\pi(s,\mathsf{T}_n^*)]\!]=\emptyset\}}(s)\ \mathbf{m}(ds)\right],\end{aligned}$$

since in the re-rooting, $d(s,\mathsf{T}_n^*)$ becomes $d(\emptyset,\mathsf{T}_n^*)=h_{\emptyset,n}$. Therefore, we get, using (2.13) again,

$$\begin{aligned}\mathbb{E}_\infty[Z_n] &= \mathbb{E}_\infty\left[\int_{\mathscr{T}}(d(\emptyset,s)-h_{\emptyset,n})^{-\alpha}\mathbf{1}_{\mathscr{T}\setminus\mathcal{E}_n}(s)\ \mathbf{m}(ds)\right]\\ &= \mathbb{E}_\infty\left[\int_{\mathscr{T}_n^{(1)}\cup\mathscr{T}_n^{(2)}} d(s_{\emptyset,n},s)^{-\alpha}\mathbf{m}(ds)\right],\end{aligned}$$

where $\mathscr{T}_n^{(1)}$ and $\mathscr{T}_n^{(2)}$ are the connected components of $\mathscr{T} \setminus (\mathscr{E}_n \cup \{s_{\emptyset,n}\})$, joined together by their common root $s_{\emptyset,n}$. We can now use the self-similarity property of the fragmentation at heights of the Brownian CRT (see Bertoin (2002)) which shows that, conditionally on $\sigma_n^{(1)} = \mathbf{m}(\mathscr{T}_n^{(1)})$ and $\sigma_n^{(2)} = \mathbf{m}(\mathscr{T}_n^{(2)})$, the trees $\mathscr{T}_n^{(1)}$ and $\mathscr{T}_n^{(2)}$ are rescaled copies of the Brownian CRT. Thus,

$$\begin{aligned}
\mathbb{E}_\infty[Z_n] &= \mathbb{E}_\infty\left[\int_{\mathscr{T}_n^{(1)}} d(s_{\emptyset,n}, s)^{-\alpha}\mathbf{m}(ds)\right] + \mathbb{E}_\infty\left[\int_{\mathscr{T}_n^{(2)}} d(s_{\emptyset,n}, s)^{-\alpha}\mathbf{m}(ds)\right] \\
&= \mathbb{E}_\infty\left[\int_{\mathscr{T}_n^{(1)}} (\sigma_n^{(1)})^{-\alpha/2}\left(\frac{d(s_{\emptyset,n}, s)}{(\sigma_n^{(1)})^{1/2}}\right)^{-\alpha} \sigma_n^{(1)}\frac{\mathbf{m}(ds)}{\sigma_n^{(1)}}\right] \\
&\quad + \mathbb{E}_\infty\left[\int_{\mathscr{T}_n^{(2)}} (\sigma_n^{(2)})^{-\alpha/2}\left(\frac{d(s_{\emptyset,n}, s)}{(\sigma_n^{(2)})^{1/2}}\right)^{-\alpha} \sigma_n^{(2)}\frac{\mathbf{m}(ds)}{\sigma_n^{(2)}}\right] \\
&= \mathbb{E}_\infty\left[\left((\sigma_n^{(1)})^{1-\alpha/2} + (\sigma_n^{(2)})^{1-\alpha/2}\right)\int_{\mathscr{T}} d(\emptyset, s)^{-\alpha}\mathbf{m}(ds)\right],
\end{aligned}$$

using the scaling invariance of the Brownian CRT. Then, as $0 < 1 - \alpha/2$, we can simply dominate $(\sigma_n^{(1)})^{1-\alpha/2}$ and $(\sigma_n^{(1)})^{1-\alpha/2}$ by 1 to get that

$$\mathbb{E}_\infty[Z_n] \leq 2 \cdot \mathbb{E}_\infty\left[\int_{\mathscr{T}} d(\emptyset, s)^{-\alpha}\mathbf{m}(ds)\right].$$

Now, since $d(\emptyset, s)$ is Rayleigh-distributed under $\mathbb{E}_\infty$, we easily see that it has moments of order $-\alpha$ for any $\alpha < 2$, which shows that $\mathbb{E}_\infty[Z_n]$ is indeed bounded, ending our proof. □

We can now turn to the proof of Proposition 2.3.

Proof of Proposition 2.3: Let $M_n = \mathbb{E}_\infty[\Theta|\mathcal{F}_n]$. We will use the fact that

$$n^{1/4}\left(\Theta - \mathbb{E}_\infty[\Theta|\mathcal{F}_n]\right) = n^{1/4}\sum_{k=n+1}^{\infty} \mathbb{E}_\infty[M_k - M_{k-1}|\mathcal{F}_{k-1}]. \tag{2.14}$$

Let $\varepsilon > 0$ be small enough, and consider the events

$$E_k^1 = \{L_k \geq k^{1/2-\varepsilon}\} \quad ; \quad E_k^2 = \{k^{-2} \leq h_{\emptyset,k} \leq 1/2\}, \quad k \geq 1.$$

Recalling that L_k^2 is distributed as the sum of k independent exponential random variables with parameter 1, a simple application of Chernoff's inequality shows that

$$\mathbb{P}_\infty\left(E_k^1\right) \geq 1 - k^{-2\varepsilon k}\exp(k - k^{1-2\varepsilon}). \tag{2.15}$$

For E_k^2, we can use the moment estimation (1.9) for $h_{\emptyset,k}$ to find that, for any $0 \leq \alpha \leq 1$, and for any $\beta > 0$,

$$\begin{aligned}
1 - \mathbb{P}_\infty\left(E_k^2\right) &= \mathbb{P}_\infty\left(\{h_{\emptyset,k} > 1/2\} \cup \{h_{\emptyset,k} < k^{-2}\}\right) \\
&\leq \mathbb{P}_\infty(h_{\emptyset,k} > 1/2) + \mathbb{P}_\infty\left(h_{\emptyset,k}^{-1} \geq k^2\right) \\
&\leq 2^\beta\mathbb{E}_\infty\left[h_{\emptyset,k}^\beta\right] + k^{-2\alpha}\mathbb{E}_\infty\left[h_{\emptyset,k}^{-\alpha}\right] \\
&\sim C \cdot k^{-\beta/2} + C' \cdot k^{-2\alpha}k^{\alpha/2}.
\end{aligned}$$

Hence, by taking $\alpha = 1 - \eta$ (and $\beta > 3\alpha$), we get, for sufficiently large k,

$$\mathbb{P}_\infty\left(E_k^2\right) \geq 1 - k^{-3/2+3/2\eta}. \tag{2.16}$$

Thus, combining equations (2.15) and (2.16), we get that

$$\sum_{k\geq 1} \mathbb{P}_\infty\left(\left(E_k^1\right)^c \cup \left(E_k^2\right)^c\right) \leq \sum_{k\geq 1} \mathbb{P}_\infty\left((E_k^1)^c\right) + \mathbb{P}_\infty\left((E_k^2)^c\right) < \infty.$$

Thus, by the Borel-Cantelli lemma, there a.s. exists $k_0 \geq 1$ such that for $k \geq k_0$, $L_k \geq k^{1/2-\varepsilon}$ and $k^{-2} \leq h_{\emptyset,k} \leq 1/2$. We will use this truncating events in the following way: since the event $E_k = E_k^1 \cap E_k^2$ is $\mathcal{F}_k$-measurable, the usual martingale computations show that

$$\mathbb{E}_\infty\left[\left(n^{1/4}\sum_{k=n}^{\infty}(M_k - M_{k-1})\mathbf{1}_{E_{k-1}}\right)^2\right] = n^{1/2}\sum_{k=n}^{\infty}\mathbb{E}_\infty[(M_k - M_{k-1})^2\mathbf{1}_{E_{k-1}}].$$

We will now give precise estimations of $\mathbb{E}_\infty[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}]$ using the disintegration formula from Lemma 2.4. By definition, for all $k \geq 1$, we can write

$$\Theta = \int_{\mathscr{T}} \theta(s)\ \mathbf{m}(ds) = \sum_{i\in I_k} \Theta_i^{(k)}.$$

Then,

$$\begin{aligned} M_k &= \mathbb{E}_\infty[\Theta|\mathcal{F}_k] \\ &= \mathbb{E}_\infty\left[\sum_{i\in I_{k-1}} \Theta_i^{(k-1)}\middle|\mathcal{F}_k\right] \\ &= \mathbb{E}_\infty[\Theta_{i_k}^{(k-1)}|\mathcal{F}_k] + \mathbb{E}_\infty\left[\sum_{i\in I_{k-1}\setminus\{i_k\}} \Theta_i^{(k-1)}\middle|\mathcal{F}_k\right], \end{aligned}$$

where i_k is the unique index in I_{k-1} such that $\mathrm{x}_k \in \mathscr{T}_{i_k}$. We then define:

$$\begin{aligned} G_k &= \mathbb{E}_\infty\left[\Theta_{i_k}^{(k-1)}|\mathcal{F}_k\right] \\ H_k &= \mathbb{E}_\infty\left[\sum_{i\in I_{k-1}\setminus\{i_k\}} \Theta_i^{(k-1)}\middle|\mathcal{F}_k\right] - \mathbb{E}_\infty\left[\sum_{i\in I_{k-1}} \Theta_i^{(k-1)}\middle|\mathcal{F}_{k-1}\right], \end{aligned}$$

so that we have $M_k - M_{k-1} = G_k + H_k$ and

$$\mathbb{E}_\infty\left[(M_k - M_{k-1})^2|\mathcal{F}_{k-1}\right] \leq 2\mathbb{E}_\infty\left[G_k^2|\mathcal{F}_{k-1}\right] + 2\mathbb{E}_\infty\left[H_k^2|\mathcal{F}_{k-1}\right].$$

As far as G_k is concerned, conditionally on $\mathcal{F}_k$, $\Theta_{i_k}^{(k-1)}$ can be expressed as the sum $\sum_{i\in I_k} \Theta_i^{(k)}\mathbf{1}_{\{s_i\in \mathsf{B}_k\}}$, so that we can use the disintegration formula of Lemma 2.4 to get:

$$G_k = \frac{1}{\sqrt{2\pi}}\int_0^1 \frac{\mathrm{e}^{-L_k^2 v/(2-2v)}}{v^{3/2}(1-v)^{3/2}}\ dv \int_{\mathsf{B}_k} \mathbb{E}_{\theta(s)}^{(v)}[\Theta]\ \ell(ds).$$

Hence, using this expression, we can now compute:

$$\mathbb{E}_\infty[G_k^2|\mathcal{F}_{k-1}] = \mathbb{E}_\infty\left[\left(\frac{1}{\sqrt{2\pi}}\int_0^1 \frac{\mathrm{e}^{-L_k^2 v/(2-2v)}}{v^{3/2}(1-v)^{3/2}}\,dv\int_{\mathsf{B}_k}\mathbb{E}^{(v)}_{\theta(s)}[\Theta]\;\ell(ds)\right)^2\Big|\mathcal{F}_{k-1}\right]$$
$$\leq \mathbb{E}_\infty\left[\left(\frac{1}{\sqrt{2\pi}}\int_0^1 \frac{\mathrm{e}^{-L_k^2 v/(2-2v)}}{v^{1/2}(1-v)^{3/2}}\,dv\int_{\mathsf{B}_k}\ell(ds)\theta(s)\right)^2\Big|\mathcal{F}_{k-1}\right],$$

since $\mathbb{E}^{(v)}_{\theta(s)}[\Theta]\leq v\theta(s)$ (Lemma A.4). Now, the measure

$$\frac{L_n\,\mathrm{e}^{-L_n^2 v/(2-2v)}}{\sqrt{2\pi}v^{1/2}(1-v)^{3/2}}dv$$

is a probability density on [0,1] (cf. (2.6)), so that we get, using the fact that $L_{k-1}<L_k$,

$$\mathbb{E}_\infty\left[G_k^2|\mathcal{F}_{k-1}\right] \leq \mathbb{E}_\infty\left[\left(\frac{1}{L_k}\int_{\mathsf{B}_k}\ell(ds)\theta(s)\right)^2\Big|\mathcal{F}_{k-1}\right]$$
$$\leq \frac{1}{L_{k-1}^2}\mathbb{E}_\infty\left[\left(\int_{\mathsf{B}_k}\ell(ds)\theta(s)\right)^2\Big|\mathcal{F}_{k-1}\right].$$

Now, conditionally on $\mathcal{F}_{k-1}$, the record process on B_k has the distribution of an independent record process on $\mathbb{R}_+$, started from $\theta(s_k)$, stopped at time $\ell(\mathsf{B}_k)$. Furthermore, it is a consequence from the stickbreaking construction of Aldous (see Aldous (1991)) that, conditionally on $\mathcal{F}_{k-1}$, the random variables s_k and $\ell(\mathsf{B}_k)$ are independent. Furthermore, s_k is distributed uniformly on T_{k-1}, and $\ell(\mathsf{B}_k)$ can be expressed as the length of the interval between the $(k-1)$th and the kth jump of a Poisson process with intensity $t\mathbf{1}_{[0,\infty)}(t)dt$. Therefore, conditionally on $\mathcal{F}_{k-1}$, $\ell(\mathsf{B}_k)$ has density

$$r_{L_{k-1}}(dx) = (L_{k-1}+x)\,\mathrm{e}^{-x^2/2-L_{k-1}x}\;\;dx. \tag{2.17}$$

Thus, using the notation $F(q,t)=\mathbf{E}_q[(\int_0^t\theta(s)ds)^2]$ for $0<q<\infty$ and $t\geq 0$, we get

$$\mathbb{E}_\infty\left[G_k^2|\mathcal{F}_{k-1}\right] \leq \frac{1}{L_{k-1}^2}\int_{\mathsf{T}_{k-1}}\frac{\ell(ds)}{L_{k-1}}\int_0^\infty r_{L_{k-1}}(dx)F(\theta(s),x).$$

We will cut the integral in two parts, according to $\mathsf{T}_{k-1}=\mathsf{T}^*_{k-1}\cup(\mathsf{T}_{k-1}\setminus\mathsf{T}^*_{k-1})$. We then use Lemma A.2 to dominate $F(\theta(s),x)$: inequality (A.2) for $s\in\mathsf{T}^*_{k-1}$ and (A.3) for $s\in\mathsf{T}_{k-1}\setminus\mathsf{T}^*_{k-1}$. This leads to:

$$\mathbb{E}_\infty\left[G_k^2|\mathcal{F}_{k-1}\right] \leq \frac{1}{L_{k-1}^2}\int_{\mathsf{T}^*_{k-1}}\frac{\ell(ds)}{L_{k-1}}\int_0^\infty r_{L_{k-1}}(dx)\left(C_1\theta(s)^{3/2}x^{3/2}+C_2\theta(s)x^2\right)$$
$$+\frac{1}{L_{k-1}^2}\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}^*_{k-1}}\frac{\ell(ds)}{L_{k-1}}\int_0^\infty r_{L_{k-1}}(dx)\left(C_3\theta(s)^{1/2}x^{1/2}+C_4\theta(s)^{-1/2}x^{1/2}\right)$$

$$
\begin{aligned}
&= \frac{C_1}{L_{k-1}^2}\left(\int_0^\infty r_{L_{k-1}}(dx)x^{3/2}\right)\left(\int_{\mathsf{T}_{k-1}^*}\frac{\ell(ds)}{L_{k-1}}\theta(s)^{3/2}\right)\\
&+ \frac{C_2}{L_{k-1}^2}\left(\int_0^\infty r_{L_{k-1}}(dx)x^{2}\right)\left(\int_{\mathsf{T}_{k-1}^*}\frac{\ell(ds)}{L_{k-1}}\theta(s)\right)\\
&+ \frac{C_3}{L_{k-1}^3}\left(\int_0^\infty r_{L_{k-1}}(dx)x^{1/2}\right)\left(\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}_{k-1}^*}\ell(ds)\theta(s)^{1/2}\right)\\
&+ \frac{C_4}{L_{k-1}^3}\left(\int_0^\infty r_{L_{k-1}}(dx)x^{1/2}\right)\left(\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}_{k-1}^*}\ell(ds)\theta(s)^{-1/2}\right).
\end{aligned}
$$

We can then compute, using Lemma A.3 for the asymptotic moments of $r_{L_{k-1}}(dx)$:

$$
\begin{aligned}
\mathbb{E}_\infty\left[G_k^2\mathbf{1}_{E_{k-1}}\right] &= \mathbb{E}_\infty\left[\mathbb{E}_\infty\left[G_k^2|\mathcal{F}_{k-1}\right]\mathbf{1}_{E_{k-1}}\right]\\
&\le \mathbb{E}_\infty\left[\int_{\mathsf{T}_{k-1}^*}\frac{\ell(ds)}{L_{k-1}}\theta(s)^{3/2}\right]\cdot O(k^{-7/4+7/2\varepsilon}) \qquad (2.18)\\
&+ \mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}^*}\frac{\ell(ds)}{L_{k-1}}\theta(s)\right)\mathbf{1}_{E_{k-1}}\right]\cdot O(k^{-2+4\varepsilon}) \qquad (2.19)\\
&+ \mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}_{k-1}^*}\ell(ds)\theta(s)^{1/2}\right)\right]\cdot O(k^{-7/4+7/2\varepsilon}) \qquad (2.20)\\
&+ \mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}_{k-1}^*}\ell(ds)\theta(s)^{-1/2}\right)\right]\cdot O(k^{-7/4+7/2\varepsilon}). \qquad (2.21)
\end{aligned}
$$

Using Lemma 2.6, we see that (2.18) is indeed of the order $k^{-7/4+7/2\varepsilon}$. As far as (2.19) is concerned, we simply use Lemma A.1, whose proof can be found in the Appendix, and which implies in particular that (2.19) is of the order $k^{-2+4\varepsilon}$.

In the two remaining terms (2.20) and (2.21), the integral is taken on a single branch; therefore, we can use the linear case to get

$$
\begin{aligned}
\mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}\setminus\mathsf{T}_{k-1}^*}\ell(ds)\theta(s)^{1/2}\right)\mathbf{1}_{E_{k-1}}\right] &= \mathbb{E}_\infty\left[\mathbf{E}_\infty\left[\int_0^{h_{\emptyset,k-1}}\theta(s)^{1/2}ds\right]\mathbf{1}_{E_{k-1}}\right]\\
&= C\cdot\mathbb{E}_\infty[h_{\emptyset,k-1}^{1/2}\mathbf{1}_{E_{k-1}}],
\end{aligned}
$$

which easily converges to 0 as $k\to\infty$. A similar argument shows that (2.21) converges to 0 as $\mathbb{E}_\infty[h_{\emptyset,k-1}^{3/2}\mathbf{1}_{E_{k-1}}]$. Putting everything together, we find that $\mathbb{E}_\infty[G_k^2\mathbf{1}_{E_{k-1}}]$ is of the order $k^{-7/4+7/2\varepsilon}$ as $k\to\infty$, so that $\sum_{k=n}^\infty\mathbb{E}_\infty[G_k^2\mathbf{1}_{E_{k-1}}]$ is of the order $n^{-3/4+7/2\varepsilon}$.

Turning to H_k, we note that $I_{k-1} \setminus \{i_k\} = \{i \in I_k,\ s_i \notin \mathsf{B}_k\}$, so that, using Lemma 2.4, we get:

$$\begin{aligned} H_k =& \mathbb{E}_\infty\left[\sum_{i\in I_k} \Theta_i^{(k)} \mathbf{1}_{\{s_i\notin \mathsf{B}_k\}} \middle| \mathcal{F}_k\right] - \mathbb{E}_\infty\left[\sum_{i\in I_{k-1}} \Theta_i^{(k-1)} \middle| \mathcal{F}_{k-1}\right] \\ =& \int_0^1 \frac{\mathrm{e}^{-L_k^2 v/(2-2v)}}{\sqrt{2\pi} v^{3/2}(1-v)^{3/2}}\, dv \int_{\mathsf{T}_k} \ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[\Theta]\mathbf{1}_{\{s\notin\mathsf{B}_k\}} \\ &- \int_0^1 \frac{\mathrm{e}^{-L_{k-1}^2 v/(2-2v)}}{\sqrt{2\pi} v^{3/2}(1-v)^{3/2}}\, dv \int_{\mathsf{T}_{k-1}} \ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[\Theta], \end{aligned}$$

thus, considering that $\mathsf{T}_k = \mathsf{T}_{k-1} \cup (\mathsf{B}_k \setminus \{s_k\})$, and that of course $\ell(\{s_k\}) = 0$,

$$H_k = \int_0^1 \frac{dv}{\sqrt{2\pi} v^{3/2}(1-v)^{3/2}} \int_{\mathsf{T}_{k-1}} \ell(ds)\mathbb{E}^{(v)}_{\theta(s)}[\Theta]\left(\mathrm{e}^{-L_k^2 v/(2-2v)} - \mathrm{e}^{-L_{k-1}^2 v/(2-2v)}\right).$$

We then use the inequality $|\,\mathrm{e}^{-at} - \mathrm{e}^{-as}\,| \le a\,\mathrm{e}^{-at}(s-t)$, valid for any $a > 0$, and $t \le s$, to find:

$$\begin{aligned} \mathbb{E}_\infty\left[H_k^2|\mathcal{F}_{k-1}\right] \le \left(\frac{1}{\sqrt{2\pi}} \int_0^1 \frac{\mathrm{e}^{-L_{k-1}^2 v/(2-2v)}}{v^{3/2}(1-v)^{3/2}} \frac{v}{2-2v}\, dv \int_{\mathsf{T}_{k-1}} \mathbb{E}^{(v)}_{\theta(s)}[\Theta]\ \ell(ds)\right)^2 \\ \times \mathbb{E}_\infty\left[(L_k^2 - L_{k-1}^2)^2|\mathcal{F}_{k-1}\right]. \quad (2.22) \end{aligned}$$

On the one hand, we will use the change of variables

$$u = L_{k-1}^2 v/(2-2v) \Leftrightarrow v = u/(L_{k-1}^2/2 + u).$$

in the integral, which gives:

$$\left(\frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\mathrm{e}^{-u}}{(L_{k-1}^2/2)^{1/2}} \frac{L_{k-1}^2/2 + u}{\sqrt{u}}\, du \int_{\mathsf{T}_{k-1}} \frac{\ell(ds)}{L_{k-1}^2} \mathbb{E}^{(u/(L_{k-1}^2/2+u))}_{\theta(s)}[\Theta]\right)^2. \quad (2.23)$$

We then cut the integral in two parts, according to $\mathsf{T}_{k-1} = \mathsf{T}^*_{k-1} \cup (\mathsf{T}_{k-1} \setminus \mathsf{T}^*_{k-1})$, and we use the simple domination $\mathbb{E}^{(v)}_{\theta(s)}[\Theta] \le v\theta(s)$ on T^*_{k-1}, and the domination $\mathbb{E}^{(v)}_{\theta(s)}[\Theta] \le \mathbb{E}^{(v)}_\infty[\Theta] = \sqrt{\pi v/2}$ on $\mathsf{T}_{k-1} \setminus \mathsf{T}^*_{k-1}$ to get

$$\begin{aligned} (2.23) \le \Bigg(&\frac{1}{\sqrt{\pi}} \int_0^\infty \frac{du}{L_{k-1}^3} \frac{L_{k-1}^2/2 + u}{\sqrt{u}}\, \mathrm{e}^{-u} \int_{\mathsf{T}^*_{k-1}} \ell(ds)\theta(s) \frac{u}{L_{k-1}^2/2 + u} \\ &+ \int_0^\infty \frac{du}{L_{k-1}^3} \frac{L_{k-1}^2/2 + u}{\sqrt{2u}}\, \mathrm{e}^{-u}\, h_{\emptyset,k-1} \frac{\sqrt{u}}{\sqrt{L_{k-1}^2/2 + u}}\Bigg)^2. \end{aligned}$$

The integrals can be computed, giving

$$\begin{aligned} (2.23) \le \Bigg(&\frac{1}{\sqrt{\pi} L_{k-1}^2} \int_0^\infty \sqrt{u}\,\mathrm{e}^{-u}\, du \int_{\mathsf{T}^*_{k-1}} \frac{\ell(ds)}{L_{k-1}} \theta(s) \\ &+ \int_0^\infty \frac{du}{\sqrt{2} L_{k-1}^2} \sqrt{1/2 + u/L_{k-1}^2}\, \mathrm{e}^{-u}\, h_{\emptyset,k-1}\Bigg)^2. \end{aligned}$$

On the other hand, the term $\mathbb{E}_\infty[(L_k^2 - L_{k-1}^2)^2|\mathcal{F}_{k-1}]$ appearing in the domination (2.22) can be expanded into

$$\mathbb{E}_\infty\left[\ell(\mathrm{B}_k)^4|\mathcal{F}_{k-1}\right] + 4L_{k-1}^2\mathbb{E}_\infty\left[\ell(\mathrm{B}_k)^2|\mathcal{F}_{k-1}\right] + 4L_{k-1}\mathbb{E}_\infty\left[\ell(\mathrm{B}_k)^3|\mathcal{F}_{k-1}\right]$$

Then, recall the density (2.17) of $\ell(\mathrm{B}_k)$ conditionally on $\mathcal{F}_{k-1}$. In the proof of Lemma A.3, we show that for any $\lambda > 0$, we have a.s.

$$\mathbb{E}_\infty\left[\ell\left(\mathrm{B}_k\right)^\lambda|\mathcal{F}_{k-1}\right] = \int r_{L_{k-1}}(dx)x^\lambda \le C_1 \cdot L_{k-1}^{-\lambda} + C_2 \cdot L_{k-1}^{-\lambda-2}$$

with C_1 and C_2 deterministic constants. Thus, $\mathbb{E}_\infty[(L_k^2 - L_{k-1}^2)^2|\mathcal{F}_{k-1}]$ is a.s. bounded by $F(L_{k-1})$, where F is a nonincreasing bounded nonnegative function. In the end, we get

$$\begin{aligned}
\mathbb{E}_\infty\left[H_k^2\mathbf{1}_{E_{k-1}}\right] &\le \mathbb{E}_\infty\Bigg[\Bigg(\frac{C}{L_{k-1}^2}\int_{\mathsf{T}_{k-1}^*}\theta(s)\frac{\ell(ds)}{L_{k-1}} \\
&\qquad + \int_0^\infty \frac{\mathrm{e}^{-u}\;du}{2L_{k-1}^2}\sqrt{1/2+u/L_{k-1}^2}h_{\emptyset,k}\Bigg)^2 F(L_{k-1})\mathbf{1}_{E_{k-1}}\Bigg] \\
&\le F(k^{2-4\varepsilon})\Bigg(C\cdot k^{-2+4\varepsilon}\mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}^*}\theta(s)\frac{\ell(ds)}{L_{k-1}}\right)^2\right] \\
&\qquad + C'\cdot k^{-2+4\varepsilon}\left(\int_0^\infty \mathrm{e}^{-u}\sqrt{1/2+u/k^{1-2\varepsilon}}\right)^2\mathbb{E}_\infty[h_{\emptyset,k}^2]\Bigg).
\end{aligned}$$

Hence, using the fact that $\int_{\mathsf{T}_{k-1}^*}\theta(s)\ell(ds)/L_{k-1}$ is bounded in L^2 (which is precisely Lemma A.1), we find that $\mathbb{E}_\infty[H_k^2\mathbf{1}_{E_{k-1}}] = O(k^{-2+4\varepsilon})$. Putting this together with the estimate on $\mathbb{E}_\infty[G_k^2\mathbf{1}_{E_{k-1}}]$, we get that $\mathbb{E}_\infty[(M_k - M_{k-1})^2\mathbf{1}_{E_{k-1}}] = O(k^{-7/4+7/2\varepsilon})$. If $\varepsilon < 1/14$,

$$\mathbb{E}_\infty[(M_k - M_{k-1})^2\mathbf{1}_{E_{k-1}}] = O(k^{-7/4+7/2\varepsilon}) = o(k^{-3/2}).$$

Hence, we get

$$\lim_{n\to\infty} n^{1/2}\sum_{k=n}^\infty \mathbb{E}_\infty\left[(M_k - M_{k-1})^2\mathbf{1}_{E_{k-1}}\right] = 0.$$

This shows that the random sequence $n^{1/4}\sum_{k=n}^\infty(M_k - M_{k-1})\mathbf{1}_{E_{k-1}}$ converges to 0 in L^2, hence in probability. But, since there a.s. exists $k_0 \ge 1$ such that $\mathbf{1}_{E_k} = 1$ for all $k \ge k_0$, the sequence $n^{1/4}\sum_{k=n}^\infty(M_k - M_{k-1})$ also converges to 0 in probability, which is what we wanted to prove. □

3. Proof of the main theorem

We can now turn to the proof of the actual convergence towards a nontrivial limit, in the asymptotic $n^{1/4}$. The main idea is to apply the Martingale Central Limit Theorem (Corollary 3.1 in Hall and Heyde (1980)) to

$$M_n^* = X_n^* - \int_{\mathsf{T}_n^*}\theta(s)\;\ell(ds).$$

We recall this theorem below for convenience:

Theorem (Hall and Heyde (1980)). Let $(M_n,\ n \geq 1)$ be a zero-mean square-integrable $(\mathcal{G}_n)$-martingale, and let η^2 be an a.s. finite random variable. Suppose that, for some sequence a_n increasing to $+\infty$, we have

(1) *(Asymptotic smallness)* For all $\varepsilon > 0$, we have the convergence in probability

$$\lim_{n\to\infty} a_n^{-2} \sum_{k=1}^{n} \mathbf{E}\left[(M_k - M_{k-1})^2 \mathbf{1}_{\{|M_k - M_{k-1}| > \varepsilon a_k\}} \middle| \mathcal{G}_{k-1} \right] = 0.$$

(2) *(Convergence of the conditional variance)* We have the convergence in probability

$$\lim_{n\to\infty} a_n^{-2} \sum_{k=1}^{n} \mathbf{E}\left[(M_k - M_{k-1})^2 | \mathcal{G}_{k-1} \right] = \eta^2.$$

Then, the sequence $(a_n^{-1} M_n,\ n \geq 1)$ converges in distribution to a random variable Z with characteristic function $\mathbf{E}[\exp(-\eta^2 t^2/2)]$.

However, M_n^* is not a martingale in the filtration $(\mathcal{F}_n,\ n \geq 1)$, because the $(n+1)$st branch B_{n+1} might be connected to T_n through a vertex on $[\![\emptyset, s_{\emptyset,n}]\!]$. In that case, $M_{n+1}^* - M_n^*$ has a nonnegative $\mathcal{F}_n$-measurable part, corresponding to the atoms on $[\![s_{\emptyset,n+1}, s_{\emptyset,n}]\!]$. For this reason, we will consider

$$\widehat{M}_n = \sum_{s \in \mathsf{T}_n \setminus \mathsf{T}_1} \mathbf{1}_{\{\theta(s-) > \theta(s)\}} - \int_{\mathsf{T}_n \setminus \mathsf{T}_1} \theta(s)\ \ell(ds), \quad n \geq 2$$

and $\widehat{M}_1 = 0$. The process $(\widehat{M}_n, n \geq 1)$ is a $(\mathcal{F}_n)$-martingale. It is actually more convenient to introduce the filtration $(\mathcal{G}_n,\ n \geq 1)$, defined by:

$$\mathcal{G}_n = \sigma(\{(\mathsf{T}_m,\ m \geq 1),\ (\theta(s),\ s \in \mathsf{T}_n)\}),$$

Notice that the branching point $s_{n+1} = \mathsf{B}_{n+1} \cap \mathsf{T}_n$, as well as $\ell(\mathsf{B}_{n+1})$ and $\theta(s_{n+1})$ are all $\mathcal{G}_n$-measurable. In this filtration, $\widehat{M}$ is also a martingale. Indeed, it is obvious that $\widehat{M}$ is $\mathcal{G}$-adapted. Furthermore, we have

$$\widehat{M}_{n+1} - \widehat{M}_n = \sum_{s \in \mathsf{B}_{n+1}} \mathbf{1}_{\{\theta(s-) > \theta(s)\}} - \int_{\mathsf{B}_{n+1}} \theta(s)\ \ell(ds),$$

which is, conditionally on $\mathcal{G}_n$, distributed as $N_{\ell(\mathsf{B}_{n+1})}$, where N is the martingale from (1.4) for a linear record process started at $\theta(s_{n+1})$. Thus, we find that $\mathbb{E}_\infty[\widehat{M}_{n+1} - \widehat{M}_n | \mathcal{G}_n] = 0$.

3.1. *Convergence of the conditional variance.* In order to get a convergence in distribution of $n^{-1/4}\widehat{M}_n$, we first need to compute the asymptotic variance of the martingale. This is done in the following proposition.

Proposition 3.1. *We have:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=2}^{n} \mathbb{E}_\infty \left[\left(\widehat{M}_k - \widehat{M}_{k-1} \right)^2 \middle| \mathcal{G}_{k-1} \right] = \sqrt{2}\Theta, \tag{3.1}$$

in probability.

Proof: Using the martingale from (1.5), in the present case of a linear record process started at $\theta(s_k)$, we easily get that, for $k \geq 2$,

$$\mathbb{E}_\infty\left[\left(\widehat{M}_k - \widehat{M}_{k-1}\right)^2 \Big| \mathcal{G}_{k-1}\right] = \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \Big| \mathcal{G}_{k-1}\right]. \tag{3.2}$$

A Law of Large Numbers argument will show that we have

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=2}^{n} \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \Big| \mathcal{G}_{k-1}\right] = \lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{\mathsf{T}_n^* \setminus [\![s_{\emptyset,n}, \mathsf{x}_1]\!]} \theta(s)\ \ell(ds). \tag{3.3}$$

We postpone the proof of this equality to the end of this section. Now, recall Proposition 4.13 in Abraham and Delmas (2013), which shows that a.s.

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \int_{\mathsf{T}_n^*} \theta(s)\ \ell(ds) = \sqrt{2}\Theta.$$

Since $\mathsf{T}_n \setminus \mathsf{B}_1 = \mathsf{T}_n^* \setminus [\![s_{\emptyset,n}, \mathsf{x}_1]\!]$, the convergence (3.1) will follow if we manage to prove that

$$S_n = \frac{1}{\sqrt{n}} \int_{[\![s_{\emptyset,n}, \mathsf{x}_1]\!]} \theta(s)\ \ell(ds)$$

converges in probability to 0. We will simply compute the first moment:

$$\begin{aligned}
\sqrt{n}\mathbb{E}_\infty[S_n] = \mathbb{E}_\infty\left[\int_{[\![s_{\emptyset,n}, \mathsf{x}_1]\!]} \theta(s)\ \ell(ds)\right] &= \mathbb{E}_\infty\left[\int_{h_{\emptyset,n}}^{L_1} \theta(s)\ ds\right] \\
&= \mathbb{E}_\infty\left[\int_0^{L_1 - h_{\emptyset,n}} \mathbf{E}_{\theta(s_{\emptyset,n})}[\theta(s)]\ ds\right],
\end{aligned}$$

by the Markov property of θ at $h_{\emptyset,n}$. We can compute this expectation using (1.3):

$$\begin{aligned}
\mathbb{E}_\infty\left[\int_0^{L_1 - h_{\emptyset,n}} \mathbf{E}_{\theta(s_{\emptyset,n})}[\theta(s)]\ ds\right] &= \mathbb{E}_\infty\left[\int_0^{L_1 - h_{\emptyset,n}} \frac{1 - \mathrm{e}^{-s\theta(s_{\emptyset,n})}}{s}\ ds\right] \\
&\leq \mathbb{E}_\infty\left[\int_0^{L_1} \frac{1}{s}(s\theta(s_{\emptyset,n}))^{1/4}\ ds\right] \\
&= 4\mathbb{E}_\infty\left[\theta(s_{\emptyset,n})^{1/4} L_1^{1/4}\right],
\end{aligned}$$

by the elementary inequality $1 - \exp(-t) \leq t^{1/4}$. The Cauchy-Schwarz inequality then gives the bound

$$\sqrt{n}\mathbb{E}_\infty[S_n] \leq C \cdot \mathbb{E}_\infty\left[\theta(s_{\emptyset,n})^{1/2}\right]^{1/2}. \tag{3.4}$$

As $\theta(s_{\emptyset,n})$ is, conditionally on $\mathscr{T}$, exponentially distributed with parameter $h_{\emptyset,n}$, we get

$$\mathbb{E}_\infty[S_n] \leq C \cdot n^{-1/2} \mathbb{E}_\infty[h_{\emptyset,n}^{-1/2}]^{1/2},$$

which converges to 0 as $n \to \infty$ by (1.9), which shows (3.1).

We still have to show (3.3) to end the proof. The process

$$\left(Q_n = \sum_{k=2}^{n} \int_{\mathsf{B}_k} \theta(s)\ell(ds) - \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ell(ds) \Big| \mathcal{G}_{k-1}\right],\ n \geq 1\right) \tag{3.5}$$

is a $\mathcal{G}$-martingale. We will write

$$\langle Q\rangle_n = \sum_{k=1}^{n} \mathbb{E}_\infty \left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \right)^2 \middle| \mathcal{G}_{k-1} \right] - \mathbb{E}_\infty \left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1} \right]^2 \tag{3.6}$$

for its quadratic variation process. Conditionally on $\mathcal{G}_{k-1}$, $(\theta(s),\ s \in \mathsf{B}_k)$ is distributed as a linear record process started from $\theta(s_k)$. Hence, using (1.9) and (1.3), we get:

$$\mathbb{E}_\infty \left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1} \right] = \mathbf{E}_{\theta(s_k)} \left[\int_0^{\ell(\mathsf{B}_k)} \theta(s)\ ds \right] = \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} \frac{1-\mathrm{e}^{-u}}{u}\ du. \tag{3.7}$$

Similarly, we have:

$$\begin{aligned} \mathbb{E}_\infty \left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \right)^2 \middle| \mathcal{G}_{k-1} \right] &= \mathbf{E}_{\theta(s_k)} \left[\left(\int_0^{\ell(\mathsf{B}_k)} \theta(s)\ ds \right)^2 \right] \\ &= 2 \cdot \mathbf{E}_{\theta(s_k)} \left[\int_0^{\ell(\mathsf{B}_k)} du \int_0^u dv\ \theta(u)\theta(v) \right]. \end{aligned}$$

The latter can be computed by applying the Markov property at u, as well as (1.3), giving

$$\begin{aligned} \mathbb{E}_\infty \left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \right)^2 \middle| \mathcal{G}_{k-1} \right] &= \frac{1}{\theta(s_k)} \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} \frac{1-\mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\ ds \\ &\quad + 2 \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} ds \int_0^s dt \frac{1}{s-t} \left(\frac{1-\mathrm{e}^{-t}}{t} - \frac{1-\mathrm{e}^{-s}}{s} \right). \end{aligned} \tag{3.8}$$

Now, putting (3.7) and (3.8) together, compensations occur, so that we get, after tedious computations:

$$\begin{aligned} \langle Q\rangle_n &= \sum_{k=1}^{n} \mathbb{E}_\infty \left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \right)^2 \middle| \mathcal{G}_{k-1} \right] - \mathbb{E}_\infty \left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1} \right]^2 \\ &= \sum_{k=1}^{n} \frac{2}{\theta(s_k)} \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} \frac{1-\mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\ ds \\ &\quad +2 \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} ds \int_0^s dt\ \frac{s\,\mathrm{e}^{-s} - t\,\mathrm{e}^{-t} - (s-t)\,\mathrm{e}^{-(s+t)}}{st(s-t)}. \end{aligned}$$

The term $s\,\mathrm{e}^{-s} - t\,\mathrm{e}^{-t} - (s-t)\,\mathrm{e}^{-(s+t)}$ being negative for $t < s$, we get

$$\begin{aligned} 0 \le \langle Q\rangle_n &\le \sum_{k=1}^{n} \frac{2}{\theta(s_k)} \int_0^{\theta(s_k)\ell(\mathsf{B}_k)} \frac{1-\mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\ ds \\ &\le \sum_{k=1}^{n} \frac{2}{\theta(s_k)} \theta(s_k)\ell(\mathsf{B}_k) = 2\sum_{k=1}^{n} \ell(\mathsf{B}_k), \end{aligned}$$

the second inequality coming from $(1-\mathrm{e}^{-s})/s - \mathrm{e}^{-s} \le 1$ if $s > 0$. Then, recall that by definition, $\sum_{k=1}^{n} \ell(\mathsf{B}_k) \le L_n$, and that L_n is the square root of a $\mathit{Gamma}(n,1)$-distributed variable (Lemma 4.8 in Abraham and Delmas (2013)). Thus, for any

$\gamma > 1/2$, we have

$$\frac{1}{n^\gamma}\mathbb{E}_\infty[\langle Q\rangle_n] \leq \frac{2}{n^\gamma}\mathbb{E}_\infty[L_n] \to 0 \tag{3.9}$$

Then, by the conditional Law of Large Numbers (Theorem 1.3.17 in Duflo (1997)), we get that $n^{-1/4-\varepsilon}Q_n$ converges a.s. to 0 for any $\varepsilon > 0$, which implies (3.3), hence ends the proof. □

3.2. *Asymptotic smallness.* We now turn to the proof of the asymptotic smallness of the sequence $(\widehat{M}_n,\ n \geq 1)$. In order to prove this, we will use a Liapounov-type criterion, which is sufficient to prove asymptotic negligibility.

Proposition 3.2. *We have the following convergence in probability:*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E}_\infty\left[(\widehat{M}_k - \widehat{M}_{k-1})^2 \mathbf{1}_{\{|\widehat{M}_k - \widehat{M}_{k-1}| > \varepsilon n^{1/4}\}} \middle| \mathcal{G}_{k-1}\right] = 0.$$

Proof: We use the standard inequality $\mathbf{1}_{\{|\widehat{M}_k - \widehat{M}_{k-1}| > \varepsilon n^{1/4}\}} \leq (\widehat{M}_k - \widehat{M}_{k-1})^2/\varepsilon^2\sqrt{n}$ to get that, for $\varepsilon > 0$:

$$\begin{aligned}\frac{1}{\sqrt{n}} \sum_{k=1}^n \mathbb{E}_\infty&\left[\left(\widehat{M}_k - \widehat{M}_{k-1}\right)^2 \mathbf{1}_{\{|\widehat{M}_k - \widehat{M}_{k-1}| > \varepsilon n^{1/4}\}} \middle| \mathcal{G}_{k-1}\right] \\ &\leq \frac{1}{\varepsilon^2 n}\sum_{k=1}^n \mathbb{E}_\infty\left[\left(\widehat{M}_k - \widehat{M}_{k-1}\right)^4 \middle| \mathcal{G}_{k-1}\right].\end{aligned}$$

Using the martingale from (1.6), we find that:

$$\begin{aligned}\frac{1}{\varepsilon^2 n}\sum_{k=2}^n \mathbb{E}_\infty\left[\left(\widehat{M}_k - \widehat{M}_{k-1}\right)^4 \middle| \mathcal{G}_{k-1}\right] &= \frac{3}{\varepsilon^2 n}\sum_{k=2}^n \mathbb{E}_\infty\left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds)\right)^2 \middle| \mathcal{G}_{k-1}\right] \\ &\quad + \frac{1}{\varepsilon^2 n}\sum_{k=2}^n \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1}\right].\end{aligned}$$

In this expression, the term $n^{-1}\sum_{k=2}^n \mathbb{E}[\int_{\mathsf{B}_k}\theta(s)\ell(ds)|\mathcal{G}_{k-1}]$ converges in probability to 0, according to (3.2) and Proposition 3.1. Furthermore, recall from (3.6) that

$$\begin{aligned}\frac{3}{\varepsilon^2 n}\sum_{k=1}^n \mathbb{E}_\infty&\left[\left(\int_{\mathsf{B}_k} \theta(s)\ \ell(ds)\right)^2 \middle| \mathcal{G}_{k-1}\right] \\ &= \frac{3\langle Q\rangle_n}{\varepsilon^2 n} + \frac{3}{\varepsilon^2 n}\sum_{k=1}^n \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1}\right]^2,\end{aligned}$$

where Q is the martingale defined in (3.5). The quadratic variation process $\langle Q\rangle_n/n$ converges in probability to 0 by (3.9). Also, applying Lemma A.5 to the sequence $a_k = \mathbb{E}_\infty[\int_{\mathsf{B}_k}\theta(s)\ell(ds)|\mathcal{G}_{k-1}]$, we find that

$$\frac{1}{n}\sum_{k=1}^n \mathbb{E}_\infty\left[\int_{\mathsf{B}_k} \theta(s)\ \ell(ds) \middle| \mathcal{G}_{k-1}\right]^2 = 0,$$

which ends the proof. □

Putting all the previous elements together, we can now prove Theorem 1.1.

Proof of Theorem 1.1: First, we write that

$$n^{1/4}\left(\frac{X_n^*}{\sqrt{2n}}-\Theta\right)=\frac{\widehat{M}_n}{\sqrt{2}n^{1/4}}+\frac{M_n^*-\widehat{M}_n}{\sqrt{2}n^{1/4}}+n^{1/4}\left(\frac{1}{\sqrt{2n}}\int_{\mathsf{T}_n^*}\theta(s)\ \ell(ds)-\Theta\right).$$

The convergence in distribution of $n^{-1/4}\widehat{M}_n$ towards a non-degenerate limit Z is a consequence of the Martingale Central Limit Theorem recalled at the beginning of this section with $a_n = n^{1/4}$, as well as the two Propositions 3.1 and 3.2. Furthermore, the limiting random variable Z is indeed distributed as announced:

$$\mathbb{E}_\infty\left[\mathrm{e}^{itZ}\right]=\mathbb{E}_\infty\left[\mathrm{e}^{-t^2\sqrt{2}\Theta/2}\right].$$

The term $e_n = M_n^* - \widehat{M}_n$ can be expressed as

$$e_n = M_n^*-\widehat{M}_n=\sum_{s\in[\![s_{\emptyset,n},\mathrm{x}_1]\!]}\mathbf{1}_{\{\theta(s-)>\theta(s)\}}-\int_{[\![s_{\emptyset,n},\mathrm{x}_1]\!]}\theta(s)\ \ell(ds).$$

Using the martingale (1.5) to compute its second moment, we get

$$\mathbb{E}_\infty\left[e_n^2\right]=\mathbb{E}_\infty\left[\int_{[\![s_{\emptyset,n},\mathrm{x}_1]\!]}\theta(s)\ \ell(ds)\right],$$

so that $n^{-1/4}(M_n^*-\widehat{M}_n)$ converges to 0 in L^2, hence in distribution as $n\to\infty$, by the previously used bound (3.4). Finally, Proposition 2.2 and Proposition 2.3 show that the term $((2n)^{-1/2}\int_{\mathsf{T}_n^*}\theta(s)\ \ell(ds)-\Theta)$ brings no contribution in the asymptotic $n^{1/4}$. This ends the proof. □

Remark 3.3. Note that, under our assumptions, since $\Theta > 0$, $\mathbb{P}_\infty$-a.s., we can actually prove that the convergence in distribution of $n^{-1/4}\widehat{M}_n$ is *mixing* (see Aldous and Eagleson (1978) for more details on mixing limit theorems). This implies in particular that we can obtain a standard normal limit by renormalizing by the random factor V_n, where V_n^2 is the conditional variance

$$V_n^2=\sum_{k=1}^n\mathbb{E}_\infty\left[\left(\widehat{M}_k-\widehat{M}_{k-1}\right)^2\Big|\mathcal{G}_{k-1}\right],$$

instead of the deterministic renormalization $n^{1/4}$. Corollary 3.2 in Hall and Heyde (1980) then shows that $V_n^{-1}\widehat{M}_n$ converges in distribution to a standard $\mathcal{N}(0,1)$ random variable.

Appendix A. Technical appendix

In this appendix, we will state and prove several lemmas that are used throughout the paper. First, we start by Lemma A.1, which is instrumental in the proof of Proposition 2.3.

Lemma A.1. $\mathbb{E}_\infty\left[\left(\int_{\mathsf{T}_{k-1}^*}\theta(s)\ell(ds)/L_{k-1}\right)^2\mathbf{1}_{E_{k-1}}\right]$ *is bounded as* $k\to\infty$.

Proof of Lemma A.1: Recall (2.7):

$$-R_{k-1}\le\mathbb{E}_\infty[\Theta|\mathcal{F}_{k-1}]-\frac{1}{L_{k-1}}\int_{\mathsf{T}_{k-1}^*}\theta(s)\ \ell(ds)\le V_{k-1}. \tag{A.1}$$

Therefore, we can write

$$\mathbb{E}_\infty\left[\left(\int_{\mathsf{T}^*_{k-1}}\frac{\ell(ds)}{L_{k-1}}\theta(s)-\mathbb{E}_\infty[\Theta|\mathcal{F}_{k-1}]\right)^2\mathbf{1}_{E_{k-1}}\right]$$
$$\le\mathbb{E}_\infty\left[(R_{k-1}\vee V_{k-1})^2\mathbf{1}_{E_{k-1}}\right]\le\mathbb{E}_\infty\left[R^2_{k-1}\mathbf{1}_{E_{k-1}}\right]+\mathbb{E}_\infty\left[V^2_{k-1}\mathbf{1}_{E_{k-1}}\right].$$

Using (2.10), we can see that, since $E_{k-1}\in\sigma(\{\mathsf{T}_n\})$,

$$\mathbb{E}_\infty\left[V^2_{k-1}\mathbf{1}_{E_{k-1}}\right]\le\mathbb{E}_\infty\left[\left(C\cdot h_{\emptyset,k-1}+\frac{\sqrt{\pi}}{\sqrt{2}L_{k-1}}\right)^2\mathbf{1}_{E_{k-1}}\right]$$
$$\le\mathbb{E}_\infty\left[(C\cdot h_{\emptyset,k-1}+\sqrt{\pi/2}/L_{k-1})^2\right].$$

Hence, as $h_{\emptyset,k-1}$ and L^{-1}_{k-1} are integrable and decrease to 0 a.s., $\mathbb{E}_\infty[V^2_{k-1}\mathbf{1}_{E_{k-1}}]$ converges to 0 by monotone convergence. As for $\mathbb{E}_\infty[R^2_{k-1}\mathbf{1}_{E_{k-1}}]$, we use the fact that, conditionally on T_{k-1}, $\theta(h_{\emptyset,k-1})$ is exponentially distributed with parameter $h_{\emptyset,k-1}$ to find

$$\mathbb{E}_\infty\left[R^2_{k-1}\mathbf{1}_{E_{k-1}}\right]=\mathbb{E}_\infty\left[\frac{1}{16}\,\mathrm{e}^{-L^2_{k-1}/2}\,h^{-4}_{\emptyset,k-1}\mathbf{1}_{E_{k-1}}\right]$$
$$\le\frac{1}{16}k^8\mathbb{E}_\infty\left[\mathrm{e}^{-L^2_{k-1}/2}\right],$$

which easily converges to 0 as $k\to\infty$. Hence, since $\mathbb{E}_\infty[\Theta|\mathcal{F}_{k-1}]$ converges in L^2 to Θ, it is of course L^2-bounded, so that $\mathbb{E}_\infty[(\int_{\mathsf{T}^*_{k-1}}\theta(s)\ell(ds)/L_{k-1})^2\mathbf{1}_{E_{k-1}}]$ is indeed bounded as $k\to\infty$, as announced. □

The following lemmata are purely analytic in nature, and their proofs are elementary, so we gather them here, for the reader's convenience. First, we prove some universal bounds on $F(q,t)=\mathbf{E}_q[(\int_0^t\theta(s)ds)^2]$.

Lemma A.2. *There exists $C_1,C_2,C_3,C_4>0$ such that*

$$F(q,t)\le C_1(qt)^{3/2}+C_2qt^2 \tag{A.2}$$
$$F(q,t)\le C_3\log^2(qt)+C_4q^{-1/2}t^{1/2} \tag{A.3}$$

Proof: First, we recall that, according to (3.8),

$$\begin{aligned}F(q,t)&=\mathbf{E}_q\left[\Big(\int_0^t\theta(s)\ ds\Big)^2\right]\\&=\frac{1}{q}\int_0^{qt}\frac{1-\mathrm{e}^{-s}}{s}-\mathrm{e}^{-s}\ \ ds+2\int_0^{qt}ds\int_0^s dt\ \frac{1}{s-t}\left(\frac{1-\mathrm{e}^{-t}}{t}-\frac{1-\mathrm{e}^{-s}}{s}\right)\\&:=\tilde{F}(q,t)+2\cdot G(qt).\end{aligned}$$

The two estimates (A.2) and (A.3) will come from an asymptotic analysis of

$$\tilde{F}(q,t)=\frac{1}{q}\int_0^{qt}\frac{1-\mathrm{e}^{-s}}{s}-\mathrm{e}^{-s}\ \ ds$$

and

$$G(qt)=\int_0^{qt}ds\int_0^s dt\ \frac{1}{s-t}\left(\frac{1-\mathrm{e}^{-t}}{t}-\frac{1-\mathrm{e}^{-s}}{s}\right).$$

Let us start with $\tilde F$. We have

$$\begin{aligned}\tilde F(q,t) &= \frac{1}{q}\int_0^{qt} \frac{1-\mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\, ds \\ &= \frac{1}{q}\lim_{\varepsilon\to 0}\left(\int_\varepsilon^{qt} \frac{1-\mathrm{e}^{-s}}{s} - \mathrm{e}^{-s}\, ds\right) \\ &= \frac{1}{q}\left(\log(qt) + \mathrm{e}^{-qt} - 1 + \int_{qt}^\infty \frac{\mathrm{e}^{-s}}{s}\, ds - \lim_{\varepsilon\to 0}\left(\log(\varepsilon) + \int_\varepsilon^\infty \frac{\mathrm{e}^{-s}}{s}\, ds\right)\right).\end{aligned}$$

Using integration by parts, we find that $\int_\varepsilon^\infty \exp(-s)/s\, ds = \int_\varepsilon^\infty \log(s)\,\mathrm{e}^{-s}\, ds - \log(\varepsilon)\,\mathrm{e}^{-\varepsilon}$, so that in the end

$$\tilde F(q,t) = \frac{1}{q}\left(\gamma + \log(qt) + \int_{qt}^\infty \frac{\mathrm{e}^{-s}}{s}\, ds + \mathrm{e}^{-qt} - 1\right),$$

where $\gamma = \int_0^\infty \log(u)\,\mathrm{e}^{-u}\, du$ is Euler's constant. It is elementary to check that the function $\gamma + \log(x) + \int_x^\infty \frac{\mathrm{e}^{-t}}{t}\, dt + \mathrm{e}^{-x} - 1$ is equivalent to $x^2/4$ at $x = 0$, and equivalent to $\log(x) = o(\sqrt{x})$ when $x \to \infty$. Since $\sqrt{x} = o(x^2)$ in the neighbourhood of $+\infty$ and $x^2 = o(\sqrt{x})$ in the neighbourhood of 0, by continuity, we can find constants C_2 and C_4 such that $\tilde F(q,t) \le C_2(qt)^2/q$ and such that $\tilde F(q,t) \le C_4(qt)^{1/2}/q$. Turning to the function G, we can write

$$\begin{aligned}G(x) &= \int_0^x ds \int_0^s dt \frac{1}{s-t}\left(\frac{1-\mathrm{e}^{-t}}{t} - \frac{1-\mathrm{e}^{-s}}{s}\right) \\ &= \int_0^1 du \int_0^u dv \frac{1}{u-v}\left(\frac{1-\mathrm{e}^{-xv}}{v} - \frac{1-\mathrm{e}^{-xu}}{u}\right),\end{aligned}$$

so that

$$G'(x) = \int_0^1 du \int_0^u dv \frac{1}{u-v}(\mathrm{e}^{-xv} - \mathrm{e}^{-xu}),$$

and that

$$G''(x) = \int_0^1 du \int_0^u dv \frac{1}{u-v}(u\,\mathrm{e}^{-xu} - v\,\mathrm{e}^{-xv}).$$

Thus, we have $G(0) = G'(0) = 0$ and $G''(0) = 1/2$. Since G is smooth, we get that $G(x) \sim x^2/4$ when $x \to 0$.

As far as the asymptotic $x \to \infty$ is concerned, we can express $G'(x)$ in terms of the exponential integral[1] function $Ei(x) = \int_{-\infty}^x \exp(t)/t\, dt$:

$$\begin{aligned}G'(x) &= \int_0^1 du \int_0^u \frac{dv}{u-v}(\mathrm{e}^{-xv} - \mathrm{e}^{-xu}) \\ &= \int_0^1 du\, \mathrm{e}^{-xu} \int_0^{xu} \frac{dv}{v}(\mathrm{e}^{v} - 1) \\ &= \int_0^1 du\, \mathrm{e}^{-xu}(Ei(xu) - \log(xu) - \gamma).\end{aligned}$$

[1] Note that this integral has to be taken in the sense of Cauchy's principal value.

When $x \to \infty$, we get

$$G'(x) \sim \int_0^1 du\, \mathrm{e}^{-xu}\, Ei(xu) = \frac{1}{x}\int_0^x du\, \mathrm{e}^{-t}\, Ei(t)$$
$$\sim \frac{\log x}{x}.$$

Integrating from 0 to x, we get $G(x) \sim \log^2 x = o(\sqrt{x})$ when $x \to \infty$. Again, $\sqrt{(}x) = o(x^2)$ in the neighbourhood of $+\infty$ and $x^2 = o(\sqrt{x})$ in the neighbourhood of 0, so that by continuity, there exist two constants C_1 and C_2 such that both $G(x) \le C_1 x^2$ and $G(x) \le C_2 x^{1/2}$. Thus, we get the two dominations (A.2) and (A.3). □

We now turn to a useful estimation of the moments of the distribution $r_a(dx)$ introduced in (2.17): $r_a(dx) = (a+x)\,\mathrm{e}^{-x^2/2-ax}\,\mathbf{1}_{(0,\infty)}(x)\; dx$.

Lemma A.3. *Let $\lambda > 0$. If $(a(n),\; n \ge 1)$ is some sequence in $\mathbb{R}_+$ increasing to $+\infty$, then, as $n \to \infty$, we have $\int_0^\infty r_{a(n)}(dx)x^\lambda = O(a(n)^{-\lambda})$.*

Proof: This is fairly easy: if $\lambda > 0$, we can write

$$\begin{aligned}\int_0^\infty r_{a(n)}(dx)x^\lambda &= \int_0^\infty x^\lambda (a(n)+x)\,\mathrm{e}^{-x^2/2-a(n)x}\; dx\\ &= \int_0^\infty \frac{u^\lambda}{a(n)^\lambda}\left(a(n)+\frac{u}{a(n)}\right)\mathrm{e}^{-u^2/(2a(n)^2)-u}\;\frac{du}{a(n)}\\ &\le \frac{1}{a(n)^\lambda}\int_0^\infty u^\lambda\,\mathrm{e}^{-u}\; du + \frac{1}{a(n)^{\lambda+2}}\int_0^\infty u^{\lambda+1}\,\mathrm{e}^{-u}\; du,\end{aligned}$$

which ends the proof. □

Lemma A.4. *For any $0 < q < \infty$ and any $v \ge 0$, we have*

$$\mathbb{E}_q^{(v)}[\Theta] \le \sqrt{\pi/2}\,\min(qv, \sqrt{v}).$$

Proof: We will use formula (36) from Abraham and Delmas (2013), stating that, in our context, if Y is a Rayleigh-distributed variable, then

$$\mathbb{E}_q^{(v)}[\Theta] = \sqrt{v}\int_0^{q\sqrt{v}} \mathbb{E}\left[\mathrm{e}^{-tY}\right]\; dt.$$

We simply expand the Laplace transform, giving

$$\begin{aligned}\mathbb{E}_q^{(v)}[\Theta] &= \sqrt{v}\int_0^{q\sqrt{v}}\int_0^\infty x\,\mathrm{e}^{-x^2/2}\,\mathrm{e}^{-tx}\; dx\; dt\\ &= \sqrt{v}\int_0^\infty \mathrm{e}^{-x^2/2}\left(1-\mathrm{e}^{-xq\sqrt{v}}\right)\; dx.\end{aligned}$$

Now, we use the obvious inequality $1-\exp(-x) \le \min(x,1)$, to get the desired domination, since $qv\int_0^\infty x\exp(-x^2/2) = qv$ and $\sqrt{v}\int_0^\infty \mathrm{e}^{-x^2/2}\,dx = \sqrt{\pi v/2}$. □

Finally, the next lemma is needed to prove the asymptotic smallness of the martingale $\widehat{M}_n$.

Lemma A.5. *Let* $(a_n, n \geq 1)$ *be a nonnegative sequence such that*

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k < \infty.$$

Then, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} a_k^2 = 0.$$

Proof: Let $s_n = n^{-1/2} \sum_{k=1}^{n} a_k$. Taking the difference $s_n - s_{n-1}$, we easily see that $n^{-1/2} a_n$ converges to 0. Then, if $\varepsilon > 0$, there exists $n_0 \geq 1$ such that for all $n \geq n_0$, $a_n < \varepsilon\sqrt{n}$. Thus, if $n \geq n_0$, we have

$$\begin{aligned} \sup_{k\leq n} a_k &\leq \sup_{k<n_0} a_k + \sup_{n_0\leq k\leq n} a_k \\ &\leq \sup_{k<n_0} a_k + \varepsilon\sqrt{n}, \end{aligned}$$

which proves that actually

$$\lim_{n\to\infty} \frac{\sup_{k\leq n} a_k}{\sqrt{n}} = 0.$$

Then, we simply write

$$\frac{1}{n} \sum_{k=1}^{n} a_k^2 \leq \left(\frac{\sup_{k\leq n} a_k}{\sqrt{n}}\right) \left(\frac{1}{\sqrt{n}} \sum_{k=1}^{n} a_k\right)$$

to conclude. □

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References

R. Abraham and J.-F. Delmas. Record process on the continuum random tree. *ALEA - Latin American Journal of Probability and Mathematical Statistics* **10**, 225–251 (2013). ALEA v10-10.

R. Abraham, J.-F. Delmas and P. Hoscheit. A note on the Gromov-Hausdorff-Prokhorov distance between (locally) compact metric measure spaces. *Electron. J. Probab.* **18**, no. 14, 21 (2013). MR3035742.

D. Aldous. The continuum random tree. I. *Ann. Probab.* **19** (1), 1–28 (1991). MR1085326.

D. Aldous. The continuum random tree. III. *Ann. Probab.* **21** (1), 248–289 (1993). MR1207226.

D. Aldous and J. Pitman. The standard additive coalescent. *Ann. Probab.* **26** (4), 1703–1726 (1998). MR1675063.

D. J. Aldous and G. K. Eagleson. On mixing and stability of limit theorems. *Ann. Probability* **6** (2), 325–331 (1978). MR0517416.

J. Bertoin. Self-similar fragmentations. *Ann. Inst. H. Poincaré Probab. Statist.* **38** (3), 319–340 (2002). MR1899456.

J. Bertoin. *Random fragmentation and coagulation processes*, volume 102 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-86728-3; 0-521-86728-2. MR2253162.

J. Bertoin and G. Miermont. The cut-tree of large galton-watson trees and the brownian crt. *ArXiv Mathematics e-prints* (2012). arXiv: 1201.4081.

R. Dong, C. Goldschmidt and J. B. Martin. Coagulation-fragmentation duality, Poisson-Dirichlet distributions and random recursive trees. *Ann. Appl. Probab.* **16** (4), 1733–1750 (2006). MR2288702.

M. Duflo. *Random iterative models*, volume 34 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin (1997). ISBN 3-540-57100-0. Translated from the 1990 French original by Stephen S. Wilson and revised by the author.MR1485774.

T. Duquesne and J.-F. Le Gall. Probabilistic and fractal aspects of Lévy trees. *Probab. Theory Related Fields* **131** (4), 553–603 (2005). MR2147221.

P. Hall and C. C. Heyde. *Martingale limit theory and its application.* Academic Press Inc. [Harcourt Brace Jovanovich Publishers], New York (1980). ISBN 0-12-319350-8. Probability and Mathematical Statistics. MR624435.

S. Janson. Random cutting and records in deterministic and random trees. *Random Structures Algorithms* **29** (2), 139–179 (2006). MR2245498.

S. Janson. Simply generated trees, conditioned Galton-Watson trees, random allocations and condensation. *Probab. Surv.* **9**, 103–252 (2012). MR2908619.

J.-F. Le Gall. The uniform random tree in a Brownian excursion. *Probab. Theory Related Fields* **96** (3), 369–383 (1993). MR1231930.

A. Meir and J. W. Moon. Cutting down random trees. *J. Austral. Math. Soc.* **11**, 313–324 (1970). MR0284370.

A. Panholzer. Cutting down very simple trees. *Quaest. Math.* **29** (2), 211–227 (2006). MR2233368.

J. Pitman. *Combinatorial stochastic processes*, volume 1875 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin (2006). ISBN 978-3-540-30990-1; 3-540-30990-X. Lectures from the 32nd Summer School on Probability Theory held in Saint-Flour, July 7–24, 2002, With a foreword by Jean Picard. MR2245368.

D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition (1999). ISBN 3-540-64325-7. MR1725357.